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Optical metrics and projective equivalence

Stephen Casey,[*] Maciej Dunajski,[†] Gary Gibbons,[‡] and Claude Warnick[§]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

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Trajectories of light rays in a static spacetime are described by unparametrized geodesics of the Riemannian optical metric associated with the Lorentzian spacetime metric. We investigate the uniqueness of this structure and demonstrate that two different observers, moving relative to one another, who both see the Universe as static may determine the geometry of the light rays differently. More specifically, we classify Lorentzian metrics admitting more than one hyper-surface orthogonal timelike Killing vector and analyze the projective equivalence of the resulting optical metrics. These metrics are shown to be projectively equivalent up to diffeomorphism if the static Killing vectors generate a group $SL(2,\mathbb{R})$, but not projectively equivalent in general. We also consider the cosmological C metrics in Einstein-Maxwell theory and demonstrate that optical metrics corresponding to different values of the cosmological constant are projectively equivalent.

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I. INTRODUCTION: NONEQUIVALENT OPTICAL METRICS

When trying to interpret the physical properties of a spacetime, of fundamental importance is the behavior of null geodesics as these correspond to the trajectories of light rays. The vast majority of measurements made of the Universe consist of observation of electromagnetic waves emitted in the past at great distances from us. The behavior of light rays as they bend around the sun gave the first observational evidence for general relativity and such gravitational lensing continues to be a significant branch of astronomy.

In the case where a spacetime is static or conformally static, a powerful approach for investigating the properties of light rays is the *optical metric*. This may be thought of as a natural Riemannian geometry experienced by light rays. It has been recently used to study light bending by a black hole in the presence of a cosmological constant [1] and to give an alternative interpretation of black hole no-hair theorems [2]. An important question one should address when introducing such a structure is to what extent it is unique; in other words, can one spacetime give rise to more than one optical metric. Physically this would mean that there exist two different observers, moving relative to one another, who both see the Universe as (possibly conformally) static and who would determine the geometry of the light rays differently. This is the question we shall address here.

Let (M, g) be a pseudo-Riemannian manifold with a metric of signature $(D, 1)$, where $D > 0$. The metric is called static if it admits a hyper-surface-orthogonal (HSO) timelike Killing vector K. Any such metric is locally of the form

$$g = V^2(-dt^2 + h), \tag{1.1}$$

where $h = h_{ij}dx^i dx^j$ is a Riemannian metric on the space of orbits Σ of $K = \partial/\partial t$ and $V = V(x^i)$ is a function on Σ. The metric h is called the optical metric of g and the motivation behind this terminology [1–3] comes from the fact that null geodesics of g project to unparamertrized geodesics of h. This can be readily verified as null geodesics of g coincide with the null geodesics of $V^{-2}g$.

It is clear from this discussion that an optical metric depends on the choice of a static timelike Killing vector (Fig. 1). Three different equivalence classes of Riemannian metrics will play a role in our discussion. Let (Σ, h) and $(\bar{\Sigma}, \bar{h})$ be two D-dimensional Riemannian manifolds, and let $\rho\colon \Sigma \rightarrow \bar{\Sigma}$ be a diffeomorphism. The metrics h and $\bar{h}$ are

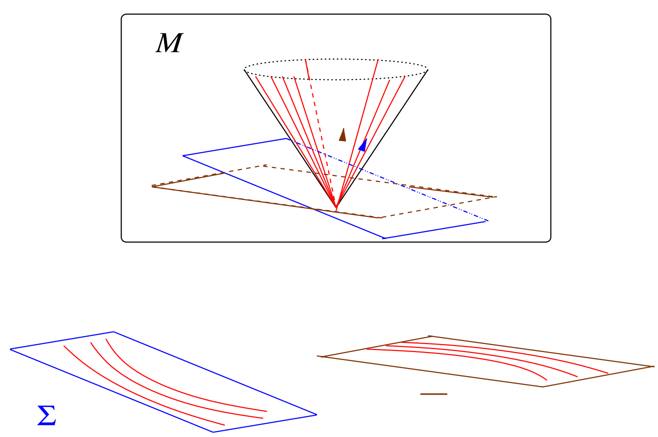

FIG. 1 (color online). Nonequivalent optical metrics.

[*] sc581@cam.ac.uk
[†] M.Dunajski@damtp.cam.ac.uk
[‡] G.W.Gibbons@damtp.cam.ac.uk
[§] cmw50@cam.ac.uk

(i) *Equivalent*, if there exists a ρ such that $\rho^*\bar{h} = h$.
(ii) *Projectively equivalent*, if there exists a ρ such that $\rho^*\bar{h}$ and h share the same unparametrized geodesics.
(iii) *Optically equivalent*, if there exists a pseudo-Riemannian $(D+1)$-dimensional manifold M with two HSO Killing vectors K and $\bar{K}$ such that Σ and $\bar{\Sigma}$ are hypersurfaces orthogonal to K and $\bar{K}$, respectively, and $(h, \bar{h})$ are optical metrics of K and $\bar{K}$, respectively.

All equivalences we shall discuss are in fact local equivalences as ρ is only required to be a smooth map between some open sets.

If two metrics are equivalent, they are also projectively equivalent, but the converse is not true in general. In this paper we shall analyze the connection between the projective equivalence and optical equivalence. It turns out that the latter does not always imply the former.

Let us assume that (M, g) admits two optical metrics h and $\bar{h}$. Thus g can be written in the form (1.1) in more than one way. Therefore there exists a diffeomorphism $f: M \to M$ such that f^*g and g are both of the form (1.1) albeit written in different coordinate systems

$$V^2(-dt^2 + h) = \bar{V}^2(-d\bar{t}^2 + \bar{h}),$$

where $\bar{V} = \bar{V}(\bar{x})$ and $\bar{x} = \bar{x}(x, t), \bar{t} = \bar{t}(x, t)$. Moreover $\bar{K} = \partial/\partial\bar{t}$ and $K = f_*(\partial/\partial t)$ are two timelike HSO Killing vectors. If one of these vectors is a constant multiple of the other then we can deduce that the optical metrics h and $\bar{h}$ are related by a constant rescaling. Let us therefore assume that these vectors are not proportional.

We emphasize that the light cone structure on M does not give rise to a canonical bijection between geodesics of h and $\bar{h}$. For example, if

$$g = -dt^2 + dx^2 + dy^2$$

is the Minkowski metric on $M = \mathbb{R}^{2,1}$ and $K = \partial/\partial t$ then the associated optical metric is $h = dx^2 + dy^2$. Setting

$$t = \bar{y}\sinh\bar{t}, \qquad x = \bar{x}, \qquad y = \bar{y}\cosh\bar{t}$$

yields $g = \bar{y}^2(-d\bar{t}^2 + \bar{h})$, where the upper half-plane metric $\bar{h} = \bar{y}^{-2}(d\bar{x}^2 + d\bar{y}^2)$ is the optical metric of $\bar{K} = \partial/\partial\bar{t}$. Now consider a geodesic γ of h given by $y = 1$. This lifts to a one parameter family of null geodesics $\{y = 1, t = x - c\}$ of g, and this family projects to a family γ_c of geodesics of $\bar{h}$ given by unit semicircles

$$\bar{y}^2 + (\bar{x} - c)^2 = 1$$

parametrized by the position of their centers on the $\bar{x}$ axis (Fig. 2).

Note that for this example h and $\bar{h}$ are projectively equivalent: there exists a diffeomorphism between the Euclidean plane $\Sigma = \mathbb{R}^2$ and the upper half plane $\bar{\Sigma} = \mathbb{H}^2$ which maps unparametrized geodesics of h to

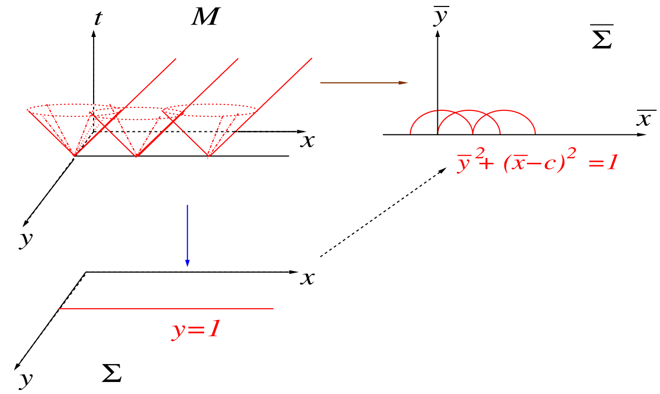

FIG. 2 (color online). No bijection between geodesics of two optical metrics.

unparametrized geodesics of $\bar{h}$. We shall demonstrate that this is not the case in general.

The paper is organized as follows: In Sec. II (Proposition 2.3) we shall find generic local forms of Lorentzian metrics which admit two nonproportional HSO timelike Killing vectors.[1] They are warped product metrics on $M = S_0 \times S_1$ given by

$$g = e^w\gamma_0 + \gamma_1, \tag{1.2}$$

where (S_0, γ_0) is a two-dimensional Lorentzian manifold of constant curvature, (S_1, γ_1) is an arbitrary two-dimensional Riemannian manifold, and $w: S_1 \to \mathbb{R}$ is an arbitrary function. We shall also show that imposing the Einstein condition on (1.2) leads to nontrivial metrics which are analytic continuations of the Kottler metric (Proposition 2.4). In Sec. III we shall compute the optical metrics associated to each Killing vector (Proposition 3.1 and Proposition 3.2). In Sec. IV we shall determine when optically equivalent metrics are projectively equivalent. If the curvature of γ_0 is nonzero, then the general HSO Killing timelike vector is a linear combination of the generators of $SL(2,\mathbb{R})$ acting isometrically on M with two-dimensional orbits S_0. In this case the resulting optical metrics are projectively equivalent to

$$h = (1 - \kappa r^2)^{-2}dr^2 + e^{-w}(1 - \kappa r^2)^{-1}\gamma_1,$$

where $\kappa = \pm 1$ is the curvature of γ_0 (Proposition 4.1). If γ_0 is flat, then the HSO Killing vector arises from the generators of the three-dimensional group *Sol* of isometries of $\mathbb{R}^{1,1}$ and the optical metrics are not projectively equivalent in general. In Sec. V we shall consider the cosmological C metrics in Einstein-Maxwell theory. These metrics fall outside of our class (1.2) and the notion of optical metric is unambiguous. We shall demonstrate that optical

[1]This problem was already addressed in [4] but our construction will be different. In the language of [4] we shall look for a special case of conformal ultrastatic transformations such that $\exp 2\Phi = \bar{V}^2/V^2$.

metrics corresponding to different values of the cosmological constant are projectively equivalent. Thus, the trajectories of light rays in the C-metric spacetimes depend on the mass and electric charge, but not on the cosmological constant.

II. MULTI-STATIC METRICS

We shall now classify local forms of pseudo-Riemannian structures (M, g) which admit more than one HSO timelike Killing vector.

Definition 2.1—A Lorentzian metric is called multistatic if it admits at least two nonproportional HSO timelike Killing vectors.

From now on we shall assume that the dimension of M is equal to four. Let (K, ξ) be two HSO timelike Killing vectors[2] on M. We can choose a local coordinate system [Note: We use the letters from the start of the alphabet $(a, b, c, \ldots)$ to run over 0,1,2,3 and letters from the middle of the alphabet $(i, j, k, \ldots)$ to run over 1, 2, 3] $x^a = (t, x^i)$, such that the metric is given by (1.1) and $K = \partial/\partial t$. In this coordinate system

$$\xi = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i},$$

where $\xi^0, \ldots, \xi^3$ are functions of (x, t). From our assumptions it follows that not all ξ^i are identically zero (if they where, then the Killing equations $\nabla_0 \xi_0 = \nabla_{(i}\xi_{0)} = 0$ would imply $\xi^0 = \text{const}$ thus contradicting our assumptions about the independence of K and ξ). Therefore, there exists t_0 such that the projection of the restriction of ξ at the surface Σ given by $t = t_0$

$$\tilde{\xi} = \xi|_{t=t_0} \tag{2.1}$$

is a nonzero vector field. Furthermore, we can make the coordinate transformation $t \rightarrow t - t_0$ while preserving the form of the metric (1.1) so that $\tilde{\xi}^i = \xi^i\,|_{t=0}$.

The HSO Killing equations for ξ imply that $\tilde{\xi}$ is a HSO Killing vector for $V^2 h$ and so there exists a function $r\colon \Sigma \rightarrow \mathbb{R}$ such that

$$V^2 h = e^w dr^2 + \gamma,$$

where $\tilde{\xi} = \partial/\partial r$, and (w, γ) are a function and a metric on a two-dimensional surface S_1 (the space of orbits of $\tilde{\xi}$ in Σ) which do not depend on r. We can use the isothermal coordinates (x, y) so that $\gamma = e^u(dx^2 + dy^2)$ and u, w are functions of (x, y). Thus, the most general Lorentzian metric which admits more than one optical metric is locally of the form

$$g = -V^2 dt^2 + e^w dr^2 + e^u(dx^2 + dy^2), \tag{2.2}$$

where $V = V(r, x, y)$, $u = u(x, y)$, and $w = w(x, y)$. We note that the function V is not arbitrary—its form is restricted by the Killing equation for ξ.

Our next step is to classify the normal forms of ξ and thus read off the canonical forms of its optical metric $\bar{h}$ on some three-manifold $\bar{\Sigma}$ where $\bar{K} = \partial/\partial\bar{t}$ giving rise to $\bar{h}$ is the push forward of ξ under some local diffeomorphism between Σ and $\bar{\Sigma}$. We shall make the additional genericity assumption.

Definition 2.2—A multistatic metric is called generic if the isometry group generated by any pair of HSO timelike Killing vectors (and their commutators) has two-dimensional orbits in M.

The genericity assumption implies that for any t_0, the HSO Killing vector ξ restricted to the surface $t = t_0$ defined by K is proportional to a fixed vector field.

Proposition 2.3—Any generic multistatic metric is locally a warped product metric on $M = S_0 \times S_1$ given by

$$g = e^w \gamma_0 + \gamma_1, \tag{2.3}$$

where (S_0, γ_0) is a two-dimensional Lorentzian manifold whose curvature is constant, (S_1, γ_1) is a two-dimensional Riemannian manifold, and $w\colon S_1 \rightarrow \mathbb{R}$ is an arbitrary function.

Proof.—First we shall show that given a pair of HSO timelike Killing vectors (K, ξ), the genericity assumption implies existence of two functions (r, t) such that the metric takes the form (2.2), and

$$K = \frac{\partial}{\partial t}, \qquad \xi = \xi^0(t, r, x, y)\frac{\partial}{\partial t} + a(t)\frac{\partial}{\partial r}, \tag{2.4}$$

where (x, y) are coordinates on surface S_1 parametrizing the 2D orbits in M, and a is a function which depends only on t. To prove this statement, note that the group generated by the Killing vectors and their commutators acts on M with two-dimensional orbits so

$$[K, \xi] = pK + q\xi, \tag{2.5}$$

where p, q are functions on M. We need to show that there exists functions α, β such that

$$[\beta^{-1}(\xi - \alpha K), K] = 0, \tag{2.6}$$

as then the local existence of r, t will follow from the Frobenius theorem. Expanding the Lie bracket (2.6) and using (2.5) gives a pair of ODEs

$$K(\beta^{-1}) = \beta^{-1} q, \qquad K(\alpha\beta^{-1}) = \beta^{-1} p.$$

The existence of α, β is a consequence of the Picard existence theorem applied to these ODEs and

$$K = \frac{\partial}{\partial t}, \qquad \xi = \alpha K + \beta\frac{\partial}{\partial r}.$$

[2]The vector field ξ is closely related to $\bar{K}$ from the previous section. These vector fields are however defined on different spaces which justifies our notation. Moreover, as we shall see in Sec. III, the general form of ξ depends on some constants of integration and thus, several different forms of $\bar{K}$ can arise.

Now consider the HSO Killing vector $\tilde{\xi}$ given by (2.1) on the surface Σ of constant t. The Killing equations on Σ imply that $\beta = \beta(r, t)$ and that for any value of t_0 the resulting vector is proportional to the same Killing vector. Thus, $\beta(r, t) = a(t)b(r)$. We now redefine the r coordinate to set $b(r) = 1$. This establishes (2.4). Therefore, for any value of t_0,

$$\xi^i \frac{\partial}{\partial x^i}\bigg|_{t=t_0} \propto \frac{\partial}{\partial r}.$$

The Killing equations $\nabla_{(2}\xi_{0)} = 0 = \nabla_{(3}\xi_{0)}$ give $\xi^0 = \xi^0(t, r)$. Using this and Eq. (2.4) above, the hypersurface orthogonality conditions $\xi_{[0}\nabla_1\xi_{2]} = 0$ and $\xi_{[0}\nabla_1\xi_{3]} = 0$ yield

$$V^2(r, x, y) = v^2(r)e^{w(x,y)} \tag{2.7}$$

for some function $v(r)$. Hence, the metric g may already be written as (2.3) where the two-dimensional metric γ_0 is given by

$$\gamma_0 = -v^2(r)dt^2 + dr^2.$$

The scalar curvature of this metric is

$$\kappa = -\frac{2v''(r)}{v(r)}. \tag{2.8}$$

This will be important later. The only remaining equations that need to be satisfied are the Killing conditions $\nabla_{(0}\xi_{0)} = 0$ and $\nabla_{(1}\xi_{0)} = 0$. These equations give

$$-v^2(r)\partial_t\xi^0 = v(r)\frac{dv(r)}{dr}a(t),$$
$$-v^2(r)\partial_r\xi^0 = -\frac{da(t)}{dt}.$$

Differentiating the first condition with respect to r and the second condition with respect to t and equating the mixed partial derivatives of ξ^0 yields

$$\frac{1}{a(t)}\frac{d^2a(t)}{dt^2} = \left(\frac{dv(r)}{dr}\right)^2 - v(r)\frac{d^2v(r)}{dr^2}. \tag{2.9}$$

The left-hand side of this equation is a function of t only. Hence

$$\left(\frac{dv(r)}{dr}\right)^2 - v(r)\frac{d^2v(r)}{dr^2} = \Omega = \text{constant.}$$

Differentiating with respect to r, we find that

$$0 = v'(r)v''(r) - v(r)v'''(r) = \frac{v^2(r)}{2}\frac{\partial}{\partial r}\left(-\frac{2v''(r)}{v(r)}\right).$$

Hence, by (2.8), the curvature of γ_1 is constant. Furthermore, if the curvature is $\kappa \neq 0$ then we can set its absolute value to one by adding a constant to the function w. ■

Einstein equations

We shall now impose the Einstein condition on (1.2) and show that the resulting metrics are analytic continuations of the cosmological Kottler solution.

Proposition 2.4—Let $\gamma_0^{(0)}$ be the two-dimensional Minkowski metric and $\gamma_0^{(\pm 1)}$ be the line element of the two-dimensional de Sitter and anti–de Sitter metrics with cosmological constant ± 1, respectively. Consider metrics of the form

$$g = \gamma_1 + w^2{\gamma_0}^{(k)}, \tag{2.10}$$

where γ_1 and w are, respectively, a metric and nonconstant function on some two-dimensional surface. Any such metric which is Einstein, with cosmological constant Λ is locally diffeomorphic to the metric

$$g = \left(k + \frac{c}{r} - \frac{\Lambda}{3}r^2\right)d\tau^2 + \frac{dr^2}{k + \frac{c}{r} - \frac{\Lambda}{3}r^2} + r^2\gamma_0^{(k)} \tag{2.11}$$

for some constant c. The case where w is constant yields that γ_1 is an Einstein metric with appropriate cosmological constant to match that of the other factor.

Proof.—The derivation of (2.11) is analogous to the proof of Birkhoff's theorem in general relativity (see, e.g., [5]), except that the constant curvature warped factor is Lorentzian rather than Riemannian. One chooses a coordinate system (r, τ) on S_1, where $r = w$, establishes the τ independence of the metric and finally examines the $(r\tau)$ component of the Einstein tensor, which gives the r dependence. ■

III. OPTICAL METRICS

To determine the optical metrics resulting from (1.2) we need to consider three cases depending on the curvature of γ_0.

A. Zero curvature case

We can find local coordinates such that $\gamma_0 = -dt^2 + dr^2$, and the general HSO Killing vector of g becomes

$$\xi = (Ar + B)\frac{\partial}{\partial t} + (At + C)\frac{\partial}{\partial r}$$

for some constants A, B, and C. If $A \neq 0$ we translate (r, t) by adding constants and rescale the Killing vector so that

$$\xi = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}.$$

Setting $t = \bar{r}\sinh(\bar{t})$, $r = \bar{r}\cosh(\bar{t})$ gives the optical metric of $\partial/\partial\bar{t}$,

$$\bar{h} = \bar{r}^{-2}(d\bar{r}^2 + e^{-w}\gamma_1). \tag{3.1}$$

If $A = 0$ then a constant rescaling of t can be used to set $\xi = \cos\theta\partial_t + \sin\theta\partial_r$, where θ is a constant in a range which makes ξ timelike. The pseudo-orthogonal transformation of (r, t) can now be used to set $\xi = \partial/\partial t$, so the optical metric in this case is

$$h = dr^2 + e^{-w}\gamma_1. \tag{3.2}$$

B. Anti–de Sitter case

Now, let us consider the case where the metric has the form (1.2), where the constant curvature of γ_0 is negative. In the AdS_2 case we can choose local coordinates so that

$$\gamma_0 = \frac{-dt^2 + dr^2}{r^2}.$$

Both γ_0 and the resulting Lorentzian metric g have three Killing vectors generating $SL(2, \mathbb{R})$. In the chosen coordinates these vectors are

$$K_1 = \frac{\partial}{\partial t}, \qquad K_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \qquad K_3 = \left(\frac{t^2 + r^2}{2}\right)\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r},$$

and

$$[K_1, K_2] = K_1, \qquad [K_2, K_3] = K_3, \qquad [K_1, K_3] = K_2.$$

Furthermore, it is easy to show that any linear combination

$$\xi = AK_1 + BK_2 + CK_3$$

is an HSO Killing vector for the metric g, which is timelike in some open set to which we restrict our attention from now on.

Proposition 3.1—For any timelike HSO Killing vector, ξ, of the metric (1.2), where γ_0 has negative constant curvature, the optical metric associated to ξ is diffeomorphic to

$$\bar{h} = \frac{1}{(\phi + \bar{r}^2)^2} d\bar{r}^2 + \frac{e^{-w}}{\phi + \bar{r}^2}\gamma_1 \tag{3.3}$$

for some constant ϕ.

Proof.—Let us first consider the HSO Killing vectors for which $C \neq 0$. Then, adding a constant to t we can set $B = 0$ without changing the metric. If $A = 0$ then divide ξ by $C/2$ to set $C = 2$. Otherwise, rescale (t, r) by the same constant factor to set $A = \pm C/2$ and then divide ξ by $C/2$. Thus, the resulting Killing vector can take one of three possible forms

$$\xi = (c + t^2 + r^2)\frac{\partial}{\partial t} + 2tr\frac{\partial}{\partial r}, \quad \text{where } c = 0, -1, 1.$$

We look for a coordinate transformation $(t, r) \to (\bar{t}, \bar{r})$ such that $\xi = \partial/\partial\bar{t}$.

(i) If $c = 1$ set

$$t = \frac{\sqrt{\bar{r}^2 + 4}\cos(2\bar{t})}{\bar{r} - \sqrt{\bar{r}^2 + 4}\sin(2\bar{t})}, \qquad r = \frac{2}{\bar{r} - \sqrt{\bar{r}^2 + 4}\sin(2\bar{t})}.$$

(ii) If $c = -1$ set

$$t = \frac{\sqrt{\bar{r}^2 - 4}(1 - e^{4\bar{t}})}{\sqrt{\bar{r}^2 - 4}(1 + e^{4\bar{t}}) - 2\bar{r}e^{2\bar{t}}}, \qquad r = \frac{4e^{2\bar{t}}}{\sqrt{\bar{r}^2 - 4}(1 + e^{4\bar{t}}) - 2\bar{r}e^{2\bar{t}}}.$$

(iii) If $c = 0$ set

$$t = \frac{\bar{r}^2\bar{t}}{1 - \bar{r}^2\bar{t}^2}, \qquad r = \frac{\bar{r}}{\bar{r}^2\bar{t}^2 - 1}.$$

This gives, in all three cases $\gamma_0 = -(\bar{r}^2 + 4c)d\bar{t}^2 + (\bar{r}^2 + 4c)^{-1}d\bar{r}^2$ and the optical metric (3.3) with $\phi = 4c$.

Now consider the case $C = 0$. Adding an appropriate constant to t sets $B = 0$ so that

$$\xi = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$$

Setting

$$t = \frac{\bar{r}}{\sqrt{\bar{r}^2 - 1}}e^{\bar{t}}, \qquad r = \frac{1}{\sqrt{\bar{r}^2 - 1}}e^{\bar{t}}$$

yields $\xi = \partial/\partial\bar{t}$ and $\gamma_0 = -(\bar{r}^2 - 1)d\bar{t}^2 + (\bar{r}^2 - 1)^{-1}d\bar{r}^2$. The optical metric in this case is (3.3) with $\phi = -1$.

Finally, suppose $C = B = 0$ so that $\xi = \frac{\partial}{\partial t}$. This gives the optical metric

$$\bar{h} = dr^2 + r^2 e^{-w(x,y)}\gamma_1.$$

A coordinate transformation $r = \bar{r}^{-1}$ puts it in the form (3.3) with $\phi = 0$. Thus, we have covered all cases. ■

C. de Sitter case

In this case γ_0 can be written in local coordinates as

$$\gamma_0 = \frac{-dt^2 + dr^2}{t^2}.$$

This switches the role of r and t in the previous section. The general HSO timelike Killing vector on g is of the form

$$\xi = AK_1 + BK_2 + CK_3,$$

where

$$K_1 = \frac{\partial}{\partial r},$$
$$K_2 = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t},$$
$$K_3 = \left(\frac{t^2 + r^2}{2}\right)\frac{\partial}{\partial r} + tr\frac{\partial}{\partial t}.$$

If $C \neq 0$, then adding a constant to r can be used to set $B = 0$. The resulting vector will be timelike (in a certain open set in M) only if $AC < 0$. In this case we can rescale (r, t) by the same constant factor to set $A = -C/2$, so that

$$\xi = (-1 + t^2 + r^2)\frac{\partial}{\partial r} + 2tr\frac{\partial}{\partial t}.$$

A coordinate transformation

$$t = \frac{\sqrt{4 - \bar{r}^2}(1 + e^{4\bar{t}})}{\sqrt{4 - \bar{r}^2}(1 - e^{4\bar{t}}) + 2\bar{r}e^{2\bar{t}}},$$
$$r = -\frac{4e^{2\bar{t}}}{\sqrt{4 - \bar{r}^2}(1 - e^{4\bar{t}}) + 2\bar{r}e^{2\bar{t}}}$$

gives $\xi = \partial/\partial\bar{t}$ and

$$\gamma_0 = -(4 - \bar{r}^2)d\bar{t}^2 + \frac{1}{4 - \bar{r}^2}d\bar{r}^2,$$

which is defined for $|\bar{r}| < 2$. The optical metric is

$$\bar{h} = \frac{1}{(4 - \bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{4 - \bar{r}^2}\gamma_1.$$

If $C = 0$ then, adding an appropriate constant to r gives $\xi = K_2$. The transformation

$$t = \frac{\bar{r}}{\sqrt{1 - \bar{r}^2}}e^{\bar{t}}, \qquad r = \frac{1}{\sqrt{1 - \bar{r}^2}}e^{\bar{t}}$$

yields $\xi = \partial/\partial\bar{t}$ and $\gamma_0 = -(1 - \bar{r}^2)d\bar{t}^2 + (1 - \bar{r}^2)^{-1}d\bar{r}^2$. The optical metric is this case is

$$\bar{h} = \frac{1}{(1 - \bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{1 - \bar{r}^2}\gamma_1. \tag{3.4}$$

Finally, if $C = B = 0$ then ξ is always spacelike and does not lead to an optical structure. Therefore we have.

Proposition 3.2—For any timelike HSO Killing vector, ξ, of the metric (1.2) where the curvature of γ_0 is positive, the optical metric associated to ξ is diffeomorphic to

$$\bar{h} = \frac{1}{(\phi - \bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{\phi - \bar{r}^2}\gamma_1 \tag{3.5}$$

for some constant $\phi > 0$.

IV. PROJECTIVE EQUIVALENCE

A. Zero curvature

We claim that $\bar{h}$ and h given by (3.1) and (3.2), respectively, are not projectively equivalent even up to diffeomorphisms: The metric (3.2) admits a nontrivial affine equivalence, i.e., there exists a covariantly constant symmetric (0, 2)-tensor h_1 that is not proportional to (3.2) (in our case $h_1 = dr^2$). The canonical forms of Levi-Civita[3] [6] implies that (3.1) admits a nonaffine geodesic equivalence, i.e., there exists a geodesically equivalent metric that is not covariantly constant in the Levi-Civita connection of (3.1). It is given by

$$h_2 = \frac{1}{\bar{r}^2 + 1}\left(\frac{\bar{r}^2}{\bar{r}^2 + 1}d\bar{r}^2 + e^{-w}\gamma_1\right).$$

Thus, if (3.1) and (3.2) were equivalent, there would exist at least three nonproportional metrics sharing the same geodesics. This in dimension three implies [7] that h has constant curvature and so it is flat.[4]

B. Nonzero curvature

Let us first consider the case when γ_0 has negative curvature.

Proposition 4.1—Let ξ_1 and ξ_2 be two timelike HSO Killing vectors for the metric g defined by (1.2) where γ_0 is AdS_2. Then, the optical metric associated to ξ_1 is projectively equivalent to the optical metric associated to ξ_2 after some diffeomorphism. Thus, all optical metrics are equivalent to (3.3) with $\phi = 1$.

Proof.—Let us first consider (3.3) By Proposition 3.1, the optical metric associated to any timelike HSO Killing vector ξ is given, after diffeomorphism, by (3.3) for some constant ϕ. For Killing vectors ξ_1 and ξ_2, let h_1, h_2 be the associated optical metrics written in the form (3.3) with corresponding constants ϕ_1 and ϕ_2, respectively. Let Γ^i_{jk}, $\tilde{\Gamma}^i_{jk}$ be the connection components of the metric connection of h_1, h_2, respectively. Then, these metrics are projectively equivalent (see, for example, [8,9]) if and only if there exists a 1-form $\omega = \omega_j dx^j$ such that

[3]The result of Levi-Civita is that the metrics

$$h = dr^2 + f(r)\gamma, \quad \text{and} \quad \tilde{h} = \frac{1}{(\kappa f(r) + 1)^2}dr^2 + \frac{f(r)}{\kappa f(r) + 1}\gamma$$

are projectively equivalent for any constant κ. Here f is an arbitrary function of r and γ is an arbitrary r-independent metric. The result holds in any dimension.

[4]This example shows that some care is needed with the projective Weyl tensor argument from [1]. Consider the metric (4.1) in this paper (numbers as in published version but r replaced by u and h replaced by $e^{-w}\gamma_1$)

$$h = \frac{du^2}{u^4 f(u)^2} + \frac{1}{f(u)}e^{-w}\gamma_1.$$

Taking $f = 1$ and setting $u = 1/r$ this gives our (3.2). Now take (4.1) with $f = 2/u$, so that $u^3 f' + (1/2)u^4 f'' = 0$ and the projective Weyl tensor is the same as that with $f = 1$. Changing variables by $u = 2/R^2$ gives (3.1). So (3.2) and (3.1) are both of the form (4.1) where the Weyl tensor only depends on h_{ij} but, as we have demonstrated, they are not projective equivalent.

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \omega_k + \delta^i_k \omega_j.$$

Working this out explicitly, we find that the 1-form

$$\omega = \frac{\bar{r}(\phi_2 - \phi_1)}{(\bar{r}^2 + \phi_1)(\bar{r}^2 + \phi_2)} d\bar{r}$$

satisfies this criteria. ■

The same argument, with

$$\omega = \frac{\bar{r}(\phi_1 - \phi_2)}{(\bar{r}^2 - \phi_1)(\bar{r}^2 - \phi_2)} d\bar{r},$$

can be used in the dS_2 case, to show that any two optical metrics (3.5) are projectively equivalent to (3.5) with $\phi = -1$.

V. *C* METRIC

The C metric represents a pair of separated black holes accelerating in opposite directions. The original solution constructed by Weyl can be generalized to the cosmological setting—the relevant line element with $\Lambda < 0$ belongs to the Plebański-Demiański class [10] and is given by

$$g = \frac{1}{A^2(x^2 + y^2)}\left(-F dt^2 + \frac{1}{F} dy^2 + \frac{1}{G} dx^2 + G d\phi^2\right), \tag{5.1}$$

where

$$F = y^2 - 2mAy^3 + e^2A^2y^4 - 1 - \frac{\Lambda}{3A^2},$$
$$G = 1 - x^2 - 2mAx^3 - e^2A^2x^4.$$

The angular coordinate ranges between $-\pi C$ and πC, where C is a positive constant. The constants A, m, and e characterize the acceleration, mass and charge, respectively, and are such that $e^2 < m^2$. The x coordinate lies in an interval between two roots of G which contains 0 and $y \in (-x, \infty)$.

The C metric solves the Einstein-Maxwell equations with the electromagnetic field $e dy \wedge dt$, or the pure Einstein equations in the limiting case $e = 0$. The case $m = e = 0$ is the space of constant curvature. If $m \neq 0$, the case $A < \sqrt{-\Lambda/3}$ corresponds to a single accelerated black hole and $A > \sqrt{-\Lambda/3}$ corresponds to infinite number of pairs of accelerating AdS black holes.

The optical metric of (5.1) is

$$h = \frac{1}{F^2} dy^2 + \frac{1}{GF} dx^2 + \frac{G}{F} d\phi^2. \tag{5.2}$$

We claim that the optical metrics corresponding to different values of Λ are projectively equivalent. To establish this it is enough calculate the Christoffel symbols of h and notice that

$$\Gamma^i_{jk} = (\Gamma_0)^i_{jk} + \delta^i{}_j \omega_k + \delta^i{}_k \omega_j,$$

where $(\Gamma_0)^i_{jk}$ is the Levi-Civita connection of (5.2) with $\Lambda = 0$ and

$$\omega = \omega_i dx^i = \tfrac{1}{2} d(\ln(F(y)|_{\Lambda=0}) - \ln F(y)).$$

This projective equivalence implies that the unparametrized geodesics of h (and so null geodesics of the C metric) are not affected by the cosmological constant. The details of this projective equivalence do not depend on the exact form of $F = F(y)$ and $G = G(x)$ and the argument above demonstrates that the projective class does not change under $F \rightarrow F + \text{const}$. Moreover, analyzing the associated Liouville system [8,9,11] it can be shown any metric which shares unparametrized geodesics with the optical metric (5.2) is a constant rescaling (5.2) possibly with a different value of Λ. Setting

$$x = \cos\theta, \qquad y = \frac{1}{Ar},$$

and taking the limit $A \rightarrow 0$ (need to rescale t) yields the Schwarzschild-de Sitter metric, and in this case we recover a known result [1,12] that the trajectories of light rays in the Schwarzschild-de Sitter metric depend on the mass but not on the cosmological constant.

VI. CONCLUSIONS

The significance of projective differential geometry in general relativity goes back at least to Weyl: an equivalence class of unparametrized geodesics can be used to describe the geometry of free falling massive particles. Various aspects of the theory have been explored—see [13,14] and references therein—but, as emphasized in [15], there is more to general relativity than projective geometry. Some cosmological observables—for example cosmic jerk and its higher order generalizations [16]—are not projectively invariant, and thus depend on a choice of the metric in a projective equivalence class.

In this paper we have explored a novel aspect of projective equivalence. The light rays in static spacetimes give rise to projective structures of optical metrics. This leads to ambiguity if a spacetime is static in more than one way, as nonproportional timelike Killing vectors lead to different optical metrics, which as we have demonstrated are not always projectively equivalent.

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APPENDIX: ULTRASTATIC METRICS

Here we shall show that in the ultrastatic case $V = 1$, we can integrate the Killing equations without making the additional genericity assumption and establish Proposition (2.3) with $w = \text{const}$ and γ_0 being flat. This is essentially the case considered by Sonego [4]. We shall however take our analysis further and consider optical metrics resulting from this construction. In the adapted coordinate system, the Killing vector $\tilde{\xi}$ on Σ satisfies

$$(\xi^1, \xi^2, \xi^3)\,|_{t=0} = (1, 0, 0). \tag{A1}$$

Now consider the Killing equations for ξ. Using $\Gamma^0_{ij} = 0$ we find that $\nabla_{(0}\xi_{0)} = 0$, $\nabla_{(0}\xi_{i)} = 0$ imply

$$\partial_t \xi^0 = 0, \qquad e^{w(x,y)}\partial_t \xi^1 = \partial_r \xi^0,$$
$$e^{u(x,y)}\partial_t \xi^2 = \partial_x \xi^0, \qquad e^{u(x,y)}\partial_t \xi^3 = \partial_y \xi^0.$$

Integrating and using the initial conditions (A1) gives

$$\xi^1 = e^{-w(x,y)}(\partial_r \xi^0)t + 1, \qquad \xi^2 = e^{-u(x,y)}(\partial_x \xi^0)t,$$
$$\xi^3 = e^{-u(x,y)}(\partial_y \xi^0)t. \tag{A2}$$

Now, let us consider the hypersurface orthogonality condition $\xi \wedge d\xi = 0$. We find

$$\begin{aligned} 0 &= \xi_{[0}\nabla_1 \xi_{2]} \\ &= -\xi^0\left((\partial_r\partial_x\xi^0)t - (\partial_x\partial_r\xi^0)t - \frac{\partial w}{\partial x}e^w\right) \\ &\quad + ((\partial_r\xi^0)t + e^w)(-2\partial_x\xi^0) + (\partial_x\xi^0)t(2\partial_r\xi^0). \end{aligned}$$

This, together with a similar condition resulting from $\xi_{[0}\nabla_1\xi_{3]} = 0$, implies after some algebra

$$\xi^0 = \alpha(r)e^{(1/2)w(x,y)}. \tag{A3}$$

The rest of the hypersurface orthogonality conditions are then satisfied. The remaining Killing equations will yield conditions on $w(x, y)$ and $u(x, y)$ as well as a condition for $\alpha(r)$ as follows: Eq. (A3) and $\nabla_{(2}\xi_{3)} = 0$ give

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right)\right]. \tag{A4}$$

Similarly, the Killing conditions $\nabla_{(2}\xi_{2)} = 0 = \nabla_{(3}\xi_{3)}$ give

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)\right], \\ \frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right)\right]. \end{aligned} \tag{A5}$$

The Killing equations $\nabla_{(1}\xi_{2)} = 0 = \nabla_{(1}\xi_{3)}$ are now satisfied and the condition $\nabla_{(1}\xi_{1)} = 0$ gives

$$\frac{\partial^2 \alpha(r)}{\partial r^2} = -\frac{1}{4}e^{w-u}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\alpha(r).$$

The left-hand side of this equation depends only on r, so the quantity

$$\mu^2 \equiv \frac{1}{4}e^{w-u}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{A6}$$

is a constant. Let us first consider the case $\mu \neq 0$. Solving (A6) for u and substituting the partial derivatives of u into (A4) and (A5) gives, after some algebra,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) = 0.$$

This means that the function $e^{w/2}$ is harmonic, thus $e^{w(x,y)/2} = G(z) + \overline{G(z)}$, where G is holomorphic in $z = x + iy$. A coordinate transformation

$$X = \frac{2}{\mu}\,\mathrm{Re}(G)\cos(\mu r), \qquad Y = \frac{2}{\mu}\,\mathrm{Re}(G)\sin(\mu r),$$
$$Z = \frac{2}{\mu}\,\mathrm{Im}(G), \qquad T = t$$

yields the Minkowski metric $g = -dT^2 + dX^2 + dY^2 + dZ^2$.

Now, let us consider the case $\mu = 0$. Equation (A6) implies that $w(x, y)$ is a constant and so the metric (2.2), after rescaling r, becomes

$$g = -dt^2 + dr^2 + \gamma_1,$$

where $\gamma_1 = e^u(dx^2 + dy^2)$. We also have $\alpha = Ar + B$, and given the initial conditions, the Killing vector ξ can be written as

$$\xi = (Ar + Be^{(1/2)w})\frac{\partial}{\partial t} + (At + e^{(1/2)w})\frac{\partial}{\partial r}.$$

If $A \neq 0$ we translate (r, t) by adding constants and rescale the Killing vector so that

$$\xi = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}.$$

Setting $t = \bar{r}\sinh(\bar{t})$, $r = \bar{r}\cosh(\bar{t})$ gives

$$g = \bar{r}^2(-d\bar{t}^2 + \bar{h}),$$

where

$$\bar{h} = \bar{r}^{-2}(d\bar{r}^2 + \gamma_1) \tag{A7}$$

is the optical metric associated to the Killing vector $\partial/\partial\bar{t}$.

If $A = 0$ then a constant rescaling of t can be used to set $\xi = \cos\theta\partial_t + \sin\theta\partial_r$, where θ is a constant in a range which makes ξ is timelike. The pseudo-orthogonal transformation of (r, t) can now be used to set $\xi = \partial/\partial t$, so the optical metric in this case is

$$h = dr^2 + \gamma_1. \tag{A8}$$

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