## Mathematics of curved spaces

## Maciej Dunajski

Clare College<br>and

Department of Applied Mathematics and Theoretical Physics University of Cambridge


## Intrinsic vs Extrinsic

- Bugs can only measure distance along curves - curvature is extrinsic

- Intelligent bugs can measure curvature of surfaces - curvature is intrinsic



## Curved Curves

- Curvature=reciprocal of the osculating circle radius. $\kappa=r^{-1}$

- Straight line: $\kappa=0$
- Circle of radius one: $\kappa=1$
- In general $y=f(x)$ (Exercise!)

$$
\kappa=\frac{f^{\prime \prime}}{\left(1+\left(f^{\prime}\right)^{2}\right)^{3 / 2}}
$$

## Gaussian curvature

- Carl Friedrich Gauss 1777-1855


$$
K=\kappa_{\max } \times \kappa_{\min }
$$



Negative Curvature


Zero Curvature


Positive Curvature

- $K=1 \times(-1)=-1 . \quad K=1 \times 0=0 . \quad K=1 \times 1=1$


## Back to bugs - geodesic triangles

- Geodesic=shortest path.

- Geodesic triangles

- Small geodesic circle of radius $r$

$$
\text { Circumference }=2 \pi r-K \pi \frac{r^{3}}{3}+\ldots
$$

## MAP MAKING

- Stereographic projection


$$
X^{2}+Y^{2}+Z^{2}=1, \quad z=\frac{X+i Y}{1-Z}, \quad S^{2}=\mathbb{R}^{2}+\{\infty\}
$$

- Gerard Mercator (1512-1594): straight lines are rhumb lines

- Both projections are conformal: angles are preserved.


## Euclid's Parallel postulate

Euclid of Alexandria (4th century BC) 'Let the following be postulated'

(1) A straight line may be drawn between any two points.
(2) A piece of straight line may be extended indefinitely.
(3) A circle may be drawn with any given radius and an arbitrary center.
(1) All right angles are equal.
(0) (The parallel postulate)


## Hyperbolic geometry

- Poincare Disc $\mathbb{D}=\{z \in \mathbb{C},|z|<1\}$.
- Geodesics



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- Hyperbolic distance

$$
d(P, Q)=\ln \left(\frac{|A Q||P B|}{|A P||P Q|}\right) .
$$

## UPPER HALF-PLANE AND THE PSEUDOSPHERE

- $S^{2}$ can be embedded in $\mathbb{R}^{3}$, but the disc can not: pseudosphere

- Upper half-plane $\mathbb{H}=\{w \in \mathbb{C}, \operatorname{Im}(w)>0\}$.



## MÖbiUs TRANFORMATION

Maurits Cornelis Escher (1898-1972)


Exercise: Show that $f: \mathbb{D} \rightarrow \mathbb{H}$.
(1) $\operatorname{lm}(w)>0$ if $|z|<1$.
(2) The map is one-to-one: any point in $\mathbb{H}$ is an image of exactly one point in $\mathbb{D}$.

## Curvature of the Universe

- One-dimensional curve, two-dimensional surface, ..., four dimensional space-time (three space+one time)
- Gravitation is an effect of the curvature

- Einstein's general relativity: Mass=Curvature.
- Regions where curvature is infinite: Big-bang, black holes
- Physics (as we know it) breaks down...

