Anti–self–dual conformal structures in split signature

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Xi transform, Banff, March 2007

Outline

- Anti–self–duality
- Spinors in split signature
- Lax pairs
  - Pseudo Hypercomplex
  - Scalar Flat Kähler
  - Null Kähler
  - Pseudo Hyper Kähler
- Twistor Theory
- Symmetries
  - Non-null symmetries and Einstein–Weyl structures
  - Null symmetries and projective structures
- Applications

Anti–Self–Duality

- (M, g) oriented 4-manifold with $(2, 2)$ metric. $* : \Lambda^2 \to \Lambda^2$ Hodge operator.

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

- $R_{abcd} = R_{[ab][cd]}, \quad \mathcal{R} : \Lambda^2 \to \Lambda^2$.
- Curvature decomposition

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ + \frac{s}{12} & \phi \\ \hline \phi & C_- + \frac{s}{12} \end{array} \right)$$

$C_\pm$ are the self-dual (SD) and anti-self-dual (ASD) parts of the Weyl tensor, ϕ is the tracefree Ricci curvature, and s is the scalar curvature.

- Conformally invariant ASD equations

$$C_+ = 0.$$

- In $(2,2)$ the equations are ultrahyperbolic, whereas in the Riemannian case they are elliptic.
- Neutral case is far less rigid than the Riemannian case. There exist 'null' vectors, wave–like solutions, non–analytic ASD structures.
- Example. The ASD Ricci flat metric

$$g = dwdx + dzdy + F(w,y)dw^2$$

is non-trivial and well defined on a compact manifold (e.g T^4). There are no known examples of such metrics in the Riemannian case (althrough it is known that the metric on $K3$ must exists).

Motivation: Integrable Systems

- ASD conformal equations are integrable by twistor transform.
- Symmetries in the form of Killing vectors the equations reduce to lower dimensional integrable systems.
- Most lower dimensional integrable systems arise from ASD Yang Mills, or ASD conformal equations in $(2,2)$ or $(4,0)$ signature.
- Different integrable systems can be obtained by combining symmetries with geometric conditions for a metric in a conformal class. Evolution equations in $2+1$ and $1+1$ dimensions are reductions from $(2,2)$.

Motivation: Integral Geometry

- Fritz John (1938). $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with decay conditions at infinity. For any oriented line $L \subset \mathbb{R}^3$ define $\theta(L) = \int_L f$, or

$$\theta(x, y, w, z) = \int_{-\infty}^{\infty} f(xs + z, ys - w, s)ds,$$

- The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on θ.
- Ultrahyperbolic wave equation for the flat $(2, 2)$ conformal structure

$$\frac{\partial^2\theta}{\partial x \partial w} + \frac{\partial^2\theta}{\partial y \partial z} = 0.$$

Conformal Compactification of $\mathbb{R}^{2,2}$

- Projective quadric in $\mathbb{RP}^5$

$$|\mathbf{x}|^2 - |\mathbf{y}|^2 = 0$$

  where $[\mathbf{x}, \mathbf{y}] \in \mathbb{R}^3 \times \mathbb{R}^3$ are homogeneous coordinates on $\mathbb{RP}^5$.

- The freedom $(\mathbf{x}, \mathbf{y}) \sim (c\mathbf{x}, c\mathbf{y})$ is fixed by $|\mathbf{x}| = |\mathbf{y}| = 1$ which is $S^2 \times S^2$. Quotient this by the antipodal map $(\mathbf{x}, \mathbf{y}) \to (-\mathbf{x}, -\mathbf{y})$ to obtain the conformal compactification

$$\overline{\mathbb{R}^{2,2}} = (S^2 \times S^2)/\mathbb{Z}_2.$$

- Use stereographic coordinates on the double cover. The flat metric $|d\mathbf{x}|^2 - |d\mathbf{y}|^2$ on $\mathbb{R}^{3,3}$ is the $(2,2)$ metric on $S^2 \times S^2$

$$g_0 = 4\frac{d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} - 4\frac{d\chi d\bar{\chi}}{(1+\chi\bar{\chi})^2}.$$

  ASD, Scalar–flat, Kähler.

- $SO(2,2) \cong (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2$.
- $TM \cong S \otimes S'$
- S and S' are real rank two vector bundles over M with parallel symplectic structures ϵ and ϵ'.
- $v_1, v_2 \in \Gamma(S)$ unprimed spinors, $w_1, w_2 \in \Gamma(S')$ primed spinors

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \epsilon(v_1, v_2)\epsilon'(w_1, w_2)$$

- $\Lambda^2_+ \cong S'^* \odot S'^*, \qquad \Lambda^2_- \cong S^* \odot S^*$.

$u \in \Gamma(S') \leftrightarrow$ simple SD two–form $\Omega_u \leftrightarrow$ rank 2 distribution Ker (Ω_u).

$$\Omega_u(v_1 \otimes w_1, v_2 \otimes w_2) = \epsilon'(u, w_1)\epsilon'(u, w_2)\epsilon(v_1, v_2).$$

- $x \in M, u \in \Gamma(S'_x)$. 2 dimensional α-plane $\mathcal{U}_x = \text{span}(\text{Ker}(\Omega_u))$.
- Totally null and SD

$$[g]_{\mathcal{U}_x} = 0, \quad * \Omega_u = \Omega_u.$$

- Totally null ASD 2-planes are β-planes.
- $(2, 2)$ version of Penrose's Theorem:
  - Each α plane is tangent to a surface iff $C_+ = 0$.
  - The space of α–surfaces in M is three dimensional.
  - There is a circle worth of α–surfaces through each $x \in M$.

Lax pair

- $(\mathbf{e}_{00'}, \mathbf{e}_{01'}, \mathbf{e}_{10'}, \mathbf{e}_{11'})$ real frame of vector fields on M (a Lax frame).
- Horizontal lift of an α–plane distribution to $\mathbb{P}(S') = M \times \mathbb{RP}^1$

$$L_0 = \mathbf{e}_{00'} + \lambda \mathbf{e}_{01'} + f_0 \frac{\partial}{\partial \lambda}, \qquad L_1 = \mathbf{e}_{10'} + \lambda \mathbf{e}_{11'} + f_1 \frac{\partial}{\partial \lambda}, \quad \lambda \in \mathbb{RP}^1$$

$(f_0, f_1) : M \times \mathbb{RP}^1 \longrightarrow \mathbb{R}$ are cubic polynomials in λ.

- Frobenius integrability $[L_0, L_1] = 0 \, (\text{mod} L_0, L_1)$ implies the anti–self–duality of the conformal structure

$$g = 2(\mathbf{e}_{00'} \otimes \mathbf{e}_{11'} - \mathbf{e}_{10'} \otimes \mathbf{e}_{01'}).$$

- Any ASD conformal structure admits a Lax frame $e_{AA'}$.

Curvature restrictions

- Pseudo hypercomplex
- Scalar–flat (pseudo) Kähler
- Null Kähler
- Ricci flat (pseudo hyper Kähler)
- Einstein

Pseudo hypercomplex is conformally invariant. Other conditions are not. Null Kähler does not have a Riemannian analogue.

- $I, S, T : TM \longrightarrow TM$,

$$S^2 = T^2 = \mathbf{1}, \quad I^2 = -\mathbf{1}, \quad ST = -TS = \mathbf{1}.$$

- Hyperboloid of almost complex structures $aI + bS + cT$ integrable for any (a, b, c) satisfying $a^2 - b^2 - c^2 = 1$.
- Hyperhermitian conformal structure: (X, SX, TX, IX) has signature $(2, 2)$ and is automatically ASD.
- Lax pair does not contain the vertical term, i.e. $f_0 = f_1 = 0$.

Curvature restrictions: Scalar–flat Kähler

- (Pseudo) Kähler $J : TM \longrightarrow TM, \quad g \in [g]$

$$J^2 = \pm \mathbf{1}, \qquad g(X,Y) = \mp g(JX,JY), \qquad \nabla J = 0$$

  - (Pseudo) Kähler+scalar–flat implies ASD.
  - (Pseudo) Kähler+ASD implies scalar–flat
  - ... but scalar flat+ASD does not imply Kähler!
- Lax pair: $\mathbf{e}_{AA'}$ are volume preserving, and the polynomials f_0, f_1 have double zero at $\lambda = 0$ and no other zeroes.

- Null Kähler: $N : TM \longrightarrow TM$,

$$N^2 = 0, \quad g(NX, Y) + g(X, NY) = 0, \quad \nabla N = 0, \quad \text{for } X, Y \in TM.$$

- $\Omega(X, Y) := g(X, NY)$, so ${\Lambda^2}_+(M) \cong \mathrm{Sym}^2(S'^*)$ implies the existence of parallel real spinor.
- There exist coordinates (x, y, w, z) and $\Theta : M \longrightarrow \mathbb{R}$ s.t. locally

$$\begin{aligned} g &= dwdx + dzdy - \Theta_{xx}dz^2 - \Theta_{yy}dw^2 + 2\Theta_{xy}dwdz, \\ N &= dw \otimes \partial/\partial y - dz \otimes \partial/\partial x. \end{aligned}$$

- Now impose ASD on g

$$\Box_g H = 0, \quad \text{where} \quad H := \Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2.$$

- ASD Null Kähler $\rightarrow$ 4th order integrable PDE with Lax pair. (Linearizes to $\Box^2\theta = 0$).

Curvature restrictions: Ricci flat

This is the special case of any of the previous there.

- Pseudo hypercomplex s.t.

$$\omega_I(.,.) = g(., I.), \quad \omega_S(.,.) = g(., S.), \quad \omega_T(.,.) = g(., T.)$$

  are closed,

- or ASD Null Kähler such that $H = 0$. 4th order PDE $\rightarrow$ Plebański's Second Heavenly Equation

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0.$$

- $(2,2)$ analog of the Riemannian hyper-Kähler structures.
- Lax pair has no vertical terms, and consists of volume preserving vector fields on M. ASDYM in 0 dimensions with G=SDiff(M). The heavenly equation is a gauge choice $M = T\Sigma$. This selects a parallel frame on the primed spin bundle $S' \rightarrow M$.

- Complexify: $(M_{\mathbb{C}}, [g_{\mathbb{C}}])$ complex four-manifold with a holomorphic ASD conformal structure. $\mathbb{P}(S') = M_{\mathbb{C}} \times \mathbb{CP}^1$ The Twistor space

$$\mathcal{PT} = \mathbb{P}(S')/\{L_0, L_1\},$$

is the three complex dimensional manifold of α–surfaces in $M_{\mathbb{C}}$.

- $x \in M_{\mathbb{C}} \longleftrightarrow l_x \cong \mathbb{CP}^1 \subset \mathcal{PT}, \quad N(l_x) = \mathcal{O}(1) \oplus \mathcal{O}(1)$
- $x, y \in M_{\mathbb{C}}$ are null separated iff $l_x, l_y \subset \mathcal{PT}$ intersect at a point.
- Real structure $\rho : M_{\mathbb{C}} \longrightarrow M_{\mathbb{C}}, \quad \rho(x) = \overline{x}$.
  - Maps α–surfaces to α–surfaces.
  - Antiholomorphic involution $\rho : \mathcal{PT} \longrightarrow \mathcal{PT}$. Fixed points on $3D$ real mfd $\mathcal{PT}_{\mathbb{R}}$
  - ρ fixes real equators of $\mathbb{RP}^1 \subset l_x$.

- Holomorphic fibration $\theta : \mathcal{PT} \to \mathbb{CP}^1$ corresponds to pseudo hypercomplex conformal structures.
- Preferred section of $\kappa^{-1/2}$ which vanishes at exactly two points on each twistor line corresponds to scalar–flat Kähler $g_\mathbb{C} \in [g_\mathbb{C}]$.
- Preferred section of $\kappa^{-1/4}$ corresponds to ASD null Kähler $g_\mathbb{C} \in [g_\mathbb{C}]$.
- Holomorphic fibration $\theta : \mathcal{PT} \to \mathbb{CP}^1$ and holomorphic isomorphism $\theta^*\mathcal{O}(-4) \cong \kappa$ correspond to Ricci–flat $g_\mathbb{C} \in [g_\mathbb{C}]$.
- These structure need to be ρ invariant for real $(2, 2)$ conformal structures.

($\kappa \longrightarrow \mathcal{PT}$ is the holomorphic canonical line bundle.)

Conformal Killing vector K

$$\mathcal{L}_K g = cg, \qquad g \rightarrow e^{2f} g, \quad c \rightarrow c + 2K(f).$$

Maps α surfaces to α surfaces.

- Non-null $g(K, K) \neq 0$. The three–dimensional space of orbits of K inherits an Einstein–Weyl structure. The resulting equations are integrable but not solvable. Leads to many known and new dispersionless integrable systems, e.g. $SU(\infty)$ Toda, dispersionless KP, ...
- Null $g(K, K) = 0$. Killing equations imply the existence of a two parameter family of β surfaces (ASD null surfaces). The ASD conformal structure in 4D gives rise to a projective structure on the 2D space of β surfaces. The problem is completely solvable. The general solution depends on wether K is twisting or not.

Non–null Symmetries

- $M \longrightarrow W = M/U(1)$. Weyl structure $(W, [h], D)$ in $(2,1)$ signature:

$$Dh = \omega \otimes h, \qquad h \longrightarrow e^{2f}h, \quad \omega \longrightarrow \omega + df.$$

- $h = |K|^{-2}g - |K|^{-4}K \otimes K, \qquad \omega = 2|K|^{-2} *_g (K \wedge dK).$
- α surfaces in $M \longrightarrow$ totally geodesic null surfaces in W.
- Conformally invatiant Einstein-Weyl equations:
  Traceless symmetrised Ricci tensor of D is proportional to h.
- Example – dispersionless Kadomtsev–Petviashvili equation.
  ASD Null Kähler with symmetry, $\mathcal{L}_K N = 0$ s.t.

$$h = dy^2 - 4dxdt - 4udt^2, \qquad \omega = -4u_x dt$$

where $u = u(x, y, t)$ satisfies the dKP equaton

$$(u_t - uu_x)_x = u_{yy}.$$

This EW structure admits a parallel weighted vector.

- 2D Projective structure $(U, [\Gamma])$. Equivalence class of torsion free connections with the same unparametrised geodesics.
- $(x, y) \in U$. The geodesic equation

$$\frac{d^2y}{dx^2} = A_3(x,y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x,y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x,y)\Big(\frac{dy}{dx}\Big) + A_0(x,y),$$

- Integral curves lift to integral curves of the spray $\boldsymbol{\Theta}$ on $\mathbb{P}(TU)$

$$\boldsymbol{\Theta} = \partial_x + z\partial_y + (A_0 + zA_1 + z^2A_2 + z^3A_3)\partial_z.$$

- Nonlinear Penrose–Radon Transform

$$U \longleftarrow \quad \mathbb{P}(TU) \quad \stackrel{\boldsymbol{\Theta}}{\longrightarrow} \mathcal{Z}$$

point	$\longleftrightarrow$	$\mathcal{O}(1)$ real line
geodesic	$\longleftrightarrow$	point.

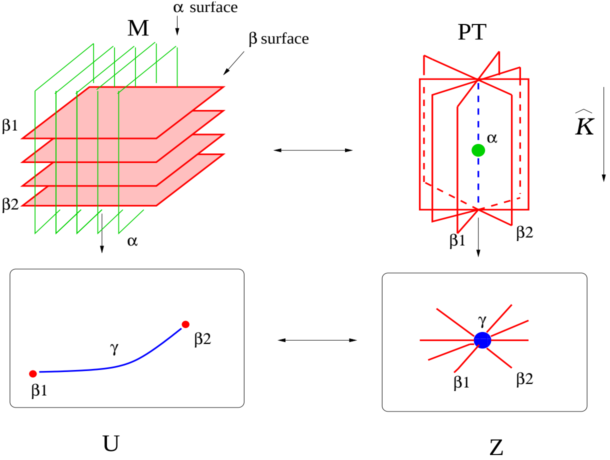
Null Symmetries

- $K = \iota \otimes o$ where $\iota \in \Gamma(S), o \in \Gamma(S')$.
- 2D Frobenius integrable distribution $\mathcal{D}_\iota = \text{Ker}\Omega_\iota$
- 2D space of special β surfaces $U = M/\mathcal{D}_\iota$
- U admits a projective structure
- Projective spray distribution $\mathbf{\Theta} = \{L_0, L_1, K\}/\mathcal{D}_\iota$ defined on $\mathbb{P}(TU) = (\mathbb{P}(S') = M \times \mathbb{RP}^1)/\mathcal{D}_\iota$.

$$\begin{array}{ccc} M & \longleftarrow & M \times \mathbb{RP}^1 \\ \mathcal{D}_\iota \downarrow & & \downarrow \{L_0, L_1, K\} \\ U & \longleftarrow & \mathbb{P}(TU) \end{array}$$

- Reconstruction of the ASD conformal structure with null symmetry: Extend the spray on $\mathbb{P}(TU)$ to a Lax pair on $\mathbb{P}(S')$.

Null Symmetries. Twistor Correspondence



Null Symmetries. Examples

- $U = T^2$, $K \wedge dK = 0$. Kodaira surface $M = \mathbb{C}^2/\Gamma \rightarrow U$.

$$g = d\phi dy - dzdx - Q(x,y)dy^2. \qquad \text{Ricci–flat Kähler.}$$

- $(U, [\Gamma])$ Flat, $K \wedge dK \neq 0$.

$$g = (d\phi + Q(x,y)dx)(dy - zdx) - dzdx. \qquad \text{Pseudo hypercomplex.}$$

- General $(U, [\Gamma])$ (given by $A_i(x,y)$). General $K \wedge dK \neq 0$.

$$\begin{aligned} g &= \partial_z^2 G(dz - (A_0 + zA_1 + z^2 A_2 + z^3 A_3)dx)dx \\ &- (d\phi + A_3 \partial_z G dy + (A_2 \partial_z G + 2A_3(z\partial_z G - G)(dy - zdx) \end{aligned}$$

where $G = G(x,y,z)$ satisfies a linear P.D.E $\mathbf{\Theta}(\partial_z G) = 0$.

Applications: Time dependent 3+1 space–times

- Lift a $(2,2)$ ASD vacuum metric with non–null S^1 symmetry $\partial/\partial\phi$ to $3+2$ dimensions with two commuting Killing vectors

$$g_{(3,2)} = g_{(2,2)} + dz^2.$$

- Perform a Kaluza–Klein reduction along the time like symmetry $\partial/\partial\phi$.
- This yields a $3+1$ dimensional solution to Einstein–Maxwell–Dilaton equations
- ... but the Maxwell field has negative energy, and both gravity and electromagnetism are attractive forces. Peculiar physical consequences: e.g. black holes can increase their mass by radiating photons out!
- Could generalise to F theory with $ds^2 = g_{(2,2)} + d\mathbf{x_8}^2$.

- $g_{(2,2)} = Vh_{(2,1)} - V^{-1}(d\theta + A)^2$, ASD+Ricci flat.
- $g_{(3,2)} = \exp(-2\Phi/\sqrt{3})G_{(3,1)} - \exp(4\Phi/\sqrt{3})(d\theta + A)^2$.
- In $(3+1)$ dimensions: metric $G_{(3,1)}$, dilaton Φ, Maxwell potential A.
- Example. $g_{(2,2)} = (2,2)$ Taub-Nut

$$
\begin{aligned}
G_{(3,1)} &= V^{-1/2}dz^2 + V^{1/2}(d\rho^2 + \rho^2 d\theta^2 - dt^2), \\
\Phi &= -\frac{\sqrt{3}}{4}\log V, \qquad A = V^{-1}dz, \qquad V = \Big(1 + \frac{m}{\sqrt{\rho^2 - t^2}}\Big)^{-1/2}
\end{aligned}
$$

- $\theta \cong \theta + 4\pi$ for regular initial data. This solution represents a charged particle moving with the along the z axis. Unstable and invariant under $\mathbb{R} \times SO(2,1)$ (Tachyon).

Outlook

- Conformal anti-self-duality in $2+2$ dimensions.
- Richer than the Riemannian case.
  - Null structures, wave–like solutions, non-analyticity.
  - Plenty of non–trivial compact and complete examples.
- Unifying framework for dispersionless integrable systems in $2+1$ dimensions.
- Reductions to $(2+1)$ dimensional Einstein–Weyl structures, and 2 dimensional projective structures.
- Connections with physics (??): $N=2$ superstring, timelike Kaluza–Klein reductions, F–theory.