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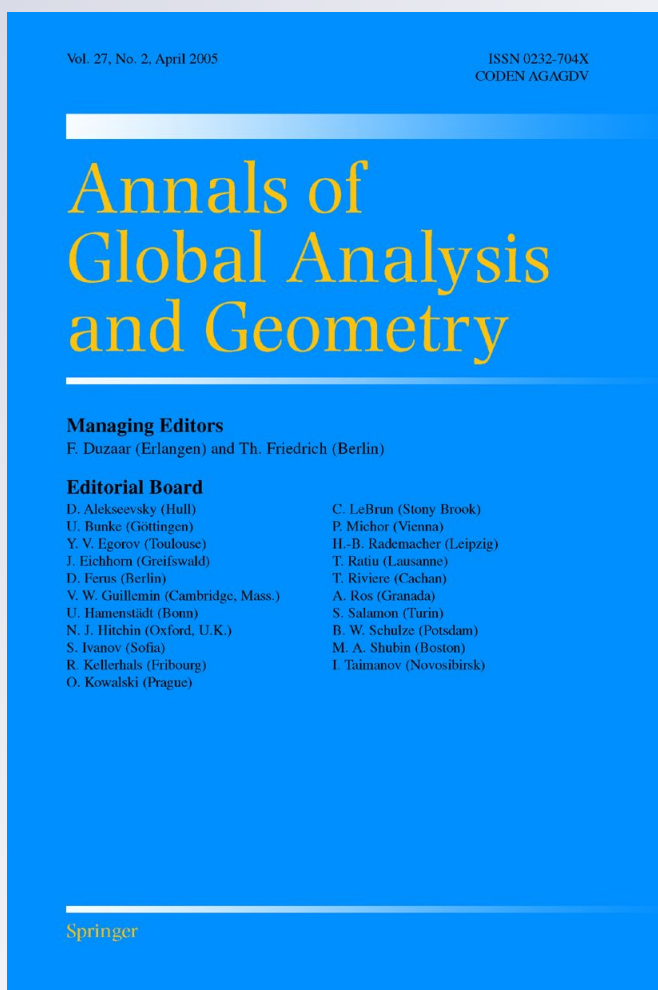
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Co-calibrated G_2 structure from cuspidal cubics

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Abstract We establish a twistor correspondence between a cuspidal cubic curve in a complex projective plane, and a co-calibrated homogeneous G_2 structure on the seven-dimensional parameter space of such cubics. Imposing the Riemannian reality conditions leads to an explicit co-calibrated G_2 structure on $SU(2, 1)/U(1)$. This is an example of an $SO(3)$ structure in seven dimensions. Cuspidal cubics and their higher degree analogues with constant projective curvature are characterised as integral curves of certain seventh order ODEs. Projective orbits of such curves are shown to be analytic continuations of Aloff–Wallach manifolds, and it is shown that only cubics lift to a complete family of contact rational curves in a projectivised cotangent bundle to a projective plane.

Keywords Co-calibrated G_2 · Twistor theory · Cubic curves

1 Introduction

Twistor theory gives rise to correspondences between global algebraic geometry of rational curves in complex two-folds or three-folds, and local differential geometry on the moduli spaces of such curves. The embedding of a rational curve L in a complex manifold \mathcal{Z} is, to the first order, described by the normal bundle $N(L) := T\mathcal{Z}/TL$. This is a holomorphic vector bundle, and thus by the Birkhoff–Grothendieck theorem, it is a direct sum of $(\dim(\mathcal{Z}) - 1)$ line bundles $\mathcal{O}(n)$ of degree n which can vary between the summands. The Kodaira deformation theorem [25] states that if $H^1(L, N(L)) = 0$, then L belongs to a locally complete family $\{L_m, m \in M\}$ where M is some complex manifold, and there exists a canonical isomorphism:

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$$T_m M \cong H^0(L_m, N(L_m)).$$

In the original Non-linear Graviton construction of Penrose [31], the twistor space \mathcal{Z} is a complex three-fold and the normal bundle is $N(L) = \mathcal{O}(1) \oplus \mathcal{O}(1)$. This gives rise to an anti-self-dual conformal structure on a four-dimensional manifold M such that the null vectors fields in M correspond to sections of $N(L)$ vanishing at one point. In the subsequent twistor constructions of Hitchin [24], the twistor space \mathcal{Z} is a two-fold, and $N(L) = \mathcal{O}(1)$ or $\mathcal{O}(2)$. In the first case M is a surface admitting a projective structure, and in the second case, M is a three-dimensional manifold with an Einstein–Weyl structure. The whole set up can be generalised to contact rational curves in complex three-folds [6]. The moduli space of such curves with normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$ admits an integrable $GL(2)$ structure [6, 17]. See [14] for other examples of twistor constructions.

The aim of this article is to use a twistor correspondence to construct seven dimensional manifolds with G_2 structure. The general theory was developed in [15], and in the present article we construct a class of explicit new examples corresponding to L being a plane cuspidal cubic:

$$y^2 - x^3 = 0 \tag{1.1}$$

in a complex two-fold $\mathcal{Z} = \mathbb{C}P^2$. In order to do this, we need to refine the twistor correspondence as outlined above, because the cuspidal cubics, although rational, are singular in the complex projective plane. This can be dealt with either by considering the contact lifts of the cuspidal cubics to $\mathbb{P}(T\mathbb{C}P^2)$, where they become smooth contact curves with normal bundle $\mathcal{O}(5) \oplus \mathcal{O}(5)$, or by working directly with singular curves. Both approaches lead to deformation theory of (1.1) as a cuspidal cubic curve (rather than as a general plane cubic). We shall find that normal vector field to a cuspidal cubic L vanishes at six general points on L away from the cusp. The parameter space of cuspidal cubics arising from this deformation theory is the seven-dimensional homogeneous space $M = PSL(3, \mathbb{C})/\mathbb{C}^*$. All cuspidal cubics are projectively equivalent and belong to the same $PSL(3, \mathbb{C})$ orbit of (1.1) in $\mathbb{C}P^2$.

To formulate our main result, recall that the $(n + 1)$ -dimensional space of holomorphic sections $H^0(\mathbb{C}P^1, \mathcal{O}(n))$ is isomorphic to the vector space $\mathcal{V}_n = \text{Sym}^n(\mathbb{C}^2)$ of homogeneous polynomials of degree n in two variables (s, t) . Any such section is of the form:

$$V(s, t) = v_0 t^n + n v_1 t^{n-1} s + \frac{1}{2} n(n-1) v_2 t^{n-2} s^2 + \dots + v_n s^n. \tag{1.2}$$

Let U, V be elements of \mathcal{V}_n . The p th transvectant of two polynomials U and V is an element of \mathcal{V}_{2n-2k} given by

$$\langle U, V \rangle_p = \frac{1}{p!} \sum_{i=0}^p (-1)^i \binom{p}{i} \frac{\partial^p U}{\partial t^{p-i} \partial s^i} \frac{\partial^p V}{\partial t^i \partial s^{p-i}}.$$

We shall first establish (Proposition 3.2) a canonical identification between vector fields on M and elements of $\text{Sym}^6(\mathbb{C}^2)$, and then use it to prove.

Theorem 1.1 *The seven-dimensional space $M = SL(3, \mathbb{C})/\mathbb{C}^*$ of plane cuspidal cubics admits a canonical complexified G_2 structure where the three form ϕ and the metric g*

$$\phi(U, V, W) = \langle \langle U, V \rangle_3, W \rangle_3, \quad g(U, U) = \langle U, U \rangle_6$$

are explicitly given by (4.19). This G_2 structure is co-calibrated, i.e.

$$d\phi = \lambda * \phi + \tau, \quad d * \phi = 0, \tag{1.3}$$

where λ is a constant, and τ is a certain three-form such that $\phi \wedge \tau = \phi * \tau = 0$.

We are ultimately interested in real G_2 , and we shall show that the structure from Theorem 1.1 admits three homogeneous real forms: two with signature $(4, 3)$, where $M = SL(3, \mathbb{R})/\mathbb{R}^*$ or $M = SU(3)/U(1)$, and one Riemannian with $M = SU(2, 1)/U(1)$.

This article is organised as follows: In the next section, we shall summarise basic facts about the nonlinear group actions on spaces of symmetric polynomials, with the particular emphasis on sextics in two variables and cubics in three variables. In Sect. 3, we shall demonstrate (Proposition 3.2) that the family M of cuspidal cubics admits a $GL(2)$ structure, so that the vectors tangent to M can be identified with elements of $\text{Sym}^6(\mathbb{C}^2)$. In Sect. 4, we shall prove Theorem 1.1 and construct the conformal structure and the associated three-form directly from the $GL(2)$ structure on M . Restricting to a real Riemannian slice will reveal a co-calibrated G_2 structure on $SU(2, 1)/U(1)$. In Sect. 5, we shall discuss the differential equations approach, where M arises as the solution space of a seventh order ODE. To find this ODE, take seven derivatives of the general form of the cuspidal cubic and express the seven parameters in terms of $y(x)$ and its first six derivatives, which leaves one condition in the form of the ODE (Proposition 5.1). From the point of view of projective geometry of curves, the cuspidal cubics belong to the class of algebraic curves with constant projective curvature [36]. We shall find (Proposition 5.2) all curves in $\mathbb{C}\mathbb{P}^2$ which give rise to seven-dimensional projective orbits. These curves have constant projective curvature and are related to Aloff–Wallach seven-manifolds. In Sect. 6 we shall introduce the generalised Wilczynski invariants, and show (Theorem 6.4) that only cubics give rise to a complete analytic family on a contact complex three-fold $\mathbb{P}(T\mathbb{C}\mathbb{P}^2)$.

The connection between the algebraic geometry of cuspidal cubics, the differential geometry of their parameter space, and the seventh order differential equations forms a part of more general theory [15]. Any seven-dimensional family of rational curves which lifts to a complete family of non-singular contact curves in contact complex three-folds gives rise to a G_2 structure. This structure in general has torsion which can be expressed in terms of contact invariants of the associated seventh order ODE characterising the curves.

2 Group actions on symmetric polynomials

Most calculations in the article rely on an explicit description of the actions of the projective linear group on spaces of homogeneous polynomials, and in this Section we shall summarise the basic facts and notation here. Let $E = \mathbb{C}^m$ be an m -dimensional complex vector space, and let $\text{Sym}^k(E)$ be the vector space of complex homogeneous polynomials of degree k in m variables. The two cases of interest will be the space of binary sextic where $(k = 6, m = 2)$, and the space of ternary cubics where $(k = 3, m = 3)$.

The general element of $\text{Sym}^k(E)$ is of the form:

$$P(Z) = \sum_{\alpha_i=1}^m P_{\alpha_1\alpha_2\dots\alpha_k} Z^{\alpha_1} Z^{\alpha_2} \dots Z^{\alpha_k},$$

where the symmetric tensor $P_{\alpha_1\alpha_2\dots\alpha_k}$ consists of the coefficients of the polynomial, and $Z^\alpha = [Z^1, Z^2, \dots, Z^k]$ are homogeneous coordinates on $\mathbb{P}(E)$. The linear action of $GL(E)$ on E is given by ordinary matrix multiplication $Z \rightarrow \hat{Z}$, where $Z^\alpha = N^\alpha_\beta \hat{Z}^\beta$ for $N \in GL(E)$. This induces a nonlinear action of $GL(E)$ on $\text{Sym}^k(E)$ given by $P(Z) = \hat{P}(\hat{Z})$, so that

$$\hat{P}_{\alpha_1\alpha_2\dots\alpha_k} = \sum_{\beta_i=1}^m N^{\beta_1}_{\alpha_1} N^{\beta_2}_{\alpha_2} \dots N^{\beta_k}_{\alpha_k} P_{\beta_1\beta_2\dots\beta_k}. \tag{2.4}$$

This action preserves the homogeneity of the polynomials, and so it induces a nonlinear projective group action of $PGL(E)$ on $\mathbb{P}(E)$. Before going any further we shall restrict ourselves to the following two cases of interest:

2.1 Binary sextics and classical invariants

Let $E = \mathbb{C}^2$ and let $\mathcal{V}_6 = \text{Sym}^6(\mathbb{C}^2)$ be the seven-dimensional space of binary sextics of the form (1.2) with $n = 6$. Let

$$V(s, t) = v_0t^6 + 6v_1t^5s + 15v_2t^4s^2 + 20v_3t^3s^3 + 15v_4t^2s^4 + 6v_5ts^5 + v_6s^6 \in \mathcal{V}_6.$$

Definition 2.1 An invariant of a binary sextic under the $GL(2, \mathbb{C})$ action (2.4) is a function $I = I(v_0, \dots, v_6)$ such that

$$I(\hat{v}_0, \dots, \hat{v}_6) = (\alpha\delta - \beta\gamma)^w I(v_0, \dots, v_6), \quad \text{where } N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

The number w is called the weight of the invariant.

One of the classical results of the invariant theory is that all invariants arise from the transvectants (see e.g. [19,29]). There are five invariants for binary sextics of degrees 2, 4, 6, 10 and 15, respectively, connected by a syzygy of degree 30. The one we will be concerned with is the quadratic invariant:

$$I_2(V) := \langle V, V \rangle_6 = v_0v_6 - 6v_1v_5 + 15v_2v_4 - 10(v_3)^2. \tag{2.5}$$

Invariants of several binary forms arise in an analogous way, and we shall make use of an invariant of three binary sextics, which should be thought of as a scalar part in the Clebsch–Gordan decomposition of $\mathcal{V}_6 \otimes \mathcal{V}_6 \otimes \mathcal{V}_6$. This is given by

$$I_3(U, V, W) := \langle \langle U, V \rangle_3, W \rangle_6. \tag{2.6}$$

The invariant I_2 defines a symmetric quadratic form. The invariant I_3 is anti-symmetric in any pair of vectors.

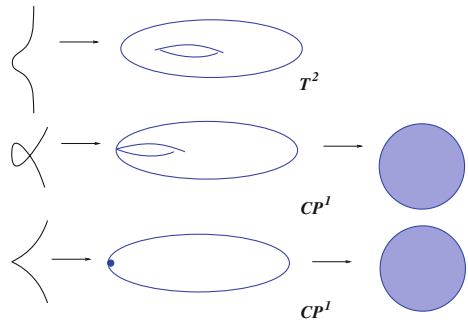
2.2 Ternary cubics and their orbits

Let $E = \mathbb{C}^3$. We shall consider the space of irreducible ternary cubics which give rise to plane cubic curves in $\mathbb{C}\mathbb{P}^2$ of the form:

$$\sum_{\alpha,\beta,\gamma=1}^3 P_{\alpha\beta\gamma} Z^\alpha Z^\beta Z^\gamma = 0.$$

There are ten coefficients $P_{\alpha\beta\gamma}$, but the overall scale is unimportant, and so the space of such cubics is $\mathbb{C}\mathbb{P}^9$. The group action (2.4) preserves the homogeneity, and so it descends to the projective action of $PGL(3, \mathbb{C})$ on $\mathbb{C}\mathbb{P}^9$. There are three types of orbits (see e.g. [23]) which we shall present in inhomogeneous coordinates $x = Z^1/Z^3, y = Z^2/Z^3$ (Fig. 1).

Fig. 1 Three types of orbits of irreducible cubics



- (1) Smooth cubic $y^2 = x(x - 1)(x - c)$.
- (2) Nodal cubic $y^2 = x^3 - x^2$.
- (3) Cuspidal cubic $y^2 = x^3$.

The first orbit corresponds to a smooth cubic which is a curve of genus one. Two smooth cubics are projectively equivalent if their j -invariants $j = (c^2 - c + 1)^3 / (c^2(c - 1)^2)$ coincide. The remaining two orbits correspond to singular rational cubics. There is the eight-dimensional orbit of the nodal cubic $(Z^2)^2 Z^3 - (Z^1)^3 + (Z^1)^2 Z^3 = 0$ which contains all nodal cubics, and finally there is the seven-dimensional orbit M of the cuspidal cubic $Z^3(Z^2)^2 - (Z^1)^3 = 0$. The one-dimensional stabiliser of this cubic is given by the projective transformations with $N = \text{diag}(a, a^4, a^{-5})$, where $a \in \mathbb{C}^*$. Thus the space of cuspidal cubics is a homogeneous manifold $M = SL(3, \mathbb{C}) / \mathbb{C}^*$.

3 Cuspidal cubics, their moduli, and the $GL(2)$ structure

In this section, we shall use the twistor correspondence to construct a $GL(2)$ structure on the space of cuspidal cubics.

3.1 $GL(2)$ structures

Definition 3.1 The $GL(2)$ structure on an $(n + 1)$ -dimensional manifold M is an isomorphism:

$$TM \cong \text{Sym}^n(\mathbb{S}), \tag{3.7}$$

where \mathbb{S} is a rank two symplectic vector bundle over M .

The $GL(2)$ structures were originally called the paraconformal structures in [17]. If M is a complex manifold, we talk about $GL(2, \mathbb{C})$ structures and \mathbb{S} is a complex vector bundle. The tangent vector fields to M are identified by (3.7) with homogeneous polynomials of degree n in two variables. There is also a unique, up to scale, symplectic structure on the fibres \mathbb{C}^2 of \mathbb{S} . The group action (2.4) with $m = 2$ and $k = n$ gives rise to an irreducible $(n + 1)$ representation of $GL(2, \mathbb{C})$, and thus to the embedding of $GL(2, \mathbb{C})$ inside $GL(n + 1, \mathbb{C})$. The image of $SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$ is contained in $Sp(n + 1, \mathbb{C})$ if n is odd, or in $SO(n + 1, \mathbb{C})$ if n is even. We shall consider the case of even n , where the representation space \mathcal{V}_n is odd-dimensional. Then we have two-real sections of the $GL(2, \mathbb{C})$ structures on real $(n + 1)$ -dimensional manifold M which correspond to two real forms of $GL(2, \mathbb{C})$.

- (1) The $GL(2, \mathbb{R})$ structures give an identification of tangent vectors with real homogeneous polynomials of degree n . The fibers of \mathbb{S} are real vector spaces \mathbb{R}^2 .
- (2) The $SO(3, \mathbb{R}) \times \mathbb{R}^*$ structures (or locally equivalent $U(2)$ structures) identify the tangent vectors to M with harmonic homogeneous polynomials in three variables. This has its roots in the isomorphism $\mathbb{C}^3 = \text{Sym}^2(\mathbb{C}^2)$ between complex vectors in \mathbb{C}^3 and symmetric 2 by 2 matrices with complex coefficients. The $(n + 1)$ -dimensional space $\text{Sym}^n(\mathbb{C}^2)$ is then identified with a subspace of $\text{Sym}^{n/2}(\mathbb{C}^3)$ which consist of harmonic ternary forms, i.e. those forms $\sum_{\alpha, \dots, \gamma=1}^{n/2} P_{\alpha\beta\dots\gamma} Z^\alpha Z^\beta \dots Z^\gamma$ which satisfy $\sum_{\alpha, \beta=1}^{n/2} \delta^{\alpha\beta} P_{\alpha\beta\dots\gamma} = 0$.

In practice, the isomorphism (3.7) is specified by a homogeneous polynomial S of degree n with values in $\Lambda^1(M)$. Given $S \in \Lambda^1(M) \otimes \mathcal{V}_n$, the homogeneous polynomial corresponding under (3.7) to a vector field $V \in TM$ is the contraction $V \lrcorner S$.

3.2 Cuspidal cubics and their deformations

A cuspidal cubic $L \subset \mathbb{C}\mathbb{P}^2$ is a singular rational curve with self-intersection number 9, as two general cuspidal cubics intersect in exactly nine points (albeit not in the general positions). The arithmetic genus is constant in algebraic families, and so this is the same as the genus of a smooth cubic curve. The arithmetic genus of a curve with a cusp is 1 plus the genus of a resolution of singularities.) The Riemann-Roch theorem for singular curves yields

$$h^0(L, N(L)) - h^1(L, N(L)) = \text{deg}(N(L)) - g(L) + 1 = 9 - 1 + 1 = 9.$$

In the case of cuspidal cubics, $h^1(L, N(L)) = 0$ and so $h^0(L, N(L)) = 9$. Here $h^0(L, N(L))$ is equal to the dimension of the space of all deformations of L as a curve in $\mathbb{C}\mathbb{P}^2$. Indeed, the space of all cubic curves in $\mathbb{C}\mathbb{P}^2$ is isomorphic to $\mathbb{C}\mathbb{P}^9$. Thus, $H^0(L, N(L))$ describes all deformations of L as a curve in $\mathbb{C}\mathbb{P}^2$, not just those as a cuspidal curve.

We want to consider the deformations of $y^2 - x^3 = 0$ as a *rational cuspidal curve*, and not allow the perturbations of cuspidal cubics to smooth curves. Thus we shall compute the Zariski tangent space to M at a point in M . This will make use of the rational parametrisation of L and lead to the first-order deformations.¹

Let $L_m \in \mathcal{Z} = \mathbb{C}\mathbb{P}^2$ be a cuspidal cubic corresponding to $m \in M$. Consider a neighbouring curve in \mathcal{Z} corresponding to a point $m + \delta m$ in M . In the proof of Proposition 3.2 where we shall show that two nearby cuspidal cubics L_m and $L_{m+\delta m}$ intersect at six points away from the triple intersection at the cusp. This will follow from the form of the normal vector field to L (Fig. 2).

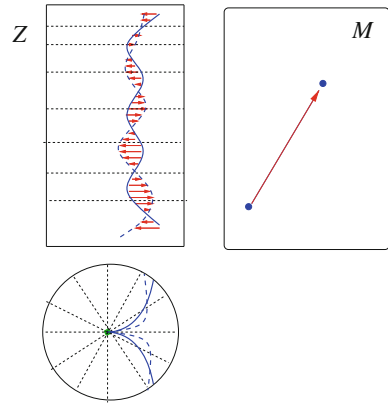
Thus, a vector in M connecting two neighbouring points corresponds to a six-order homogeneous polynomial (defined by its six roots) in two variables which gives rise to an isomorphism:

$$T_m M \cong \text{Sym}^6(\mathbb{C}^2). \tag{3.8}$$

To make contact with the notation in formula (2.4) we let $\hat{Z}^\alpha = [\hat{Z}^1, \hat{Z}^2, \hat{Z}^3]$ be homogeneous coordinates on $\mathbb{C}\mathbb{P}^2$.

¹ An alternative construction based on contact resolution of the cusp will be presented in Theorem 6.4.

Fig. 2 Normal vector field to $L_m \subset \mathcal{Z}$ and tangent vector at $m \in M$.



Proposition 3.2 *The seven-dimensional space of cuspidal cubics $M = SL(3, \mathbb{C})/\mathbb{C}^*$ admits a $GL(2)$ structure (3.8). If*

$$\sum_{\alpha, \beta, \dots, \phi=1}^3 P_{\alpha\beta\gamma} N^\alpha_\delta N^\beta_\epsilon N^\gamma_\phi \hat{Z}^\delta \hat{Z}^\epsilon \hat{Z}^\phi = 0, \tag{3.9}$$

where $P_{111} = -1, P_{223} = P_{232} = P_{322} = 1/3$ (and other components of P vanish) is the general element of M then the $GL(2)$ structure is given by $S \in \Lambda^1(M) \otimes \text{Sym}^6(\mathbb{C}^2)$

$$S(s, t) = 2\sigma^2_3 s^6 - 3\sigma^1_3 t s^5 + 2\sigma^2_1 t^2 s^4 + (\sigma^3_3 + 2\sigma^2_2 - 3\sigma^1_1) t^3 s^3 - 3\sigma^1_2 t^4 s^2 + \sigma^3_1 t^5 s + \sigma^3_2 t^6, \tag{3.10}$$

where $\sigma = N^{-1}dN$ is the Maurer–Cartan one-form on $SL(3, \mathbb{C})$ with values in the Lie algebra of traceless 3×3 matrices.

Proof Consider a rational parametrisation of (1.1)

$$x = t^2, \quad y = t^3,$$

or, in homogeneous coordinates $Z^\alpha = T^\alpha(t) = [t^2, t^3, 1]$. To establish the isomorphism (3.8) we shall construct a binary sextic in (s, t) with values in T^*M . The $PSL(3)$ orbit of (1.1) is seven-dimensional, and is parametrised by the components of the matrix $N \in SL(3, \mathbb{C})$. We identify two such matrices if they differ by a multiplication by the stabiliser $\text{diag}(a, a^4, a^{-5})$.

Formula (2.4) with $d = 3$ thus implies that the homogeneous form of the general cuspidal cubics is (3.9), as this is the general orbit of $y^2 - x^3 = 0$. To construct the normal vector field to this family, differentiate (3.9) with respect to the moduli parameters N^α_β , and substitute the rational parametrisation:

$$\hat{Z}^\alpha(t) = \sum_{\beta=1}^3 (N^{-1})^\alpha_\beta T^\beta(t), \quad \text{where } T^\alpha = (t^2, t^3, 1).$$

In general, if the family of rational curves $f(x, y; m) = 0$ parametrised by $m \in M$ admits a rational parametrisation $x = x(t, m), y = y(t, m)$, then the polynomial is given by [15]

$$\sum_{k=1}^{\dim M} \frac{\partial f}{\partial m^k} \Big|_{\{x=x(t,m), y=y(t,m)\}} dm^k.$$

In our case this gives a polynomial of degree 9 in t

$$\sum_{\alpha, \beta, \gamma, \delta=1}^3 P_{\alpha\beta\gamma} \sigma^\gamma_\delta T^\alpha T^\beta T^\delta, \tag{3.11}$$

where $\sigma = N^{-1}dN$ is the Maurer–Cartan one-form on $SL(3, \mathbb{C})$ with values in the Lie algebra of traceless 3×3 matrices. Pulling out the overall scalar factor of t^3 and introducing the homogeneous coordinates (s, t) in place of an affine coordinate t yields the T^*M -valued sextic polynomial (4.13) given by (3.10). Its roots depend on coordinates on M . \square

In particular, the proof above shows that the normal vector field to a cuspidal cubics vanishes to the third order at the cusp $t = 0$, and at six smooth points on the cubics in general positions.

4 Construction of the G_2 structure

In this section we shall present the proof of Theorem 1.1 and show that the space of cuspidal cubics M is equipped with a conformal G_2 structure (a good reference to G_2 structures is [33]). In our discussion of $GL(2, \mathbb{C})$ structures, we have noted the existence of the embedding of $SL(2, \mathbb{C})$ in $SO(7, \mathbb{C})$. The representation theoretic argument of [35], or the explicit construction in [15] shows that this leads to an intermediate embedding $SL(2, \mathbb{C}) \subset G_2^{\mathbb{C}} \subset SO(7, \mathbb{C})$ which we shall now explore.²

Let $V, U, W \in TM$. The $GL(2)$ structure allows the identification of vector fields with binary sextics, and therefore the invariants (2.5) and (2.6) give rise to a non-degenerate symmetric quadratic form and a skew-symmetric three-form on M given by

$$g(U, V) = \langle U \lrcorner S, V \lrcorner S \rangle_6, \quad \phi(U, V, W) = \langle \langle U \lrcorner S, V \lrcorner S \rangle_3, W \lrcorner S \rangle_3. \tag{4.12}$$

For a general $GL(2)$ structure the isomorphism (3.8) is specified by a T^*M valued sextic polynomial

$$S(s, t) = a^0 t^6 + 6a^1 t^5 s + 15a^2 t^4 s^2 + 20a^3 t^3 s^3 + 15a^4 t^2 s^4 + 6a^5 t s^5 + a^6 s^6, \tag{4.13}$$

for linearly independent one-forms a^0, \dots, a^6 on M . Given $S(s, t)$, a sextic polynomial corresponding to a vector $V \in TM$ is given by the contraction $V \lrcorner S(s, t)$. The transvectant formulae for the invariants (2.5) and (2.6), and formula (4.12) imply that the quadratic form g and the three-form ϕ are given by

$$g = a^0 \odot a^6 - 6a^1 \odot a^5 + 15a^2 \odot a^4 - 10(a^3)^2, \tag{4.14}$$

and

$$\phi = \sqrt{\frac{5}{2}} \left(3(a^1 \wedge a^2 \wedge a^6 + a^0 \wedge a^4 \wedge a^5) + a^3 \wedge (a^0 \wedge a^6 + 6a^1 \wedge a^5 - 15a^2 \wedge a^4) \right). \tag{4.15}$$

² Dynkin has shown (see e.g. [35]) that for general n , there is no proper Lie subgroup G of $Sp(n+1, \mathbb{C})$ or $SO(n+1, \mathbb{C})$ such that $SL(2, \mathbb{C}) \subset G$. The exception is $n = 6$, where $G = G_2^{\mathbb{C}}$. Thus, in all dimensions apart from seven, the $GL(2)$ structure does not induce any additional G structure on M apart from a conformal structure if n is even, or a symplectic structure, if n is odd.

The quadratic invariant g induces a conformal structure on M and ϕ (the overall multiple $\sqrt{5/2}$ has been chosen for later convenience) endows M with a three-form compatible with g in a sense that

$$(V \lrcorner \phi) \wedge (V \lrcorner \phi) \wedge \phi = 0 \quad \text{iff} \quad g(V, V) = 0.$$

The invariants (2.5) and (2.6) have weights six and nine, respectively, and thus g gives rise to a conformal structure, but not a metric. Changing a metric in the conformal structure has to be complemented by changing the three-form according to

$$g \longrightarrow \Omega^6 g, \quad \phi \longrightarrow \Omega^9 \phi,$$

where Ω is a non-vanishing function on M . Thus, the structure group of TM reduces to the complexification of conformal G_2 [15]. We shall now find this structure explicitly, and demonstrate that it is co-calibrated.

Proof of Theorem 1.1 Using the form of the $GL(2)$ structure given by (3.10) together with formulae (4.14) and (4.15) gives rise to a conformal structure represented by the metric

$$g = 2\sigma^3_2 \odot \sigma^2_3 + \frac{1}{2}\sigma^3_1 \odot \sigma^1_3 - \frac{2}{5}\sigma^1_2 \odot \sigma^2_1 - \frac{1}{40}(4\sigma^1_1 - \sigma^2_2)^2. \quad (4.16)$$

This holomorphic conformal structure on $SL(3, \mathbb{C})/\mathbb{C}^*$ admits three real forms, all leading to co-calibrated G_2 structures.

- If the components of σ are all real, then g is a metric of signature (3, 4) on the non-compact manifold $SL(3, \mathbb{R})/\mathbb{R}^*$.
- If σ is anti-hermitian then g is a metric of signature (4, 3) on a compact seven-manifold $M = SU(3)/U(1)$, where $U(1)$ is the group of diagonal matrices:

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{4i\theta} & 0 \\ 0 & 0 & e^{-5i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R}. \quad (4.17)$$

Thus, M is the Aloff–Wallach space $N(1, 4)$. The homogeneous co-calibrated G_2 structures of Riemannian signature on some Aloff–Wallach spaces have appeared before in [1, 4, 7, 10, 32].

- If the diagonal components of σ are imaginary and the reality conditions

$$\sigma^2_1 = -\overline{\sigma^1_2}, \quad \sigma^3_1 = \overline{\sigma^1_3}, \quad \sigma^3_2 = \overline{\sigma^2_3} \quad (4.18)$$

hold, then the metric (4.16) has Riemannian signature. The relations (4.18) imply that σ takes values in $\mathfrak{su}(2, 1)$. Thus we obtain a homogeneous Riemannian conformal class on the non-compact seven-manifold $M = SU(2, 1)/U(1)$, where $U(1)$ is given by (4.17).

We shall now work out the details of the G_2 structure (4.16) associated with the Riemannian reality conditions (4.18), and show that it is co-calibrated with the conformal factor equal to 1. Let

$$e_8 = \begin{pmatrix} i & 0 & 0 \\ 0 & 4i & 0 \\ 0 & 0 & -5i \end{pmatrix}$$

span the one dimensional Lie algebra of the $U(1)$ stabiliser (4.17). We choose the following basis for the invariant complement of e_8 (the various square roots multiples are chosen such that the dual one-forms in the resulting metric have length one)

$$\begin{aligned}
 e_3 &= \frac{\sqrt{10}}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_7 = \frac{\sqrt{10}}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \sqrt{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
 e_6 &= \sqrt{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \\
 e_4 &= \frac{\sqrt{10}}{7} \begin{pmatrix} -3i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & i \end{pmatrix}.
 \end{aligned}$$

The Maurer–Cartan one-form on $SU(2, 1)$ is

$$\sigma = N^{-1}dN = \begin{pmatrix} \sigma^1_1 & \sigma^1_2 & \sigma^1_3 \\ \sigma^2_1 & \sigma^2_2 & \sigma^2_3 \\ \sigma^3_1 & \sigma^3_2 & -\sigma^1_1 - \sigma^2_2 \end{pmatrix} = \sum_{k=1}^8 e_k \otimes \theta^k,$$

where θ^k are the left-invariant one-forms on the group. Thus,

$$\begin{aligned}
 \sigma^1_2 &= \frac{\sqrt{10}}{2}(-\theta^3 + i\theta^7), \quad \sigma^1_3 = \sqrt{2}(\theta^2 - i\theta^6), \quad \sigma^2_3 = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^5), \\
 \sigma^2_2 - 4\sigma^1_1 &= 2i\sqrt{10}\theta^7
 \end{aligned}$$

together with the reality conditions (4.18). The metric (4.16) and the G_2 three-form (4.15) become

$$\begin{aligned}
 g &= (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + (\theta^4)^2 + (\theta^5)^2 + (\theta^6)^2 + (\theta^7)^2, \\
 \phi &= \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356},
 \end{aligned} \tag{4.19}$$

where $\theta^{jkl} = \theta^j \wedge \theta^k \wedge \theta^l$. The relations

$$d\sigma + \sigma \wedge \sigma = 0, \quad [e_j, e_k] = \sum_{l=1}^3 c_{jkl} e_l$$

give

$$\begin{aligned}
 d\theta^1 &= \sqrt{10}\theta^2 \wedge \theta^3 + \frac{\sqrt{10}}{7}\theta^4 \wedge \theta^5 - 9\theta^5 \wedge \theta^8 + \sqrt{10}\theta^6 \wedge \theta^7, \\
 d\theta^2 &= -\frac{\sqrt{10}}{4}\theta^1 \wedge \theta^3 + \frac{4\sqrt{10}}{7}\theta^4 \wedge \theta^6 - \frac{\sqrt{10}}{4}\theta^5 \wedge \theta^7 + 6\theta^6 \wedge \theta^8, \\
 d\theta^3 &= -\frac{\sqrt{10}}{5}\theta^1 \wedge \theta^2 + \frac{5\sqrt{10}}{7}\theta^4 \wedge \theta^7 + \frac{\sqrt{10}}{5}\theta^5 \wedge \theta^6 - 3\theta^7 \wedge \theta^8, \\
 d\theta^4 &= \frac{\sqrt{10}}{20}\theta^1 \wedge \theta^5 + \frac{4\sqrt{10}}{5}\theta^2 \wedge \theta^6 - \frac{5\sqrt{10}}{4}\theta^3 \wedge \theta^7, \\
 d\theta^5 &= \frac{\sqrt{10}}{7}\theta^1 \wedge \theta^4 + 9\theta^1 \wedge \theta^8 + \sqrt{10}\theta^2 \wedge \theta^7 + \sqrt{10}\theta^3 \wedge \theta^6, \\
 d\theta^6 &= -\frac{\sqrt{10}}{4}\theta^1 \wedge \theta^7 + \frac{4\sqrt{10}}{7}\theta^2 \wedge \theta^4 - 6\theta^2 \wedge \theta^8 - \frac{\sqrt{10}}{4}\theta^3 \wedge \theta^5, \\
 d\theta^7 &= -\frac{\sqrt{10}}{5}\theta^1 \wedge \theta^6 + \frac{\sqrt{10}}{5}\theta^2 \wedge \theta^5 + \frac{5\sqrt{10}}{7}\theta^3 \wedge \theta^4 + 3\theta^3 \wedge \theta^8,
 \end{aligned}$$

$$d\theta^8 = \frac{3}{14}\theta^1 \wedge \theta^5 - \frac{4}{7}\theta^2 \wedge \theta^6 - \frac{5}{14}\theta^3 \wedge \theta^7$$

and finally (1.3). Thus, the G_2 structure is co-calibrated. □

The real form $SU(2, 1)/U(1)$ which we have explored in the proof corresponds to the second ‘real form’ of the $GL(2, \mathbb{C})$ structure (see the discussion of the real forms in Sect. 3). Thus it is an example of the $SO(3)$ structure in seven dimensions [2].

A similar construction applied to nodal cubics would instead lead to a symplectic structure on the eight-dimensional $PSL(3)$ orbit. To make contact with conformal geometry one needs to blow up a point in $\mathbb{C}P^2$ to get a seven-dimensional family M . This will admit a non-homogeneous G_2 structure.

5 ODEs for cuspidal curves

All rational curves with seven-dimensional orbits are projectively equivalent to the cuspidal curves

$$y^p - x^q = 0 \tag{5.20}$$

with (p, q) integers. This fact follows from a more general result established in [13]. In this section, we shall present a direct proof based on the projective curvature. We shall also characterise the family of cuspidal cubics and their higher degree generalisations (5.20) as integral curves of the seventh order ODEs.

In Wilczyński’s approach to projective differential geometry [36] each curve $\mathbb{C} \rightarrow \mathbb{C}P^2$ (or $\mathbb{R} \rightarrow \mathbb{R}P^2$) corresponds to a unique third order homogeneous linear ODE

$$\frac{d^3Y}{dx^3} + 3p_1(x)\frac{d^2Y}{dx^2} + 3p_2(x)\frac{dY}{dx} + p_3(x)Y = 0 \tag{5.21}$$

such that given a curve $x \rightarrow [y_1(x), y_2(x), y_3(x)]$, the functions $\mathbf{y} = [y_1(x), y_2(x), y_3(x)]$ span the solution space of (5.21). To find this ODE, substitute each $y_i(x)$ into (5.21) and solve the resulting system of linear algebraic equations for each of the smooth functions p_i s. Linear transformations of the basis \mathbf{y} correspond to projective transformations of the curve. These transformations do not change the ODE (5.21). The combinations of the coefficients which only depend on the ratios of the solutions, i.e. are unchanged by transformations $\mathbf{y} \rightarrow \gamma(x)\mathbf{y}$ are the semi-invariants

$$P_2 = p_2 - (p_1)^2 - (p_1)_x, \quad P_3 = p_3 - 3p_1p_2 + 2(p_1)^3 - (p_1)_{xx}, \tag{5.22}$$

where the subscripts stand for partial derivatives. The lowest-order relative projective invariant is given by

$$\Theta_3(x) = P_3 - \frac{3}{2}(P_2)_x.$$

The cubic differential $\Theta_3(x)dx^3$ is invariant under an overall scaling of homogeneous coordinates and reparametrisation of the curve,

$$(x, \mathbf{y}) \longrightarrow (\xi(x), \gamma(x)\mathbf{y}).$$

Invariants of the ODE (5.21) under this class of transformations are also projective invariants of the curve. Using these transformations, one can set two out of the three functions p_i to zero.

- The Laguerre–Forsyth canonical form is achieved by setting $p_1 = p_2 = 0$ in which case $\Theta_3 = p_3$. Thus, if $\Theta_3 = 0$, the solution space is $\mathbf{y} = [1, x, x^2]$ and the curve is a conic.
- Consider a curve $y = y(x)$, so that $\mathbf{y} = [1, x, y(x)]$. This gives $p_2 = p_3 = 0$ and $p_1 = -y_{xxx}/(3y_{xx})$. Then,

$$\Theta_3 = \frac{9(y^{(2)})^2 y^{(5)} - 45y^{(2)}y^{(3)}y^{(4)} + 40(y^{(3)})^3}{(y^{(2)})^3}. \tag{5.23}$$

This, as we have just shown, vanishes for conics which gives a characterisation of the five-dimensional space of plane conics by a fifth order ODE originally due to Halphen [22].

We will say that $\Theta_r(x)$ is a relative invariant of weight r if $\Theta_r(x)dx^r$ is an invariant. Wilczynski shows that given Θ_r , the quantity

$$\Theta_{2r+2} = 2r\Theta_r(\Theta_r)_{xx} - (2r + 1)((\Theta_r)_x)^2 - 3r^2 P_2(\Theta_r)^2 \tag{5.24}$$

is a relative invariant of weight $(2r + 2)$. Thus, Θ_3 gives rise to Θ_8 , and we can define the projective curvature to be an absolute invariant

$$\kappa = \frac{(\Theta_8)^3}{(\Theta_3)^8}. \tag{5.25}$$

This is the lowest-order absolute projective invariant.³ If a parametrisation with $p_2 = p_3 = 0$ is chosen, then the expression for κ depends on y and its first seven derivatives.

Proposition 5.1 (Wilczynski [36], Sylvester [34]) *The cuspidal cubics have constant projective curvature, and are characterised by the seventh order ODE*

$$\kappa(y, y', \dots, y^{(7)}) = \frac{3^9 7^3}{2^4 5^2}. \tag{5.26}$$

Proof This can be seen directly parametrising the cuspidal cubic by $[1, x, x^{3/2}]$ so that Eq. 5.21 becomes

$$\frac{d^3 Y}{dx^3} + \frac{1}{2x} \frac{d^2 Y}{dx^2} = 0,$$

and κ can be found directly by substituting $p_1 = 1/(6x)$ in the formulae above. □

It is easy to find all other curves with constant projective curvature. A curve $[1, x, x^\gamma]$, where $\gamma \neq 0, 1$ is characterised by

$$\kappa = 3^9 \frac{(1 + \gamma^2 - \gamma)^3}{(\gamma - 2)^2 (2\gamma - 1)^2 (\gamma + 1)^2}.$$

The case

$$\kappa = 3^9/2^2$$

which corresponds to $\gamma = 0$ or $\gamma = 1$ has to be considered separately, as for these two values the solutions to Wilczyński's ODE (5.21) are not independent. We verify that this special case corresponds to a curve $[1, x, \ln(x)]$. Therefore, we have established.

³ The fact that there are no invariants of order lower than seven can also be seen by direct counting. The prolongations of the eight generators of $PSL(3)$ from the (x, y) plane to the sixth jet J^6 are vector fields which are independent almost everywhere, and thus span TJ^6 at almost every point. Therefore, the only functions of $(x, y, y', \dots, y^{(6)})$ constant along the flows generated by the lifts are constant identically.

Proposition 5.2 *All curves of constant projective curvature are projectively equivalent to*

$$y = x^\gamma, \quad \gamma \neq 0, 1, -1, 2, 1/2 \quad \text{or} \quad y = \ln x.$$

This is in agreement with [13], where the same class of curves arose as the most general homogeneous curves in $\mathbb{C}P^2$. All algebraic curves in this class are projectively equivalent to the rational cuspidal curves (5.20). The stabiliser of (5.20) is the one-dimensional group of matrices

$$\begin{pmatrix} a^{q-2p} & 0 & 0 \\ 0 & a^{p-2q} & 0 \\ 0 & 0 & a^{p+q} \end{pmatrix}, \quad a \in \mathbb{C}^*.$$

There are three real forms of the space of orbits, as before. One of these is the Aloff-Wallach space $N(k, l) = SU(3)/U(1)$, where the $U(1)$ subgroup consists of matrices of the form [3]

$$\begin{pmatrix} e^{il\theta} & 0 & 0 \\ 0 & e^{ik\theta} & 0 \\ 0 & 0 & e^{-i(k+l)\theta} \end{pmatrix}, \quad \theta \in \mathbb{R}$$

and k, l are integers such that

$$p = \frac{2l + k}{3}, \quad q = -\frac{l + 2k}{3}.$$

The space $N(1, 1)$ is special from this perspective. It corresponds to curves of the form $xy = 1$. These curves form a five-dimensional orbit $SL(3)/SL(2)$ which is the space of all conic sections.

6 Generalised Wilczynski invariants for the constant projective curvature

6.1 Classical Wilczynski invariants

Equation 5.25 with constant κ is a seventh order ODE, solutions of which are curves with constant projective curvature κ . Therefore, for any given κ we have a seven-dimensional space of such curves and each of the curves has a one-dimensional stabiliser. All these solution spaces can be identified with homogeneous spaces $PSL(3, \mathbb{R})/H$, where H is one of the one-dimensional subgroups in $PSL(3, \mathbb{R})$ described above.

In [15], it has been demonstrated that a solution space to a seventh order ODE admits a G_2 -structure if all its *generalised Wilczynski invariants* vanish identically. To introduce these invariants, recall that given an arbitrary linear differential equation

$$Y^{(n)} + \binom{n}{1} p_1(x) Y^{(n-1)} + \binom{n}{2} p_2(x) Y^{(n-2)} + \dots + p_n(x) Y(x) = 0, \quad (6.27)$$

with real smooth or complex holomorphic coefficients $p_i(x), i = 1, \dots, n$ the classical Wilczynski invariants are constructed as follows: First, we can always bring this equation to the so-called semi-canonical form:

$$Y^{(n)} + \binom{n}{2} P_2(x) Y^{(n-2)} + \binom{n}{3} P_3(x) Y^{(n-3)} + \dots + P_n(x) Y(x) = 0. \quad (6.28)$$

This is achieved by

$$Y \mapsto \lambda Y, \quad \text{where } \lambda = \exp\left(-\int p_1(x) dx\right). \quad (6.29)$$

It is easy to check that the new coefficients $P_i(x)$, $i = 2, \dots, n$ are polynomial expressions in terms of $p_i(x)$, $i = 1, \dots, n$ and their derivatives. They also do not depend on the integration constant in (6.29). For example, P_2 and P_3 are given by (5.22). Next, the semi-canonical form (6.28) can be brought to the Laguerre–Forsyth canonical form:

$$Y^{(n)} + \binom{n}{3}q_3(x)Y^{(n-3)} + \dots + \binom{n}{n-1}q_{n-1}(x)Y'(x) + q_n(x)Y(x) = 0, \quad (6.30)$$

by means of the following change of variables:

$$(x, Y) \mapsto (\xi(x), (\xi')^{(n-1)/2}Y), \quad (6.31)$$

where ξ satisfies the third-order ODE reduced to the Riccati equation:

$$\eta' - 1/2\eta^2 = \frac{6}{n+1}P_2, \quad \eta = \xi''/\xi'. \quad (6.32)$$

In general, the coefficients $q_i(x)$, $i = 3, \dots, n$ of the canonical form (6.30) depend on the choice of the solution $\xi(x)$ of defined by (6.32).

Theorem 6.1 (Wilczynski [36]) *The expressions:*

$$\Theta_r = \frac{1}{2} \sum_{s=0}^{r-3} (-1)^s \frac{(r-2)!r!(2r-s-2)!}{(r-s-1)!(r-s)!(2r-3)!s!} q_{r-s}^{(s)} \quad (6.33)$$

are relative invariants of the linear differential equation (6.27) with respect to the transformations $(x, y) \mapsto (\xi(x), \lambda(x)y)$, i.e. $\Theta_r \mapsto (\xi')^r \Theta_r$.

The expressions (6.33) were introduced by Wilczynski in [36] and are called the linear (relative) invariants of the equation (6.27). Wilczynski also shows that all other invariants (relative and absolute) can be defined from them via differentiation and algebraic operations. In particular, equation (6.27) can be transformed to the trivial equation $Y^{(n)} = 0$ if and only if $\Theta_r = 0$ for all $r = 3, \dots, n$.

We note that the invariants Θ_r do not depend on the choice of η in (6.32) and can be expressed explicitly in terms of coefficients $p_i(x)$, $i = 1, \dots, n$, of the initial equation (6.27). To do it in practice, bring (6.27) to the form (6.28) and calculate the coefficients q_i using (6.31). Now q_1 vanishes identically, but q_2 , does not, unless the Riccati equation holds. We nevertheless formally compute the expressions (6.33) with substitutions $\xi'' = \xi'\eta$ and $\eta' = 1/2\eta + \frac{6}{n+1}P_2$. The coefficients in (6.33) are chosen in such a way that the resulting expression will no longer depend on η . For example, the explicit expression for Θ_3 does not depend on the order n and has the form:

$$\Theta_3 = p_3 - 3p_1p_2 + 2p_1^3 + 3p_1p_1' - \frac{3}{2}p_2' + \frac{1}{2}(p_1')^2.$$

For $n = 3$ we arrive at the invariant Θ_3 defined in the previous section.

Other approaches to define Wilczynski invariants (6.33) are presented in some other studies [8,30]. In particular, R. Chalkley provides an algorithm for computing Wilczynski invariants in a way that avoids the Laguerre–Forsyth canonical form. He also gives alternative proofs of the above results.

We note that Wilczynski invariants can also be computed in the cases when the coefficients of the initial equation have isolated singularities. In particular, they are well-defined meromorphic functions if the linear equation is defined on complex domain with all coefficients being meromorphic functions on this domain.

6.2 Generalized Wilczynski invariants

The generalised Wilczynski invariants [11, 12] of an arbitrary nonlinear ODE

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}) \tag{6.34}$$

are defined as classical Wilczynski invariants of its linearisation. Analytically, they are computed by substituting $-\binom{n}{r}^{-1} D_x^k \left(\frac{\partial F}{\partial y^{(n-r)}} \right)$ in place of $p_r^{(k)}(x)$ in the classical Wilczynski invariants. Here by D_x , we denote the operator of total derivative:

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \dots + y^{(n-1)} \frac{\partial}{\partial y^{(n-2)}} + F \frac{\partial}{\partial y^{(n-1)}}.$$

We denote by Θ_i the generalised Wilczynski invariant that we obtain from Θ_i by this formal substitution.

Theorem 6.2 ([11]) *Generalised Wilczynski invariants are (relative) contact invariants of nonlinear ordinary differential equations.*

Generalised Wilczynski invariants were defined and studied in [11, 12]. They are closely related to Wünschmann conditions defined in [17] and further explored in [18]. For $n = 7$, these Wünschmann conditions are explicitly computed in [15]. They consist of five expressions W_1, \dots, W_5 , which should vanish identically to guarantee the existence of a natural $GL(2, \mathbb{R})$ -structure (3.7) on the solution space⁴ M of the equation, such that normal vector to a hypersurface given by fixing (x, y) in the general solution to (6.34) corresponds to a sextic polynomial with a root of multiplicity 6. For $n = 7$ these Wünschmann conditions are related to the generalised Wilczynski invariants as follows:

$$W_i = a_i \left(\Theta_{i+2} + \sum_{j=3}^{i+1} b_i^j \Theta_j \right),$$

where a_i are constants and b_i^j are linear differential operators which are polynomials in the total derivative operator D_x of order at most 4. For example, $a_1 = -3430, a_2 = -240100$ and $b_2^3 = \frac{2}{5} D_x - \frac{12}{35} \frac{\partial F}{\partial y^{(6)}}$ so that:

$$\begin{aligned} W_1 &= -3430 \Theta_3, \\ W_2 &= -240100 \left(\Theta_4 + \frac{2}{5} D_x(\Theta_3) - \frac{12}{35} \frac{\partial F}{\partial y^{(6)}} \Theta_3 \right). \end{aligned}$$

In particular, it immediately follows that vanishing of $\Theta_i, i = 3, \dots, 7$ is equivalent to vanishing of $W_i, i = 1, \dots, 5$.

Lemma 6.3 *Let $(\Theta_8)^3 = \kappa(\Theta_3)^8$ be the differential equation (5.25) defining all curves on projective plane with the constant projective curvature $\kappa \neq 0$. All generalized Wilczynski invariants vanish for this equation if and only if $\kappa = \frac{3^9 7^3}{2^4 5^2}$. It corresponds exactly to the family of all cuspidal cubics.*

Proof The explicit expression for Θ_3 is given by equation (5.23). The explicit expression for Θ_8 is given by (5.24) and is linear in the highest derivative $y^{(7)}$. Resolving the equation

⁴ Not all $GL(2)$ structures come from ODEs. Those which do have been partially characterised in [26].

$(\Theta_8)^3 = \kappa(\Theta_3)^8$ with respect $y^{(7)}$ and using the explicit formulae for Wilczynski invariants (6.33), we get that the generalised Wilczynski invariants for equation (5.25) are given by

$$\Theta_3 = \Theta_4 = \Theta_5 = \Theta_7 = 0,$$

$$\Theta_6 = -\frac{(2^4 5^2 \kappa - 3^9 7^3) \left(9(y^{(2)})^2 y^{(5)} - 45 y^{(2)} y^{(3)} y^{(4)} + 40(y^{(3)})^3 \right)^2}{2^2 3^{12} 7^4 (y'')^6}.$$

Thus, we see that $\kappa = \frac{3^9 7^3}{2^4 5^2}$ is the only value of projective curvature, for which all generalised Wilczynski invariants vanish identically. □

6.3 Orbits of general cuspidal curves

Theorem 6.4 *Let $\mathcal{C}_{(p,q)} = SL(3, \mathbb{C})/\mathbb{C}^*$ be a seven-dimensional family of all plane curves projectively equivalent to the curve*

$$y^p - x^q = 0,$$

where (p, q) are coprime positive integers with $\text{Max}(p, q) > 2$. The following statements are equivalent

- (1) *The seven-dimensional family $\mathcal{C}_{(p,q)}$ is a complete contact deformation family of non-singular Legendrian curves in $\mathbb{P}(T\mathbb{C}\mathbb{P}^2)$.*
- (2) *Generalised Wilczynski invariant of the ODE (5.25) vanish.*
- (3) *$\mathcal{C}_{(p,q)}$ is projectively equivalent to the family of cuspidal cubics $\mathcal{C}_{(2,3)}$.*

Proof Let us consider the $SL(3)$ orbits of general cuspidal curves (5.20). The reason for non-vanishing of Θ_6 is that the $SL(3)$ orbit is not a complete analytic family in the sense of [25] unless the curve is cubic. Repeating the steps leading to (3.10), with $T^\alpha = (t^p, t^q, 1)$ gives

$$S(s, t) = (q - p)\sigma^3_2 t^{2q} + (q - p)\sigma^3_1 t^{q+p} s^{q-p} - q\sigma^1_2 t^{2q-p} s^p + ((q - p)\sigma^3_3 + p\sigma^2_2 - q\sigma^1_1) t^q s^q + p\sigma^2_1 t^p s^{2q-p} - q\sigma^1_3 t^{q-p} s^{q+p} + p\sigma^2_3 s^{2q}.$$

The form of this polynomial is not preserved by the rational transformation of t unless $\text{Max}(p, q) = 3$.

Let us also clarify this fact from the point of view of singularities of Legendrian curves. Passing to the homogeneous coordinates $[X : Y : Z]$ on $\mathbb{C}\mathbb{P}^2$ and permuting the coordinates if needed, we can always assume that the equation of the curve is written as $Y^p Z^{q-p} = X^q$, where $p \leq q$ and $q \geq 3$. In this case, the point $[0 : 0 : 1]$ is always singular, so that all curves in our family will be singular as well.

In order to apply Bryant's generalisation [6] of Kodaira deformation theory we need all curves of the family $\mathcal{C}_{(p,q)}$ to be non-singular, or at least their lifts to the projectivized cotangent bundle $\mathbb{P}(T\mathbb{C}\mathbb{P}^2)$ to be non-singular. Elementary computation shows that the lift of the curve $Y^p Z^{q-p} = X^q$ to $P(T\mathbb{C}\mathbb{P}^2)$ is a non-singular curve if and only if $p = 1$ or $q = p + 1$. Indeed, the parametrisation of the affine coordinates $(x, y) = (t^p, t^q)$ lifts to a rational parametrisation of the curves $\gamma(t) = (x, y, \zeta) = (t^p, t^q, (q/p)t^{q-p})$ which are Legendrian with respect to the contact form $dy - \zeta dx$. We find that $\gamma = \dot{\gamma} = 0$ at $t = 0$ unless $p = 1$, or $q = p + 1$.

Both cases are projectively equivalent via the change of coordinates Y and Z . Hence, we shall treat only the first case and assume that $p = 1$. Then, only for $q = 3$ (the cuspidal cubics case), the lifts of all curves in the family $\mathcal{C}_{(p,q)}$ constitute the complete deformation family of the curve $YZ^2 = X^3$. Indeed, Lemma 6.3 shows that for $q > 3$, the projective curvature (5.25) is different from the distinguished value $\frac{3^{973}}{245^2}$ and the generalized Wilczynski invariant Θ_6 does not vanish for the seventh order ODE defining the family of curves projectively equivalent to $YZ^{q-1} = X^q$. \square

The fact that the generalised Wilczynski invariant Θ_6 does not vanish on curves $YZ^{q-1} = X^q$ for $q > 3$, although all these curves are rational can be explained as follows. Consider the seventh order ODE defining the family of curves $\mathcal{C}_{(1,q)}$. Formulae (5.24) with $r = 3$ and (5.25) imply that the coefficient of the leading term $y^{(7)}$ equal exactly to the nominator of the classical Wilczynski invariant Θ_3 (see (5.23)). Thus, the equation of plane conics

$$9(y^{(2)})^2 y^{(5)} - 45y^{(2)}y^{(3)}y^{(4)} + 40(y^{(3)})^3 = 0$$

defines at the same time the set of singular points for the equation of curves of constant projective curvature. This is exactly the set where all generalized Wilczynski invariants have singularities as well. Now computing the expression (5.23) for the curve $YZ^{q-1} = X^q$, $q \geq 3$, we find that it is equal to X^{3q-9} up to a non-zero constant. Thus, the lifts of curves projectively equivalent to $YZ^{q-1} = X^q$ cross the set of singular points transversally whenever $q > 3$ and do not intersect this set at all if $q = 3$. In other words, the generalized Wilczynski invariants have no singularities on the curves $YZ^{q-1} = X^q$ if and only if $q = 3$.

7 Outlook. ODEs for rational curves

We have constructed a co-calibrated G_2 structure on the moduli space M of cuspidal cubics. Co-calibrated G_2 structures play a role in theoretical physics: they give rise to solutions of IIB supergravity for which the only flux is the self-dual five-form [20]. They also appear in the context of near-horizon geometries in heterotic supergravity. In particular, the eight-dimensional spatial cross-sections of the horizon are $U(1)$ fibrations over a conformally co-calibrated G_2 structures on a seven-manifold M [21]. There is also a connection with $SU(3)$ structures [9].

In this study, the manifold M arises as the solution space of the seventh order ODE (5.26). This is an example of the general construction of [15] which associates G_2 structures with 7th order ODEs with general solutions given by rational curves.

There are few known ODE with that property—they are partially characterised by the vanishing of the Wilczynski invariants [11,12,17,18]. They also correspond to projective differential invariants [28].

- The ODE

$$25y^{(7)}(y^{(4)})^2 - 105y^{(6)}y^{(5)}y^{(4)} + 84(y^{(5)})^3 = 0$$

describes rational sextics with two cusps and admits eight-dimensional group of symmetries (this group is different than $PSL(3)$). The corresponding G_2 structure is closed:

$$d\phi = 0, \quad d * \phi = \tau \wedge \phi,$$

where τ is some two-form [15].

- The ODE

$$10(y^{(3)})^3 y^{(7)} - 70(y^{(3)})^2 y^{(4)} y^{(6)} - 49(y^{(3)})^2 (y^{(5)})^2 + 280(y^{(3)})(y^{(4)})^2 y^{(5)} - 175(y^{(4)})^4 = 0$$

is (together with $y''' = 0$) the unique ODE admitting ten-dimensional group of contact symmetries [27]. The general solution is given by certain family of rational sextics [16]. The symmetry group is isomorphic to $Sp(2)$, and the seven-dimensional solution space $M = Sp(2)/SL(2)$ admits a nearly integrable G_2 structure:

$$d\phi = \lambda * \phi, \quad d * \phi = 0,$$

where λ is a constant. If the real form $SO(5)/SO(3)$ is chosen, then the G_2 structure is Riemannian [5].

There are also some lower order examples.

- The fifth order ODE for conics

$$9(y^{(2)})^2 y^{(5)} - 45y^{(2)} y^{(3)} y^{(4)} + 40(y^{(3)})^3 = 0$$

is equivalent to the vanishing of Θ_3 given by (5.23). This ODE goes back at least to Halphen [22]. The resulting solution space $SL(3)/SL(2)$ admits a homogeneous metric which can be read off from the general conic in a way analogous to our construction of (4.16).

- Considering all conics passing through two points $[1, 0, 0]$, $[0, 1, 0]$ in $\mathbb{C}\mathbb{P}^2$ gives a three-dimensional family

$$y = \frac{ax + b}{cx + d}.$$

This arises from the Schwartzian ODE

$$2y^{(3)}y^{(1)} - 3(y^{(2)})^2 = 0.$$

The problem of classifying ODEs, all solutions of which are rational curve, remains open.

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