Part II Integrable Systems, Sheet Two

Professor Maciej Dunajski, Lent Term 2026

1. **Lax pair.** Consider a one-parameter family of self–adjoint operators $L(t)$ in some complex inner product space such that
$$L(t) = U(t)L(0)U(t)^{-1}$$
where $U(t)$ is a unitary operator, i.e. $U(t)U(t)^\dagger = 1$ where $U^\dagger$ is the adjoint of U.

   Show that $L(t)$ and $L(0)$ have the same eigenvalues. Show that there exist an anti-self-adjoint operator A such that $U_t = -AU$ and
$$L_t = [L, A].$$

2. **Lax representation of ODEs**. Let $L(t), A(t)$ be complex valued n by n matrices such that
$$\dot{L} = [L, A].$$

   Deduce that $\text{Trace}(L^p), p \in \mathbb{Z}$ does not depend on t.

   [It is possible to show that systems integrable in a sense of Arnold–Liouville's theorem can be put in this form, with the Poisson commuting first integrals given by traces of powers of L].

   Assume that
$$\begin{aligned} L &= (\Phi_1 + i\Phi_2) + 2\Phi_3\lambda - (\Phi_1 - i\Phi_2)\lambda^2, \\ A &= -i\Phi_3 + i(\Phi_1 - i\Phi_2)\lambda \end{aligned}$$

   where λ is a parameter and find the system of ODEs satisfied by matrices $\Phi_j(t), j = 1, 2, 3$.

   [The Lax relations should hold for any value of the parameter λ. The system you are asked to find known as Nahm's equations. It underlies the construction of non–abelian magnetic monopoles.]

   Now take $\Phi_j(t) = -i\sigma_j w_j(t)$ (no summation) where σ_j are matrices
$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   which satisfy $[\sigma_j, \sigma_k] = i\sum_{k=1}^3 \varepsilon_{jkl}\sigma_l$. Show that the system reduces to the Euler equations
$$\dot{w}_1 = w_2 w_3, \qquad \dot{w}_2 = w_1 w_3, \qquad \dot{w}_3 = w_1 w_2.$$

   Use $\text{Trace}(L^p)$ to construct first integrals of this system.

3. **Toda equation.** Write down the Hamiltonian equations for the Toda Hamiltonian for N particles moving in one dimension, $H = \frac{1}{2}\sum_{j=1}^N p_j^2 + \sum_{j=1}^{N-1}\exp(q_j - q_{j+1})$ and show that with the definitions $a_j = \frac{1}{2}\exp[(q_j - q_{j+1})/2]$ and $b_j = -\frac{1}{2}p_j$ they imply the Toda equations
$$\dot{a}_j = a_j(b_{j+1} - b_j), \qquad \dot{b}_j = 2(a_j^2 - a_{j-1}^2). \tag{1}$$

   (Use the convention that $q_0 = -\infty, e^{q_0} = 0, q_{N+1} = +\infty, e^{-q_{N+1}} = 0$.)

   Verify that the Toda problem with $N = 2$ can be written as the Lax pair $\dot{L} = [B, L]$ with
$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}.$$

   Express the eigenvalues of L in terms of the total momentum $p_1 + p_2$ and the energy H, check they are in involution.

   Obtain the general solution to the system.

4. **Lax pair for KdV.** Show that the The KdV equation is equivalent to

$$L_t = [L, A]$$

where the Lax operators are

$$L = -\frac{d^2}{dx^2} + u, \quad A = 4\frac{d^3}{dx^3} - 3\Big(u\frac{d}{dx} + \frac{d}{dx}u\Big), \qquad u = u(x,t).$$

5. **Review of IB quantum mechanics** Let $L = -\frac{d^2}{dx^2} + u(x)$ be the one dimensional Schrödinger operator with potential u, assumed to decay rapidly at infinity. Show that if $L\psi = \lambda\psi$ and $L\psi' = \lambda\psi'$ then the Wronskian $W(\psi, \psi') \equiv \psi\psi'_x - \psi'\psi_x$ is constant.

   Show that if ψ and ψ' are bound states corresponding to the same discrete eigenvalue then $\psi \propto \psi'$. Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.

6. **Evolution of scattering data.** Referring to the operators L, A defining the Lax structure of KdV in Q4, show that L is selfadjoint and A is skew-adjoint: $\langle\varphi, L\psi\rangle = \langle L\varphi, \psi\rangle$, $\langle\varphi, A\psi\rangle = -\langle A\varphi, \psi\rangle$ for any smooth, rapidly decaying functions ψ and φ. If ψ is a real function with $\|\psi(t)\| = 1$ for all t and $\tilde{\psi}(t) = \psi_t(t) + A\psi(t)$, show that ψ and $\tilde{\psi}$ are orthogonal, i.e. $\langle\tilde{\psi}, \psi\rangle = 0$. Conclude that if u satisfies the KdV equation and ψ is a bound state for L then $\psi_t + A\psi = 0$ and obtain the time dependence of the discrete part of the scattering (b_n, χ_n) data associated to the potential u. [*Hint*: use question Q5. Take the definition of b_m to be

$$\phi(x) \approx b_n e^{-\chi_n x} \quad (x \to +\infty)$$

   where $E_n = -\chi_n^2$ is the nth energy level.]

7. **2–soliton solution.** Assume that the scattering data consists of two energy levels $E_1 = -\chi_1^2, E_2 = -\chi_2^2$ where $\chi_1 > \chi_2$ and a vanishing reflection coefficient. Solve the Gelfand–Levitan–Marchenko equation to find the 2-soliton solution.

   [Follow the derivation of the 1–soliton in the Notes but try not to look at the N–soliton unless you really get stuck.]

8. **Integral equation.** Let $L\psi = k^2\psi$ where $L = -\partial_x^2 + u$. Consider ψ of the form

$$\psi(x) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz$$

   where $K(x,z), \partial_z K(x,z) \to 0$ as $z \to \infty$ for any fixed x. Use integration by parts to show

$$\psi = e^{ikx}\Big(1 + \frac{i\hat{K}}{k} - \frac{\hat{K}_z}{k^2}\Big) - \frac{1}{k^2}\int_x^\infty K_{zz}e^{ikz}dz,$$

   where $\hat{K} = K(x,x)$ and $\hat{K}_z = (\partial_z K)|_{z=x}$. Deduce that the Schrödinger equation is satisfied if

$$u(x) = -2(\hat{K}_x + \hat{K}_z), \qquad \text{and}$$

$$K_{xx} - K_{zz} - uK = 0 \quad \text{for} \quad z > x.$$

9. **Initial data for KdV solitons.** Recall from lectures that if $A = \partial_x + \chi\tanh\chi x$ and $A^\dagger = -\partial_x + \chi\tanh\chi x$, then

$$-\partial_x^2 + \chi^2 \;=\; A\,A^\dagger \quad \text{and} \quad -\partial_x^2 + \chi^2 - 2\chi^2 sech^2\chi x \;=\; A^\dagger\,A\,, \tag{2}$$

   from which we found the bound state for the potential $-2\chi^2 sech^2\chi x$ with energy $E_1 = -\chi^2$. Now by considering $B = \partial_x + 2\chi\tanh\chi x$ and $B^\dagger = -\partial_x + 2\chi\tanh\chi x$, and computing $BB^\dagger$ and $B^\dagger B$, find the bound states for the potential $-6\chi^2 sech^2\chi x$ and their energy levels.

10. **First integrals for KdV.** Consider the Riccati equation

$$\frac{dS}{dx} - 2ikS + S^2 = u.$$

for the first integrals of KdV. Assume that

$$S = \sum_{n=1}^{\infty} \frac{S_n(x)}{(2ik)^n}$$

and find the recursion relations

$$S_1(x,t) = -u(x,t), \quad S_{n+1} = \frac{dS_n}{dx} + \sum_{m=1}^{n-1} S_m S_{n-m}.$$

Solve the first few relations to show that

$$S_2 = -\frac{\partial u}{\partial x}, \quad S_3 = -\frac{\partial^2 u}{\partial x^2} + u^2, \quad S_4 = -\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial}{\partial x}u^2.$$

and find S_5. Use the KdV equation to verify directly that

$$\frac{d}{dt}\int_{\mathbb{R}} S_3 dx = 0, \qquad \frac{d}{dt}\int_{\mathbb{R}} S_5 dx = 0.$$