MATHEMATICS OF CURVED SPACES

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Curved Spaces

INTRINSIC VS EXTRINSIC

• Bugs can only measure distance along curves - curvature is extrinsic

• Intelligent bugs can measure curvature of surfaces - curvature is intrinsic



CURVED CURVES

• Curvature=reciprocal of the osculating circle radius. $\kappa = r^{-1}$



- Straight line: $\kappa = 0$
- Circle of radius one: $\kappa = 1$
- In general y = f(x) (Exercise!)

$$\kappa = \frac{f''}{(1 + (f')^2)^{3/2}}.$$

GAUSSIAN CURVATURE

• Carl Friedrich Gauss 1777-1855



 $K = \kappa_{\max} \times \kappa_{\min}.$



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BACK TO BUGS - GEODESIC TRIANGLES

• Geodesic=shortest path.



Geodesic triangles



• Small geodesic circle of radius r

Circumference =
$$2\pi r - K\pi \frac{r^3}{3} + ...$$

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Curved Spaces

• Stereographic projection



$$X^{2} + Y^{2} + Z^{2} = 1, \quad z = \frac{X + iY}{1 - Z}, \quad S^{2} = \mathbb{R}^{2} + \{\infty\}$$

• Gerard Mercator (1512-1594): straight lines are rhumb lines



• Both projections are conformal: angles are preserved.

EUCLID'S PARALLEL POSTULATE

Euclid of Alexandria (4th century BC) 'Let the following be postulated'



- A straight line may be drawn between any two points.
- A piece of straight line may be extended indefinitely.
- **③** A circle may be drawn with any given radius and an arbitrary center.
- All right angles are equal.
- (The parallel postulate)



HYPERBOLIC GEOMETRY

- Poincare Disc $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}.$
- Geodesics



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• Hyperbolic distance

$$d(P,Q) = \ln\left(\frac{|AQ||PB|}{|AP||PQ|}\right).$$

UPPER HALF-PLANE AND THE PSEUDOSPHERE

• S^2 can be embedded in \mathbb{R}^3 , but the disc can not: pseudosphere



• Upper half-plane $\mathbb{H} = \{ w \in \mathbb{C}, \operatorname{Im}(w) > 0 \}.$



Möbius tranformation

Maurits Cornelis Escher (1898-1972)



$$z \in \mathbb{C}, \quad w = f(z) = \frac{i+iz}{1-z}$$

Exercise: Show that $f : \mathbb{D} \to \mathbb{H}$.

- $\operatorname{Im}(w) > 0$ if |z| < 1.
- ② The map is one-to-one: any point in 𝔄 is an image of exactly one point in 𝔅.

CURVATURE OF THE UNIVERSE

- One-dimensional curve, two-dimensional surface, ..., four dimensional space-time (three space+one time)
- Gravitation is an effect of the curvature



- Einstein's general relativity: Mass=Curvature.
- Regions where curvature is infinite: Big-bang, black holes
- Physics (as we know it) breaks down...

Thank You