
1a. Show, by exhibiting the coordinate charts, that the real projective space $\mathbb{RP}^n$ is a manifold. Show that $\mathbb{RP}^n$ may be regarded as the n–sphere S^n with antipodal points identified. Prove that $\mathbb{RP}^3 \equiv SO(3)$.

1b. Consider three one–parameter groups of transformations of $\mathbb{R}$

$$x \to x + \varepsilon_1, \quad x \to e^{\varepsilon_2} x, \quad x \to \frac{x}{1 - \varepsilon_3 x},$$

and find the vector fields V_1, V_2, V_3 generating these groups. Deduce that these vector fields generate a three-parameter group of transformations

$$x \to \frac{ax + b}{cx + d}, \qquad ad - bc \neq 0.$$

Show that the vector fields V_α generate the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ and thus deduce that $\mathfrak{sl}(2, \mathbb{R})$ is a subalgebra of the infinite dimensional Lie algebra vect(Σ) of vector fields on $\Sigma = \mathbb{R}$. Find all other finite dimensional subalgebras of vect(Σ).

2. Let $\hat{T} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}^*_p(\mathcal{M})$ be a linear map. Show that one can define a tensor T of type $(0, 2)$ as the map $T : (X, Y) \mapsto (\hat{T}(Y))(X)$. Similarly show that a linear map $\mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$ defines a tensor of type $(1, 1)$. What tensor δ arises from the identity map?

3. Let V^{ab} be an arbitrary $\binom{2}{0}$ tensor, and let S_{ab}, A_{ab} be symmetric and antisymmetric $\binom{0}{2}$ tensors, i.e., $S_{ab} = S_{ba}$, $A_{ab} = -A_{ba}$. Show that $V^{ab} S_{ab} = V^{(ab)} S_{ab}$ and $V^{ab} A_{ab} = V^{[ab]} A_{ab}$.

4. You are given a $\binom{2}{0}$ tensor K. Working first in some basis devise a criterion to test whether it is the *direct product* of two vectors A, B, i.e., $K^{ab} = A^a B^b$. (You can, but do not need to, use determinants.) Can you express the test in a manifestly basis-invariant manner?
Show that the general $\binom{2}{0}$ tensor in n dimensions cannot be written as a direct product, but can be expressed as a sum of many direct products.

5. Let $\mathcal{M}$ be a manifold and $f : \mathcal{M} \to \boldsymbol{R}$ be a smooth function such that $\mathrm{d}f = 0$ at some point $p \in \mathcal{M}$. Let $\{x^\mu\}$ be a coordinate chart defined in a neighbourhood of p. Define

$$F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \; .$$

By considering the transformation law for components show that $F_{\mu\nu}$ defines a $\binom{0}{2}$ tensor, the *Hessian* of f at p. Construct also a coordinate-free definition and demonstrate its tensorial properties.

6. Let g_{ab} be a $\binom{0}{2}$ tensor. In a basis, one can regard the components $g_{\mu\nu}$ as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{\mu\nu})$. How does g transform under a change of basis?

7. Let $\{e_\mu\}$ be a basis for vectors and set

$$[e_\mu, e_\nu] = \gamma^\rho{}_{\mu\nu}\, e_\rho.$$

(The $\gamma^\rho{}_{\mu\nu}$ are the *commutator components*.) Let $\{\omega^\mu\}$ be the dual basis of covectors. Let $\{x^\alpha\}$ be coordinates (here we use early Greek indices α, β, ... to denote coordinate basis components and distinguish them from components in the generic bases denoted by late Greek indices μ, ν, ...). Expanding e_μ and ω^μ in the coordinate basis gives

$$e_\mu = e_\mu{}^\alpha \frac{\partial}{\partial x^\alpha}, \qquad \omega^\mu = \omega^\mu{}_\alpha \, \mathrm{d}x^\alpha ,$$

where $e_\mu{}^\alpha \omega^\nu{}_\alpha = \delta_\mu{}^\nu$. Show first that

$$e_\mu{}^\alpha \frac{\partial e_\nu{}^\beta}{\partial x^\alpha} - e_\nu{}^\alpha \frac{\partial e_\mu{}^\beta}{\partial x^\alpha} = \gamma^\rho{}_{\mu\nu}\, e_\rho{}^\beta ,$$

and deduce that

$$e_\mu{}^\alpha e_\nu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} - e_\nu{}^\alpha e_\mu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} = -\gamma^\sigma{}_{\mu\nu} ,$$

and finally that

$$\frac{\partial \omega^\sigma{}_\delta}{\partial x^\gamma} - \frac{\partial \omega^\sigma{}_\gamma}{\partial x^\delta} = -\gamma^\sigma{}_{\mu\nu}\, \omega^\mu{}_\gamma\, \omega^\nu{}_\delta . \qquad (\dagger)$$

In certain circumstances there may exist coordinates $\{y^\mu\}$ such that

$$\omega^\mu = \mathrm{d}y^\mu, \qquad e_\mu = \frac{\partial}{\partial y^\mu} .$$

(We would then say that the bases are *coordinate induced*.) Show that if the bases are coordinate induced then $[e_\mu, e_\nu] = 0$, $\forall \mu$, ν. Use (†) to show also the converse, i.e. if $[e_\mu, e_\nu] = 0$ $\forall \mu$, ν then the basis is coordinate induced. Deduce that bases being coordinate induced is equivalent to vanishing commutator components.

8. In inertial frame coordinates, the metric of Minkowski spacetime is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

(a) Show that if we replace (x, y, z) with spherical polar coordinates (r, θ, ϕ) defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \tan\phi = \frac{y}{x} ,$$

then the metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(b) Find the components of the metric and inverse metric in "rotating coordinates" defined by

$$\tilde{t} = t, \quad \tilde{x} = \sqrt{x^2 + y^2}\, \cos(\phi - \omega t), \quad \tilde{y} = \sqrt{x^2 + y^2}\, \sin(\phi - \omega t), \quad \tilde{z} = z ,$$

where $\tan\phi = y/x$.

9. The *Schwarzschild metric* in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \,.$$

Write out the components of the geodesic equation for this metric and hence determine the Christoffel symbols. Note that the Lagrangian from which geodesics are obtained is independent of t and ϕ. What consequence does this have?

10. Obtain the form of the general timelike geodesic in a two-dimensional spacetime with metric

$$ds^2 = t^{-2}(-dt^2 + dx^2) \,.$$

Hint: You should use the symmetries of the Lagrangian, and you will probably find the following indefinite integrals useful:

$$\int \frac{dt}{t\sqrt{1 + C^2t^2}} = \frac{1}{2} \ln \left(\frac{\sqrt{1 + C^2t^2} - 1}{\sqrt{1 + C^2t^2} + 1} \right) \,, \quad \int \frac{ds}{\sinh^2 s} = -\coth s \,,$$