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Inverse Problems in Imaging

Example sheet **4** Presentation **30. November 2016**, **2-3pm**, **MR15**.

Exercise 1 (Convexity of a sum)
Let $\alpha, \beta \geq 0$ and $E, F \colon \mathcal{U} \to \mathbb{R}_\infty$ be two convex functions. Prove the following two statements.

(a) The sum of the two functions $\alpha E + \beta F \colon \mathcal{U} \to \mathbb{R}_\infty$ is convex

(b) If, in addition $\beta > 0$ and F is strictly convex, then $\alpha E + \beta F$ is strictly convex.

Exercise 2 (Convexity of data term)
Let $\mathcal{U}$ be a Banach space and and $\mathcal{V}$ a Hilbert space. Furthermore, let $K \in \mathcal{L}(\mathcal{U}, \mathcal{V}), f \in \mathcal{V}$ and $D \colon \mathcal{V} \to \mathbb{R}_\infty$ be defined as $D(u) := \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2$. Prove the following two statements.

(a) The data term D is convex.

(b) The data term D is strictly convex if and only if the operator K is injective.

Exercise 3 (Differentiation)
Let $\mathcal{U}$ be a Banach space and $E : \mathcal{U} \to \mathbb{R}$ be a convex functional that is Fréchet-differentiable in $u \in \mathcal{U}$. Then

$$\partial E(u) = \{E'(u)\} .$$

Exercise 4 (Exact recovery)
Let $u_\lambda \neq 0$ be a generalised singular vector satisfying $\lambda K^* K u_\lambda \in \partial J(u_\lambda)$ and $\|Ku_\lambda\|_{\mathcal{V}} = 1$, for some constant λ. Further let J be absolutely one-homogeneous, i.e. $J(cu) = |c| J(u)$ for all constants $c \in \mathbb{R}$.

(a) Compute λ.

(b) Argue why this definition is a generalisation of singular vectors.

(c) Show that $u_\alpha = \max\{\gamma - \lambda\alpha, 0\} u_\lambda$ is a global minimiser of $\Phi_{\alpha,f}$ as introduced in the lecture, for $f = \gamma K u_\lambda$.

(d) Let $\lambda_0 := \min_{\|Ku\|_{\mathcal{V}}=1} J(u)$. Prove that minimisers u_α of $\Phi_{\alpha,f}$ have a systematic bias of the form

$$\|Ku_\alpha - f\|_{\mathcal{V}} \geq \alpha\lambda_0$$

if $\|f\|_{\mathcal{V}} \geq \alpha\lambda_0$.

Please turn over!

Exercise 5 (Total variation regularised deconvolution)
The discrete two-dimensional convolution of a signal $u \in \mathbb{R}^{n_1 \times n_2}$ with a kernel $g \in \mathbb{R}^{(2r_1+1)\times(2r_2+1)}$ can be described via

$$f_{i_1,i_2} = (Cu)_{i_1,i_2} := \sum_{k_1=-r_1}^{r_1} \sum_{k_2=-r_2}^{r_2} u_{k_1,k_2}\, g_{i_1-k_1,i_2-k_2}\,, \tag{1}$$

for $i_1, r_i \in \{1, \ldots, n_1\}$, $i_2, r_2 \in \{1, \ldots, n_2\}$.

(a) **Regularised inverse problem**: Implement the following discrete deconvolution model

$$u_\alpha \in \arg\min_{u \in \mathbb{R}^n} \left\{ \frac{1}{2}\|Cu - f\|_2^2 + \alpha \max_{\substack{p \in \mathbb{R}^n \times \mathbb{R}^n \\ \||p\|_2\|_\infty \leq 1}} \langle \nabla u, p \rangle \right\} \tag{2}$$

via the primal-dual hybrid gradient method introduced in the lecture. Here C denotes the discrete convolution (1), $n = n_1 n_2$, and ∇ is a forward finite-difference approximation of the gradient.

(*) **Bonus exercise**: Implement the discrete two-dimensional convolution operator (with periodic boundary conditions) as a MATLAB©-class named `convolution`. An object `C` of this class should be initialised via the command `C = convolution(imsize, kernel);`. Here `imsize` denotes the image dimensions of the two-dimensional image to be convolved (n_1 and n_2 in (1)), and `kernel` is the convolution kernel. Moreover, the object should be able to compute the convolution via the call `f = C*u;`, where `f` and `u` are column vector representations of the convolved image f and the original image u. Further should it be possible to compute the adjoint (transpose) convolution via `u = C'*f;`.

Hint: Make use of the discrete convolution theorem that allows to diagonalise the convolution operator in the discrete Fourier transform domain.

(b) Take your favourite picture, load it into MATLAB© via `imread` and convert it into a gray-value image with `double`-precision intensity values. Either create an object of the class you have implemented in Exercise 5 (*) to convolve this image with a convolution kernel `kernel`, or use the MATLAB©-command `imfilter` (with appropriate boundary conditions) instead. Use for instance the MATLAB©-command `fspecial` to create a convolution kernel of your choice. Subsequently, create different noisy versions of your blurry image for different noise levels.

(c) Use your noisy, blurry images of Exercise 5 (b) as the input for your deconvolution algorithm of Exercise 5 (a). Compute reconstructions of (2) for different choices of α and visualise your results.

Exercise 6 (Bregman iteration)

With the help of the generalised Bregman distance we can define the following iterative regularisation scheme (also known as Bregman iteration)

$$\begin{aligned} u_\alpha^{k+1} &\in \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2 + \alpha D_J^{p_\alpha^k}(u, u_\alpha^k) \right\}, \\ p_\alpha^{k+1} &= p_\alpha^k + \frac{1}{\alpha}K^*(f - Ku_\alpha^{k+1}) \end{aligned} \tag{3}$$

for $\alpha > 0$ and with $u_\alpha^0 = p_\alpha^0 = 0$ and $p_\alpha^k \in \partial J(u_\alpha^k)$ for all $k \in \mathbb{N}$.

(a) Show that the iterates (3) satisfy $\|Ku_\alpha^{k+1} - f\|_{\mathcal{V}} \leq \|Ku_\alpha^k - f\|_{\mathcal{V}}$.

(b) Show that iteration (3) can also be written as

$$\begin{aligned} u_\alpha^{k+1} &\in \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2}\|Ku - f_\alpha^k\|_{\mathcal{V}}^2 + \alpha J(u) \right\}; \\ f_\alpha^{k+1} &= f_\alpha^k + f - Ku_\alpha^{k+1}, \end{aligned}$$

for $f_\alpha^0 = f$.

(c) Show that if there exists a k_* such that $u_\alpha^{k_*}$ satisfies $Ku_\alpha^{k_*} = f$, then $u_\alpha^{k_*}$ is a J-minimising solution (of all elements in the set $\{u \mid Ku = f\}$).

Hint: you can make use of Exercise 6 (b).

(d) Argue similar to Exercise 1 d) of Exercise Sheet 3 why in case of noisy data the Bregman iteration together with Morozov's discrepancy principle as a stopping criterion is a useful strategy to find J-minimising solutions. Give also an estimate for the parameter η in the discrepancy principle.

(e) Verify the estimate

$$D_J^{p_\alpha^k}(u^\dagger, u_\alpha^k) \leq \frac{\alpha\|w\|_{\mathcal{V}}^2}{2k}$$

for $k \in \mathbb{N} \setminus \{1\}$, exact data ($\delta = 0$) and the existence of an element w such that the source condition $K^*w \in \partial J(u^\dagger)$ is satisfied.

(f) Implement the Bregman iteration (3) for the deconvolution problem (2) of Exercise 5. Use the discrepancy principle as a stopping criterion, and compute reconstructions for different choices of α.

Hint: you can make use of Exercise 6 (b).