

Twistor Geometry of a Pair of Second Order ODEs

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Dedicated to Mike Eastwood on the occasion of his 60th birthday

Abstract: We discuss the twistor correspondence between path geometries in three dimensions with vanishing Wilczynski invariants and anti-self-dual conformal structures of signature $(2, 2)$. We show how to reconstruct a system of ODEs with vanishing invariants for a given conformal structure, highlighting the Ricci-flat case in particular. Using this framework, we give a new derivation of the Wilczynski invariants for a system of ODEs whose solution space is endowed with a conformal structure. We explain how to reconstruct the conformal structure directly from the integral curves, and present new examples of systems of ODEs with point symmetry algebra of dimension four and greater which give rise to anti-self-dual structures with conformal symmetry algebra of the same dimension. Some of these examples are $(2, 2)$ analogues of plane wave space-times in General Relativity. Finally we discuss a variational principle for twistor curves arising from the Finsler structures with scalar flag curvature.

1. Introduction

A path geometry on an open set $\mathcal{T} \subset \mathbb{R}^n$ consists of unparametrised curves, one curve through each point in \mathcal{T} in each direction. Regarding these curves as solutions to a system of $(n - 1)$ second order ODEs, one can give an alternative definition of the path geometry as an equivalence class of systems of second order ODEs, where two systems are regarded as equivalent if they can be mapped into each other by a change of dependent and independent variables. In this paper we shall study three-dimensional path geometries encoded into a system of two second order ODEs,

$$Y'' = F(X, Y, Z, Y', Z'), \quad Z'' = G(X, Y, Z, Y', Z'), \quad (1.1)$$

where $Y' = dY/dX$ etc, and (F, G) are arbitrary functions on an open set in \mathbb{R}^5 which we assume to be of class C^5 . Two such systems are locally equivalent if they are related by a point transformation $(X, Y, Z) \rightarrow (\bar{X}(X, Y, Z), \bar{Y}(X, Y, Z), \bar{Z}(X, Y, Z))$. A natural question arises: Given a system (1.1) is it equivalent to a pair of trivial ODEs

$Y'' = 0$, $Z'' = 0$? The answer comes down to constructing a set of invariants for (1.1). These invariants can be divided into two groups \mathcal{C} and \mathcal{P} . Vanishing of the invariants from each of these groups selects a subclass of three-dimensional path geometries.

- The system (1.1) belongs to the *conformal branch* if its four-dimensional solution space M admits a conformal structure such that the two-dimensional surfaces in M corresponding to points in \mathcal{T} are totally isotropic (this condition uniquely determines the conformal structure). The conformal branch is characterised by the vanishing of three fundamental Wilczynski invariants \mathcal{C} given by the expressions (2.1). Systems with vanishing Wilczynski invariants are also referred to as *torsion-free* [17] and we will use this terminology several times throughout.
- The system (1.1) belongs to the *projective branch* if its integral curves in \mathcal{T} are unparametrised geodesics of some projective connection. The projective branch is characterised by the vanishing of eight invariants \mathcal{P} given by the expressions (2.2).

A system is point equivalent to a trivial one if and only if it belongs to both the conformal and projective branches [17]. In this case the symmetry algebra of (1.1) is isomorphic to $\mathfrak{sl}(4, \mathbb{R})$. The corresponding Lie group $PSL(4, \mathbb{R})$ acts projectively on $\mathcal{T} = \mathbb{RP}^3$ preserving the unparametrised geodesics of the flat projective connection.

In this paper we shall concentrate on the conformal branch, and make use of the local isomorphism,

$$PSL(4, \mathbb{R}) \cong SO(3, 3),$$

where $SO(3, 3)$ is the conformal group of the flat conformal structure on the solution space M . If the curvature of the conformal structure does not vanish, then the conformal symmetry group is a proper subgroup of $SO(3, 3)$, and conformal Killing vectors on M give rise to point symmetries of the system (1.1) (see Lemma 6.1). We shall exploit this correspondence to construct several examples of torsion-free ODE systems and the corresponding conformal structures which admit a large symmetry group, 9-dimensional symmetry being the sub-maximal case.

In the rest of the paper we shall call \mathcal{T} the *twistor space*, and the integral curves of (1.1) the *twistor lines*. In the next section we shall give explicit expressions for point invariants of (1.1). In Sect. 3 we summarise the twistor correspondence anti-self-dual conformal structures which underlies all constructions in this paper. In Sect. 4, we describe how, given an ASD conformal structure of $(2, 2)$ signature, to construct a system of ODEs on the twistor space \mathcal{T} whose integral curves are the twistor lines. Using this framework, in Sect. 5 we present a new derivation for the condition of vanishing Wilczynski invariants assuming solely that the space of solutions be endowed with a conformal structure. We demonstrate how this construction proceeds in the Ricci-flat case, and show (Theorem 5.1) that in this case the system of ODEs can be read-off directly from the solution of Plebanski's heavenly equation. In Sect. 6, we explore the local isomorphism between groups of point symmetries of (1.1) and conformal symmetries of the corresponding conformal structure. In Sect. 7 we construct examples of torsion-free ODE systems in three dimensions with point symmetry algebras of dimensions between nine and two. Finally in Sect. 8 we explore the connection between systems of ODEs with vanishing Wilczynski invariants, and unparametrised geodesics of Finsler structures with scalar flag curvature. This correspondence gives rise to yet another class of examples. Some of these examples have gravitational analogues in the theory of plane wave space-times, and the whole construction sheds new light on variational aspects of the Nonlinear Graviton theorem.

It is a pleasure to dedicate this paper to Mike Eastwood on the occasion of his 60th birthday. Mike has made important contributions to twistor theory and conformal geometry over the last thirty five years. He and his coworkers [3] have developed an approach to symmetry of differential equations which underlies some of our work.

2. Systems of ODEs and Their Invariants

To present the invariants in a compact form set $Y^A = (Y, Z)$ and $F^A = (F, G)$, where the capital letter indices A, B, C, \dots take values 0, 1. Moreover we set

$$P^A = \frac{\partial Y^A}{\partial X}, \quad \text{and} \quad \frac{d}{dX} = \frac{\partial}{\partial X} + P^A \frac{\partial}{\partial Y^A} + F^A \frac{\partial}{\partial P^A}.$$

The three Wilczynski conditions \mathcal{C} are given by

$$T - \frac{1}{2} \text{Trace}(T) \mathbb{I} = 0, \quad \text{where} \quad T_B^A = -\frac{\partial F^A}{\partial Y^B} - \frac{1}{4} \frac{\partial F^A}{\partial P^C} \frac{\partial F^C}{\partial P^B} + \frac{1}{2} \frac{d}{dX} \frac{\partial F^A}{\partial P^B}. \quad (2.1)$$

Our terminology is motivated by the work of Doubrov [8,9], who has demonstrated that the conditions (2.1) for systems of non-linear ODEs are equivalent to the classical Wilczynski invariants [30] on the linearisations of these ODEs.

The complementary set of invariant conditions \mathcal{P} characterising the projective branch is

$$S^A_{(BCD)} = 0, \quad (2.2)$$

where [15]

$$S^A_{BCD} = \frac{\partial^3 F^A}{\partial P^B \partial P^C \partial P^D} - \frac{3}{4} \frac{\partial^3 F^E}{\partial P^E \partial P^B \partial P^C} \delta_D^A.$$

The following result gives an interpretation of the zero locus of Wilczynski invariants.

Theorem 2.1 (Grossman [17]). *There is one-to-one correspondence between equivalence classes of systems of second order ODEs with vanishing Wilczynski invariants (2.1) and conformal structures of signature (2, 2) with vanishing self-dual Weyl curvature.*

A point $p \in M$ corresponds to an integral curve L_p of the system (1.1) – a *twistor line*. Moreover, if the Wilczynski invariants vanish then M is endowed with a conformal structure and points in \mathcal{T} correspond to special null two-surfaces in M called α -surfaces (see next section). In Grossman's terminology M acquires a *Segre structure* from the torsion-free system of ODEs [17,21]. In four dimensions the Segre structure is the splitting of the tangent bundle into a product of rank two vector bundles. This is equivalent to the existence of a conformal structure on M – see formula (3.3) in the next section – so from now on we shall use the terminology of conformal geometry without referring to the Segre structures.

In the original Penrose twistor correspondence [24], the four-manifold M is assumed to be real analytic, and the correspondence extends to the complexified category. The curves L_p in a complex three-fold \mathcal{T} corresponding to points in M are globally characterised by the type of their normal bundle $N(L_p) = \mathcal{O}(1) \oplus \mathcal{O}(1)$, where $\mathcal{O}(k) \rightarrow \mathbb{CP}^1$ is a holomorphic line bundle with Chern class k . In the present context we shall replace this global condition on L_p by local-differential conditions (2.1).

3. Twistor Correspondence

Let $[g] = \{cg, c : M \rightarrow \mathbb{R}^+\}$ be a conformal structure on a 4-dimensional manifold M consisting of an equivalence class of metrics of signature $(2, 2)$ and let $g \in [g]$. A vector field $V \subset TM$ is called null if $g(V, V) = 0$. The null property of vector fields is invariant under rescalings of g , therefore it makes sense to talk about a null vector of a conformal structure. Conversely any conformal structure is completely characterised by specifying null vector fields.

Locally there exist real rank two vector bundles \mathbb{S}, \mathbb{S}' (called spin-bundles) over M equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that

$$TM \cong \mathbb{S} \otimes \mathbb{S}' \quad (3.3)$$

is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2)$$

for $v_1, v_2 \in \Gamma(\mathbb{S})$ and $w_1, w_2 \in \Gamma(\mathbb{S}')$. Under this decomposition, any null vector field is of the form $V = \kappa \otimes \pi$ for some $\kappa \in \Gamma(\mathbb{S})$ and $\pi \in \Gamma(\mathbb{S}')$. Also, the Riemann tensor can be decomposed as

$$\begin{aligned} R_{abcd} = & \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ & + \phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ & + \frac{R}{12}(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}), \end{aligned} \quad (3.4)$$

where ψ_{ABCD} and $\psi_{A'B'C'D'}$ are ASD and SD Weyl spinors which are symmetric in their indices and $\phi_{A'B'CD} = \phi_{(A'B')(CD)}$ is the traceless Ricci spinor.

An α -plane is a two-dimensional plane in $T_p M$ spanned by null vectors of the above form with π fixed, and an α -surface is a two-dimensional surface in $\zeta \subset M$ such that its tangent plane at every point is an α -plane. Examining the Frobenius integrability conditions for the existence of α -surfaces leads to the seminal result of Penrose [24]: A maximal, three dimensional, family of α -surfaces exists in M iff the Weyl tensor of g is anti-self-dual.

The anti-self-duality of the Weyl tensor is a conformally invariant property, therefore one can talk about ASD conformal structures. Let us assume that the conformal structure is real-analytic and therefore can be complexified. Thus $(M, [g])$ is a holomorphic four-manifold with a holomorphic conformal structure.

Definition 3.1. *The twistor space \mathcal{T} of a holomorphic conformal structure $(M, [g])$ with anti-self-dual Weyl curvature is the manifold of α -surfaces in M .*

The twistor correspondence puts restrictions on the twistor space \mathcal{T} :

Theorem 3.2 (Penrose [24]). *There is a one-to-one correspondence between ASD conformal structures $(M, [g])$ and three-dimensional complex manifolds \mathcal{T} with the four parameter family of sections of μ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

The additional structures on the twistor space which give Ricci flat metric $g \in [g]$ are

- (1) A projection $\mu : \mathcal{T} \rightarrow \mathbb{CP}^1$, such that the four parameter family of curves above are sections of μ .
- (2) Symplectic structure with values in $\mathcal{O}(2)$ on the fibers of μ .

The points in the twistor space \mathcal{T} correspond to totally null self-dual two dimensional surfaces (which coincide with the α -surfaces) in M , and points in M correspond to rational curves (in the Ricci flat case these are sections of μ) in \mathcal{T} . The relation between \mathcal{T} and M is best revealed by exploiting the double fibration picture

$$M \xleftarrow{r} \mathcal{F} \xrightarrow{s} \mathcal{T}, \quad (3.5)$$

where the correspondence space

$$\mathcal{F} = \mathcal{T} \times M|_{\zeta \in L_p} = M \times \mathbb{CP}^1$$

can be identified with the projective primed spin bundle $P(\mathbb{S}')$. A point in \mathcal{F} corresponds to a point in $p \in M$ together with one α -surface containing p .

The holomorphic curves $s(\mathbb{CP}_p^1)$, where $\mathbb{CP}_p^1 = r^{-1}p$, $p \in M$, have normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Two points p_1 and p_2 in M are null separated iff the corresponding curves L_{p_1} and L_{p_2} intersect at one point.

In the double fibration approach, the twistor space arises as a quotient of \mathcal{F} by a two-dimensional integrable distribution (called the Lax pair) defined by two vector fields L_0, L_1 which at each point of \mathcal{F} are horizontal lifts to $P(\mathbb{S}')$ of vectors spanning an α surface at a point in M . The existence of the Lax pair for an arbitrary conformal structure with vanishing self-dual Weyl curvature has the following consequence [13, 20].

Theorem 3.3. *Let E_1, \dots, E_4 be four real vector fields on M and let e^1, \dots, e^4 be the corresponding dual one-forms. The conformal structure defined by*

$$g = e^1 \odot e^2 - e^3 \odot e^4$$

is ASD if and only if there exist functions f_0, f_1 on $M \times \mathbb{RP}^1$ depending on $\lambda \in \mathbb{RP}^1$ such that the distribution

$$L_0 = E_1 - \lambda E_3 + f_0 \frac{\partial}{\partial \lambda}, \quad L_1 = E_4 - \lambda E_2 + f_1 \frac{\partial}{\partial \lambda} \quad (3.6)$$

is Frobenius integrable, that is, $[L_0, L_1] = 0$ modulo L_0 and L_1 .

The real $(2, 2)$ ASD conformal structures are obtained by introducing an involution $\tau : \mathcal{T} \rightarrow \mathcal{T}$ on the twistor space given by complex conjugation. The fixed points correspond to real α -surfaces in M . The involution acts on the twistor lines, thus giving rise to maps from \mathbb{CP}^1 to \mathbb{CP}^1 which swaps the lower and upper hemispheres preserving the real equator. The moduli space of τ -invariant twistor curves is a real four-manifold.

4. From ASD Conformal Structures to Systems of ODEs

In this section we shall present an algorithm for reconstructing a system (1.1) of torsion-free ODEs from a given ASD conformal structure $[g]$. Given $(M, [g])$ we shall construct a real projective line parametrised by $\lambda \in \mathbb{RP}^1$ of real α -surfaces through a point in $p \in M$. This is a curve in $M \times \mathbb{RP}^1$ which also depends on the four coordinates of p in M . We shall regard this as a parametrised integral curve of a system of two second order ODEs. The system (1.1) will arise by eliminating the coordinates of p and removing the parametrisation. To implement this procedure, consider the Lax pair (3.6) (see Sect. 3) of vector fields L_0, L_1 on $M \times \mathbb{RP}^1$. Now proceed as follows:

- (1) Find three independent functions $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ on $\mathcal{F} = M \times \mathbb{RP}^1$ which satisfy

$$L_0 f = 0, \quad L_1 f = 0.$$

These functions descend to the twistor space \mathcal{T} where they provide a local coordinate system (X, Y, Z) . A point in M corresponds to a curve in \mathcal{T} . The pull back of the four-parameter family of curves to \mathcal{F} can be parametrised as

$$\lambda \longrightarrow (\mathcal{X}(\lambda, p), \mathcal{Y}(\lambda, p), \mathcal{Z}(\lambda, p)),$$

where $p = (w, z, x, y)$ are local coordinates on M .

- (2) Use the implicit function theorem to solve the equation $X = \mathcal{X}(\lambda, p)$ for $\lambda = \lambda(X, p)$. Then solve the relations

$$Y = \mathcal{Y}, \quad Z = \mathcal{Z}, \quad Y' = \frac{\partial \mathcal{Y}}{\partial X}, \quad Z' = \frac{\partial \mathcal{Z}}{\partial X}$$

to express (w, z, x, y) as functions of (Y, Z, Y', Z') .

- (3) Differentiating once more, and substituting for (w, z, x, y) from above gives a pair of second order ODEs (1.1).

As an example of this construction, let us consider the ASD Ricci-flat metric which, in local coordinates may be written as

$$g = dw dx + dz dy - \Theta_{xx} dz^2 - \Theta_{yy} dw^2 + 2\Theta_{xy} dw dz, \quad (4.1)$$

where $\Theta_x = \partial_x \Theta$ etc, and $\Theta = \Theta(w, x, y, z)$ is a function satisfying the *second heavenly equation*

$$\Theta_{xw} + \Theta_{yz} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2 = 0. \quad (4.2)$$

It follows from the work of Plebański [25] that all (2, 2) ASD Ricci flat metrics locally arise from some solution to this equation. The only non-vanishing part of curvature is given by the ASD Weyl spinor which in the spin-frame adapted to the heavenly equation is a section of $\text{Sym}^4(\mathbb{S}) \rightarrow M$ given by

$$\psi_{ABCD} = \frac{\partial^4 \Theta}{\partial x^A \partial x^B \partial x^C \partial x^D},$$

where the indices A, B, \dots take values 0, 1 and $x^A = (y, -x)$. The Lax pair for this system may be written as

$$\begin{aligned} L_0 &= \partial_y - \lambda(\partial_w - \Theta_{xy} \partial_y + \Theta_{yy} \partial_x), \\ L_1 &= \partial_x + \lambda(\partial_z + \Theta_{xx} \partial_y - \Theta_{xy} \partial_x). \end{aligned} \quad (4.3)$$

A curve $L_p \subset \mathcal{T}$ corresponding to a point $p \in M$ is parametrised by choosing a two-dimensional fiber of $\mu : \mathcal{T} \rightarrow \mathbb{CP}^1$ and defining (w, z) to be the coordinates of the initial point of the curve, and $(y, -x)$ to be the tangent vector to the curve. Thus the pulled back curve is $\lambda \rightarrow (X = \lambda, Y = \mathcal{Y}(\lambda, p), Z = \mathcal{Z}(\lambda, p))$, where the functions $(\mathcal{Y}, \mathcal{Z})$ admit the following expansion [10]:

$$\begin{aligned} \mathcal{Y} &= w + \lambda y - \Theta_x \lambda^2 + \Theta_z \lambda^3 + \dots, \\ \mathcal{Z} &= z - \lambda x - \Theta_y \lambda^2 - \Theta_w \lambda^3 + \dots, \end{aligned} \quad (4.4)$$

and the higher order terms can be determined by recursion on successive powers of λ from $L_A(\mathcal{Y}) = L_A(\mathcal{Z}) = 0$, where $A = 0, 1$.

Example 1. Consider a solution to (4.2),

$$\Theta = \frac{1}{4}y^4.$$

The corresponding system of ODEs is

$$Y'' = 0, \quad Z'' = -2(Y')^3. \quad (4.5)$$

The system has a nine-dimensional symmetry group (see formula (7.1)), which is also the largest symmetry group of a non-trivial ASD conformal structure. The corresponding pp -wave metric has constant Weyl curvature with only one non-vanishing component $\psi_{0000} = 6$.

Example 2. This example, when analytically continued to the Riemannian signature, is relevant in the theory of gravitational instantons. The second heavenly equation (4.2) with $\Theta_z = 0$ reduces to a wave equation on a flat $(2 + 1)$ -dimensional background. Introduce $t := \Theta_x$ and perform a Legendre transform

$$H(t, y, w) := tx(w, y, t) - \Theta(w, y, x(w, y, t)).$$

Then $x = H_t$, $\Theta_y = -H_y$ and (4.2) yields the wave equation

$$H_{tw} + H_{yy} = 0, \quad (4.6)$$

together with the metric

$$\begin{aligned} g &= H_{tt} \left(\frac{1}{4} dy^2 + dw dt \right) - \frac{1}{H_{tt}} \left(dz - \frac{H_{tt}}{2} dy + H_{ty} dw \right)^2 \\ &= V \left(\frac{1}{4} dy^2 + dw dt \right) - V^{-1} (dz + A)^2, \end{aligned} \quad (4.7)$$

where $V = H_{tt}$ and $A = H_{ty} dw - (H_{tt}/2) dy$ satisfy the monopole equation $*dV = dA$, where $*$ is the Hodge operator on $\mathbb{R}^{2,1}$ with its flat metric. Thus (4.7) is an analytic continuation of the Gibbons–Hawking metric [16]. The vanishing of Θ_z implies that the whole series (4.4) for \mathcal{Y} truncates at 2nd order. From (4.4), we get

$$\begin{aligned} \mathcal{Y} &= w + \lambda y - \lambda^2 t, \\ \mathcal{Z} &= z - \lambda H_t + \lambda^2 H_y + \lambda^3 H_w + \dots, \end{aligned} \quad (4.8)$$

where $H = H(w, y, t)$, from which we can obtain the corresponding path geometry. An example with $H(w, y, t) = yt^2$ is

$$\begin{aligned} Y'' &= X^{-1} Y' - \sqrt{(X^{-1} Y')^2 - 2X^{-1} Z'}, \\ Z'' &= (X^{-1} Y' - \sqrt{(X^{-1} Y')^2 - 2X^{-1} Z'})^2 / 2. \end{aligned} \quad (4.9)$$

The potential in the Gibbons–Hawking metric is linear in the flat coordinates on $\mathbb{R}^{2,1}$.

5. Wilczynski Invariants

We shall now demonstrate a new way of deriving the Wilczynski invariants (2.1), from the double fibration picture (3.5). This will provide a converse of the construction presented in the last section. Our procedure is analogous to the recursive construction of the Wilczynski invariants of a single n^{th} order ODE [11].

Each point in \mathcal{T} corresponds to an α -plane in M , and the functions $P^A = (dY/dX, dZ/dX)$ are null coordinates which are mutually orthogonal and thus $a_0 P^0 + a_1 P^1$ is null for arbitrary constants a_0, a_1 . Furthermore, by differentiation, the one-form $a_0 dY^0 + a_1 dY^1$ is orthogonal to $a_0 dP^0 + a_1 dP^1$.

In the derivation below we shall regard (Y^A, P^A, X) as coordinates on the five dimensional correspondence space \mathcal{F} from the double fibration picture (3.5), and define the degenerate metric g on this five-dimensional space. We then demand that this quadratic form Lie derives up to scale along the total derivative d/dX , and so it gives a conformal structure on M . The metric g necessarily takes the form

$$g = \varepsilon_{AB} dY^A dP^B + \phi_{AB} dY^A dY^B, \quad (5.1)$$

where $\phi_{AB} = \phi_{(AB)}$. The conformal structure of M is invariant along the fibres of s in the fibration (3.5), and therefore

$$\frac{dg}{dX} = \Omega^2 g$$

for some function Ω . Plugging in the expression (5.1) and comparing coefficients on both sides we obtain the equations

$$2\phi_{AB} + \frac{\partial F^C}{\partial P^B} \varepsilon_{AC} = \Omega^2 \varepsilon_{AB}, \quad (5.2)$$

$$\frac{d\phi_{AB}}{dX} + \frac{\partial F^C}{\partial Y^{(B}} \varepsilon_{A)C} = \Omega^2 \phi_{AB}. \quad (5.3)$$

Taking the trace of the (5.2) (using ε to raise and lower indices),

$$\frac{\partial F^C}{\partial P^C} = 2\Omega^2,$$

and substituting back into (5.2) for Ω we get

$$\phi_{AB} = -\frac{1}{2} \varepsilon_{AC} \frac{\partial F^C}{\partial P^B} + \frac{1}{4} \varepsilon_{AB} \frac{\partial F^C}{\partial P^C}.$$

Then, from (5.3), we obtain the Wilczynski invariants (2.1),

$$-\frac{1}{2} \frac{d}{dX} \frac{\partial F^A}{\partial P^B} + \frac{\partial F^A}{\partial Y^B} + \frac{1}{4} \frac{\partial F^C}{\partial P^B} \frac{\partial F^A}{\partial P^C} \sim \delta_B^A,$$

where here we have used the fact that

$$\frac{\partial F^C}{\partial P^B} \frac{\partial F^A}{\partial P^C} - \frac{\partial F^C}{\partial P^C} \frac{\partial F^A}{\partial P^B} \sim \delta_B^A$$

to rewrite the third term.

Note here that the expression for the conformal metric (5.1) actually gives a metric if $\Omega = 0$ i.e. $\frac{\partial F^C}{\partial P^C} = 0$ which implies

$$F^A = 2\varepsilon^{AB} \frac{\partial \Lambda}{\partial P^B} \quad (5.4)$$

for some function Λ . The metric (5.1) then resembles the heavenly form (4.1) of the Ricci-flat metric. The exact equivalence arises from evaluating (5.1) at $X = 0$, where $Y^A = (w, z)$, $P^A = (-y, x)$ and $\Lambda(Y^A, P^A) = -\Theta(w, z, x, y)$.

Example 3. One example is given by

$$Y'' = 0 \quad Z'' = \beta(Y') \quad (5.5)$$

for some arbitrary function β . The ASD conformal structure on the solution space M is type-N and Ricci-flat. Notice here that the system (4.5) of sub-maximal point symmetry is just a special case of this solution.

The last example was Ricci-flat. One may seek conditions on F^A for this to be true more generally. In other words one seeks to obtain (4.2) as well as (4.1). For this, one may proceed as follows: introduce the two-form

$$\Sigma = dY \wedge dZ = \frac{1}{2} \varepsilon_{AB} dY^A \wedge dY^B,$$

and then calculate

$$\begin{aligned} \frac{d\Sigma}{dX} &= \varepsilon_{AB} dY^A \wedge dP^B, \\ \frac{d^2\Sigma}{dX^2} &= \varepsilon_{AB} (dP^A \wedge dP^B + dY^A \wedge dF^B), \end{aligned}$$

and

$$\frac{d^3\Sigma}{dX^3} = \varepsilon_{AB} (3dP^A \wedge dF^B + dY^A \wedge \frac{d}{dX} dF^B),$$

all modulo dX and where d/dX may be thought of as Lie-derivative.

In the context of the Nonlinear Graviton Theorem 3.2 the two-form Σ is the pull back of the $\mathcal{O}(2)$ -valued symplectic structure on the fibres of $\mathcal{T} \rightarrow \mathbb{CP}^1$ to the correspondence space. Thus, up to an overall scale, this two-form is given by a quadratic polynomial in the base coordinate X . We can therefore examine the consequences of imposing the requirement

$$\frac{d^3\Sigma}{dX^3} \sim \Sigma \text{ modulo } dX.$$

The coefficient of $dP^0 \wedge dP^1$ on the left-hand-side must vanish, which implies $\partial F^B / \partial P^B = 0$, and we may introduce Λ as in (5.4). Then the vanishing of terms in $dY^A \wedge dP^B$ reduces to a single equation

$$\varepsilon^{AB} \frac{\partial^2 \Lambda}{\partial Y^A \partial P^B} + \frac{1}{2} \varepsilon^{AD} \varepsilon^{BE} \frac{\partial^2 \Lambda}{\partial P^A \partial P^E} \frac{\partial^2 \Lambda}{\partial P^B \partial P^D} = 0,$$

which is precisely (4.2) in this notation. The remaining condition (5.3) can now be viewed as the initial value problem for Λ with the initial data given by the solution of the heavenly equation. This proves the following

Theorem 5.1. *Let $\Theta = \Theta(w, z, x, y)$ be a solution to the heavenly equation (4.2) which gives the Ricci-flat ASD metric (4.1). The corresponding system of ODEs with vanishing Wilczynski invariants is*

$$Y'' = 2 \frac{\partial \Lambda}{\partial Z'}, \quad Z'' = -2 \frac{\partial \Lambda}{\partial Y'}, \quad (5.6)$$

where $\Lambda|_{X=X_0} = -\Theta(Y(X_0), Z(X_0), Z'(X_0), -Y'(X_0))$ and the X -dependence of Λ is determined by (5.3).

The formula (5.1) gives a way to reconstruct the conformal structure directly from the system of ODEs. An alternative procedure based on the original non-linear graviton construction of Penrose [24] gives the conformal structure directly from the integral curves of the system of ODEs. This procedure will be implemented in the example (7.2).

6. Symmetries of Torsion-free Path Geometries

In the twistor approach, we observed there is a one-to-one correspondence between points in \mathcal{T} and α -surfaces in M . Therefore, any transformation which preserves points in \mathcal{T} will give rise to a transformation of M which preserves the ASD conformal structure.

Lemma 6.1. *There is a one-to-one correspondence between conformal Killing vectors of a (2, 2) ASD conformal structure $(M, [g])$ and point symmetries of the torsion-free system of ODEs whose integral curves are twistor lines for $(M, [g])$.*

Proof. We shall use the double fibration picture (3.5). Given a conformal Killing vector K of $(M, [g])$ we can lift it to the vector field \tilde{K} on the correspondence space \mathcal{F} , so that $[L_0, \tilde{K}] = 0$ and $[L_1, \tilde{K}] = 0$, where L_0, L_1 is the twistor distribution (3.6), and the commutators vanish modulo a linear combination of L_0 and L_1 . The lift is explicitly given by $\tilde{K} = K + Q\partial_\lambda$, where Q is a quadratic polynomial in λ with coefficients depending on coordinates on M . In the spinor notation $Q = \nabla_{AA'} K^A{}_{B'} \pi^{A'} \pi^{B'}$, where $\pi^{A'} = (1, -\lambda)$.

The space \mathcal{T} is a quotient of \mathcal{F} by the distribution spanned by L_0 and L_1 and thus $\mathcal{K} = s_* \tilde{K}$ is a vector field on \mathcal{T} . Therefore it generates a one-parameter group of transformations of \mathcal{T} which therefore takes α -surfaces in M to α -surfaces. As K generates diffeomorphisms of M and integral curves of the system (1.1) in \mathcal{T} correspond to points in M , then the action generated by \mathcal{K} preserves the integral curves of (1.1). Thus it is a point symmetry on (1.1).

Conversely, a point symmetry of (1.1) corresponds to a transformation of M which maps α -surfaces to α -surfaces. Therefore it gives a conformal Killing vector on M . \square

In the trivial case ($Y'' = 0, Z'' = 0$), the path geometry has point symmetry group $SL(4, \mathbb{R})$ which is isomorphic to the symmetry group of the flat conformal structure $SO(3, 3)$. The sub-maximal case of a torsion-free system of ODEs with nine-dimensional point symmetry algebra (4.5) which corresponds to a Ricci-flat ASD conformal structure with only one constant non-vanishing component of the ASD Weyl tensor. Therefore the ‘gap’ in (2, 2) conformal geometry equals to $6 = 15 - 9$. This, by Lemma 6.1 coincides with the gap of path geometries in three dimensions, as it is known that the sub-maximal symmetry in the projective class \mathcal{P} is only eight dimensional. This follows from the general result of Egorov [14] that all sub-maximally symmetric projective

connections on \mathbb{R}^n are transitive and have symmetry algebra of dimension $n^2 - 2n + 5$. See [19] for an account of the general theory of such ‘gaps’.

Example (4.5) is the unique (up to diffeomorphism) torsion-free path geometry with point symmetry algebra of sub-maximal size and we expect systems with 6, 7, 8 symmetries to be also comparatively rare. To see explicitly why this occurs, consider the lift of the integral curves of an arbitrary path geometry (1.1) to the second jet bundle $J^2(\mathcal{T}, \mathbb{R})$ which is a seven-dimensional manifold with local coordinates given by $(X, Y, Z, Y', Z', Y'', Z'')$. Any point symmetry of (1.1) can locally be described by some vector field χ on \mathcal{T} which we can prolong to a vector field $\text{pr}^{(2)}\chi$ over some open set in $J^2(\mathcal{T}, \mathbb{R})$. Then, the functions

$$\Delta^1 = Y'' - F(X, Y, Z, Y', Z'), \quad \Delta^2 = Z'' - G(X, Y, Z, Y', Z')$$

are constant along $\text{pr}^{(2)}\chi$.

Now suppose that the given path geometry admits a Lie point symmetry algebra of dimension five, generated by five vector fields which prolong to an integrable distribution \tilde{L} of $J^2(\mathcal{T}, \mathbb{R})$ on rank at most five. In the generic case, when the rank is five, these vector fields will lie in the tangent bundle of some five-dimensional submanifold $U \subset J^2(\mathcal{T}, \mathbb{R})$. Given that the codimension of U is 2, we can construct two functions Δ^0, Δ^1 which are invariant with respect to the action of \tilde{L} , i.e., given a five-dimensional Lie algebra L of vector fields over \mathcal{T} , we can find a (non-trivial) path geometry (1.1) which has L as its point symmetry algebra. This statement is true subject to some regularity conditions. For example it is sufficient to demand that the first prolongation of the vector fields to the first jet bundle $J^1(\mathcal{T}, \mathbb{R})$ must not be contained within some four-dimensional submanifold of the tangent bundle $T(J^1(\mathcal{T}, \mathbb{R}))$.

Lie algebras L of dimension six or greater will generically not give rise to functions Δ^0, Δ^1 , and there will be a constraint on finding non-trivial path geometries with point symmetry algebras of this size. In particular, we must require that the prolonged algebra forms a distribution of rank lower than six. If additionally, we impose the constraint that the Wilczynski conditions (2.1) hold, then this sudden decline of examples will be observed sooner (at dimension four rather than six). In the next section, we outline the prolongation procedure and present some examples of path geometries with 4, 5, 6, 7, 8, 9 point symmetries together with some details of the Lie algebraic structure. Where possible, we also make use of the twistor approach to say something about the corresponding conformal structure on the space of solutions.

7. Examples

For a given path geometry in three dimensions (1.1) the point symmetries may be found by the following well known procedure:

- (1) Let the generator of a point symmetry be

$$\chi = \chi_{-1}\partial_X + \chi_0\partial_Y + \chi_1\partial_Z,$$

where the χ_i are functions of (X, Y, Z) for which we must solve.

- (2) Determine the first and second prolongations by

$$\eta_A^{(1)} = \frac{d\chi_A}{dX} - P^A \frac{d\chi_{-1}}{dX}, \quad \eta_A^{(2)} = \frac{d\eta_A^{(1)}}{dX} - Q^A \frac{d\chi_{-1}}{dX},$$

$$A = 0, 1, \text{ where } Q^A := \frac{d^2 Y^A}{dX^2}.$$

(3) The prolongation of the vector field χ to the jet bundle $J^2(\mathcal{T}, \mathbb{R})$ is given by

$$\text{pr}^{(2)}(\chi) = \chi + \sum_{B=0}^1 \eta_B^{(1)} \frac{\partial}{\partial P^B} + \sum_{B=0}^1 \eta_B^{(2)} \frac{\partial}{\partial Q^B}.$$

(4) Then we determine point symmetries by finding functions χ_i from

$$\text{pr}^{(2)}(\chi)(\Delta^0)|_{\Delta^0=\Delta^1=0} = \text{pr}^{(2)}(\chi)(\Delta^1)|_{\Delta^0=\Delta^1=0} = 0.$$

7.1. Special Ricci-Flat Case. Consider the system (5.5) with β real analytic. The sub-maximal torsion-free system (4.5) lies in this class so we might expect it to yield more examples of systems with a high number of point symmetries. The corresponding ASD conformal structure on the moduli space of solutions is given by

$$g = dw dx + dz dy - \frac{1}{2} \beta'(y) dw^2$$

and is Ricci-flat, as it corresponds to (4.1) with $\Theta = (1/2) \int \beta(y) dy$.

Under these conditions, it is not difficult to show that the point symmetry algebra contains a six-dimensional subalgebra $L_6 \subset \mathfrak{sl}(4, \mathbb{R})$. It transpires that L_6 is solvable and in terms of point symmetries we may write it down explicitly,

$$L_6 = \text{span}\{\mathbf{e}_1 = \partial_X, \mathbf{e}_2 = \partial_Y, \mathbf{e}_3 = \partial_Z, \mathbf{e}_4 = X\partial_Y, \mathbf{e}_5 = Z\partial_Y, \mathbf{e}_6 = X\partial_X + 2Y\partial_Y + Z\partial_Z\}.$$

However, as we have seen, there will be some special cases of (5.5) for which the point symmetry algebra is larger but will contain L_6 as a subalgebra.

Proposition 7.1. *Consider a system of two second order ODEs of the form (5.5) for some function β of the form*

$$\beta(Y') = \sum_{k=0}^{\infty} \xi_k (Y')^k.$$

If β is a quadratic function then the system (5.5) is diffeomorphic to a trivial one (and the symmetry group is 15-dimensional). Otherwise, the symmetry algebra has dimension

$$12 - \text{Rank}(M_1) - \text{Rank}(M_2),$$

where M_1 is a matrix with rows

$$(\xi_k \ \xi_{k+1}), \quad k \geq 3$$

and M_2 is a matrix with rows

$$(\xi_k \ (k-2)\xi_k \ (k-3)\xi_{k-1} \ \dots \ -(k+1)\xi_{k+1}), \quad k \geq 3.$$

Proof. Without loss of generality, let us simplify the problem by making the diffeomorphism

$$Z \rightarrow Z + \frac{1}{2}\xi_0 X^2 + \xi_1 XY + \frac{1}{2}\xi_2 Y^2$$

so that we obtain the system

$$Y'' = 0, \quad Z'' = \sum_{k=3}^{\infty} \xi_k (Y')^k,$$

which has the same number of point symmetries as the original.

For a given vector field,

$$\chi = \chi_{-1} \frac{\partial}{\partial X} + \chi_0 \frac{\partial}{\partial Y} + \chi_1 \frac{\partial}{\partial Z},$$

the expressions

$$\text{pr}^{(2)}(\chi)(\Delta^0)|_{\Delta^0=\Delta^1=0} \text{ and } \text{pr}^{(2)}(\chi)(\Delta^1)|_{\Delta^0=\Delta^1=0}$$

are real analytic in P^0 and P^1 with coefficients which are functions of X , Y and Z . For χ to be a symmetry of the system (4.5), each of these coefficients must vanish separately. This leads to the following system of differential equations in χ_{-1} , χ_0 and χ_1 :

$$\begin{aligned} \frac{\partial^2 \chi_0}{\partial X^2} &= \frac{\partial^2 \chi_1}{\partial X^2} = \frac{\partial^2 \chi_0}{\partial X \partial Z} = \frac{\partial^2 \chi_1}{\partial X \partial Y} = \frac{\partial^2 \chi_{-1}}{\partial Y \partial Z} = \frac{\partial^2 \chi_{-1}}{\partial Z^2} = \frac{\partial^2 \chi_0}{\partial Z^2} = 0, \\ 2 \frac{\partial^2 \chi_0}{\partial X \partial Y} &= \frac{\partial^2 \chi_{-1}}{\partial X^2}, \quad \frac{\partial^2 \chi_0}{\partial Y^2} = 2 \frac{\partial^2 \chi_{-1}}{\partial X \partial Y}, \quad \frac{\partial^2 \chi_0}{\partial Y \partial Z} = \frac{\partial^2 \chi_{-1}}{\partial X \partial Z}, \\ 2 \frac{\partial^2 \chi_1}{\partial X \partial Z} &= \frac{\partial^2 \chi_{-1}}{\partial X^2}, \quad \frac{\partial^2 \chi_1}{\partial Y \partial Z} = \frac{\partial^2 \chi_{-1}}{\partial X \partial Y}, \quad \frac{\partial^2 \chi_1}{\partial Z^2} = 2 \frac{\partial^2 \chi_{-1}}{\partial X \partial Z}, \\ \frac{\partial^2 \chi_1}{\partial Y^2} &= 3\xi_3 \frac{\partial \chi_0}{\partial X}, \quad \xi_3 \frac{\partial \chi_0}{\partial Z} = \frac{\partial^2 \chi_{-1}}{\partial Y^2}, \quad \frac{\partial^2 \chi_{-1}}{\partial Y^2} + 3\xi_3 \frac{\partial \chi_0}{\partial Z} = 0, \end{aligned}$$

and, for $k \geq 3$,

$$\begin{aligned} (k-3)\xi_k \frac{\partial \chi_{-1}}{\partial Z} - (k+1)\xi_{k+1} \frac{\partial \chi_0}{\partial Z}, \quad \xi_{k+1} \frac{\partial \chi_0}{\partial Z} &= \xi_k \frac{\partial \chi_{-1}}{\partial Z}, \\ (k-2)\xi_k \frac{\partial \chi_{-1}}{\partial X} + \xi_k \frac{\partial \chi_1}{\partial Z} + (k-3)\xi_{k-1} \frac{\partial \chi_{-1}}{\partial Y} - (k+1)\xi_{k+1} \frac{\partial \chi_0}{\partial X} - k\xi_k \frac{\partial \chi_0}{\partial Y} &= 0. \end{aligned}$$

Enforcing the system to be non-trivial (i.e., not all $\xi_k = 0$) we obtain

$$\frac{\partial \chi_{-1}}{\partial Z} = \frac{\partial \chi_0}{\partial Z} = \frac{\partial^2 \chi_{-1}}{\partial Y^2} = 0.$$

Then, the most general solution for χ is

$$\begin{aligned} \chi_{-1} &= a_1 X^2 + a_2 XY + (a_3 + a_4)X + a_5 Y + a_6, \\ \chi_0 &= a_1 XY + a_7 X + a_2 Y^2 + 2a_3 Y + a_8, \\ \chi_1 &= a_1 XZ + a_9 X + a_2 YZ + (a_3 + a_{10})Z + \frac{\xi_3}{2} a_1 Y^2 + \left(\frac{3\xi_3}{2} a_7 + a_{11} \right) Y + a_{12}, \end{aligned}$$

where, for all $k \geq 3$,

$$\xi_k a_2 + \xi_{k+1} a_1 = 0$$

and

$$\xi_k a_{10} + (k-2)\xi_k a_4 + (k-3)\xi_{k-1} a_5 - (k+1)\xi_{k+1} a_7 - (k+1)\xi_k a_3 = 0,$$

and the result follows.

Thus, in the non-trivial case, the point symmetry algebra of (5.5) has at most dimension 9. A little more work shows us that systems of dimension 8 are not attainable for this example (if the rank of M_2 is 1, then the rank of M_2 is 2 and if the rank of M_1 is 2 then the rank of M_2 is at least 3). \square

Dimension 9. A system with point symmetry algebra of dimension 9 is given by (4.5),

$$Y'' = 0, \quad Z'' = -2(Y')^3.$$

A quick check of the expressions for M_1 and M_2 will show that the system

$$Y'' = 0, \quad Z'' = \frac{(Y')^3}{1 - Y'}$$

also admits a nine-dimensional point symmetry algebra but these systems can be shown to be diffeomorphic. The associated Lie algebra in the first case can be written as

$$L_9 = L_6 \oplus \text{span}\{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9\} \quad (7.1)$$

with

$$\begin{aligned} \mathbf{e}_7 &= 3Y\partial_Y + Z\partial_Z, \quad \mathbf{e}_8 = \frac{3}{2}Z^2\partial_Y + X\partial_Z, \\ \mathbf{e}_9 &= \frac{1}{2}X^2\partial_X + \left(\frac{1}{2}XY + \frac{1}{4}Z^3\right)\partial_Y + \frac{1}{2}XZ\partial_Z. \end{aligned}$$

This has Levi-decomposition

$$L_9 = \tilde{L}_6 \ltimes \mathfrak{sl}(2, \mathbb{R}) = \text{span}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8\} \ltimes \text{span}\{\mathbf{e}_1, \mathbf{e}_6 - \frac{1}{2}\mathbf{e}_7, \mathbf{e}_9\}.$$

Dimension 7. A path geometry with point symmetry algebra of dimension 7 is given by

$$Y'' = 0, \quad Z'' = (Y')^k$$

for $k \geq 4$. This will be a solvable Lie algebra for any value of k and is obtained by adding the vector field $\mathbf{e}_7 = kY\partial_Y + Z\partial_Z$ to L_6 .

7.2. *ASD Einstein - dimension 8.* Consider the system

$$Y'' = 0, \quad Z'' = \frac{2(Z')^2 Y'}{ZY' - 1}. \quad (7.2)$$

This system has been suggested to us by Boris Doubrov, who has also pointed out its appearance in the theory of chains in the homogeneous contact geometry [4]. The Wilczynski invariants vanish for (7.2).

The corresponding conformal structure is found by demanding that two neighbouring integral curves in \mathcal{T} intersect at one point – this is the condition which selects null vectors in M . This, in the complexified setting, reduces the normal bundle of the integral curve L_p to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. To perform this calculation explicitly, solve (7.2) for (Y, Z) and demand that equations $\delta Y = 0$ and $\delta Z = 0$ for X have a common solution. We apply this procedure to the integral curves

$$\mathcal{Y} = w + yX, \quad \mathcal{Z} = \frac{1}{y} + \frac{1}{y^2(z - xX)},$$

where $\delta Y = \sum_a \partial_a \mathcal{Y} \delta x^a$, $\delta Z = \sum_a \partial_a \mathcal{Z} \delta x^a$, and $x^a = (w, z, x, y)$. Then the discriminant condition for two curves to intersect gives the metric

$$g = dwdx + dzdy + x^2 dw^2 + \left(z^2 + \frac{2z}{y}\right) dy^2 + 2\left(zx + \frac{x}{y}\right) dwdy,$$

which is ASD and Einstein, with scalar curvature equal to -24 . Both the system of ODEs and the metric have eight-dimensional symmetry group $SL(3, \mathbb{R})$ which acts isometrically on M . The homogeneous model for the space of solutions is $SL(3, \mathbb{R})/GL(2, \mathbb{R})$, which is a $(2, 2)$ real form of the Fubini-Study metric on $\mathbb{CP}^2 = SU(3)/U(2)$.

7.3. *Symmetry algebra of dimension 5.* The example discussed in the Gibbons-Hawking context (4.9) has five-dimensional point symmetry algebra which we can write as

$$L_5 = \text{span}\{\partial_Y, \partial_Z, X\partial_X + Y\partial_Y, -\frac{1}{2}X\partial_X + Z\partial_Z, X^2\partial_Y + 2Y\partial_Z\}.$$

The algebra L_5 is solvable and contains both Bianchi II and Bianchi V as three-dimensional subalgebras.

7.3.1. *Sparling-Tod solution* Another way to construct a torsion-free path geometry with a given point symmetry algebra is to determine an ASD metric with that conformal symmetry algebra and determine the system of ODEs governing twistor lines via the method given in Sect. 5. The metric of [28] given by

$$g = dwdx + dzdy - \frac{2}{(xw + yz)^3} (wdz - zdw)^2$$

is an ASD Ricci-flat metric (4.1) with $\Theta = \frac{1}{xw + yz}$ and has five-dimensional conformal symmetry algebra. The conformal Killing vectors are

$$\mathbf{k}_1 = z\partial_w - x\partial_y, \mathbf{k}_2 = -2w\partial_w + x\partial_x - y\partial_y, \mathbf{k}_3 = w\partial_w + z\partial_z, \mathbf{k}_4 = -y\partial_x + w\partial_z, \\ \mathbf{k}_5 = z\partial_x - w\partial_y.$$

We should note here that the resulting system of ODEs does not coincide with the previous example as the conformal symmetry algebra of the Sparling-Tod solution is not solvable. It has an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra generated by $(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{k}_3, \mathbf{k}_4)$. Its Levi decomposition can be expressed as the semi-direct product of an $\mathfrak{sl}(2, \mathbb{R})$ with the 2-dimensional non-Abelian Lie algebra.

The system of ODEs can be read-off directly using Theorem 5.1 and setting $\Lambda = -\Theta(Y, Z, Z', -Y')$. This yields $\Lambda = (Y'Z - YZ')^{-1}$ and (5.4) gives

$$Y'' = \frac{2Y}{(Y'Z - YZ')^2}, \quad Z'' = \frac{2Z}{(Y'Z - YZ')^2}. \quad (7.3)$$

The integral curves are

$$\mathcal{Y} = Ae^{\gamma X} + Be^{-\gamma X}, \quad \mathcal{Z} = Ce^{\gamma X} + De^{-\gamma X},$$

where (A, B, C, D) are constants of integrations, and $\gamma^2 = (AD - BC)^{-1}\sqrt{2}^{-1}$. The original metric can be recovered from the twistor lines by setting

$$w = A + B, \quad z = (C + D), \quad y = \gamma(B - A), \quad x = \gamma(C - D).$$

7.4. Symmetry algebra of dimension 4. Consider the system

$$Y'' = 0, \quad Z'' = -\left(Z' + \sqrt{(Y')^2 - 1}\right)^2. \quad (7.4)$$

This example was found by constructing the most general system of ODEs with Lie point symmetry algebra

$$L_4 = \text{span}\{\partial_X, \partial_Y, \partial_Z, Y\partial_X + X\partial_Y\}$$

and imposing the torsion-free conditions. The algebra L_4 above is a particular realisation of the abstract algebra¹ $A_{4,1}$ as given by Patera and Winternitz [23]. The integral curves of the system (7.4) are given by

$$\mathcal{Y} = w + Xy, \quad \mathcal{Z} = \log(X - x) - \sqrt{y^2 - 1}X + z.$$

Following the procedure applied in case of the system (7.2) we find the ASD conformal structure

$$g = dx dy + (dw + x dy)(dz + y\sqrt{y^2 - 1}^{-1} dw).$$

This admits a null Killing vector $\partial/\partial z$ and thus fits into the classification of [12].

¹ The Referee has pointed out that there are two more nonequivalent representations of this algebra:

$$L_{4a} = \text{span}\{\partial_Y, -X\partial_Y, \partial_Z, \partial_X - XZ\partial_Y\}$$

and

$$L_{4b} = \text{span}\{2\partial_X, \partial_Z, -Y^2\partial_X - y\partial_Z, 2Z\partial_X + \partial_Y\}.$$

We have checked that there is no torsion-free system of ODEs with symmetry L_{4b} , and there exists a torsion-free system of ODEs with a seven-dimensional symmetry algebra which contains L_{4a} as a sub-algebra.

7.5. Symmetry algebra of dimension 3 or less. If the symmetry algebra has dimension 3, then it necessarily belongs to the Bianchi classification of three-dimensional Lie algebras. There are many examples in this case which are analytic continuations of Riemannian metrics. See [29] for a discussion of these examples. Examples of an ASD conformal class with one or two symmetries can be found in the Gibbons–Hawking class (4.7). An axisymmetric solution H to the wave equation (4.6) gives a metric with two Killing vectors. Another example with two-dimensional symmetry is the system (8.5) discussed in the next section. A general solution of (4.6) with no symmetries gives a metric which only admits one Killing vector $\partial/\partial z$. Gravitational instantons of class D_k are examples of ASD conformal structures with no symmetries.

8. Finsler Structures with Scalar Flag Curvature

For a given n -dimensional manifold \mathcal{T} with coordinates X^k ($k = 1, \dots, n$), a *Finsler metric* is a positive continuous function $\mathcal{F} : \mathcal{T} \rightarrow [0, \infty)$ such that

- \mathcal{F} is smooth on $T\mathcal{T} \setminus 0 = \{(X^k, P^k) \in T\mathcal{T} | P \neq 0\}$,
- $\mathcal{F}(X^k, cP^k) = c\mathcal{F}(X^k, P^k)$ for $c > 0$,
- The tensor $f_{ij} = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial P^i \partial P^j}$ is positive definite for all $(X^k, P^k) \in T\mathcal{T} \setminus 0$.

We shall consider the case $n = 3$ and set $X^k = (X, Y, Z)$ and $P^k = \dot{X}^k = \frac{dX^k}{dt}$ for some parameter t . Finsler geometry generalizes the notion of Riemannian geometry in that the norm on each tangent space $\mathcal{F}(X^k, \cdot)$ is not necessarily induced by a metric tensor. This makes these metrics useful in the study of problems involving paths of least time. The metric tensor f_{ij} allows us to define Finslerian geodesics, which are integral curves of the system

$$\ddot{X}^i + \gamma_{jk}^i \dot{X}^j \dot{X}^k = 0, \quad i, j, k = 1, \dots, n,$$

where

$$\gamma_{jk}^i = \frac{1}{2} f^{il} (f_{lk, X^j} + f_{jl, X^k} - f_{jk, X^l}).$$

It was argued in [5] that, for $n > 2$, given any system of ODEs with vanishing Wilczynski invariants its integral curves arise as the set of unparametrised geodesics of a Finsler function of scalar flag curvature. In this context, the torsion-free path geometries are viewed as projective equivalence classes of *isotropic* sprays on $T\mathcal{T}$. Given a spray

$$S = P^i \frac{\partial}{\partial X^i} - 2\Gamma^i \frac{\partial}{\partial P^i}, \quad i = 1, \dots, n,$$

we can define its Riemann curvature by

$$R_{kij}^l = H_i \left(\Gamma_{jk}^l \right) - H_j \left(\Gamma_{ik}^l \right) + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m,$$

where $\Gamma_{jk}^i = \frac{\partial^2 \Gamma^i}{\partial P^j \partial P^k}$, etc and the H_i form a horizontal distribution determined by S :

$$H_i = \frac{\partial}{\partial X^i} - \Gamma_i^j \frac{\partial}{\partial P^j}.$$

Then the spray S is said to be *isotropic* if its Ricci curvature is given by

$$R_j^i = R_{jkl}^i P^k P^l = \rho \delta_j^i + \tau_j P^i$$

for some function ρ and covector τ_i . It was shown in [6], that for $n > 2$, a spray is isotropic if and only if it is projectively equivalent to one whose Riemann curvature vanishes (so-called R-flat) and in [7], that any R-flat spray arises as the geodesic spray of some Finsler function. More importantly, in [27] it is shown that the geodesic sprays of a Finsler function are isotropic if and only if the Finsler function has scalar flag curvature. Here, the flag curvature of a Finsler metric is defined in terms of its Riemann curvature by

$$K(X^k, P^k, V^k) = \frac{V^i (P^j R_{jkl}^i P^l) V^k}{f(P, P)f(V, V) - (f(P, V))^2},$$

where V is a section of the tangent bundle transverse to P and the indices are raised/lowered with the tensor f_{ij} . The flag curvature is scalar if

$$K(X^k, P^k, V^k) = K(X^k, P^k).$$

Thus, we could construct all systems of ODEs with vanishing Wilczynski invariants if we know how to characterize the Finsler functions of scalar flag curvature. Although a lot of work has been done in this area, a complete characterization of such functions has not yet been achieved. Most success has come with a special type of Finsler function

$$\mathcal{F} = \sqrt{a_{ij}(X^k) P^i P^j} + b_i(X^k) P^i, \quad i, j, k = 1, \dots, n.$$

This is known as a *Randers metric*. If we restrict our attention to Randers metrics of constant flag curvature, then we can use the following

Theorem 8.1 (Bao–Robels–Shen [2]). *A Randers metric \mathcal{F} has constant flag curvature if and only if the corresponding Zermelo data (h, W) satisfy the following:*

- h is a Riemannian metric with constant sectional curvature.
- W is a Killing vector or homothety of h .

Here, the Randers data can be expressed in terms of the Zermelo data as follows [26]:

$$a_{ij} = \frac{\lambda h_{ij} + W_i W_j}{\lambda^2}, \quad b_i = -\frac{W_i}{\lambda},$$

where $\lambda = 1 - h_{ij} W^i W^j$ and $W_i = h_{ij} W^j$. This gives rise to a procedure for constructing torsion-free systems of ODEs from a three-dimensional Riemannian metric of constant sectional curvature and a homothety of this metric. The geodesic spray coefficients of such systems was worked out in [26].

Example 4. Consider the Zermelo data

$$h = dX^2 + dY^2 + Y^2 dZ^2, \quad W = \frac{\partial}{\partial Z}.$$

The geodesics of the Randers metric associated to this Zermelo data are the integral curves of the systems of ODEs with vanishing Wilczynski invariants,

$$\begin{aligned} Y'' &= \frac{2YZ' \sqrt{1 + (Y')^2 + Y^2 ((Z')^2 - (Y')^2 - 1)}}{Y(Y^2 - 1)} \\ &\quad + \frac{Y(1 + (Y')^2 + (Z')^2 Y^2 ((Z')^2 - (Y')^2 - 1))}{(Y^2 - 1)^2}, \\ Z'' &= \frac{2Y' \left(Z' + \sqrt{1 + (Y')^2 + Y^2 ((Z')^2 - (Y')^2 - 1)} \right)}{Y(Y^2 - 1)}. \end{aligned} \quad (8.5)$$

This system has two symmetries $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Z}$.

We can also view this correspondence in the other direction i.e, given a system of ODEs with vanishing Wilczynski invariants, we can construct a Finsler metric of scalar flag curvature.

Example 5. To illustrate this point, let us consider the submaximal system (4.5) corresponding to ASD Ricci-flat pp waves with constant Weyl curvature. Then, using the procedure in [22], this system describes the unparametrised geodesics of the Finsler function

$$\mathcal{F} = \dot{X} \mathcal{G} \left(\frac{\dot{Y}}{\dot{X}}, 2 \frac{\dot{Z}}{\dot{X}} - 2 \frac{X \dot{Y}^3}{\dot{X}^3} + \frac{6Y \dot{Y}^2}{\dot{X}^2} \right)$$

in some open domain of \mathcal{T} where f_{ij} is positive definite, where \mathcal{G} is any function of two variables. In particular, let $\mathcal{G}(x, y) = \sqrt{xy}$ and consider the Lagrangian $L = \frac{1}{2} \mathcal{F}^2$ in unparametrised form, where $\dot{X} = 1$,

$$L = 2Y'Z' - 2X(Y')^4 + 6Y(Y')^3.$$

The Euler-Lagrange equations give (4.5). For this example, the flag curvature of \mathcal{F} vanishes.

Similarly other systems with vanishing Wilczynski invariants arise from a variational principle induced by a Finsler structure. Thus all these systems fit into the formalism of [1].

9. Conclusions and Outlook

We have constructed several examples of systems of two second order ODEs with vanishing Wilczynski invariants. The integral curves of such systems are twistor curves which correspond to points in a four-manifold M with ASD conformal structure. The twistor curves can also be viewed as unparametrised geodesics of Finsler structures with scalar flag curvature.

The maximally symmetric, non-trivial example is a Ricci-flat ASD pp -wave with 9-dimensional group of conformal symmetries, and we have given examples of systems with symmetry groups of dimensions between 9 and 5. Some of the examples have a special form $Y'' = 0$, $Z'' = G(X, Y, Z, Y', Z')$. This is what Grossman calls ‘a weaker

form of integrability for the second ruling' [17]. It corresponds to the existence of a two parameter family of β -surfaces (these are null ASD surfaces). This family gives a surface in the twistor space \mathcal{T} , and it is known [12] any ASD $(2, 2)$ conformal structure with a conformal null Killing vector admits such structure.

In the case of a trivial ODE with integral curves being straight lines in $\mathcal{T} = \mathbb{R}^3$ is a starting point for John's integral transform [18]. Given a function $\phi : \mathcal{T} \rightarrow \mathbb{R}$ with appropriate decay conditions at infinity and a straight line $L \subset \mathcal{T}$, define a function $\hat{\phi}$ on the space M of straight lines

$$\hat{\phi}(L) = \int_L \phi.$$

The result of John is that the range of this transform is characterised by an ultra-hyperbolic wave equation on M , where the wave operator is induced by a flat $(2, 2)$ metric on $M = \mathbb{R}^4$. If the line L is parametrised by $X \rightarrow (X, Y = w + Xy, Z = z - Xx)$, then

$$\hat{\phi}(w, z, x, y) = \int_{\mathbb{R}} \phi(X, w + Xy, z - Xx) dX \quad \text{and} \quad \hat{\phi}_{wx} + \hat{\phi}_{zy} = 0.$$

It may be interesting to develop a non-linear version of John's transform applicable to non-trivial systems of ODEs with vanishing Wilczynski invariants. Integral curves of (4.5) could be a good starting point for this construction.

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References

1. Anderson, I., Thompson, G.: The Inverse Problem in the Calculus of Variations for Ordinary Differential Equations. *Memoirs of the AMS*, **473**, 1982
2. Bao, D., Robles, C., Shen, Z.: Zermelo Navigation on Riemannian Manifolds. *J. Diff. Geom.* **66**, 391–449 (2004)
3. Branson, T.P., Čap, A., Eastwood, M.G., Gover, A.R.: Prolongations of geometric overdetermined systems. *Internat. J. Math.* **17**, 641–664 (2006)
4. Čap, A., Zadnik, V.: Contact projective structures and chains. *Geom. Dedicata* **146**, 67–83 (2010)
5. Crampin, M., Saunders, D.J.: Path Geometries and almost Grassmann Structures. *J. Math. Phys* **18**, 1449 (2005)
6. Crampin, M.: Isotropic and R-flat sprays. *Houst. J. Math.* **33**, 451–459 (2005)
7. Crampin, M.: On the inverse problem for sprays. *Publ. Math-Debr.* **70**, 319–335 (2005)
8. Doubrov, B.: Contact trivialization of ordinary differential equations. *Diff. Geom. Appl. Proc. Conf. Opava-(Czech Repub.)* 73–84 (2001)
9. Doubrov, B.: Generalized Wilczynski invariants for non-linear ordinary differential equations. *The IMA Vol in Maths. and its Appl.* **144**, 25–40 (2008)
10. Dunajski, M., Mason, L.J.: Hyper-Kähler Hierarchies and Their Twistor Theory. *Commun. Math. Phys* **213**, 641–672 (2000)
11. Dunajski, M., Tod, K.P.: Paraconformal geometry of n th order ODEs, and exotic holonomy in dimension four. *J. Geom. Phys.* **56**, 1790–1809 (2006)
12. Dunajski, M., West, S.: Anti-self-dual conformal structures from projective structures. *Commun. Math. Phys.* **272**, 85–118 (2007)
13. Dunajski, M.: Solitons, Instantons, and Twistors. Oxford: Oxford University Press, 2009 **Ch.10**.
14. Egorov, I.P.: Collineations of projectively connected spaces. (Russian) *Dokl. Akad. Nauk SSSR (N.S.)* **80**, 709–712 (1951)

15. Fels, M.: The equivalence problem for systems of second-order ordinary differential equations. Proc. London Math. Soc. **71**, 221–240 (1995)
16. Gibbons, G.W., Hawking, S.W.: Gravitational multi-instantons. Phys. Lett. **78B**, 430–432 (1978)
17. Grossman, D.: Torsion-free path geometries and integrable second order ODE systems. Selecta Math. (N.S.) **6**, 399–442 (2000)
18. John, F.: The ultrahyperbolic differential equation with four independent variables. Duke Math. Journ **4**, 300–322 (1938)
19. Kruglikov, B.: The gap phenomenon in the dimension study of finite type systems. <http://arXiv.org/abs/1111.6315v2> [math.DG], 2012
20. Mason, L.J., Woodhouse, N.M.J.: *Integrability, selfduality, and twistor theory*. Oxford, UK: Clarendon LMS monographs, new series: **15**, 1996
21. Mettler, T.: Reduction of beta-integrable 2-Segre structures. Comm. Anal. Geom. **21**(2), 331–353 (2013)
22. Muzsnay, Z.: The Euler-Lagrange PDE and Finsler Metrizability. Houst. J. Math. **32**, 1 (2006)
23. Patera, J., Winternitz, P.: Subalgebras of real three- and four-dimensional Lie algebras. J. Math. Phys **18**, 1449 (1977)
24. Penrose, R.: Nonlinear gravitons and curved twistor theory. Gen. Rel. Grav. **7**, 31–52 (1976)
25. Plebański, J.F.: Some solutions of complex Einstein Equations. J. Math. Phys. **16**, 2395–2402 (1975)
26. Robles, C.: Geodesics in Randers Spaces of Constant Curvature. Tran. Amer. Math. Soc. **359**, 1633 (2007)
27. Shen, Z.: *Differential Geometry of Spray and Finsler Spaces*. Dordrecht: Kluwer, 2001
28. Sparling, G.A.J., Tod, K.P.: An Example of an H-space. J. Math. Phys. **22**, 331–332 (1981)
29. Tod, K.P.: *Cohomogeneity-one metrics with self-dual Weyl tensor*. Twistor Theory. Ed. S. Huggett. Lect. Notes Pure Appl. Math., **169**, New York: Marcel Dekker, 1995
30. Wilczynski, E.J.: *Projective differential geometry of curves and ruled surfaces*. Leipzig: Teubner, 1905

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