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First integrals of affine connections and Hamiltonian systems of hydrodynamic type

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We find necessary and sufficient conditions for a local geodesic flow of an affine connection on a surface to admit a linear first integral. The conditions are expressed in terms of two scalar invariants of differential orders 3 and 4 in the connection. We use this result to find explicit obstructions to the existence of a Hamiltonian formulation of Dubrovin–Novikov type for a given one-dimensional system of hydrodynamic type. We give several examples including Zoll connections, and Hamiltonian systems arising from two-dimensional Frobenius manifolds.

Keywords: affine connections; Hamiltonian systems; hydrodynamic type.

1. Introduction

The existence of a first integral of a geodesic flow of an affine connection puts restrictions on the form of the connection. A generic connection admits no first integrals. If the connection arises from a metric, and the first integral is linear in velocities, then the metric admits a one-parameter group of isometries generated by a Killing vector field. Characterizing metrics which admit Killing vectors by local tensor obstructions is a classical problem which goes back at least to Darboux [1], and can be solved completely in two dimensions. The analogous characterization of non-metric affine connections has not been carried over in full.[1] It is given in Theorem 1.1, where we construct two invariant scalar obstructions to the existence of a linear first integral. A non-metric connection can (unlike a Levi–Civita connection) admit precisely two independent linear local first integrals. This case will also be characterized by a tensor obstruction.

As an application of our results we shall, in Section 3, characterize one-dimensional systems of hydrodynamic type which admit a Hamiltonian formulation of the Dubrovin–Novikov type [4]. The existence of such formulation leads to an over-determined system of PDEs, and we shall show (Theorem 1.2) that this system is equivalent to a condition that a certain non-metric affine connection admits a linear first integral. This, together with Theorem 1.1 will lead to a characterization of Hamiltonian, bi-Hamiltonian and tri-Hamiltonian systems of hydrodynamic type. In Section 4 we shall give examples of

[1] The remarkable exception is the paper of Levine [2] and its extension [3] where the necessary condition for the existence of a first integral was found, albeit not in a form involving the Schouten and Cotton tensors. The sufficient conditions found in [2] are not all independent. Levine gives seven tensor conditions, where in fact two scalar conditions suffice.

connections resulting from hydrodynamic type systems. In particular we shall show that systems arising from two-dimensional Frobenius manifolds are tri-Hamiltonian.

In the remaining part of the Introduction we shall state our main results. Let ∇ be a torsion-free affine connection of differentiability class C^4 on a simply connected orientable surface Σ (so we require the transition functions of Σ to be of class at least C^6). A curve $\gamma : \mathbb{R} \to \Sigma$ is an affinely parameterized geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$, or equivalently if

$$\ddot{X}^a + \Gamma^a_{bc}\dot{X}^b\dot{X}^c = 0, \quad a, b, c = 1, 2, \tag{1.1}$$

where $X^a = X^a(\tau)$ is the curve γ expressed in local coordinates X^a on an open set $U \subset \Sigma$, τ is an affine parameter, Γ^c_{ab} are the Christoffel symbols of ∇ and we use the summation convention. A linear function on $T\Sigma$ given by $\kappa = K_a(X)\dot{X}^a$ is called a *first integral* if $d\kappa/d\tau = 0$ when (1.1) holds, or equivalently if

$$\nabla_{(a}K_{b)} = 0. \tag{1.2}$$

The following Theorem gives local necessary and sufficient conditions for a connection to admit one, two or three linearly independent solutions to the Killing equation (1.2). The necessary conditions involve vanishing of obstructions I_N and T given by (2.7) and (2.11)—for these to make sense the connection needs to be at least three times differentiable.

THEOREM 1.1 The necessary condition for a C^4 torsion-free affine connection ∇ on a surface Σ to admit a linear first integral is the vanishing, on Σ, of invariants I_N and I_S given by (2.7) and (2.9), respectively. For any point $p \in \Sigma$ there exists a neighbourhood $U \subset \Sigma$ of p such that conditions $I_N = I_S = 0$ on U are sufficient for the existence of a first integral on U. There exist precisely two independent linear first integrals on U if and only if the tensor T given by (2.11) vanishes and the skew part of the Ricci tensor of ∇ is non-zero on U. There exist three independent first integrals on U if and only if the connection is projectively flat and its Ricci tensor is symmetric.

This Theorem will be established by constructing (Proposition 2.1) a prolongation connection D on a rank-three vector bundle $\Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma)$ for the over-determined system (1.2), and restricting the holonomy of its curvature when one, two or three parallel sections exist. In Proposition 2.2 we shall find all local normal forms of connections from Theorem 1.1 which admit precisely two linear first integrals.

Finally we shall consider one-dimensional systems of hydrodynamic type. Any such system with two dependent variables (X^1, X^2) and two independent variables (x, t) can be written in the so-called Riemann invariants as

$$\frac{\partial X^1}{\partial t} = \lambda^1(X^1, X^2)\frac{\partial X^1}{\partial x}, \quad \frac{\partial X^2}{\partial t} = \lambda^2(X^1, X^2)\frac{\partial X^2}{\partial x}, \tag{1.3}$$

where $\lambda^1 \neq \lambda^2$ at a generic point. This system admits a Hamiltonian formulation of the Dubrovin–Novikov type, if it can be written as

$$\frac{\partial X^a}{\partial t} = \Omega^{ab}\frac{\delta H}{\delta X^b}, \tag{1.4}$$

where $H[X^1, X^2] = \int_{\mathbb{R}} \mathcal{H}(X^1, X^2)dx$ is the Hamiltonian of hydrodynamic type, and the Poisson structure on the space of maps $\mathrm{Map}(\mathbb{R}, U)$ is given by

$$\Omega^{ab} = g^{ab}\frac{\partial}{\partial x} + b^{ab}_c\frac{\partial X^c}{\partial x}.$$

The Jacobi identity imposes severe constraints on $g(X^a)$ and $b(X^a)$—see Section 3 for details. We shall prove

THEOREM 1.2 The hydrodynamic type system (1.3) admits one, two or three Hamiltonian formulations with hydrodynamic Hamiltonians if and only if the affine torsion-free connection ∇ defined by its non-zero components

$$\Gamma^1_{11} = \partial_1 \ln A - 2B, \quad \Gamma^2_{22} = \partial_2 \ln B - 2A, \quad \Gamma^1_{12} = -\Big(\frac{1}{2}\partial_2 \ln A + A\Big), \quad \Gamma^2_{12} = -\Big(\frac{1}{2}\partial_1 \ln B + B\Big),$$
$$\text{where} \quad A = \frac{\partial_2\lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1\lambda^2}{\lambda^1 - \lambda^2} \quad \text{and} \quad \partial_a = \partial/\partial X^a \tag{1.5}$$

admits one, two or three independent linear first integrals, respectively.

This Theorem, together with Theorem 1.1 leads to explicit obstructions for the existence of a Hamiltonian formulation (1.4).

2. Killing operator for affine connection

Given an affine connection ∇ on a surface Σ, its curvature is defined by

$$[\nabla_a, \nabla_b]X^c = R_{ab}{}^c{}_d X^d.$$

In two dimensions the projective Weyl tensor vanishes, and the curvature can be uniquely decomposed as

$$R_{ab}{}^c{}_d = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + B_{ab}\delta_d{}^c, \tag{2.1}$$

where P_{ab} is the Schouten tensor related to the Ricci tensor $R_{ab} = R_{ca}{}^c{}_b$ of ∇ by $\mathrm{P}_{ab} = (2/3)R_{ab} + (1/3)R_{ba}$ and $B_{ab} = \mathrm{P}_{ba} - \mathrm{P}_{ab} = -2\mathrm{P}_{[ab]}$. We shall assume that Σ is orientable, and choose a volume form ϵ_{ab} on Σ. We shall also introduce ϵ^{ab} such that $\epsilon^{ab}\epsilon_{cb} = \delta^a_c$. These skew-symmetric tensors are used to raise and lower indices according to $V^a = \epsilon^{ab}V_b$ and $V_a = \epsilon_{ba}V^b$. Then

$$\nabla_a \epsilon_{bc} = \theta_a \epsilon_{bc},$$

where $\theta_a = (1/2)\epsilon^{bc}\nabla_a\epsilon_{bc}$. Set $\beta = B_{ab}\epsilon^{ab}$.

PROPOSITION 2.1 There is a one-to-one correspondence between solutions to the Killing equations (1.2), and parallel sections of the prolongation connection D on a rank-three vector bundle $E = \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma) \to \Sigma$ defined by

$$D_a \begin{pmatrix} K_b \\ \mu \end{pmatrix} = \begin{pmatrix} \nabla_a K_b - \epsilon_{ab}\mu \\ \nabla_a \mu - \left(\mathrm{P}^b{}_a + \frac{1}{2}\beta\delta^b{}_a\right)K_b + \mu\theta_a \end{pmatrix}. \tag{2.2}$$

Proof. Dropping the symmetrization in (1.2) implies the existence of μ such that $\nabla_a K_b = \mu\epsilon_{ab}$. Differentiating this equation covariantly, skew-symmetrizing over all indices and using the curvature decomposition (2.1) together with the Bianchi identity yields the statement of the Proposition. □

The connection D is related to the standard tractor connection in projective differential geometry (see e.g. [6]). In the proof of Theorem 1.1 we shall find the integrability conditions for the existence of parallel sections of this connection. This will lead to a set of invariants of an affine connection ∇.

Proof of Theorem 1.1. The integrability conditions $(\nabla_a\nabla_b - \nabla_b\nabla_a)\mu = 0$ give the algebraic condition

$$F^a K_a + \beta\mu = 0, \quad \text{where} \quad F^a = \frac{1}{3}\epsilon^{ab}(L_b - \epsilon^{cd}\nabla_b B_{cd}) \tag{2.3}$$

and $L_b \equiv \epsilon^{cd}\nabla_c \mathrm{P}_{db}$ is the Cotton tensor of ∇. Geometrically the condition (2.3) means that the curvature of D has rank at most one, and annihilates a parallel section of D. Applying ∇_a to the condition (2.3), and using the vanishing of (2.2) leads to two more algebraic conditions

$$M_a{}^b K_b + N_a\mu = 0, \tag{2.4}$$

where

$$M_a{}^b = \nabla_a F^b + \left(\mathrm{P}^b{}_a + \frac{1}{2}\delta^b{}_a\beta\right)\beta, \quad N_a = -F_a + \nabla_a\beta - \beta\theta_a.$$

Multiplying the equation (2.3) by $2\theta_a$, and adding the resulting expression to (2.4) results in $M_a{}^b \to M_a{}^b + 2\theta_a F^b$ and $N_a \to N_a + 2\theta_a\beta$. We can use this freedom to get rid of θ^a from the expressions for M and N. This yields

$$M_a{}^b = \frac{1}{3}\epsilon^{bc}\epsilon^{de}(\nabla_a Y_{dec} - \nabla_a\nabla_c B_{de}) + \beta\, \mathrm{P}^b{}_a + \frac{1}{2}\beta^2\delta^b{}_a, \quad N_a = -F_a + \epsilon^{bc}\nabla_a B_{bc}, \tag{2.5}$$

where $Y_{cdb} = \nabla_{[c}\mathrm{P}_{d]b}$. Therefore a parallel section $\Psi \equiv (K_1, K_2, \mu)^T$ of D must satisfy a system of three linear algebraic equations which we write in a matrix form as

$$\mathcal{M}\Psi \equiv \begin{pmatrix} F^1 & F^2 & \beta \\ M_1{}^1 & M_1{}^2 & N_1 \\ M_2{}^1 & M_2{}^2 & N_2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \mu \end{pmatrix} = 0. \tag{2.6}$$

A necessary condition for the existence of a non-zero parallel section Ψ is therefore the vanishing of the determinant of the matrix $\mathcal{M}$. This gives the first obstruction which we write as a vanishing of the relative scalar invariant

$$I_N = \epsilon_{cd}\epsilon^{be}M_e{}^c\Big(N_bF^d - \frac{1}{2}\beta M_b{}^d\Big). \tag{2.7}$$

This invariant has weight -5: if we replace ϵ_{ab} by $e^f\epsilon_{ab}$, where $f:\Sigma\to\mathbb{R}$, then $I_N\to e^{-5f}I_N$. Thus $I_N\otimes(\epsilon_{ab}dX^a\wedge dX^b)^{\otimes 5}$ is an invariant. The vanishing of I_N is not sufficient for the existence of a non-zero parallel section. To assure sufficiency assume that $I_N=0$. Rewrite (2.3) and (2.4) as

$$V^\alpha\Psi_\alpha = 0, \quad (D_aV^\alpha)\Psi_\alpha = 0, \quad \alpha = 1,\dots,3,$$

where $V=(F^1,F^2,\beta)$ in the formula above is a section of the dual bundle E^*, and D_a is the dual connection inherited from (2.2). We continue differentiating, and adding the linear equations on Ψ. The Frobenius theorem tells us that the process terminates once a differentiation does not add any additional independent equations, as then the rank of the matrix of equations on Ψ stabilizes and does not grow. The space of parallel sections of D has dimension equal to 3 (the rank of the bundle E) minus the number of independent equations on Ψ. Therefore the sufficient condition for the existence of a Killing form assuming that $I_N=0$ is

$$\operatorname{rank}\begin{pmatrix} V \\ D_1V \\ D_2V \\ D_1D_1V \\ D_{(1}D_{2)}V \\ D_2D_2V \end{pmatrix} < 3. \tag{2.8}$$

If $I_N=0$ and $V\neq 0$, then

$$cV + c_1D_1V + c_2D_2V = 0,$$

where (c,c_1,c_2) are some functions on U, and (c_1,c_2) are not both zero. This implies that the term $D_{(1}D_{2)}V$ in (2.8) is a linear combination of all other terms, and can be disregarded. Now, suppose that $D_1V=0$. Then (2.8) becomes $\det(V,D_2V,D_2D_2V)=0$. Equivalently, if $D_2V=0$ then (2.8) becomes $\det(V,D_1V,D_1D_1V)=0$. We conclude that (2.8) is equivalent to

$$I_s = W_{abc} \equiv \det(V, D_aV, D_{(b}D_{c)}V) = 0 \tag{2.9}$$

as it is easy to show that the condition above implies (2.8). In fact vanishing of W_{abc} gives just one independent condition: If $c_2\neq 0$, then the sufficient condition is $W_{111}=0$, and if $c_1\neq 0$, then it is $W_{222}=0$. The explicit tensor form of the obstruction W is

$$W_{acd} = F_bM_a{}^bV_{(cd)} - F_bU^b_{(cd)}N_a + \beta M_{ab}U^b_{(cd)}, \quad \text{where} \tag{2.10}$$

$$U^b_{ca} = \epsilon^{bd}\epsilon^{ef}[\frac{1}{3}(\nabla_c\nabla_a Y_{efd} - \nabla_c\nabla_a\nabla_d B_{ef}) + \nabla_c(B_{ef}P_{da})]$$
$$+\frac{1}{2}\epsilon^{ef}\epsilon^{gh}\nabla_c(B_{ef}B_{gh})\delta^b_a + \epsilon^{bd}N_a(P_{dc} + \frac{1}{2}\beta\epsilon_{cd}) \quad \text{and}$$
$$V_{ca} = -M_{ac} - \frac{1}{3}\epsilon^{de}(\nabla_c\nabla_d P_{ea} - \nabla_c\nabla_a B_{de}).$$

We shall now consider the case when there exist precisely two independent solutions to the Killing equation (1.2) (note that this situation does not arise if ∇ is a Levi–Civita connection of some metric, as then the number of Killing vectors can be 0, 1 or 3 - the last case being projectively flat). Therefore the rank of the matrix $\mathcal{M}$ in (2.6) is equal to one. We find that this can happens if and only if $\beta \neq 0$ and

$$T_a{}^b = 0, \quad \text{where} \quad T_a{}^b \equiv N_a F^b - \beta M_a{}^b. \tag{2.11}$$

This condition guarantees the vanishing of all two-by-two minors of $\mathcal{M}$.

Finally, there exist three independent parallel sections of D iff the curvature of D vanishes, or equivalently if the matrix $\mathcal{M}$ vanishes. This condition is equivalent to the projective flatness of the connection ∇ together with the condition $\beta = 0$. □

Remarks

- If the connection ∇ is special (i.e. the Ricci tensor is symmetric, or equivalently $\beta = 0$) then $I_N = -3^{-3}\nu_5$, where

$$\nu_5 \equiv L^a L^b \nabla_a L_b$$

  is the Liouville projective invariant [7, 8], and the indices are rised with a parallel volume form. Note that, unlike ν_5, the obstruction I_N is not invariant under the projective changes of connection (see equation (3.4) in Section 3). The sufficient condition (2.9) is then equivalent to the vanishing of the invariant w_1 constructed by Liouville for second order ODEs in [7].
- Theorem 1.1 generalizes a well known characterization of metrics which admit a Killing vector as those with functionally dependent scalar invariants. See [9] or [10] where a 3 by 3 matrix analogous to $\mathcal{M}$ has been constructed. In this case $N = -F = \frac{1}{3} * dR$, where R is the scalar curvature, and $*$ is the Hodge operator of the metric g. The invariant (2.7) reduces to[2]

$$I_N := *\frac{1}{432} dR \wedge d(|\nabla R|^2).$$

- Any affine connection ∇ on Σ corresponds to a family of neutral signature anti-self-dual Einstein metrics on a certain rank-two affine bundle $M \to \Sigma$, given by [11]

$$G = \left(d\xi_a - \left(\Gamma^c_{ab}\xi_c - \Lambda\xi_a\xi_b - \Lambda^{-1}\mathrm{P}_{ba}\right) dX^b\right) \odot dX^a,$$

[2] The prolongation procedure in [10] has been carried over in the Riemannian case. The additional subtlety in the Lorentzian signature arises if ∇R is a non-zero null vector. We claim that no non-zero Killing vectors exist in this case. To see it, assume that a Lorentzian metric admits a Killing vector K. If K is null, then the metric is flat with $R = 0$. Otherwise it can locally be put in the form $dY^2 - f(Y)^2 dX^2$ for some $f = f(Y)$. Imposing the condition $|\nabla R|^2 \equiv 0$ leads to $R = \text{const}$.

where ξ_a are local coordinates on the fibres of M, and -24Λ is the Ricci scalar. These metrics admit a linear first integral iff ∇ admits a projective vector field.

In the metric case, a Levi–Civita connection cannot admit precisely two local linear first integrals, as β (which is proportional to the skew part of the Ricci tensor) vanishes. In the following Proposition we shall explicitly find all local normal forms of non-metric affine connections which admit two first integrals.

PROPOSITION 2.2 Let ∇ be an affine connection on a surface Σ which admits exactly two non-proportional linear first integrals which are independent at some point $p \in \Sigma$. Local coordinates (X, Y) can be chosen on an open set $U \subset \Sigma$ containing p such that

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{c}{2}, \quad \Gamma^2_{11} = \frac{P_X}{Q}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{P_Y + Q_X - cP}{2Q}, \quad \Gamma^2_{22} = \frac{Q_Y}{Q}, \tag{2.12}$$

and all other components vanish, where c is a constant equal to 0 or 1, and (P, Q) are arbitrary functions of (X, Y).

Proof. Let the one-forms K and L be two solutions to the Killing equation. If K is closed, then there exist local coordinates (X, Y) on U such that $K = dX$, and the corresponding first integral is $\dot{X}$. Therefore $\ddot{X} = 0$ and the connection components Γ^1_{ab} vanish. Let the second solution of the Killing equation be of the form $L = PdX + QdY$ for some functions (P, Q). Imposing

$$\frac{d}{d\tau}(P\dot{X} + Q\dot{Y}) = 0$$

yields the non-zero components of the connection given by (2.12) with $c = 0$. If $dK \neq 0$, then coordinates (X, Y) can be chosen so that $K = e^Y dX$. The condition $d/d\tau(e^Y \dot{X}) = 0$ gives $\Gamma^1_{12} = 1/2$. Imposing the existence of the second integral $(P\dot{X} + Q\dot{Y})$ yields the connection (2.12) with $c = 1$. □

Note that in both cases the ODEs for the unparameterized geodesics also admit a first integral, given by $e^{-cY}(P + Y'Q)$, where $' = d/dX$. Conversely if a second order ODE cubic in Y' representing projective equivalence class $[\nabla]$ of affine connections admits a first integral linear in Y', then $[\nabla]$ contains a connection of the form (2.12) with $c = 0$. To see it consider a second order ODE of the form $(P+Y'Q)' = 0$, where (P, Q) are arbitrary functions of (X, Y) and write it in the form

$$Y'' = \Gamma^1_{22}(Y')^3 + (2\Gamma^1_{12} - \Gamma^2_{22})(Y')^2 + (\Gamma^1_{11} - 2\Gamma^2_{12})Y' - \Gamma^2_{11}. \tag{2.13}$$

Equation (2.13) arises from eliminating the affine parameter τ between the two ODEs (1.1). Thus its integral curves are unparameterized geodesics of the affine connection ∇.

3. Hamiltonian systems of hydrodynamic type

An n-component (1+1) system of hydrodynamic type has the form $\partial_t u^a = v^a{}_b(u)\partial_x u^b$, where $u^a = u^a(x, t)$ and $a, b = 1, \dots, n$. From now on we shall assume that $n = 2$ and that the matrix v is diagonalizable at some point with distinct eigenvalues, in which case there always exists (in a neighbourhood of this point) two distinct functions (called the Riemann invariants) X^1 and X^2 of (u^1, u^2) such that the system is diagonal, i.e. takes the form (1.3) for some $\lambda^a(X^b)$ and can be linearized by a hodograph transformation interchanging dependent (X^1, X^2) and independent (x,t) coordinates.

The hydrodynamic type system is said to admit a local Hamiltonian formulation with a Hamiltonian of hydrodynamic type [4, 12], if there exists a functional $H[X^1, X^2] = \int_{\mathbb{R}} \mathcal{H}(X^1, X^2)dx$, where the density $\mathcal{H}$ does not depend on the derivatives of X^a and such that (1.4) holds for some functions $g^{ab}(X)$ and $b^{ab}_c(X)$. If the matrix g^{ab} is non-degenerate, then the Poisson bracket

$$\{F,G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta X^a}\Big(g^{ab}\frac{\partial}{\partial x} + b^{ab}_c \frac{\partial X^c}{\partial x}\Big)\frac{\delta G}{\delta X^b}dx$$

is skew-symmetric if g^{ab} is symmetric and the metric $g = g_{ab}dX^a dX^b$, where $g_{ab}g^{bc} = \delta_a{}^c$ is parallel with respect to the connection with Christoffel symbols γ^c_{ab} defined by $b^{ab}_c = -g^{ad}\gamma^b{}_{dc}$. The Jacobi identity then holds iff the metric g is flat, and the connection defined by γ^c_{ab} is torsion-free. The hydrodynamic type systems which admit a Hamiltonian of hydrodynamic type possess infinitely many Poisson commuting first integrals, and are integrable in the Arnold–Liouville sense [13].

Proof of Theorem 1.2. It was shown in [4] that a hydrodynamic type system in Riemann invariants is Hamiltonian in the sense defined above if and only if there exists a flat diagonal metric

$$g = k^{-1}d(X^1)^2 + f^{-1}d(X^2)^2 \tag{3.1}$$

on a surface U with local coordinates (X^1, X^2) such that

$$\partial_2 k + 2Ak = 0, \quad \partial_1 f + 2Bf = 0, \tag{3.2}$$

where f, k are functions of (X^1, X^2), and (A, B) are given by (1.5). Flatness of the metric g yields

$$(\partial_2 A + A^2)f + (\partial_1 B + B^2)k + \frac{1}{2}A\partial_2 f + \frac{1}{2}B\partial_1 k = 0. \tag{3.3}$$

We verify that equations (3.2) and (3.3) are equivalent to the Killing equations (1.2) for an affine torsion-free connection ∇ on U defined by (1.5) where $K_1 = Af, K_2 = Bk$. □

Computing the relative invariants I_N and I_S gives explicit but complicated (albeit perfectly manageable by MAPLE) obstructions given in terms of (λ^1, λ^2) and their derivatives of order up to 6. These obstructions, together with the tensor (2.11) and the Cotton tensor of ∇ characterize Hamiltonian, bi-Hamiltonian and tri-Hamiltonian systems of hydrodynamic type. The tri-Hamiltonian systems have been previously characterized by Ferapontov in [12] in terms of two differential forms he called ω and Ω. We shall now show how Ferapontov's formalism relates to our connection (1.5). We shall find that Ω is proportional to the skew-symmetric part of the Ricci tensor of ∇, and ω is the volume form of the (generically) unique Lorentzian metric on U which shares its unparameterized geodesics with ∇.

We say that a symmetric affine connection ∇ is *metric*, iff it is the Levi–Civita connection of some (pseudo)-Riemannian metric. An affine connection ∇ is *metrizable* iff it shares its unparameterized geodesic with some metric connection. Thus in the metrizable case there exists a one-form Υ and a metric h such that the Levi–Civita connection of h is given by

$$\Gamma^a{}_{bc} + \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b, \tag{3.4}$$

where $\Gamma^a{}_{bc}$ are the Christoffel symbols of ∇. Not all affine connections on a surface are metrizable. The necessary and sufficient conditions for metrizability have been found in [8].

PROPOSITION 3.1 The connection (1.5) from Theorem 1.2 is generically not metric but is metrizable by the metric

$$h = AB(dX^1) \odot (dX^2). \tag{3.5}$$

Proof. The connection is generically not metric, as its Ricci tensor R_{ab} is in general not symmetric. The skew part of R_{ab} is given by

$$(R_{21} - R_{12})dX^1 \wedge dX^2 = 3d\Upsilon, \quad \text{where} \quad \Upsilon = \Big(\frac{1}{2}\partial_1 \ln B + B\Big)dX^1 + \Big(\frac{1}{2}\partial_2 \ln A + A\Big)dX^2. \tag{3.6}$$

The unparameterized geodesics of this connection are integral curves of a second order ODE

$$Y'' = (\partial_X Z)Y' - (\partial_Y Z)(Y')^2, \quad \text{where} \quad Z = \ln(AB), \tag{3.7}$$

and $(X^1, X^2) = (X, Y)$. The ODE (3.7) is also the equation for unparameterized geodesics of the pseudo-Riemannian metric (3.5) (it can be found directly by solving the metricity equations as in [14]). The Levi–Civita connection of h is given by (3.4), where Υ is given by (3.6). Therefore ∇ is projectively equivalent to a metric connection. □

Remarks

- The pseudo-Riemannian metric (3.5) depends only on the product AB, so the transformation ($A \to \gamma A$, $B \to \gamma^{-1}B$), where $\gamma = \gamma(X^a)$ is a non-vanishing function, does not change unparameterized geodesics. It corresponds to a projective change of connection (3.4) by a one-form

$$\Upsilon = \Big((1-\gamma^{-1})B + \frac{1}{2}\partial_1 \ln \gamma\Big)dX^1 + \Big((1-\gamma)A - \frac{1}{2}\partial_2 \ln \gamma\Big)dX^2.$$

  This transformation can be used to set $R_{[ab]}$ to zero, but it does not preserve (1.3).
- As the Ricci tensor R_{ab} is in general not symmetric, the connection (1.5) does not admit a volume form on Σ which is parallel w.r.t ∇. Therefore the Killing equations (1.2) do not imply the existence of a Killing vector for the metric h.
- The two-form Ω in Theorem 9 of [12] equals $2d\Upsilon$, while ω in [12] is given by the volume form of h. In the tri-Hamiltonian case the connection ∇ is projectively flat. Equivalently the metric (3.5) has constant Gaussian curvature, i.e.

$$(AB)^{-1}\partial_1\partial_2 \ln(AB) = \text{const.} \tag{3.8}$$

  This is the Liouville equation from Section 5 in [12].
- If $n \geq 3$, there is always a discrepancy between the number of equation for a Killing tensor of any given rank and a number of conditions for a HT system to admit a Hamiltonian formulation. Therefore Theorem 1.2 does not generalize to higher dimensions in any straightforward way.

4. Examples

In the examples below we set $X^1 = X, X^2 = Y$.

Example 1

Consider an affine connection (1.5) corresponding to a system of hydrodynamic type with

$$A = cX + Y, \quad B = X + cY, \quad \text{where} \quad c = \text{const}.$$

This connection admits a parallel volume form iff $c = 0$ or $c = 1$. If $c = 0$ then the connection is projectively flat, and so the system of hydrodynamic type is tri-hamiltonian. Calculating the obstruction (2.11) yields

$$T = \frac{8c^2(c^2-9)}{9(cX+Y)^3(X+cY)^3}\Big(dY \otimes \partial_Y - dX \otimes \partial_X + \frac{X+cY}{cX+Y} dY \otimes \partial_X - \frac{cX+Y}{X+cY} dX \otimes \partial_Y\Big).$$

Therefore, if $c = 3$ or $c = -3$ then the connection admits precisely two linear first integrals, so the system is bi-Hamiltonian. Finally for any c not equal to $0, \pm 3$ the system admits a unique Hamiltonian.

Example 2

One-dimensional non-linear elastic medium is governed by the system of PDEs [15, 16]

$$u_t = h^2(v)v_x, \quad v_t = u_x,$$

where $h(v)$ is a function characterizing the type of fluid. This system is Hamiltonian with $\mathcal{H} = u^2/2 + F(v)$, where $F'' = h^2$. We find the Riemann invariants (X, Y) such that

$$u = X + Y, \quad v = G(X - Y), \quad \text{where} \quad G'h(G) = 1 \quad \text{and} \quad \lambda^1 = -\lambda^2 = \frac{1}{G'}.$$

Therefore $A = -B = -G''/(2G')$ and we find $\beta = 0$ and so the Ricci tensor of the associated connection (1.5) is symmetric. In particular, Theorem 1.1 implies that the system cannot admit precisely two Hamiltonian structures.

The projective flatness (3.8) of the connection (1.5) reduces to $(\ln A^2)'' = \text{const}.A^2$ which can be solved explicitly, and leads to a four-parameter family of tri-Hamiltonian systems. The singular solution $A = 1/(2z)$ corresponds to the Toda equation $v_{tt} = (\ln v)_{xx}$.

Example 3

We consider the system of hydrodynamic type (1.3) with

$$\lambda^1 = -\lambda^2 = (X - Y)^n (X + Y)^m.$$

Examining the conditions of Theorem 1.1 for the resulting connection (1.5) we find that this system is always bi-Hamiltonian. It is tri-Hamiltonian iff $nm(n^2 - m^2) = 0$.

Example 4. Frobenius manifolds

In this example we shall consider Hamiltonian systems of hydrodynamic type which arise from two-dimensional Frobenius manifolds. Recall [17, 18] that Σ is a two-dimensional Frobenius manifold if the tangent space to Σ at each point admits a structure of a commutative algebra $\mathbf{A}$ with a unity $\mathbf{e}$, and symmetric tensors fields $C \in C^\infty(\mathrm{Sym}^3(T^*\Sigma))$ and $\eta \in C^\infty(\mathrm{Sym}^2(T^*\Sigma))$ such that locally, in $U \subset \Sigma$, there exist a coordinate system $u^a = (u, v)$ and a function $F : U \to \mathbb{R}$ where

$$C = \frac{\partial^3 F}{\partial u^a \partial u^b \partial u^c} du^a du^b du^c, \quad \mathbf{e} = \frac{\partial}{\partial u^1}, \quad \eta = \frac{\partial^3 F}{\partial u^1 \partial u^a \partial u^b} du^a du^b.$$

The non-degenerate symmetric form η is a flat (pseudo) Riemannian metric with constant coefficients on Σ such that $\mathbf{e}$ is covariantly constant, and $C^a{}_{bc} := \eta^{ad} C_{bcd}$ are the structure constants for $\mathbf{A}$. Moreover there exists an Euler vector field $\mathbf{E}$ such that $\mathcal{L}_{\mathbf{E}}\mathbf{e} = -\mathbf{e}$, and $\mathcal{L}_{\mathbf{E}} C = (m+3)C, \mathcal{L}_{\mathbf{E}}\eta = (m+2)\eta$ for some constant m.

In dimensions higher that two the function F must satisfy a non-linear PDE resulting from the associativity conditions. In two dimensions the associativity always holds, and F can be found only from the homogeneity condition. If we assume that the identity vector field $\mathbf{e}$ is null with respect to the metric η, then we can set $\eta = du \odot dv$, and find [17, 18] that $F(u, v) = \frac{1}{2}u^2 v + f(v)$, where f (which we assume not to be zero) is given by one of the four expressions

$$f = v^k,\ k \neq 0, 2, \quad f = v^2 \ln v, \quad f = \ln v, \quad f = e^{2v}.$$

In all cases the corresponding Hamiltonian system of hydrodynamic type is

$$u_t = f'''(v) v_x, \quad v_t = u_x. \tag{4.1}$$

The characteristic velocities are $\lambda^1 = -\lambda^2 = \lambda \equiv \sqrt{f'''(v)}$, and the Riemann invariants $X^a = (X, Y)$ are

$$X = u + \int \sqrt{f'''(v)} dv, \quad Y = u - \int \sqrt{f'''(v)} dv.$$

The corresponding affine connection (1.5) is projectively flat (the Cotton tensor $\nabla_{[a} \mathrm{P}_{b]c}$ vanishes), and special (the Ricci tensor is symmetric). Therefore Theorem 1.1 implies that the system (4.1) is tri-Hamiltonian. The corresponding three-parameter family of flat metrics (3.1) is

$$g(c_1, c_2, c_3) = \lambda^{-1}\Big(\frac{dX^2}{c_1 + c_2 X + c_3 X^2} - \frac{dY^2}{c_1 + c_2 Y + c_3 Y^2}\Big), \quad \lambda \equiv \sqrt{f'''(v)}, \tag{4.2}$$

where (c_1, c_2, c_3) are arbitrary constants not all zero. We find that $\eta \equiv g(1, 0, 0)$ is the flat metric in the definition of the Frobenius manifold. The second metric $I \equiv g(0, 1, 0)$ is the so-called intersection form (see [17]). The third metric is $J \equiv g(0, 0, 1)$. It can be constructed directly from η and I as $J_{ab} = I_{ac} I_{bd} \eta^{cd}$ in agreement with [5, 19]. It would be interesting to analyse the existence of Hamiltonians for HT type systems arising from submanifolds of Frobenius manifolds [20].

Example 5. Zoll connections

Recall that a Riemannian metric h on a surface Σ is *Zoll* if all geodesics are simple closed curves of equal length. A two-dimensional sphere admits a family of axisymmetric Zoll metrics given by

$$h = (F(X) - 1)^2 dX^2 + \sin^2 X dY^2, \tag{4.3}$$

where (X, Y) are spherical polar coordinates on $\Sigma = S^2$, and $F : [0, \pi] \to [0, 1]$ is any function such that $F(0) = F(\pi) = 0$ and $F(\pi - X) = -F(X)$. A projective structure $[\nabla]$ on Σ is *Zoll* if its unparameterized geodesics are simple closed curves. The general projective structure admitting a projective vector field, and close to the flat structure of the round sphere is given by the second order ODE [21]

$$Y'' = A_3(Y')^3 + A_2(Y')^2 + A_1 Y', \quad \text{where} \tag{4.4}$$

$$A_1 = \frac{F'}{F-1} - 2\cot X, \quad A_2 = \frac{H' \sin X \cos X - 2H}{\cos X(F-1)}, \quad A_3 = -\frac{(H^2+1)\sin X \cos X}{(F-1)^2},$$

where $F = F(X)$ is as before, and $H = H(X)$ satisfies $H(0) = H(\pi) = H(\pi/2) = 0$ and $H(\pi - X) = H(X)$. The metric case (4.3) arises if $H = 0$. A general connection ∇ in this projective class with $\beta \neq 0$ will not admit even a single first integral. We use Theorem 1.1 together with (2.13) to verify that the following choice of the representative connection

$$\Gamma^1_{11} = A_1, \quad \Gamma^1_{22} = A_3, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}A_2 \tag{4.5}$$

admits a first integral for any F and H. To find a (necessarily non-metric) Zoll connection with precisely two linear first integrals we use Proposition (2.2) and match the connection (4.5) with the connection (2.12) (with the roles of X and Y reversed). This, for any given H, leads to a one-parameter family of examples

$$F = 1 + c(H^2 + 1)\cot X$$

which does not satisfy the boundary conditions. The existence of a non-metric Zoll structure on S^2 with precisely two first integrals is an interesting open problem.

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