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Anti-self-dual four-manifolds with a parallel real spinor

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Anti-self-dual metrics in the $(++--)$ signature that admit a covariantly constant real spinor are studied. It is shown that finding such metrics reduces to solving a fourth-order integrable partial differential equation (PDE), and some examples are given. The corresponding twistor space is characterized by existence of a preferred non-zero real section of $\kappa^{-1/4}$, where κ is the canonical line bundle of the twistor space. It is demonstrated that if the parallel spinor is preserved by a Killing vector, then the fourth-order PDE reduces to the dispersionless Kadomtsev–Petviashvili equation and its linearization. Einstein–Weyl structures on the space of trajectories of the symmetry are characterized by the existence of a parallel weighted null vector.

Keywords: twistor theory; differential geometry; integrable systems

1. Introduction

Constraints on a (pseudo) Riemannian geometry imposed by the existence of a parallel spinor essentially depend on the properties of the Clifford algebra and the spin group associated with the metric. There has been an interest in such geometries in pure mathematics because they extend a list of holonomy groups of Riemannian manifolds (where the existence of a parallel spinor implies Ricci flatness). Pseudo-Riemannian metrics in various dimensions with a covariantly constant spinor have also attracted a lot of attention in physics, as such spinors play a central role in supersymmetry.

Bryant (2000) analysed all cases up to six dimensions, together with some higher-dimensional examples of particular interest. In this paper I shall concentrate on the four-dimensional case.

Let $(\mathcal{M}, g)$ be a (pseudo) Riemannian spin four-manifold. Therefore, there exist complex two-dimensional vector bundles $S_\pm$ (spin-bundles) equipped with parallel symplectic structures $\varepsilon_\pm$ such that

(i) $\mathbb{C} \otimes T\mathcal{M} \cong S_+ \otimes S_-$ is a canonical bundle isomorphism;

(ii) $g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon_+(v_1, v_2)\varepsilon_-(w_1, w_2)$ for $v_1, v_2 \in \Gamma(S_+)$ and $w_1, w_2 \in \Gamma(S_-)$.

I shall assume that there exists a spinor ι, parallel with respect to a Levi-Civita connection ∇ of g,

$$\iota = (p, q) \in \Gamma(S_+), \quad \nabla\iota = 0.$$

There are three possible situations, depending on the signature of the metric.

(i) In the Lorentzian signature $(+---)$,

$$\mathrm{Spin}(3,1) = SL(2,\mathbb{C}), \quad \iota = (p,q) \to \bar{\iota} = (\bar{p},\bar{q}) \in \Gamma(S_-), \quad \nabla\bar{\iota} = 0.$$

Therefore, $l = \iota\otimes\bar{\iota}$ is a parallel null vector. This condition has been extensively studied in general relativity (Kramer *et al*. 1980): there exist two real functions u, v and one complex function ξ such that

$$g = \mathrm{d}u\mathrm{d}v - \mathrm{d}\xi\mathrm{d}\bar{\xi} + H(u,\xi,\bar{\xi})\,\mathrm{d}u^2.$$

The Ricci flat condition implies that $H(u,\xi,\bar{\xi}) = \mathrm{Re}(f(u,\xi))$, where f is holomorphic in ξ. These solutions are known as *p.p waves*. Analysis of curvature invariants shows that the function $H(u,\xi,\bar{\xi})$ cannot be eliminated by a coordinate transformation.

(ii) In the Euclidean signature $(++++)$,

$$\mathrm{Spin}(4,0) = SU(2)\times\widetilde{SU}(2), \quad \iota = (p,q) \to \bar{\iota} = (\bar{q},-\bar{p}) \in \Gamma(S_+), \quad \nabla\bar{\iota} = 0.$$

A spinor and its complex conjugate form a basis of a spin space S_+. A four-dimensional Riemannian manifold that admits a covariantly constant spinor must therefore be hyper-Kähler. Hyper-Kähler four-manifolds have been much studied for the last 25 years; see Dancer (2000) and the references therein.

(iii) In the split signature $(++--)$ (also called ultra-hyperbolic, Kleinian or neutral),

$$\mathrm{Spin}(2,2) = SL(2,\mathbb{R})\times\widetilde{SL}(2,\mathbb{R}), \quad \iota = (p,q) \to \bar{\iota} = (\bar{p},\bar{q}) \in \Gamma(S_+), \quad \nabla\bar{\iota} = 0,$$

and the representation space of the spin group splits into a direct sum of two real two-dimensional spin spaces S_+ and S_-. The conjugation of spinors is involutive and maps each spin space onto itself, and there exists an invariant notion of *real spinors*.

One can therefore look for $(++--)$ metrics with a parallel real spinor (which we choose to be $\iota\in\Gamma(S_+)$). These metrics do not have to be Ricci flat. The resulting geometry will be studied in the rest of this paper. The isomorphism ${\Lambda^2}_+(\mathcal{M}) \cong \mathrm{Sym}^2(S_+)$ between the bundle of self-dual two-forms and the symmetric tensor product of two spin bundles implies that the real self-dual two-form $\Sigma = \iota\otimes\iota\otimes\varepsilon_+$ is covariantly constant and null (i.e. $\Sigma\wedge\Sigma = 0$), which motivates the following definition.

Definition 1.1. A null-Kähler structure on a four-manifold consists of a metric of signature $(++--)$ and a real spinor field parallel with respect to this inner product. A null-Kähler structure is anti-self-dual (ASD) if the self-dual part of the Weyl spinor vanishes.

The ASD condition on null-Kähler structures is worth studying for at least two reasons. Firstly, four-dimensional vacuum metrics in signature $(++--)$ appeared in describing the bosonic sector of the $N = 2$ superstring (Barrett *et al*. 1994; Ooguri & Vafa 1990). Secondly, many integrable systems in dimensions two and three arise (together with their twistor description) as symmetry reductions of anti-self-duality equations on $(++--)$ background (Mason & Woodhouse 1996; Ward 1985).

The first aspect will be not discussed in the present paper, but I shall reveal some connections with integrable systems in §§ 3 and 5.

In the next section, the ASD null-Kähler condition will be related to Einstein–Maxwell equations. In § 3 the ASD null-Kähler condition will be reduced to a single fourth-order integrable partial differential equation (PDE). Explicit solutions to these equations will provide some examples of ASD null-Kähler structures. The resulting twistor theory will be described in § 4. The existence of a parallel real spinor will be characterized by a real structure preserving a preferred non-zero section of $\kappa^{-1/4}$, where κ is the canonical line bundle of the twistor space. In § 5 it will be shown that ASD null-Kähler structures with a symmetry that preserves the parallel spinor are locally given by solutions to the dispersionless Kadomtsev–Petviashvili equation and its linearizations (Dunajski 2000). Einstein–Weyl (EW) structures on the space of trajectories of the symmetry will be characterized by the existence of a parallel weighted null vector. The two-component spinor notation will used in the paper. The spin spaces S_- and S_+ will be denoted by S^A and $S^{A'}$, respectively.

From now on, the parallel real spinor and the corresponding null-Kähler two-form will be denoted by $\iota^{A'} \in \Gamma(S^{A'})$ and $\Sigma^{0'0'} = \iota_{A'}\iota_{B'}\Sigma^{A'B'} \in \Lambda^2{}_+(\mathcal{M})$. The notation is summarized in the appendix.

2. Null-Kähler metrics in four dimensions

It is well known (Pontecorvo 1992) that Kähler four-manifolds with vanishing scalar curvature are necessarily ASD. This is not true for scalar-flat null-Kähler four-manifolds. Instead, one has the following result.

Proposition 2.1. *Let $\iota_{A'}$ be a parallel real spinor on a four-dimensional, ultra-hyperbolic manifold. Then the scalar curvature vanishes, the Ricci tensor is null and the self-dual Weyl spinor is given by*

$$C_{A'B'C'D'} = c\iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'} \tag{2.1}$$

for some function c such that $\iota^{A'}\nabla_{AA'}c = 0$.

Proof. Vanishing of the scalar curvature follows from the second Ricci identity given by (A 5). The first identity (A 4) implies that $\Phi_{ABA'B'}\iota^{B'} = 0$, and so $\Phi_{ABA'B'} = F_{AB}\iota_{A'}\iota_{B'}$ for some F_{AB}. The formula (2.1) is a direct consequence of (A 5) and (A 6) applied to a covariantly constant spinor. ∎

Now I shall show that ASD null-Kähler metrics can be viewed as solutions to Einstein equations with electromagnetic stress-energy tensors. The ASD part of the Maxwell field is given by the Ricci form, and the self-dual (SD) part is given by the null-Kähler form.

Proposition 2.2. *There is a one-to-one correspondence between ASD metrics with a constant real spinor and ASD Einstein–Maxwell spaces for which the SD part of Maxwell field is null and covariantly constant.*

Proof. Let $(\mathcal{M}, g)$ be an ASD manifold with a covariantly constant spinor $\iota_{A'}$. Proposition 2.1 implies that

$$\Phi_{ABA'B'} = F_{AB}\iota_{A'}\iota_{B'}, \tag{2.2}$$

and the spinor Bianchi identities (A 6) yield $\nabla^{AA'}F_{AB} = 0$. Therefore,

$$F_{ab} = F_{AB}\varepsilon_{A'B'} + \iota_{A'}\iota_{B'}\varepsilon_{AB} \tag{2.3}$$

is a Maxwell field, and the formula (2.2) can be read off as Einstein equations with a Maxwell stress energy tensor

$$T_{ab} = \tfrac{1}{2}(\tfrac{1}{4}g_{ab}F_{cd}F^{cd} - F_{ac}F_b{}^c) = F_{AB}\iota_{A'}\iota_{B'}.$$

Conversely, consider a Maxwell field F_{ab} (2.3) on an ASD background, such that its SD part is null and constant. The Maxwell equations give

$$\nabla^A_{A'}F_{AB} = 0, \qquad \nabla_{AA'}\iota_{B'} = 0$$

and the Einstein equations with the stress energy tensor $T_{ab} = F_{AB}\iota_{A'}\iota_{B'}$ yield (2.2). ∎

The following result will be used in §§ 3 and 4.

Proposition 2.3. *Let $\Sigma^{A'B'} = (\Sigma^{0'0'}, \Sigma^{0'1'}, \Sigma^{1'1'})$ be a basis of normalized real SD two-forms on an ASD scalar-flat manifold such that*

$$\mathrm{d}(\iota_{A'}\iota_{B'}\Sigma^{A'B'}) = \mathrm{d}(o_{A'}\iota_{B'}\Sigma^{A'B'}) = 0. \tag{2.4}$$

Then there exists a covariantly constant real section of $S_{A'}$. Conversely, let $\iota_{A'}$ be a covariantly constant section of $S_{A'}$ on an ASD four-manifold. Then it is possible to find another section $o_{A'}$ such that $(o_{A'}, \iota_{A'})$ forms a normalized spin frame, and equations (2.4) hold.

Proof. Let $(o_{A'}, \iota_{A'})$ be a normalized spin basis. The covariant derivatives of the basis can be expressed as

$$\nabla_a\iota_{B'} = U_a\iota_{B'} + V_a o_{B'}, \qquad \nabla_a o_{B'} = W_a\iota_{B'} - U_a o_{B'}.$$

The first condition in (2.4) can be rewritten as $\nabla_A{}^{A'}(\iota_{A'}\iota_{B'}) = 0$, which implies

$$V_a = 2U_{AB'}\iota^{B'}\iota_{A'}.$$

The second condition in (2.4) yields $U_{AA'} = \alpha_A\iota_{A'}$, $W_{AA'} = \beta_A\iota_{A'}$ for some α_A, β_A. Therefore,

$$\nabla_{AA'}\iota_{B'} = \alpha_A\iota_{A'}\iota_{B'}. \tag{2.5}$$

Contracting the right-hand side of the above equation with $\nabla^A{}_{C'}$ and symmetrizing over $(A'B')$ gives 0, since g is ASD and scalar-flat. As a consequence, $\nabla^A{}_{(C'}[\iota_{A')}\iota_{B'}\alpha_A] = 0$, which gives $\nabla^A{}_{A'}\alpha_A = 0$. Consider the real spinor $\hat{\iota}_{A'} := \iota_{A'}\exp f$, where $\iota^{A'}\nabla_{AA'}f = 0$ in order to preserve $\mathrm{d}\Sigma^{0'0'} = 0$. Integrability conditions for $\nabla_{AA'}f = \alpha_A\iota_{A'}$ are satisfied, as α_A solves the neutrino equation. Therefore, we can find f for each α_A, and equation (2.5) implies that $\hat{\iota}_{A'}$ is covariantly constant.

Converse. Now assume $\nabla_{AA'}\iota_{B'}=0$. Consider $\hat{o}_{A'}$ such that $\hat{o}_{A'}\iota^{A'}=1$. The normalization condition implies $\nabla_{AA'}\hat{o}_{B'}=\gamma_A\iota_{A'}\iota_{B'}+\rho_A\hat{o}_{A'}\iota_{B'}$ for some γ_A, ρ_A. Contracting with $\nabla^A{}_{C'}$ and using Ricci identities (A 4) yields $\nabla^A{}_{A'}\rho_A=0$, and so $\rho_A=\iota^{A'}\nabla_{AA'}\phi$. Therefore, in the null-rotated normalized spin frame $\iota_{A'}$, $\hat{o}_{A'}-\phi\iota_{A'}$, we have $\rho_A=0$. Therefore,

$$\nabla_A{}^{A'}(o_{A'}\iota_{B'}+o_{B'}\iota_{A'})=2W_A{}^{A'}\iota_{A'}\iota_{B'}=\rho_A\iota_{B'}=0$$

and $\mathrm{d}\Sigma^{0'1'}=0$. ∎

3. The ASD null-Kähler condition as an integrable system

I shall now construct a local coordinate system adapted to the parallel spinor, and reduce the ASD null-Kähler condition to a pair of coupled PDEs. Integrability of these PDEs will be established using the Lax formulation (i.e. showing that they arise as the integrability conditions to an overdetermined system of linear equations).

Let $S^{A'}=\mathcal{M}\times\mathbb{C}^2$ be the bundle of complex primed spinors. The natural context for introducing the Lax pair is the geometry of the projective primed spin bundle (also called the correspondence space) $\mathcal{F}=\mathbb{P}(S^{A'})=\mathcal{M}\times\mathbb{CP}^1$. It is coordinatized by (x^a,λ), where x^a denotes the coordinates on $\mathcal{M}$ and λ is the coordinate on $\mathbb{CP}^1$ that parametrizes the α surfaces through x in $\mathcal{M}$. We relate the fibre coordinates $\pi^{A'}$ on $S^{A'}$ to λ by $\lambda=\pi_{0'}/\pi_{1'}$.

Let $\nabla_{AA'}$ be a null tetrad of vector fields for the metric g on $\mathcal{M}$ and let $\Gamma_{AA'B'C'}$ be the components of the spin connection in the associated spin frame. A horizontal lift of $\nabla_{AA'}$ to $S_{A'}$ is given by

$$\tilde{\nabla}_{AA'}=\nabla_{AA'}+\Gamma_{AA'B'C'}\pi^{B'}\frac{\partial}{\partial\pi_{C'}}.$$

Its horizontality implies $\tilde{\nabla}_{AA'}\pi_{B'}=0$.

The space $\mathcal{F}$ possesses a natural two-dimensional distribution called the twistor distribution, or Lax pair to emphasize the analogy with integrable systems. The Lax pair arises as the image under the projection $TS^{A'}\to T\mathcal{F}$ of the distribution spanned by $L_A=\pi^{A'}\tilde{\nabla}_{AA'}$, and is given by

$$L_0=\nabla_{00'}-\lambda\nabla_{01'}+l_0\partial_\lambda,\qquad L_0=\nabla_{10'}-\lambda\nabla_{11'}+l_1\partial_\lambda,\tag{3.1}$$

where $l_A=\Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\pi^{C'}$ are cubic polynomials in λ (note that $\pi_{1'}=1$ in these formulae).

Theorem 3.1 (cf. Penrose 1976). *The twistor distribution on $S_{A'}$ given by (3.1) is integrable if and only if the Weyl curvature of g is ASD, i.e. $C_{A'B'C'D'}=0$.*

We are now ready to reformulate the ASD null-Kähler condition as an integrable system.

Theorem 3.2. *Real coordinates (w,z,x,y) can be chosen such that all ASD null-Kähler metrics are locally given by*

$$g=\mathrm{d}w\mathrm{d}x+\mathrm{d}z\mathrm{d}y-\Theta_{xx}\,\mathrm{d}z^2-\Theta_{yy}\,\mathrm{d}w^2+2\Theta_{xy}\,\mathrm{d}w\mathrm{d}z,\tag{3.2}$$

where $\Theta(w,z,x,y)$ is a solution to a fourth-order PDE (which we write as a system of two second-order PDEs),

$$\Theta_{wx}+\Theta_{zy}+\Theta_{xx}\Theta_{yy}-\Theta_{xy}^2=f, \tag{3.3}$$

$$\Box f=f_{xw}+f_{yz}+\Theta_{yy}f_{xx}+\Theta_{xx}f_{yy}-2\Theta_{xy}f_{xy}=0. \tag{3.4}$$

Moreover, equations (3.3), (3.4) arise as an integrability condition for the linear system $L_0\Psi=L_1\Psi=0$, where $\Psi=\Psi(w,z,x,y,\lambda)$ and

$$\left.\begin{aligned} L_0&=(\partial_w-\Theta_{xy}\partial_y+\Theta_{yy}\partial_x)-\lambda\partial_y+f_y\partial_\lambda,\\ L_1&=(\partial_z+\Theta_{xx}\partial_y-\Theta_{xy}\partial_x)+\lambda\partial_x-f_x\partial_\lambda.\end{aligned}\right\} \tag{3.5}$$

Proof. Let $e^{AA'}$ be a tetrad of real independent one-forms. The parallel spinor $\iota_{A'}$ enables us to choose coordinates $w^A=(w,z)$ such that $e^{A0'}=-\iota_{A'}e^{AA'}=\mathrm{d}w^A$. Proposition 2.3 implies that we can choose $o_{A'}$ such that $o_{A'}\iota^{A'}=1$ and

$$\Sigma^{0'1'}=\tfrac{1}{2}\varepsilon_{AB}o_{A'}\iota_{B'}e^{AA'}\wedge e^{BB'}=o_{A'}e^{AA'}\wedge \mathrm{d}w_A$$

is a closed two-form. Therefore, the Frobenius theorem guarantees the existence of coordinates $x_A=(x,y)$ such that

$$e_A{}^{1'}=o_{A'}e_A{}^{A'}=\mathrm{d}x_A+\Theta_{AB}\,\mathrm{d}w^B,$$

where $\Theta_{AB}=\Theta_{AB}(w,z,x,y)$ is symmetric in A and B. With this choice, we have $\Sigma^{0'1'}=\mathrm{d}x_A\wedge\mathrm{d}w^A$ and the metric is given by

$$g=\mathrm{d}x_A\,\mathrm{d}w^A+\Theta_{AB}\,\mathrm{d}w^A\,\mathrm{d}w^B.$$

Calculating the components of the spin connection yields $\Gamma_{AA'B'C'}=A_{AA'}\iota_{B'}\iota_{C'}$. The residual conformal freedom is used to set $A_{AA'}=\beta_A\iota_{A'}$.

The tetrad of vector fields $\nabla_{AA'}$ dual to $e^{AA'}$ is

$$\nabla_{A1'}=\iota^{A'}\nabla_{AA'}=\frac{\partial}{\partial x^A},\qquad \nabla_{A0'}=o^{A'}\nabla_{AA'}=\frac{\partial}{\partial w^A}+\Theta_{AB}\frac{\partial}{\partial x_B},$$

and the Lax pair (3.1) is

$$L_A=\frac{\partial}{\partial x^A}-\lambda\left(\frac{\partial}{\partial w^A}+\Theta_{AB}\frac{\partial}{\partial x_B}\right)+l_A\frac{\partial}{\partial\lambda},$$

where, in the chosen spin frame, $l_A=\beta_A$ do not depend on λ. Consider the Lie bracket

$$[L_0,L_1]=\left(\frac{\partial\Theta^{CD}}{\partial w^C}+\Theta_{AC}\frac{\partial\Theta^{AD}}{\partial x_C}-\beta^D-\lambda\frac{\partial\Theta^{AD}}{\partial x^A}\right)\frac{\partial}{\partial x^D} +\left(\frac{\partial\beta^A}{\partial w^A}+\Theta_{AB}\frac{\partial\beta^A}{\partial x_B}+\lambda\frac{\partial\beta_A}{\partial x_A}\right)\frac{\partial}{\partial\lambda}.$$

The ASD condition is equivalent to integrability of the distribution L_A. In fact, $[L_0,L_1]=0$, since there is no $\partial/\partial\omega^A$ term in the Lie bracket above. We deduce that

$$\Theta_{AB}=\delta_A\delta_B\Theta,\quad \beta_A=\delta_Af,\quad \text{where } \delta_A:=\iota^{A'}\nabla_{AA'}=\frac{\partial}{\partial x^A},$$

and $f = f(w, z, x, y)$ and $\Theta(w, z, x, y)$ satisfy

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f + \mathcal{F}(w, z), \quad \Box f = 0.$$

To obtain (3.3), we absorb $\mathcal{F}(w, z)$ into f without changing (3.4). ∎

Remark 3.3. One-forms $(e^{00'}, e^{10'})$ span a differential ideal and

$$\Sigma^{0'0'} = e^{00'} \wedge e^{10'} = \mathrm{d}w \wedge \mathrm{d}z$$

is the null-Kähler form (it is covariantly constant with respect to the metric (3.2)). On the other hand, $(e^{01'}, e^{11'})$ do not span an ideal unless $f = 0$, in which case g is pseudo-hyper-Kähler.

Remark 3.4. The components of the Ricci and Weyl curvatures, and the Levi-Civita spinor connection, are

$$C_{ABCD} = \delta_A\delta_B\delta_C\delta_D\Theta, \quad C_{A'B'C'D'} = \iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}\Box f = 0, \quad R = 0,$$
$$\Phi_{ABA'B'} = \iota_{A'}\iota_{B'}\delta_A\delta_B f, \quad \Gamma_{AB} = \delta_A\delta_B\delta_C\Theta\,\mathrm{d}w^C, \quad \Gamma_{A'B'} = \iota_{A'}\iota_{B'}\delta_A f\,\mathrm{d}w^A.$$

The Bianchi identity (A 6) is satisfied as a consequence of equation (3.4). To sum out f is a potential for a null Maxwell field (the so-called Hertz potential), and Θ is a nonlinear potential for a metric.

Potential forms of complexified null Einstein–Maxwell equations were given by Garcia (1977) and Robinson (2000). It will be instructive to look for a non-trivial overlap between them and the one given above.

(*a*) *Examples*

Example 3.5. Consider a class of metrics given by $\Theta_x = 0$. Equations (3.3), (3.4) reduce to

$$\Theta_{yz} = f, \qquad f_{yz} = 0.$$

The general solution is given by

$$\Theta = B(w, y) + z\int A(w, y)\,\mathrm{d}y, \qquad f = A(w, y), \tag{3.6}$$

where $A(w, y)$ and $B(w, y)$ are arbitrary functions. (In fact, there are other terms linear in y and depending on arbitrary functions of w, z. These terms can be gauged away, as they do not change the metric.) We have

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - (zA_y + B_{yy})\,\mathrm{d}w^2.$$

If $A_y = 0$, then g is pseudo-hyper-Kähler.

Example 3.6. Consider solutions with $f = \Theta_v$, where v is one of (w, z, x, y). Equation (3.4) implies

$$\frac{\partial}{\partial v}(\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2) = 0,$$

and differentiating equation (3.3) yields $\Theta_{vv} = 0$. It is enough to consider two sub-cases: $\Theta_{xx} = 0$ and $\Theta_{ww} = 0$, where the former one can be integrated explicitly,

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - (xP_y + z(P - P_w + 2PP_y) + Q)\,\mathrm{d}w^2 + 2P\,\mathrm{d}w\mathrm{d}z, \tag{3.7}$$

where $P(w, y)$ and $Q(w, y)$ are arbitrary functions.

Example 3.7. We have

$$\Theta = A\left(\frac{x}{y}\right),$$
$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \frac{1}{y^2}\left(\left(\frac{x^2}{y^2}A'' + 2\frac{x}{y}A'\right)\mathrm{d}w^2 + A''\,\mathrm{d}z^2 + 2\left(\frac{x}{y}A'' + A'\right)\mathrm{d}w\mathrm{d}z\right),$$

where A is an arbitrary function, and A' is its derivative.

Example 3.8. We have

$$\Theta = xA(y) + B(y),$$
$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - (xA_{yy} + B_{yy})\,\mathrm{d}w^2 + 2A_y\,\mathrm{d}w\mathrm{d}z,$$

where A and B are arbitrary functions of one variable.

4. Twistor theory of ASD null-Kähler metrics

All ASD null-Kähler metrics locally arise from solutions to (3.3), (3.4). Non-analytic solutions are generic in $(2, 2)$ signature. However, in order to find a twistor description, in this section I shall restrict myself to real-analytic solutions.

Given an analytic solution to (3.3), (3.4), one can obtain the corresponding twistor space by equipping $\mathcal{M} \times \mathbb{CP}^1$ with an integrable complex structure: the basis of $[0, 1]$ vectors is $(L_0, L_1, \partial_{\bar{\lambda}})$, where (L_0, L_1) are given by (3.5). The parallel spinor $\iota^{A'}$ gives rise to the section $l = \iota^{A'}\pi_{A'}$ of $\kappa^{-1/4}$. In this section I shall perform this construction (together with its converse) in a coordinate-independent way.

Definition 4.1. An α-surface is a totally null two-dimensional surface, such that a two-form orthogonal to its tangent plane is SD.

There are Frobenius integrability conditions for the existence of such α-surfaces through each point and these are equivalent, by theorem 3.1, to the vanishing of the SD part of the Weyl curvature, $C_{A'B'C'D'}$. Thus, given $C_{A'B'C'D'} = 0$, we can define a twistor space $\mathcal{PT}$ to be the three-complex-dimensional manifold of α-surfaces in $\mathcal{M}$.

A tangent space to an α-surface is spanned by null vectors of the form $\lambda^A\pi^{A'}$, with $\pi^{A'}$ fixed and λ^A arbitrary. As mentioned in § 1, in the split signature, any spinor has an invariant decomposition into its real and imaginary part. A *real α-surface* corresponds to both λ^A and $\pi^{A'}$ being real.

In general, $\pi^{A'} = \mathrm{Re}\,\pi^{A'} + \mathrm{i}\,\mathrm{Im}\,\pi^{A'}$, and the correspondence space $\mathcal{F}$ defined in the last section decomposes into two open sets,

$$\mathcal{F}_+ = \{(x^a, [\pi^{A'}]) \in \mathcal{F};\ \mathrm{Re}(\pi_{A'})\,\mathrm{Im}(\pi^{A'}) > 0\} = \mathcal{M} \times D_+,$$
$$\mathcal{F}_- = \{(x^a, [\pi^{A'}]) \in \mathcal{F};\ \mathrm{Re}(\pi_{A'})\,\mathrm{Im}(\pi^{A'}) < 0\} = \mathcal{M} \times D_-,$$

where $D_\pm$ are two copies of a Poincaré disc. These complex submanifolds are separated by a real correspondence space,

$$\mathcal{F}_0 = \{(x^a, [\pi^{A'}]) \in \mathcal{F};\ \mathrm{Re}(\pi_{A'})\,\mathrm{Im}(\pi^{A'}) = 0\} = \mathcal{M} \times \mathbb{RP}^1.$$

The vector fields (3.1), together with the complex structure on the $\mathbb{CP}^1$, give $\mathcal{F}$, a structure of a complex manifold $\mathcal{PT}$: the integrable sub-bundle of $T\mathcal{F}$ is spanned by L_0, L_1, $\partial_{\bar\lambda}$. The distribution (3.1) with $\lambda \in \mathbb{RP}^1$ define a foliation of $\mathcal{F}_0$ with a quotient $\mathcal{PT}_0$, which leads to a double fibration,

$$\mathcal{M} \xleftarrow{p} \mathcal{F}_0 \xrightarrow{q} \mathcal{PT}_0. \tag{4.1}$$

The twistor space $\mathcal{PT}$ is a union of two open subsets $\mathcal{PT}_+ = (\mathcal{F}_+)$ and $\mathcal{PT}_- = (\mathcal{F}_-)$ separated by a three-dimensional real boundary† (*real twistor space*) $\mathcal{PT}_0 := q(\mathcal{F}_0)$.

The *real structure* $\sigma(x^a) = \bar{x}^a$ maps α-surfaces to α-surfaces, and therefore induces an anti-holomorphic involution $\sigma : \mathcal{PT} \to \mathcal{PT}$. The fixed points of this involution correspond to real α-surfaces in $\mathcal{M}$. There is an $\mathbb{RP}^1$ worth of such α-surfaces through each point of $\mathcal{M}$. The set of fixed points of σ in $\mathcal{PT}$ is $\mathcal{PT}_0$.

Each point $x \in \mathcal{M}$ determines a sphere l_x made up of all the α-surfaces through x. The normal bundle of l_x in $\mathcal{PT}$ is $N = T\mathcal{PT}|_{l_x}/Tl_x$. This is a rank-two vector bundle over $\mathbb{CP}^1$, therefore it has to be one of the standard line‡ bundles $\mathcal{O}(n) \oplus \mathcal{O}(m)$.

Lemma 4.2. *Let $p : \mathcal{F} = \mathcal{M} \times \mathbb{CP}^1 \to \mathcal{M}$. The holomorphic curves $l_x := p^{-1}(x)$, $x \in \mathcal{M}$, have normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. The bundle N can be identified with the quotient $p^*(T_x\mathcal{M})/\{\operatorname{span} L_0, L_1\}$. In their homogeneous form, the operators L_A have weight one, so the distribution spanned by them is isomorphic to the bundle $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. The definition of the normal bundle as a quotient gives a sequence of sheaves over $\mathbb{CP}^1$,

$$0 \to \mathbb{C}^2 \otimes \mathcal{O}(-1) \to \mathbb{C}^4 \to N \to 0,$$

and we see that $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$, because the last map, in the spinor notation, is given explicitly by $V^{AA'} \mapsto V^{AA'}\pi_{A'}$ projecting onto $\mathcal{O}(1) \oplus \mathcal{O}(1)$. ■

If $\mathcal{M}$ is ASD null-Kähler, then $\mathcal{PT}$ has an additional structure.

Theorem 4.3. *Let $\mathcal{PT}$ be a three-dimensional complex manifold with*

(i) *a four-parameter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$;*

(ii) *a preferred section of $\kappa^{-1/4}$, where κ is the canonical bundle of $\mathcal{PT}$;*

(iii) *an anti-holomorphic involution $\rho : \mathcal{PT} \to \mathcal{PT}$ fixing a real equator of each rational curve, and leaving the section of $\kappa^{-1/4}$ above invariant.*

Then the real moduli space $\mathcal{M}$ of the ρ-invariant curves is equipped with a restricted conformal class $[g]$ of ASD null-Kähler metric: if $g \in [g]$ and $\Sigma^{0'0'}$ is a null-Kähler two-form, then $\hat{g} = \Omega^2 g \in [g]$ for any Ω such that $\mathrm{d}\Omega \wedge \Sigma^{0'0'} = 0$. Conversely, given a real analytic ASD null-Kähler metric, there exists a corresponding twistor space with the above structures.

Proof. Let g be a real analytic ASD metric with a covariantly constant real spinor $\iota_{A'}$.

† Woodhouse (1992) performed a careful analysis of the twistor correspondence for flat $(++--)$ metrics and showed how functions on $\mathcal{PT}_0 = \mathbb{RP}^3$ can be used to construct smooth solutions to the ultra-hyperbolic wave equation.

‡ Here, $\mathcal{O}(n)$ denotes the line bundle over $\mathbb{CP}^1$, with transition functions λ^{-n} from the set $\lambda \neq \infty$ to $\lambda \neq 0$ (i.e. Chern class n).

From $C_{A'B'C'D'} = 0$, it follows that there exist coordinates $\pi^{A'}$ on the fibres of $S^{A'} \to \mathcal{M}$ such that $\pi^{A'}\tilde{\nabla}_{AA'}\pi^{B'} = 0$. Therefore, a parallel section $\iota_{A'}$ of $S_{A'}$ determines a function $l = \pi^{A'}\iota_{A'}$ constant along the twistor distribution. The line bundle $\pi^{A'}\iota_{A'} = 0$ on $\mathcal{PT}$ is isomorphic to $\kappa^{-1/4}$, where $\kappa = \Omega^3\mathcal{PT}$ is the canonical bundle.

Converse. The global section l of $\kappa^{-1/4}$, when pulled back to $S_{A'}$, determines a homogeneity degree-one function on each fibre of $S_{A'}$ and so must, by Hartog's theorem, be given by $l = \iota^{A'}\pi_{A'}$, and since l is pulled back from twistor space, it must satisfy $\pi^{A'}\tilde{\nabla}_{AA'}l = 0$. This implies

$$\nabla_{AA'}\iota_{B'} = \varepsilon_{A'B'}\alpha_A \tag{4.2}$$

for some α_A. Choose a representative in $[g]$ with $R = 0$. Contracting (4.2) with $\nabla^A{}_{C'}$ and using the spinor Bianchi identity gives

$$\nabla^A{}_{C'}\nabla_{AA'}\iota_{B'} = C_{A'B'C'D'}\iota^{D'} - \tfrac{1}{12}R\varepsilon_{C'(B'}\iota_{A')} = 0 = \varepsilon_{A'B'}\nabla^A{}_{C'}\alpha_A,$$

so α_A is a solution to the ASD spin-$\frac{1}{2}$ equation $\nabla^{AA'}\alpha_A = 0$ (the so-called neutrino equation). It can be written in terms of a potential

$$\alpha_A = \iota^{A'}\nabla_{AA'}\phi, \tag{4.3}$$

since the integrability conditions $\iota^{A'}\iota^{B'}\nabla^A{}_{A'}\alpha_A = \alpha_A\iota^{A'}\nabla^A{}_{A'}\iota^{B'}$ are satisfied. Here, ϕ is a real analytic function that satisfies

$$\nabla^a\nabla_a\phi + \nabla_a\phi\nabla^a\phi = 0, \tag{4.4}$$

as a consequence of the neutrino equation. Consider a conformal rescaling

$$\hat{g} = \Omega^2 g, \quad \hat{\varepsilon}_{A'B'} = \Omega\varepsilon_{A'B'}, \quad \hat{\iota}_{A'} = \Omega\iota_{A'}, \quad \hat{\iota}^{A'} = \iota^{A'}, \quad \hat{R} = R + \tfrac{1}{4}\Omega^{-1}\Box\,\Omega.$$

The twistor equation (4.2) is conformally invariant as $\hat{\nabla}_A^{(A'}\hat{\iota}^{B')} = \Omega^{-1}\nabla_A{}^{(A'}\iota^{B')} = 0$. Choose $\Omega \in \ker\Box$, so that $\hat{R} = 0$. Let $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$. Then

$$\hat{\nabla}_{AA'}\hat{\iota}^{B'} = \nabla_{AA'}\iota^{B'} + \varepsilon_{A'}{}^{B'}\Upsilon_{AB'}\iota^{C'} = \varepsilon_{A'}{}^{B'}(\iota^{C'}\nabla_{AC'}(\phi + \ln\Omega)),$$

where we have used equations (4.2) and (4.3). Notice that, as a consequence of (4.4), $\exp(\phi) \in \ker\Box$, and we can choose $\ln\Omega = -\phi$ and

$$\hat{\nabla}_{AA'}\hat{\iota}^{B'} = 0. \tag{4.5}$$

We can still use the residual gauge freedom and add to ϕ and an arbitrary function Ω constant along $\iota^{A'}\nabla_{AA'}$, which, by the Frobenius theorem, implies $\mathrm{d}\Omega \wedge \Sigma^{0'0'} = 0$. This means (4.5) is invariant under a conformal rescaling by functions constant along the leaves of the congruence defined by $\hat{\iota}^{A'}$. Such conformal transformations do not change $\hat{R} = 0$. ∎

5. ASD null-Kähler metrics with symmetry

In this section I shall consider ASD null-Kähler metrics that admit a Killing vector preserving the parallel spinor. Let us call them *ASD null-Kähler metrics with symmetry.* I shall show that all such metrics are (at least in the real analytic case) locally determined by solutions to a certain integrable equation and its linearization.

Before establishing this result, I shall review some facts about EW spaces that admit a parallel weighted vector (Dunajski *et al*. 2001).

(*a*) *Three-dimensional EW spaces with a parallel weighted vector*

Let $\mathcal{W}$ be a three-dimensional real manifold with a torsion-free connection D and a conformal metric $[h]$. We shall call $\mathcal{W}$ a Weyl space if the null geodesics of $[h]$ are also geodesics for D. This condition is equivalent to

$$Dh = \nu \otimes h \tag{5.1}$$

for some one-form ν. Here, h is a representative metric in the conformal class. If we change this representative by $h \to \phi^2 h$, then $\nu \to \nu + 2\,\mathrm{d}\ln\phi$. A tensor object T that transforms as $T \to \phi^m T$ when $h \to \phi^2 h$ is said to be conformally invariant of weight m. The covariant derivative of a one-form β of weight m can be expressed in terms of the Levi-Civita connection of h,

$$\tilde{D}\beta = \nabla\beta - \tfrac{1}{2}(\beta \otimes \nu + (1-m)\nu \otimes \beta - h(\nu, \beta)h). \tag{5.2}$$

The conformally invariant EW condition on $(\mathcal{W}, h, \nu)$ is

$$W_{(ij)} = \tfrac{1}{3}\Lambda h_{ij},$$

where Λ and W_{ij} are the scalar curvature and the Ricci tensor of the Weyl connection, respectively.

Three-dimensional EW structures are related to four-dimensional ASD conformal structures by the Jones–Tod correspondence (Jones & Tod 1985).

Proposition 5.1 (cf. Jones & Tod 1985). *Let $(\mathcal{M}, \hat{g})$ be an ultra-hyperbolic four-manifold with ASD conformal curvature and a conformal Killing vector K. The EW structure in indefinite signature on the space $\mathcal{W}$ of trajectories of K is defined by*

$$h := |K|^{-2}\hat{g} - |K|^{-4}\boldsymbol{K} \odot \boldsymbol{K}, \qquad \nu := 2|K|^{-2} *_{\hat{g}} (\boldsymbol{K} \wedge \mathrm{d}\boldsymbol{K}), \tag{5.3}$$

where $|K|^2 := \hat{g}_{ab}K^aK^b$, $\boldsymbol{K}$ is the one-form dual to K and $*_{\hat{g}}$ is taken with respect to $\hat{g}$. All three-dimensional EW structures arise in this way.

Conversely, let (h, ν) be a three-dimensional EW structure with a signature $(++-)$ on $\mathcal{W}$, and let (V, α) be a pair consisting of a function of weight -1 and a one-form on $\mathcal{W}$ that satisfies the generalized monopole equation

$$*_h(\mathrm{d}V + (\tfrac{1}{2})\nu V) = \mathrm{d}\alpha, \tag{5.4}$$

where $*_h$ is taken with respect to h. Then

$$g = Vh - V^{-1}(\mathrm{d}z + \alpha)^2 \tag{5.5}$$

is an ASD metric with an isometry $K = \partial_z$.

Dunajski *et al*. (2001) demonstrated that if an EW space admits a parallel weighted vector, the coordinates can be found in which the metric and the one-form are given by

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}x\mathrm{d}t - 4u\,\mathrm{d}t^2, \qquad \nu = -4u_x\,\mathrm{d}t, \qquad u = u(x, y, t), \tag{5.6}$$

and the EW equations reduce to the dispersionless Kadomtsev–Petviashvili equation

$$(u_t - uu_x)_x = u_{yy}. \tag{5.7}$$

If $u(x, y, t)$ is a smooth real function of real variables, then (5.6) has signature $(++-)$. It has also been shown that there exists a twistor construction of EW spaces (5.6) given by the following theorem.

Theorem 5.2 (cf. Dunajski *et al*. 2001). *There is a one-to-one correspondence between EW spaces (5.6) obtained from solutions to the equation (5.7) and two-dimensional complex manifolds with*

(i) *a three-parameter family of rational curves with normal bundle $\mathcal{O}(2)$;*

(ii) *a global section l of $\kappa^{-1/4}$, where κ is the canonical bundle;*

(iii) *an anti-holomorphic involution fixing a real slice, leaving a rational curve and the preferred section of $\kappa^{-1/4}$ invariant.*

(*b*) *Symmetry reduction*

Now we are ready to establish the main result of this section.

Theorem 5.3. *Let $H = H(x, y, t)$ and $W = W(x, y, t)$ be smooth real-valued functions on an open set $\mathcal{W} \subset \mathbb{R}^3$, which satisfy*†

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \tag{5.8}$$
$$W_{yy} - W_{xt} + (H_x W_x)_x = 0. \tag{5.9}$$

Then

$$g = W_x(\mathrm{d}y^2 - 4\,\mathrm{d}x\mathrm{d}t - 4H_x\,\mathrm{d}t^2) - W_x^{-1}(\mathrm{d}z - W_x\,\mathrm{d}y - 2W_y\,\mathrm{d}t)^2 \tag{5.10}$$

is an ASD null-Kähler metric on a circle bundle $\mathcal{M} \to \mathcal{W}$. All real analytic ASD null-Kähler metrics with symmetry arise from this construction.

Proof. Let (h, ν) be a three-dimensional EW structure given by (5.6) (with $u = H_x$) and let (V, α) be a pair consisting of a function and a one-form that satisfy the generalized monopole equation (5.4). The ultra-hyperbolic metric

$$g = V(\mathrm{d}y^2 - 4\,\mathrm{d}x\mathrm{d}t - 4H_x\,\mathrm{d}t^2) - V^{-1}(\mathrm{d}z + \alpha)^2 \tag{5.11}$$

is therefore ASD. It satisfies $\mathcal{L}_K g = 0$, where $K = \partial_z$. Using the relations

$$*_h\,\mathrm{d}t = \mathrm{d}t \wedge \mathrm{d}y, \qquad *_h\,\mathrm{d}y = 2\,\mathrm{d}t \wedge \mathrm{d}x, \qquad *_h\,\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}x + 2H_x\,\mathrm{d}y \wedge \mathrm{d}t,$$

we verify that equation (5.9) is equivalent to $\mathrm{d} *_h (\mathrm{d} + \nu/2)(W_x) = 0$. Therefore,

$$W_{xx}\,\mathrm{d}y \wedge \mathrm{d}x + (2(H_x W_x)_x - W_{tx})\,\mathrm{d}y \wedge \mathrm{d}t + 2W_{xy}\,\mathrm{d}t \wedge \mathrm{d}x = \mathrm{d}\alpha,$$

and we deduce that $V = W_x$ is a solution to the monopole equation (5.4) on the EW background given by (5.6). We choose a gauge in which $\alpha = Q\,\mathrm{d}y + P\,\mathrm{d}t$. This yields

$$Q_x = -W_{xx}, \qquad P_x = -2W_{xy}, \qquad P_y - Q_t = 2(H_x W_x)_x - W_{xt}, \tag{5.12}$$

so

$$Q = -W_x + A(y, t), \qquad P = -2W_y + B(y, t), \qquad \alpha = -W_x\,\mathrm{d}y - 2W_y\,\mathrm{d}t + A\,\mathrm{d}y + B\,\mathrm{d}t.$$

The integrability conditions $P_{xy} = P_{yx}$ are given by (5.9) and $A_t = B_y$. Therefore, there exists $C(y, t)$ such that $A = C_y$, $B = C_t$. We now replace z by $z - C$ and the

† With definition $u = H_x$, the x derivative of equation (5.8) becomes the dispersionless Kadomtsev–Petviashvili equation (5.7) originally used by Dunajski *et al*. (2001). There are some computational advantages in working with the 'potential' form (5.8).

metric (5.11) becomes (5.10). This proves that (5.10) is ASD. It is also scalar-flat, because, as a consequence of (5.9),

$$R = 8(W_{xyy} - W_{xxt} + (H_x W_x)_{xx})W_x = 0. \tag{5.13}$$

We now choose the null tetrad

$$e^{00'} = -2W_x\,\mathrm{d}t, \qquad e^{10'} = \frac{\mathrm{d}z - 2W_y\,\mathrm{d}t}{2W_x},$$
$$e^{01'} = \mathrm{d}z - 2W_x\,\mathrm{d}y - 2W_y\,\mathrm{d}x + ze^{00'}, \qquad e^{11'} = \mathrm{d}x + H_x\,\mathrm{d}y + ze^{10'}$$

such that $g = 2(e^{00'}e^{11'} - e^{10'}e^{01'})$. The basis of SD two-forms $\Sigma^{A'B'}$ is given by

$$\begin{aligned}
\Sigma^{0'0'} &= \iota_{A'}\iota_{B'}\Sigma^{A'B'}\\
&= e^{00'} \wedge e^{10'}\\
&= \mathrm{d}z \wedge \mathrm{d}t,\\
\Sigma^{0'1'} &= \iota_{A'}o_{B'}\Sigma^{A'B'}\\
&= e^{10'} \wedge e^{01'} - e^{00'} \wedge e^{11'}\\
&= \mathrm{d}t \wedge \mathrm{d}(z^2) + 2\,\mathrm{d}t \wedge \mathrm{d}W + \mathrm{d}y \wedge \mathrm{d}z,\\
\Sigma^{1'1'} &= o_{A'}o_{B'}\Sigma^{A'B'}\\
&= e^{01'} \wedge e^{11'}\\
&= 2W_x\,\mathrm{d}x \wedge \mathrm{d}y + 2(zW_x + W_y)\,\mathrm{d}x \wedge \mathrm{d}t - \mathrm{d}x \wedge \mathrm{d}z\\
&\qquad + (2H_xW_x - 2zW_y)\,\mathrm{d}t \wedge \mathrm{d}y + z\,\mathrm{d}z \wedge \mathrm{d}y + (H_x + z^2)\,\mathrm{d}z \wedge \mathrm{d}t.
\end{aligned}$$

These two-forms satisfy

$$-2\Sigma^{0'0'} \wedge \Sigma^{1'1'} = \Sigma^{0'1'} \wedge \Sigma^{0'1'}, \qquad \mathrm{d}\Sigma^{0'0'} = 0, \qquad \mathrm{d}\Sigma^{0'1'} = 0$$

and

$$\mathrm{d}\Sigma^{1'1'} = \mathrm{d}(H_x - 2W) \wedge \mathrm{d}t \wedge \mathrm{d}z + (W_{xt} - W_{yy} - (H_xW_x)_x)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t. \tag{5.14}$$

Therefore, proposition 2.3 implies that the metric (5.10) admits a constant spinor that is preserved by $K = \partial_z$.

Converse. Let g be a real analytic ASD metric with a covariantly constant spinor $\iota_{A'}$, which is Lie derived along a Killing vector K. Theorem 4.3 implies that the corresponding twistor space $\mathcal{PT}$ is equipped with $l \in \Gamma(\kappa^{-1/4})$. The Killing vector K gives rise to a holomorphic vector field on $\mathcal{PT}$ that preserves l. Therefore, the mini-twistor space $\mathcal{Z}$ (the space of trajectories of K in $\mathcal{PT}$) also admits a preferred real section of the $-\frac{1}{4}$ power of its canonical bundle. The mini-twistor space $\mathcal{Z}$ satisfies the assumptions of theorem 5.2 and the corresponding EW metric is of the form $\hat{g} = \Omega^2 g$, where g is given by (5.10). Both $\hat{g}$ and g are scalar flat (this follows from the spinor Ricci identities and from equation (5.13), respectively). As a consequence, we deduce that $\Omega = \Omega(t)$. Now we can use the coordinate freedom (Dunajski *et al.* 2001) to absorb Ω in the solution to the equation (5.8). ∎

Corollary 5.4. *Let $(\mathcal{M}, g)$ be an ASD null-Kähler manifold with symmetry. Then the EW structure induced by (5.3) on the space of orbits of this symmetry is locally of the form (5.6).*

Remark 5.5. Theorem 5.3 is analogous to a result of LeBrun (1991), who constructs all scalar-flat Kähler metrics with symmetry in Euclidean signature from solutions to the $SU(\infty)$ Toda equation and its linearization.

Remark 5.6. A class of solutions to the monopole (5.9) can be obtained from vectors tangent to a space of solutions to (5.8) generated by

$$x \rightarrow a(\epsilon)x + b(\epsilon), \qquad t \rightarrow a(\epsilon)t + c(\epsilon), \qquad y \rightarrow a(\epsilon)y + e(\epsilon),$$

where $a(0) = a$, $b(0) = b$, $c(0) = c$, $e(0) = e$. The corresponding linearized solution is given by

$$W(x, y, t) = \left.\frac{\mathrm{d}H}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = a(xH_x + yH_y + tH_t) + bH_x + cH_t + eH_y.$$

Remark 5.7. If $H = \text{const.}$, then (5.9) reduces to the wave equation in $2+1$ dimensions and, consequently, the metric (5.10) is the $(++--)$ Gibbons–Hawking solution (Gibbons & Hawking 1978).

Remark 5.8. Note that $\mathrm{d}\Sigma^{1'1'} \neq 0$ unless $W = H_x/2 + f(t)$, in which case

$$\mathrm{d}\Sigma^{1'1'} = \mathrm{d}(H_{xt} - H_x H_{xx} - H_{yy}) \wedge \mathrm{d}y \wedge \mathrm{d}t = 0,$$

and we are working in a covariantly constant real spin frame. The metric

$$g = \tfrac{1}{2}H_{xx}(\mathrm{d}y^2 - 4\,\mathrm{d}x\mathrm{d}t - 4H_x\,\mathrm{d}t^2) - \frac{2}{H_{xx}}(\mathrm{d}z - \tfrac{1}{2}H_{xx}\,\mathrm{d}y - H_{xy}\,\mathrm{d}t)^2 \tag{5.15}$$

is therefore pseudo-hyper-Kähler. Dunajski *et al.* (2001) showed that all pseudo-hyper-Kähler metrics with a symmetry satisfying $\mathrm{d}K_+ \wedge \mathrm{d}K_+ = 0$ are locally given by (5.15). Here, $\mathrm{d}K_+$ is a self-dual part of $\mathrm{d}K$.

Remark 5.9. If $W_x \neq \frac{1}{2}H_{xx}$, then (5.10) is not Ricci flat. This can be verified by a direct calculation. It also follows from more geometric reasoning. The Killing vector $K = \partial_z$ acts on SD two-forms by a Lie derivative. One can choose a basis $\Sigma^{A'B'}$ such that one element of this basis is fixed, and the Killing vector rotates the other two. The components of the SD derivative of K are coefficients of these rotations. Therefore, $(\mathrm{d}K)_+ = \text{const.}$ if g is pseudo-hyper-Kähler. In our case, $\mathrm{d}K_+ = (H_{xx}/W_x)\,\mathrm{d}z \wedge \mathrm{d}t$. Therefore, H_{xx}/W_x must be constant for (5.10) to be Ricci flat. An example of a non-vacuum metric is given by $W = \frac{1}{2}H_y$.

(c) *Pseudo-hyper-Kähler metrics with symmetry*

In this subsection I shall assume that an ASD null-Kähler structure $(\mathcal{M}, g)$ admits an additional parallel spinor $o_{A'}$ such that $o_{A'}\iota^{A'} = 1$. Now there exists a covariantly constant basis of the spin space $S^{A'}$, and $(\mathcal{M}, g)$ is pseudo-hyper-Kähler. In the split signature we can arrange for one of the complex structures to be real and for the other two to be purely imaginary,

$$-I^2 = S^2 = T^2 = 1, \qquad IST = 1,$$

and S and T determine a pair of transverse null foliations. Now

$$g(X, Y) = g(IX, IY) = -g(SX, SY) = -g(TX, TY)$$

for any pair of real vectors X, Y. The endomorphism I endows $\mathcal{M}$ with the structure of a two-dimensional complex Kähler manifold, as does every other complex structure $aI + bS + cT$ parametrized by the points of the hyperboloid $a^2 - b^2 - c^2 = 1$. Using the identification between the two-forms, and endomorphisms given by g, we can write

$$S = \Sigma^{0'0'} - \Sigma^{1'1'}, \qquad I = \Sigma^{0'0'} + \Sigma^{1'1'}, \qquad T = \Sigma^{0'1'}.$$

Killing vectors on pseudo-hyper-Kähler spaces give rise to a homomorphism

$$S^1 \to SO(2,1).$$

Therefore, K induces an S^1 action on the hyperboloid. Assume that $g(K,K) \neq 0$. If the action is trivial, then we are dealing with the $(++--)$ Gibbons–Hawking ansatz (Gibbons & Hawking 1978). Otherwise, there always exists a fixed point of the S^1 action. If the two-form corresponding to this fixed point is non-degenerate, then g is given in terms of solutions to the $SU(\infty)$ equation (Finley & Plebański 1979; LeBrun 1991; Ward 1990). Finally, if the fixed point corresponds to a degenerate two-form, then g is given by (5.15). Pseudo-hyper-Kähler metrics that admit K such that $g(K,K) = 0$ and $\mathcal{L}_K g = \mathcal{L}_K I = \mathcal{L}_K S = \mathcal{L}_K T = 0$ have been found by Barrett *et al.* (1994). Conformal symmetries of pseudo-hyper-Kähler metrics with non-null self-dual derivative have been classified by Dunajski & Tod (2001)

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Appendix A. Spinor notation

Let $\mathcal{M}$ be a real four-manifold equipped with a $(++--)$ metric g and compatible volume form ν. We use the conventions of Penrose & Rindler (1986). $a, b, \dots$ are four-dimensional vector indices and $A, B, \dots, A', B', \dots$ are two-dimensional spinor indices. They have ranges 0, 1 and $0'$, $1'$, respectively. The tangent space at each point of $\mathcal{M}$ is isomorphic to a tensor product of the two real spin spaces

$$T\mathcal{M} = S^A \otimes S^{A'}. \tag{A 1}$$

This isomorphism is given by

$$V^a \to V^{AA'} = \begin{pmatrix} V^0 + V^3 & V^1 + V^2 \\ V^1 - V^2 & V^0 - V^3 \end{pmatrix}.$$

Orthogonal transformations decompose into products of ASD and SD rotations,

$$SO(2,2) = (SL(2,\mathbb{R}) \times \widetilde{SL}(2,\mathbb{R}))/\mathbb{Z}_2. \tag{A 2}$$

The Lorentz transformation $V^a \to \Lambda^a{}_b V^b$ is equivalent to

$$V^{AA'} \to \lambda^A{}_B V^{BB'} \lambda^{A'}{}_{B'},$$

where $\lambda^A{}_B$ and $\lambda^{A'}{}_{B'}$ are elements of $SL(2,\mathbb{R})$ and $\widetilde{SL}(2,\mathbb{R})$.

Spin dyads (o^A, ι^A) and $(o^{A'}, \iota^{A'})$ span S^A and $S^{A'}$, respectively. The spin spaces S^A and $S^{A'}$ are equipped with parallel symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ such that

$\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These antisymmetric objects are used to raise and lower the spinor indices via $\iota_A = \iota^B \varepsilon_{BA}$, $\iota^A = \varepsilon^{AB}\iota_B$. We shall use normalized spin frames, so that

$$o^B \iota^C - \iota^B o^C = \varepsilon^{BC}, \qquad o^{B'} \iota^{C'} - \iota^{B'} o^{C'} = \varepsilon^{B'C'}.$$

Let $e^{AA'}$ be a null tetrad of one-forms on $\mathcal{M}$, i.e.

$$g = \varepsilon_{AB}\varepsilon_{A'B'} e^{AA'} e^{BB'} = 2(e^{00'}e^{11'} - e^{10'}e^{01'}),$$

and let $\nabla_{AA'}$ be the frame of vector fields dual to $e^{AA'}$. The orientation is given by fixing the volume form

$$\nu = e^{01'} \wedge e^{10'} \wedge e^{11'} \wedge e^{00'}.$$

Apart from orientability, $\mathcal{M}$ must satisfy some other topological restrictions for the $(++--)$ metric, and a global spinor fields to exist. We shall not take them into account, as we work locally in $\mathcal{M}$.

Any two-form Ω_{ab} can be written as

$$\Omega_{ABA'B'} = \Omega_{AB}\varepsilon_{A'B'} + \tilde{\Omega}_{A'B'}\varepsilon_{AB},$$

where Ω_{AB} are $\tilde{\Omega}_{A'B'}$ symmetric in their indices since Ω_{ab} is skew. This is the decomposition of a two-form into its ASD and SD parts. The space of SD two-forms is therefore isomorphic to a symmetric tensor product of two primed spin spaces.

The local basis Σ^{AB} and $\Sigma^{A'B'}$ of spaces of ASD and SD two-forms are defined by

$$e^{AA'} \wedge e^{BB'} = \varepsilon^{AB}\Sigma^{A'B'} + \varepsilon^{A'B'}\Sigma^{AB}. \tag{A 3}$$

The first Cartan structure equations are

$$\mathrm{d}e^{AA'} = e^{BA'} \wedge \Gamma^A{}_B + e^{AB'} \wedge \Gamma^{A'}{}_{B'},$$

where Γ_{AB} and $\Gamma_{A'B'}$ are the $SL(2,\mathbb{R})$ and $\tilde{SL}(2,\mathbb{R})$ spin connection one-forms. They are symmetric in their indices and

$$\Gamma_{AB} = \Gamma_{CC'AB}e^{CC'}, \qquad \Gamma_{A'B'} = \Gamma_{CC'A'B'}e^{CC'},$$
$$\Gamma_{CC'A'B'} = o_{A'}\nabla_{CC'}\iota_{B'} - \iota_{A'}\nabla_{CC'}o_{B'}.$$

The curvature of the spin connection,

$$R^A{}_B = \mathrm{d}\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B,$$

decomposes as

$$R^A{}_B = C^A{}_{BCD}\Sigma^{CD} + \tfrac{1}{12}R\Sigma^A{}_B + \Phi^A{}_{BC'D'}\Sigma^{C'D'},$$

and similarly for $R^{A'}{}_{B'}$. Here, R is the Ricci scalar, $\Phi_{ABA'B'}$ is the trace-free part of the Ricci tensor R_{ab},

$$-2\Phi_{ABA'B'} = R_{ab} - \tfrac{1}{4}Rg_{ab},$$

and C_{ABCD} is the ASD part of the Weyl tensor,

$$C_{abcd} = \varepsilon_{A'B'}\varepsilon_{C'D'}C_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}C_{A'B'C'D'}.$$

A conformal structure is called ASD if and only if $C_{A'B'C'D'} = 0$. Define the operators $\triangle_{AB}$ and $\triangle_{A'B'}$ by

$$[\nabla_a, \nabla_b] = \varepsilon_{AB}\triangle_{A'B'} + \varepsilon_{A'B'}\triangle_{AB}.$$

The spinor Ricci identities are

$$\triangle_{AB}\iota_{A'} = \Phi_{ABA'B'}\iota^{B'}, \tag{A 4}$$

$$\triangle_{A'B'}\iota_{C'} = [C_{A'B'C'D'} - \tfrac{1}{12}R\varepsilon_{D'(A'}\varepsilon_{B')C'}]\iota^{D'} \tag{A 5}$$

(and analogous equations for unprimed spinors). Bianchi identities translate to

$$\nabla^{A'}{}_{A}C_{A'B'C'D'} = \nabla^{B}_{(B'}\Phi_{C'D')AB}, \qquad \nabla^{AA'}\Phi_{ABA'B'} + \tfrac{1}{8}\nabla_{BB'}R = 0. \tag{A 6}$$

Let K be a pure Killing vector. Then $\nabla^{(A'}_{(A}K^{B')}_{B)} = 0$, $\nabla^a K_a = 0$. This implies

$$\nabla_a K_b = \phi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB},$$

where $\psi_{A'B'}$ and ϕ_{AB} are symmetric spinors. The well-known identity $\nabla_a\nabla_b K_c = R_{bcad}K^d$ implies

$$\begin{aligned}\nabla^{A}{}_{A'}\psi_{B'C'} = -2C^{D'}_{A'B'C'}K^{A}_{D'} - 2K^{B}{}_{(A'}\Phi^{A}_{B'C')B}\\ + \tfrac{1}{6}R\varepsilon_{A'(B'}K^{A}{}_{C')} - \tfrac{4}{3}\varepsilon_{A'(B'}\Phi^{D'}_{C')}{}^{DA}K_{DD'}.\end{aligned} \tag{A 7}$$

Therefore, in an ASD vacuum, $\psi_{A'B'} =$ const. A Lie derivative of a spinor along a Killing vector K is given by

$$\mathcal{L}_K\iota^{A'} = K^b\nabla_b\iota^{A'} - \psi^{A'}_{B'}\iota^{B'}.$$

Note that if $\iota_{A'}$ is covariantly constant and the Killing vector K preserves the $\iota_{A'}$, we deduce that $\psi_{A'B'} = \psi\iota_{A'}\iota_{B'}$ for some function ψ. The identity (A 7) yields

$$\nabla_{AA'}\psi = \tfrac{1}{2}F_{AB}K^{B}_{A'},$$

so $\psi \neq$ const. unless g is hyper-Kähler.

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