Classical Dynamics: Example Sheet 1

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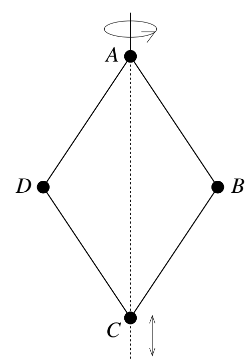
1. Practice in applications of Variational Calculus (*optional - certainly not at an expense of other questions*).

(a) Prove that the shortest distance between two points in (Euclidean) space is a straight line.

(b) Show that the geodesics (i.e. shortest distances between two points) of a spherical surface are great circles, i.e. circles whose centres lie at the centre of the sphere.

(c) Show that the solution to the *brachistochrone problem* is an (inverted) cycloid with a cusp at the initial point at which the particle is released. *Hint: the differential equation of a cycloid created by a circle of radius a is $dy/dx = \sqrt{(2a-y)/y}$.*

2. Four equal light rods of length l are hinged together to form a rhombus $ABCD$, which lies in the vertical plane. Each of the vertices has mass m. The vertex A is fixed, while C lies directly beneath it and is free to slide up and down. The whole system rotates with frequency ω around the vertical axes AC.



Identify suitable generalized coordinate(s) and write down the Lagrangian of the system.

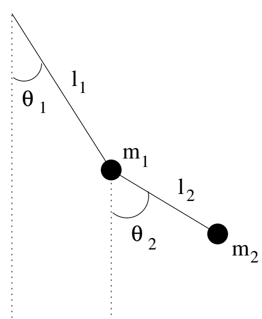
3. The circular hoop of radius a lies in the vertical plane. The hoop rotates with frequency ω around fixed vertical axis which goes through its centre, O. The bead, of mass m, is threaded on the hoop and moves without friction. Its location is denoted by A. The angle between the line OA and the downward vertical is $\psi(t)$.

(a) Using Lagrangian formalism derive second order differential equation for parameter ψ.

(b) Derive the same differential equation using Newtonian formalism. Compare the two methods.

(c) Assume now that the hoop is free to rotate about the vertical axis without friction. Write down the Lagrangian of the system. Find the additional conserved quantity.

4. A double pendulum is drawn below. Two light rods, of lengths l_1 and l_2 , oscillate in the same plane. Attached to them are masses m_1 and m_2 . How many degrees of freedom does the system have? Write down the Lagrangian describing the dynamics. Derive the equations of motion.

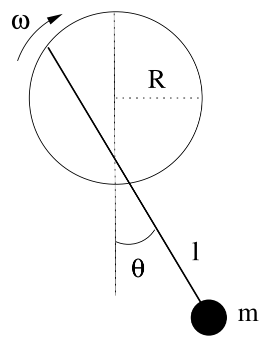


5. The pivot of a simple pendulum is attached to a disc of radius R, which rotates in the plane of the pendulum, with angular velocity ω . (See the diagram below). Write down the Lagrangian, and derive the equations of motion for the dynamical variable θ .

6. A particle moves in one dimension, in a potential $V(x)$, where x is the spatial co–ordinate. The dynamics is governed by the Lagrangian

$$L = \frac{1}{12} m^2 \dot{x}^4 + m \dot{x}^2 V - V^2 \ . \tag{1}$$

Show that the resulting equation of motion is identical to that which arises from the more traditional Lagrangian, $L = \frac{1}{2} m \dot{x}^2 - V$.



7. The Lagrangian for a relativistic point particle, of mass m, is

$$L = -mc^2 \sqrt{1 - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/c^2} - V(\mathbf{r}), \qquad (2)$$

where c is the speed of light. Derive the equation of motion, and show that it reduces to Newton's equation of motion in the limit $|\dot{\mathbf{r}}| \ll c$.

8. An electron, of mass m and charge $-e$, moves in a magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})$. The Lagrangian for the motion is

$$L = \tfrac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - e \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}). \qquad (3)$$

Show that Lagrange's equations reproduce the Lorentz force law for the electron. Then:

(a) With respect to cylindrical polar coordinates, (r, θ, z), consider the vector potential,

$$\mathbf{A} = (0, f(r)/r, 0). \qquad (4)$$

At some initial time, the electron is at a distance r_0 from the z–axis; its velocity is then in the (r, z)–plane. Show that the electron's angular velocity about the z–axis is given by

$$\dot{\theta} = \frac{e}{mr^2} [f(r) - f(r_0)]. \qquad (5)$$

(b) [Again, with respect to cylindrical polar coordinates.] Consider the (different) vector potential,

$$\mathbf{A} = (0, r\, g(z), 0), \qquad (6)$$

where $g(z) > 0$. Find two constants of the motion. The electron is projected from a point, $(r_0\,, \theta_0\,, z_0)$, with velocity $(0\,, 2\,e\,r_0\,g(z_0)/m\,, 0)$. Show that the electron will then describe a circular orbit, provided that $g'(z_0) \;=\; 0$. Show that these orbits are stable against small translations in the z–direction, provided that $g'' > 0$.

9. A particle, of mass m_1 , is restricted to move on a circle of radius R_1 in the plane $z\;=\;0$, with centre at $(x, y) = (0, 0)$. A second particle, of mass m_2 , is restricted to move on a circle of radius R_2 in the plane $z\;=\;c$, with centre at $(x, y) = (0, a)$. The two particles are connected by a spring; the resulting potential is

$$V \;=\; \tfrac{1}{2}\,\omega^2\,d^2 \;\;,$$

where d is the distance between the particles.

(a) Identify the two generalised coordinates and write down the Lagrangian of the system.

(b) Write down the Lagrangian in the case the circles lie directly beneath each other, $a = 0$, and identify a conserved quantity that appears in this case.

10. Two particles, each of mass m , are connected by a light rope, of length l . One particle sits on a smooth horizontal table at a distance r from a hole, through which the rope is threaded. The second particle hangs straight beneath the hole.

(a) Assume that the second particle hangs straight beneath the hole. Write down the Lagrangian of the system in terms of r and a variable ψ , describing the angle that the first particle makes, with respect to a fixed axis. Identify an ignorable coordinate. Write down the equation of motion for the remaining co–ordinate, assuming that the rope remains taut.

(b) Assume now that the second particle oscillates beneath the table, as a spherical pendulum. How many degrees of freedom does the system now have? Write down the Lagrangian describing this motion, assuming that the rope remains taut at all times. How many ignorable coordinates are there?

11. Consider a system with n dynamical degrees of freedom, and generalised co–ordinates denoted by q^a , $a = 1\,,\;\ldots\,,\,n$. The most general form for a purely kinetic Lagrangian is

$$L \;=\; \tfrac{1}{2}\,g_{ab}(q^1, ..., q^n)\,\dot{q}^a\,\dot{q}^b \;\;, \qquad (7)$$

where the summation convention is being used. The functions $g_{ab} = g_{ba}$ depend on the generalised co–ordinates. Assume that $\det(g_{ab}) \neq 0$, whence, the inverse matrix, g^{ab}, exists (obeying $g^{ab}\, g_{bc} = \delta^a{}_c$). Show that Lagrange's equations for this system are given by

$$\ddot{q}^a + \Gamma^a_{bc}\, \dot{q}^b\, \dot{q}^c = 0 , \tag{8}$$

where one defines

$$\Gamma^a_{bc} = \tfrac{1}{2}\, g^{ad} \left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d} \right) . \tag{9}$$

Side Remark: The functions g_{ab} define a *metric* on the configuration space, and the equations (**??**) are known as the *geodesic* equations. In addition to appearing naturally in differential geometry, these equations arise in general relativity, describing the motion of a particle falling freely under gravity (where a gravitational field is described by a curved space–time). Lagrangians of the form (**??**) appear in many other areas of physics, such as the study of solids, of nuclear forces and of string theory. In these 'physics' contexts, systems with a Lagrangian of the form (**??**) are known as *sigma models.*