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GL(2, $\mathbb{R}$) STRUCTURES, G_2 GEOMETRY AND TWISTOR THEORY

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Abstract

A GL(2, $\mathbb{R}$) structure on an $(n+1)$-dimensional manifold is a smooth point-wise identification of tangent vectors with polynomials in two variables homogeneous of degree n. This, for even $n = 2k$, defines a conformal structure of signature $(k, k+1)$ by specifying the null vectors to be the polynomials with vanishing quadratic invariant. We focus on the case $n = 6$ and show that the resulting conformal structure in seven dimensions is compatible with a conformal G_2 structure or its non-compact analogue. If a GL(2, $\mathbb{R}$) structure arises on a moduli space of rational curves on a surface with self-intersection number 6, then certain components of the intrinsic torsion of the G_2 structure vanish. We give examples of simple seventh-order ordinary differential equations whose solution curves are rational and find the corresponding G_2 structures. In particular we show that Bryant's weak G_2 holonomy metric on the homology seven-sphere SO(5)/SO(3) is the unique weak G_2 metric arising from a rational curve.

1. Introduction

Consider the three-dimensional space M of holomorphic parabolas in $\mathbb{C}^2$. Each parabola is of the form

$$y = ax^2 + 2bx + c$$

and (a, b, c) serve as local holomorphic coordinates on M. Two parabolas generically intersect at two points, and we can define a holomorphic conformal structure on M by declaring two points p and $\tilde{p}$ to be null separated if and only if the corresponding parabolas are tangent. The tangency condition is equivalent to a polynomial equation

$$v^3x^2 + 2v^2x + v^1 = 0$$

having a double root. Here $(v^1, v^2, v^3) = (\tilde{c} - c, \tilde{b} - b, \tilde{a} - a)$ is the vector connecting p and $\tilde{p}$. Calculating the discriminant shows that this vector is null if $(v^2) - v^1v^3 = 0$. This quadratic condition defines a flat conformal structure on $M = \mathbb{C}^3$.

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An immediate question is whether this approach can be generalized to curved conformal structures. One answer goes back to Wünschmann [29] who worked in the real category. The parabolas are integral curves of a third-order ordinary differential equation (ODE) $y''' = 0$. Wünschmann has found the necessary and sufficient condition for a general third-order ODE so that the conformal structure induced on the solution space by the tangency condition is well defined. The question has also been considered in the context of twistor theory [14] where one is not concerned with differential equations but rather with the algebro-geometric properties of rational curves in a complex 2-fold.

How about higher dimensions? It turns out that one can define conformal structures on certain odd-dimensional moduli spaces of rational curves, but the discriminant (which is not quadratic for higher degree curves) needs to be replaced by another invariant. In this paper we shall consider the seven-dimensional case and answer the following questions:

(i) Given a seven-dimensional family of rational curves, can one define a conformal complexified G_2 structure on the moduli space M on these curves? Does this structure admit a real form of Riemannian signature?
(ii) Can one characterize the curves and the corresponding G_2 structures in terms of a seventh-order ODE

$$y^{(7)} = F(x, y, y', \dots, y^{(6)})$$

with M as its solution space?

The methods employed in this paper form a mixture of 'old' and 'new'. To define the conformal structure on M we use the nineteenth century classical invariant theory (formula (7) in Section 3), but the characterization of curves builds on the Penrosean holomorphic twistor methods. The allowed rational curves must (after complexification) have self-intersection number 6 in some complex 2-fold or a normal bundle $\mathcal{O}(5) \oplus \mathcal{O}(5)$ in a complex contact 3-fold. This allows a point-wise identification of tangent vectors in M with sextic homogeneous polynomials in two variables. Now the invariant theory can be applied to construct a conformal structure, and the associated G_2 3-form ϕ (formulae (9) and (12) in Section 4). The ODE approach gives a good handle on the local differential geometry on M and allows expressing the components of intrinsic torsion of the G_2 structure (as well as the torsion of the associated Cartan connection) in terms of the contact invariants of the corresponding ODE (Theorems 5.1 and 5.2 formulated in Section 5 and proved in Section 10). Here we make an extensive use of the Tanaka–Morimoto theory of normal Cartan's connection (Sections 8 and 9). These methods allow us to show that if the component of the intrinsic G_2 torsion taking value in the 27-dimensional irreducible representation $\Lambda^3(\mathbb{R}^{7*})$ vanishes, then the resulting G_2 geometry admits a Riemannian real form and (up to diffeomorphisms) it is either flat, or is given by Bryant's weak G_2 holonomy [2] on $\mathrm{SO}(5)/\mathrm{SO}(3)$, or is given by a seven-parameter family of curves

$$(y + Q(x))^2 + P(x)^3 = 0,$$

where the polynomials $(Q(x), P(x))$ are the general cubic and quadratic, respectively. These curves have degree 6, but we shall find that they are rational and form a complete analytic family. The corresponding seventh-order ODE is

$$y^{(7)} = \frac{21}{5}\frac{y^{(6)}y^{(5)}}{y^{(4)}} - \frac{84}{25}\frac{(y^{(5)})^3}{(y^{(4)})^2}, \quad \text{where } y^{(k)} = \frac{\partial^k y}{\partial x^k}$$

and the associated conformal structure is given by (9) and (22). There exists a choice of the conformal factor such that corresponding G_2 structure is closed, that is

$$\mathrm{d}\phi = 0, \quad \mathrm{d} * \phi = \tau \wedge \phi$$

for some 2-form τ on M.

Most calculations in the second half of the paper were performed using MAPLE. In particular proving Theorem 5.2 required solving a system of over 600 quadratic equations for components of curvature and torsion of Cartan's normal connection. The resulting expressions are usually long and unilluminating and we have not included all of them in the manuscript. Readers who want to verify our calculations can obtain the MAPLE codes from us.

2. GL(2, ℝ) structures

DEFINITION 2.1 A GL(2, ℝ) structure on a smooth $(n+1)$-dimensional manifold M is a smooth bundle isomorphism

$$TM \cong \mathbb{S} \odot \mathbb{S} \odot \cdots \odot \mathbb{S} = S^n(\mathbb{S}), \tag{1}$$

where $\mathbb{S} \to M$ is a real rank-2 vector bundle, and $\odot$ denotes symmetric tensor product.

The isomorphism (1) identifies each tangent space $T_{\mathbf{t}}M$ with the space of homogeneous nth-order polynomials in two variables. The vectors corresponding to polynomials with repeated root of multiplicity n are called maximally null. A hypersurface in M is maximally null if its normal vector is maximally null.

In practice the isomorphism (1) giving rise to a GL(2, ℝ) structure is specified by a binary quantic with values in T^*M

$$Q(X_1, X_2) = \sum_{i=0}^{n} \binom{n}{i} \theta^{i+1} (X_1)^i (X_2)^{n-i}, \quad \binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}. \tag{2}$$

Here (X_1, X_2) are coordinates on $\mathbb{R}^2$, and the 'coefficients' in the quantic are given by linearly independent 1-forms $\theta^1, \theta^2, \ldots, \theta^{n+1}$ on M. If V is a vector field on M, then the corresponding polynomial is given by $V \lrcorner\, Q$, where $\lrcorner$ denotes the contraction of a 1-form with a vector field. If $V = \sum_i v^i \theta_i$ is expressed in a basis θ_i of TM such that $\theta_i \lrcorner\, \theta^j = \delta_i^j$, the polynomial is

$$\sum_{i=0}^{n} \binom{n}{i} v^{i+1} (X_1)^i (X_2)^{n-i}, \tag{3}$$

with the coefficients v^i being smooth functions on M.

Consider a general ODE of order $(n+1)$

$$\frac{\mathrm{d}^{n+1} y}{\mathrm{d}x^{n+1}} = F(x, y, y', \ldots, y^{(n)}), \tag{4}$$

where $y' = \mathrm{d}y/\mathrm{d}x$ etc, whose general solution is of the form $y = Z(x, \mathbf{t})$ where $\mathbf{t}$ are constants of integration. Assume that the space of solutions to (4) is equipped with a GL(2, ℝ) structure (1) such that the two-parameter family of hypersurfaces given by fixing (x, y) are maximally null. It has been shown in [10] that this imposes conditions on F which are expressed by vanishing of $(n-1)$ expressions

$$W_\alpha[F], \quad \alpha = 1, 2, \dots, n-1 \tag{5}$$

for the ODE (4). Each expression W_α is a polynomial in the derivatives of F. The simplest of these is the contact invariant

$$W_1[F] = \mathcal{D}^2 F_n - \frac{6}{n+1} F_n \mathcal{D} F_n + \frac{4}{(n+1)^2}(F_n)^3 - \frac{6}{n}\mathcal{D}F_{n-1} + \frac{12}{n(n+1)} F_n F_{n-1} + \frac{12}{n(n-1)} F_{n-2},$$

where

$$F_k = \frac{\partial F}{\partial y^{(k)}} \quad \text{and} \quad \mathcal{D} = \frac{\partial}{\partial x} + \sum_{k=1}^{n} y^{(k)} \frac{\partial}{\partial y^{(k-1)}} + F \frac{\partial}{\partial y^{(n)}}.$$

Moreover if $W_1[F] = W_2[F] = \cdots = W_{m-1}[F] = 0$ then $W_m[F]$ is a contact invariant of the ODE (4). The explicit expressions for W_α are unilluminating, but for completeness we list the five invariants (in the form given in [13]) of seventh-order ODEs in Appendix A.

The same invariants have also arisen in other related contexts [6, 7, 13]. The description given by Doubrov is particularly clear. First note that a linearization of the ODE (4) around any of its solutions is a linear homogeneous ODE of the form

$$(\delta y)^{(n+1)} = p_n(x)\delta y^{(n)} + \cdots + p_0(x)\delta y, \tag{6}$$

where $p_k = \partial F/\partial y^{(k)}$ is evaluated at the solution.

THEOREM 2.2 [6, 7] *The expressions (5) vanish if and only if the linear homogeneous* ODE *(6) can be brought to a form $\delta y^{(n+1)} = 0$ by a coordinate transformation $(x, y) \to (\beta(x), \gamma(x)y)$ for some functions β and γ. Vanishing of (5) is invariant under the contact transformations of the non-linear* ODE *(4).*

The linear homogeneous ODEs of the form (6) have been studied by Wilczynski [28] who gave explicit conditions for their trivializability in terms of the functions p_k and their derivatives.

In the simplest non-trivial case $n = 2$ the corresponding invariant was already known to Wünschmann [29]. In the case $n = 3$ the invariants have been implicitly constructed by Bryant in his study of exotic holonomy [3] and developed by Nurowski [20].

One source of ODEs for which these contact invariants vanish comes from twistor theory [3, 10]. Let $\mathcal{Y}$ be a complex contact 3-fold with an embedded rational Legendrian curve with a normal bundle $N = \mathcal{O}(n-1) \oplus \mathcal{O}(n-1)$. The moduli space of such curves is $(n+1)$-dimensional and carries a natural (complexified) $GL(2, \mathbb{R})$ structure.

The special case is $\mathcal{Y} = P(T\mathbb{T})$, where $\mathbb{T}$ is a complex 2-fold $\mathbb{T}$ containing embedded rational curve L with self-intersection number n. Such curve has a natural lift $\hat{L}$ to $\mathcal{Y}$, given by $z \in L \to (z, \dot{z} \in T_z L)$. The lifted curves are Legendrian with respect to the canonical contact structure on the projectivized tangent bundle. The ODE whose integral curves are given by holomorphic deformations of L satisfies the $GL(2, \mathbb{R})$ conditions.

3. GL(2, ℝ) conformal structure

In this section we shall associate a conformal structure to a GL(2, ℝ) structure. From now we assume that $n = 2k$ is even. We shall first recall some classical theory of invariants [12]. Let $V_n \subset \mathbb{R}[X_1, X_2]$ be the $(n + 1)$-dimensional space of homogeneous polynomials of degree n. Consider the linear action of GL(2, ℝ) on $\mathbb{R}^2$ given by

$$\tilde{X}_1 = \alpha X_1 + \beta X_2, \quad \tilde{X}_2 = \gamma X_1 + \delta X_2, \quad \alpha\delta - \gamma\beta \neq 0.$$

Given a binary quantic $Q(X_1, X_2)$ (whose coefficients may be numbers, functions, 1-forms, etc.) let $\tilde{Q}(\tilde{X}_1, \tilde{X}_2)$ be a binary quantic such that

$$\tilde{Q}(\tilde{X}_1, \tilde{X}_2) = Q(X_1, X_2).$$

This induces an embedding GL(2, ℝ) ⊂ GL($n + 1$, ℝ), as the coefficients $\tilde{\theta} = (\tilde{\theta}^1, \ldots, \tilde{\theta}^{n+1})$ are linear homogeneous functions of the coefficients of Q. Recall that an *invariant* of a binary quantic is a function $I(\theta)$ depending on the coefficients $\theta = (\theta^1, \theta^2, \ldots, \theta^{n+1})$ such that

$$I(\theta) = (\det A)^w I(\tilde{\theta}), \quad \text{where } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}).$$

The number w is called the weight of the invariant. For example, if $n = 2$, the discriminant $\theta^1\theta^3 - (\theta^2)^2$ is an invariant with weight 2.

One of the classical results of the invariant theory is that all invariants arise from the *transvectants* [12].

Definition 3.1 For any homogeneous polynomials $Q \in V_n$, $R \in V_m$ the pth transvectant is

$$\langle Q, R\rangle_p = \frac{1}{p!}\sum_{i=0}^{p}\binom{p}{i}\frac{\partial^p Q}{\partial(X_1)^{p-i}\partial(X_2)^i}\frac{\partial^p R}{\partial(X_1)^i\partial(X_2)^{p-i}} \in V_{n+m-2p}.$$

In particular specifying $Q = R$ the successive transvectant operations reduce to elements of V_0 which are invariants. The simplest of these is

$$I_0 = \langle Q, Q\rangle_n.$$

It vanishes if n is odd, and for even $n = 2k$ it has weight n and is given by

$$I_0 = \left\{2\sum_{i=0}^{k-1}(-1)^i\binom{2k}{i}\theta^{i+1}\theta^{2k+1-i}\right\} + \binom{2k}{k}(-1)^k(\theta^{k+1})^2. \tag{7}$$

In particular if I_0 is evaluated for the binary quantic (2) defining the GL(2, ℝ) structure where θ^i are 1-forms, then I_0 should be regarded as a section of $S^2(T^*M)$. It is well known that a conformal structure $[g]$ is determined by specifying null vectors, i.e. sections $V \in \Gamma(TM)$ such that $g(V, V) = 0$ for $g \in [g]$. This gives the following proposition.

PROPOSITION 3.2 *A* GL(2, $\mathbb{R}$) *structure on a* $(2k+1)$*-dimensional manifold* M *induces a conformal structure* $[g]$ *of signature* $(k+1, k)$ *or* $(k, k+1)$*. A vector field is null with respect to* $[g]$ *if and only if the corresponding polynomial has* $I_0(V) = 0$*.*

Proof. The 'nullness' of a vector is a quadratic condition and thus leads to a quadratic bilinear form up to scale. Let a vector V correspond to a polynomial (3). The condition $I_0(V) = 0$, where I_0 is given by (7) is indeed a quadratic and leads to a symmetric bilinear form $g(X, Y) = \langle X, Y\rangle_n$ of signature $(k+1, k)$ or $(k, k+1)$.

In general conformal structures induced by GL(2, $\mathbb{R}$) structures form a subclass of all conformal structures of signature $(k+1, k)$, except when $k = 1$ in which case the two notions are equivalent. For n odd (that is, for even-dimensional M), the bilinear form (compare formula (7)) resulting from this definition is anti-symmetric, so does not lead to conformal structures.

EXAMPLE 3.3 In three dimensions GL(2, $\mathbb{R}$) structures are the same as conformal structures of Lorentzian signature. This is related to the isomorphism

$$\mathrm{SL}(2, \mathbb{R})/\mathbb{Z}_2 \cong \mathrm{SO}(2, 1)$$

which underlies the existence of spinors. Let a conformal structure be represented by a metric $g = \eta_{ij} e^i e^j$ where $\eta = \mathrm{diag}(1, -1, -1)$ and e^i, $i = 1, 2, 3$, is an orthonormal basis of 1-forms. The GL(2, $\mathbb{R}$) structure is defined by (2) with $\theta^1 = e^1 + e^3, \theta^2 = e^2, \theta^3 = e^1 - e^3$. A vector $V = v^i \theta_i$ corresponds to a polynomial

$$v^1 + 2xv^2 + x^2 v^3,$$

where $x = X_2/X_1$. The nullness condition

$$I_0(V) = v^1 v^3 - (v^2)^2 = 0$$

is given by vanishing of the discriminant. Thus a vector is null if and only if the corresponding polynomial has a repeated root. In the standard approach to spinors in three dimensions one represents a vector by a symmetric 2×2 matrix V^{AB} where $A, B = 1, 2$, such that $g(V, V) = \det(V^{AB})$. The non-zero null vectors correspond to matrices with vanishing determinant, which therefore must have rank 1. Any such matrix is of the form $V^{AB} = p^A p^B$. In our approach the matrix V^{AB} gives rise to a homogeneous polynomial $V^{AB} X_A X_B$ which, in case of null vectors, has a repeated root $x = -p^1/p^2$.

EXAMPLE 3.4 The five-dimensional GL(2, $\mathbb{R}$) structures correspond to special conformal structure in signature (3, 2). The nullness condition can also be described geometrically in this case and the following interpretation is well known in the context of classical invariant theory [**12**, **22**]. In the five-dimensional case vectors correspond to binary quartics. A generic quartic will have four distinct roots, and the nullness condition $I_0(V) = 0$ implies that their cross-ratio is a cube root of unity. This is the equianharmonic condition. The roots of the quartic, when viewed as points on the Riemann sphere, can in this case be transformed into vertices of a regular tetrahedron by Möbius transformation. Riemannian analogues of such geometries have been studied in [**1**].

We have been unable to find a geometric interpretation of the null condition $I_0(V) = 0$ in the case of seven-dimensional GL(2, $\mathbb{R}$) conformal structures which will play a role in the rest of the paper.

The vectors correspond to binary sextics which generically admit six distinct roots $z_1, z_2, \ldots, z_6$. In this case one can also form an SL(2, ℂ) invariant multi cross-ratio

$$\frac{(z_1 - z_2)(z_3 - z_4)(z_5 - z_6)}{(z_2 - z_3)(z_4 - z_5)(z_6 - z_1)}.$$

Let $z_1, z_2, \ldots, z_6$ denote positions of six points on a plane. Given a triangle with vertices (z_1, z_3, z_5), and three points (z_2, z_4, z_6) on the lines $(z_3 z_5)$, $(z_5 z_1)$ and $(z_1 z_3)$, respectively, the lines $(z_1 z_2)$, $(z_3 z_4)$ and $(z_5 z_6)$ are concurrent if and only if the multi cross-ratio is equal to 1. This is the Ceva theorem.

The theorem of Menelaus states that the points (z_2, z_4, z_6) are collinear if the multi cross-ratio is equal to -1.

We have expressed the invariant I_0 in terms of the roots, hoping to characterize its vanishing by the Menelaus–Ceva conditions, but found that the invariant does not vanish in either of these two cases.

3.1. *Twistor theory*

If the GL(2, ℝ) structure comes from an ODE, then induced conformal structure (7) arises from the twistor correspondence described at the end of Section 2. Here we shall concentrate on the special case when the Legendrian curves on a complex 3-fold are lifts of rational curves from a 2-fold.

Let

$$x \longrightarrow (x, y = Z(x, t_1, t_2, \ldots, t_{2k+1}))$$

be a graph of a rational curve L in a complex surface $\mathbb{T}$ with a normal bundle $N(L) = \mathcal{O}(2k)$. The cohomological obstruction group $H^1(L, N(L))$ vanishes, and therefore the Kodaira theorems [**16**] imply that the curve belongs to a $(2k+1)$-dimensional complete family $\{L_t, \mathbf{t} \in M\}$ parametrized by points in a $(2k+1)$-dimensional complex manifold (the space of solutions to (4)). Moreover there exists a canonical isomorphism

$$T_t M \cong H^0(L_t, N(L_t))$$

which associates a tangent vector at $\mathbf{t} \in M$ to a global holomorphic section of a normal bundle $N(L_t) = \mathcal{O}(2k)$. Such sections are given by homogeneous polynomials of degree $2k$ which establishes the existence of a GL(2, ℝ) structure.

The curve L_t has self-intersection number $2k$, i.e.

$$\delta y = \frac{\partial Z}{\partial \mathbf{t}} \delta \mathbf{t}$$

vanishes at the zeros of a polynomial of degree $2k$ in $x = X_2/X_1$. In its homogeneous form this polynomial is a binary form (3) with coefficients v^α, $\alpha = 1, \ldots, 2k+1$ which depend on t_α and are linear in δt_α. A vector at a point in $\mathbf{t} \in M$ corresponds to a normal vector field to the rational curve L_t, i.e. a section of $N(L_t) = \mathcal{O}(2k)$ which is the same as a homogeneous polynomial of degree $2k$. The corresponding invariant I_0 gives a quadratic form on M up to a multiple and its vanishing selects the null vectors. This determines the conformal structure.

In practice one proceeds as follows: if the rational curve is given by

$$F(x, y, t_\alpha) = 0$$

and its rational parametrization is

$$x = p(\lambda, t_\alpha), \quad y = q(\lambda, t_\alpha),$$

where p, q are functions rational in $\lambda \in \mathbb{CP}^1$, then the polynomial in λ giving rise to a null vector is given by the polynomial part of

$$\sum_\alpha \frac{\partial F}{\partial t_\alpha}\bigg|_{\{x=p, y=q\}} \delta t_\alpha. \tag{8}$$

4. G_2 structures from GL(2, $\mathbb{R}$) conformal structures

We shall now restrict to the case $n = 6$, and demonstrate that the seven-dimensional GL(2, $\mathbb{R}$) manifolds admit a conformal structure with a compatible G_2 structure. If the associated 3-form is closed and co-closed, then the conformal structure is necessarily flat. We will however find examples of non-trivial G_2 structures where some components of the torsion vanish. In particular there is a non-trivial example of weak G_2 holonomy compatible with the GL(2, $\mathbb{R}$) structure. This example is originally by Bryant [2]. In Theorem 5.2 we shall show that this example is essentially unique.

Consider a GL(2, $\mathbb{R}$) structure given by the binary form

$$Q(x) = \theta^1 x^6 + 6\theta^2 x^5 + 15\theta^3 x^4 + 20\theta^4 x^3 + 15\theta^5 x^2 + 6\theta^6 x + \theta^7,$$

with the corresponding quadratic invariant (conformal structure) (7)

$$I_0 = \theta^1\theta^7 - 6\theta^2\theta^6 + 15\theta^3\theta^5 - 10(\theta^4)^2. \tag{9}$$

Here $x = X_2/X_1$ is an inhomogeneous coordinate on the projective line $\mathbb{RP}^1$.

Use a combination of transvectants to construct a 3-form (Some readers may prefer the two component spinor notation [22]. The capital letter indices $A, B, \ldots$ take values 1, 2. They are raised and lowered by a symplectic form represented by an anti-symmetric matrix ε_{AB} on $\mathbb{R}^2$ such that $\varepsilon_{12} = 1$. The homogeneous polynomials are of the form $Q = Q_{AB\ldots C}\pi^A\pi^B \ldots \pi^C$, where $\pi^A = (X_1, X_2)$. Then $\langle Q, P\rangle_n = Q_{AB\ldots C}P^{AB\ldots C}$. The conformal structure and the 3-form are given by

$$I_0 = e_{ABCDEF} \odot e^{ABCDEF},$$

$$\phi = e^{ABC}{}_{DEF} \wedge e^{DEF}{}_{GHI} \wedge e^{GHI}{}_{ABC},$$

where $e^{ABCDEF} = e^{(ABCDEF)}$ is $\mathbb{R}^7$ valued 1-form such that

$$e^{111111} = \theta^1, e^{111112} = \theta^2, e^{111122} = \theta^3, e^{111222} = \theta^4, e^{112222} = \theta^5, e^{122222} = \theta^6, e^{222222} = \theta^7.)$$

$$\phi(X, Y, Z) = \langle\langle X, Y\rangle_3, Z\rangle_6, \tag{10}$$

PROPOSITION 4.1 *The 3-form ϕ is compatible with the conformal structure I_0: The vector V is null with respect to I_0 if and only if*

$$(V \lrcorner \phi) \wedge (V \lrcorner \phi) \wedge \phi = 0, \tag{11}$$

where $(V \lrcorner \phi)(X, Y) := \phi(V, X, Y)$.

Proof. Consider the conformal structure induced by the vanishing of (7). Calculating the components of the 3-form ϕ given by (10) and its dual with respect to I_0 gives

$$\begin{aligned}\phi &= 3(\theta^2\wedge\theta^3\wedge\theta^7+\theta^1\wedge\theta^5\wedge\theta^6)+\theta^4\wedge(\theta^1\wedge\theta^7+6\,\theta^2\wedge\theta^6-15\theta^3\wedge\theta^5),\\ *\phi &= -20\theta^1\wedge\theta^4\wedge\theta^5\wedge\theta^6+5\theta^1\wedge\theta^3\wedge\theta^5\wedge\theta^7-20\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^7\\ &\quad -2\theta^1\wedge\theta^2\wedge\theta^6\wedge\theta^7+30\theta^2\wedge\theta^3\wedge\theta^5\wedge\theta^6.\end{aligned} \tag{12}$$

This is in fact the non-compact form G_2^{split} of the G_2 structure, as these forms agree with the more usual orthonormal frame formulae (see [2])

$$\begin{aligned} I_0 &= (e^1)^2+(e^2)^2+(e^3)^2-(e^4)^2-(e^5)^2-(e^6)^2-(e^7)^2,\\ \phi &= e^{123}-e^{145}-e^{167}-e^{246}+e^{257}+e^{347}+e^{356},\\ *\phi &= e^{4567}-e^{2367}-e^{2345}-e^{1357}+e^{1247}+e^{1256},\end{aligned} \tag{13}$$

provided that

$$\begin{aligned} &e^1=\tfrac{1}{2}(\theta^1+\theta^7),\quad e^5=\tfrac{1}{2}(-\theta^1+\theta^7),\quad e^2=\tfrac{\sqrt{6}}{2}(\theta^2-\theta^6)\\ &e^6=\tfrac{\sqrt{6}}{2}(\theta^2+\theta^6),\quad e^3=\tfrac{\sqrt{15}}{2}(\theta^3+\theta^5),\\ &e^7=\tfrac{\sqrt{15}}{2}(-\theta^3+\theta^5),\quad e^4=\sqrt{10}\theta^4.\end{aligned}$$

(here $e^{ijk}=e^i\wedge e^j\wedge e^k$ etc.). The condition (11) can now be verified directly. Conversely, given a 3-form ϕ, the conformal structure defined by (11) is represented by I_0 as shown in [2].

We have therefore explicitly demonstrated that $\mathfrak{gl}(2,\mathbb{C})$ can be embedded in the complexification ${\mathfrak{g}_2}^{\mathbb{C}}\oplus\mathbb{C}$ of $\mathfrak{g}_2\oplus\mathbb{R}$, or equivalently that $\mathfrak{sl}(2,\mathbb{C})$ can be embedded in ${\mathfrak{g}_2}^{\mathbb{C}}$. This follows more abstractly from a theorem of Morozov [26] which says that for any nilpotent element e of a complex semi-simple Lie algebra $\mathfrak{g}$ there exist $f,h\in\mathfrak{g}$ and a homomorphism $\rho:\mathfrak{sl}(2,\mathbb{C})\to\mathfrak{g}$ such that $\rho(\mathbf{e})=e$, $\rho(\mathbf{f})=f$, $\rho(\mathbf{h})=h$, where $\mathbf{e},\mathbf{f},\mathbf{h}$, is the basis of $\mathfrak{sl}(2,\mathbb{C})$ such that

$$[\mathbf{e},\mathbf{f}]=\mathbf{h},\quad [\mathbf{h},\mathbf{e}]=2\mathbf{e},\quad [\mathbf{h},\mathbf{f}]=-2\mathbf{f}.$$

4.1. *Fernandez–Gray types*

In this paper we follow the standard terminology of G-structures and define a G_2 structure on a seven-dimensional manifold to be a reduction of the frame bundle from $\mathrm{GL}(7,\mathbb{R})$ to G_2 (or its non-compact analogue G_2^{split}). This structure is represented by a 3-form ϕ in the open orbit of $\mathrm{GL}(7,\mathbb{R})$ in $\Lambda^3(M)$. The 3-form induces a metric [2] on M. If ϕ is given by (13) then the metric is given by I_0. It is Riemannian if the forms $e^1,\dots,e^3$ are real and $e^4,\dots,e^7$ are imaginary and has signature $(3,4)$ if all 1-forms are real. (Another equivalent definition [11] is to start form a Riemannian (respectively, signature $(3,4)$) metric g and define a G_2 structure to be a cross product $P:T_tM\times T_tM\to T_tM$ on each tangent space which varies smoothly with $t \in M$ and such that P is a bilinear map satisfying

$$g(P(X, Y), X) = 0, \quad |P(X, Y)|^2 = |X|^2|Y|^2 - g(X, Y)^2, \quad \forall X, Y,$$

where $|X|^2 = g(X, X)$. One then *defines* the associated 3-form by

$$\phi(X, Y, Z) = g(P(X, Y), Z).$$

This cross product equips each tangent space with the algebraic structure of pure octonions (respectively, pure split octonions)). The latter case corresponds to the non-compact form G_2^{split}. Proposition 4.1 shows that seven-dimensional GL(2, $\mathbb{R}$) structure is equivalent to a further reduction of the frame bundle from $\mathbb{R}^+ \times G_2 \subset \mathbb{R}^+ \times \text{SO}(3, 4)$ to GL(2, $\mathbb{R}$).

We do not assume anything about the closure of the 3-form ϕ or its dual. Various types of G_2 structures (or their non-compact analogues) are characterized by a representation theoretic decomposition of $\nabla\phi$, where ∇ is the Levi-Civita connection of the metric induced by ϕ. Following [4, 11] we have

$$\begin{aligned} \mathrm{d}\phi &= \lambda * \phi + \tfrac{3}{4}\Theta \wedge \phi + *\tau_3 \\ \mathrm{d} * \phi &= \Theta \wedge *\phi - \tau_2 \wedge \phi, \end{aligned} \tag{14}$$

where λ is a scalar, Θ is a 1-form, τ_2 is a 2-form such that $\tau_2 \wedge \phi = - * \tau_2$ and τ_3 is a 3-form such that $\tau_3 \wedge \phi = \tau_3 \wedge *\phi = 0$. The forms $(\lambda, \Theta, \tau_2, \tau_3)$ can be interpreted as components of intrinsic torsion of a natural connection of G_2 structure. To define this connection apply the canonical decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathbb{R}^7$ (or its $\mathfrak{so}(3, 4)$ analogue) to the Levi-Civita connection. If γ is the $\mathfrak{so}(7)$-valued connection 1-form, then writing $\gamma = \hat{\gamma} + \tau$ defines a connection with torsion (not to be confused with the torsion T of the $\mathfrak{gl}(2, \mathbb{R})$-valued connection Γ studied in Sections 7 and 10) represented by a 1-form $\hat{\gamma}$ with values in $\mathfrak{g}_2$. See [4] for details. If the 3-form is only defined up to a multiple by a non-zero function (as it is the case in this paper) then λ, τ_2 and τ_3 scale with appropriate weights and Θ transforms like a Maxwell field. More precisely conformal rescaling $g \to e^{2f} g$ leaves (14) invariant if

$$\phi \longrightarrow e^{3f}\phi, \quad \lambda \longrightarrow e^{-f}\lambda, \quad \Theta \longrightarrow \Theta + 4\,\mathrm{d}f, \quad \tau_2 \longrightarrow e^{f}\tau_2, \quad \tau_3 \longrightarrow e^{2f}\tau_3. \tag{15}$$

If all components of the torsion vanish, then the G_2 structure gives rise to G_2 holonomy and the resulting metric is Ricci-flat. Such G_2 structures are sometimes called *integrable* or more correctly *torsion-free*. If $\Theta = \tau_2 = \tau_3 = 0$, then the metric is Einstein with non-zero Ricci scalar and one speaks of *weak G_2 holonomy*. If $\lambda = \Theta = \tau_3 = 0$ then the G_2 structure is *closed* (see [5]).

The representation theoretic decomposition of the torsion is as follows.

(i) λ is a function and $\lambda\phi$ belongs to the one-dimensional irreducible representation $\mathcal{W}_1 \subset \Lambda^3\mathbb{R}^{*7}$ of G_2.
(ii) The 2-form τ_2 belongs to the 14-dimensional irreducible representation $\mathcal{W}_2 \subset \Lambda^2\mathbb{R}^{*7}$.
(iii) The 3-form τ_3 belongs to the 27-dimensional irreducible representation $\mathcal{W}_3 \subset \Lambda^3\mathbb{R}^{*7}$.
(iv) The *Lee 1-form* Θ belongs to the seven-dimensional representation $\mathcal{W}_4 = \mathbb{R}^{7*}$.

Equations (14) uniquely define λ, τ_2, τ_3 and Θ. Vanishing of these objects defines the Fernandez–Gray $\mathcal{W}$ type of G_2 geometry: if none of them vanishes the geometry is of generic type $\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3+\mathcal{W}_4$, if $\lambda=0$ then the geometry is of type $\mathcal{W}_2+\mathcal{W}_3+\mathcal{W}_4$, when $\tau_2=0$ we have the type $\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_4$ and so on. There are sixteen $\mathcal{W}$ types.

Proposition 4.1 demonstrates that the seven-dimensional GL(2, $\mathbb{R}$) geometry is a special case of conformal split G_2 geometry as $\mathrm{GL}(2,\mathbb{R})\subset\mathbb{R}\times G_2^{\mathrm{split}}$. The representations of G_2^{split} decompose into irreducible representations of GL(2, $\mathbb{R}$) as follows:

$$\begin{aligned}\mathcal{W}_1&=V^1,\\ \mathcal{W}_2&=V^3\oplus V^{11},\\ \mathcal{W}_3&=V^5\oplus V^9\oplus V^{13},\\ \mathcal{W}_4&=V^7,\end{aligned}\tag{16}$$

where V^k is the k-dimensional representation space $\mathrm{S}^{k-1}(\mathbb{S})$. Hence τ_2 and τ_3 have a priori two and three irreducible components under action of GL(2, $\mathbb{R}$).

A GL(2, $\mathbb{R}$) geometry defines a whole conformal class of G_2^{split} geometries, hence we may only talk about those $\mathcal{W}$ types which are invariant with respect to conformal rescalings (15). In particular vanishing of Θ is not conformally invariant. However, $\mathrm{d}\Theta$ is a well-defined 2-form, in particular the condition $\mathrm{d}\Theta=0$ means that in some conformal gauge Θ vanishes locally. In general $\mathrm{d}\Theta$ is a 2-form decomposing according to

$$\Lambda^2\mathbb{R}^7=V^3\oplus V^7\oplus V^{11}.\tag{17}$$

5. G_2 structures from ODEs

Now we are ready to give the relations between the intrinsic torsion of the split G_2 structure and the contact invariants of the seventh-order ODE

$$y^{(7)}=F(x,y,y',\dots,y^{(6)}).\tag{18}$$

In the following theorems (which will be established in Section 10) we shall assume the vanishing of the conditions W_α (Appendix A) which are necessary and sufficient for an ODE to give rise to a GL(2, $\mathbb{R}$) geometry.

THEOREM 5.1 *Let the seventh-order* ODE (*18*) *admit the* GL(2, $\mathbb{R}$) *geometry on the solution space. Then the following conditions hold*:

no $\mathcal{W}_1$ *component* ($\lambda=0$) $\iff$ $F_{66}(9\mathcal{D}F_6-\frac{9}{7}F_6^2-15F_5)+12F_{65}F_6+14F_{55}-\frac{84}{5}F_{64}=0.$

no $\mathcal{W}_2$ *component* ($\tau_2=0$) $\iff$ $21\mathcal{D}F_{66}+14F_{65}+15F_6F_{66}=0.$

no $\mathcal{W}_3$ *component* ($\tau_3=0$) $\iff$ $F_{66}=0.$

The 2-form $\mathrm{d}\Theta$ *falls into components in irreducible representations* V^3, V^7 *and* V^{11}. *The* V^3*-part is expressed algebraically by* λ, τ_2 *and* τ_3. *In particular it vanishes if* τ_2 *vanishes. The* V^7*-part of* $\mathrm{d}\Theta$ *vanishes if and only if*

$$(\mathcal{D}F)_{66}F_{66} + \tfrac{3}{2}(\mathcal{D}F)_6 F_{666} - \tfrac{12}{7}F_{666}F_6^2 - 4F_{666}F_5 + 2F_{665}F_6 - \tfrac{14}{5}F_{664} + \tfrac{7}{3}F_{655} - \tfrac{4}{3}F_{66}F_{65} - \tfrac{16}{7}F_{66}^2F_6 = 0.$$

The V^{11}*-part of* $\mathrm{d}\Theta$ *vanishes if and only if*

$$F_{666} = 0.$$

The non-generic $\mathcal{W}$ types are characterized by the following result.

THEOREM 5.2 *There are only three conformal split* G_2 *geometries from ODEs of type* $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4$.

(1) *The flat geometry of* $y^{(7)} = 0$, *which is the only case admitting holonomy* G_2^{split}.
(2) *The geometry of*

$$y^{(7)} = 7\frac{y^{(6)}y^{(4)}}{y^{(3)}} + \frac{49}{10}\frac{(y^{(5)})^2}{y^{(3)}} - 28\frac{y^{(5)}(y^{(4)})^2}{(y^{(3)})^2} + \frac{35}{2}\frac{(y^{(4)})^4}{(y^{(3)})^3}. \tag{19}$$

This is the only geometry of type $\mathcal{W}_1 + \mathcal{W}_4$. *The Lee form is closed, so that in certain conformal gauge it is the nearly parallel* ($\mathcal{W}_1$) *geometry of* $\mathrm{SO}(3,2)/\mathrm{SO}(2,1)$.
(3) *The geometry of*

$$y^{(7)} = \frac{21}{5}\frac{y^{(6)}y^{(5)}}{y^{(4)}} - \frac{84}{25}\frac{(y^{(5)})^3}{(y^{(4)})^2}, \tag{20}$$

which is of type $\mathcal{W}_2 + \mathcal{W}_4$. *The Lee form is closed, so in certain conformal gauge it is a closed* G_2 *structure* ($\mathcal{W}_2$).

The G_2 geometry associated with (19) has two real forms: The homogeneous space $\mathrm{SO}(3,2)/\mathrm{SO}(2,1)$ which yields a weak G_2 metric in signature (3, 4) and $\mathrm{SO}(5)/\mathrm{SO}(3)$ which gives a Riemannian metric. The later metric was first constructed by Bryant in his seminal paper [**2**] without using the ODE or twistor techniques. Theorem 5.2 implies that up to diffeomorphisms of M this is the only weak G_2 metric arising from an ODE. The ODE (19) has appeared in several other contexts. See [**21**, **23**].

The ODE (20) has an elementary solution given by certain rational curves which will be analysed in the next section. The solution to (19) can also be constructed in terms of rational curves, but explicit description in this case is more involved [**9**].

We note that there exists at least one more connection between differential equations and non-compact G_2: the holonomy of an ambient metric associated to Nurowski's (3, 2) conformal structure is contained in G_2^{split} (see [**19**]).

6. Examples

In this section we give some examples. The first three arise on moduli spaces of rational curves by twistor theoretic techniques. The last one comes from an ODE satisfying the Wünshmann conditions (5).

We shall write the general seventh-order ODE (18) as

$$y^{(7)} = F(x, y, p, q, r, s, t, u),$$

where $p = y', q = y'', r = y^{(3)}, s = y^{(4)}, t = y^{(5)}, u = y^{(6)}$.

The examples below can be partially classified by the dimension of the group of contact symmetries (recall that the maximal symmetry group of the trivial ODE $y^{(7)} = 0$ is 11-dimensional and given by $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^7$).

EXAMPLE 6.1 Consider a hyperelliptic curve of degree 6 with 2 cusps. The general sextic has genus 10 and so is not rational, but in our case the genus is 0 and a rational parametrization exist. To see it write the curve as

$$(y + Q(x))^2 + P(x)^3 = 0, \tag{21}$$

where (Q, P) are general cubic and quadratic, respectively, which we write as

$$Q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3, \quad P(x) = p_3(x - p_2)(x - p_1).$$

This has three singular points. Two double points at $(p_1, -Q(p_1))$ and $(p_2, -Q(p_2))$ of type [2, 1, 1] (see [27]) and one point of order 4 at infinity of type [4, 8, 2], which can be seen by writing (21) in the homogeneous coordinates. Calculating the genus yields

$$\mathrm{g} = \frac{5 \cdot 4}{2} - 1 - 1 - 8 = 0,$$

as the quadruple point at infinity is not ordinary and has the δ-invariant equal to 8. The rational parametrization can now be found

$$x(\lambda) = \frac{p_1 + p_2\lambda^2}{\lambda^2 + 1},$$
$$y(\lambda) = {p_3}^{3/2}(p_1 - p_2)^3 \frac{\lambda^3}{(\lambda^2 + 1)^3} - Q(x(\lambda)).$$

Eliminating the parameters $(p_1, p_2, p_3, q_0, \ldots, q_3)$ between (21) and its six derivatives yields the seventh-order ODE characterizing the sextic (21)

$$\frac{\mathrm{d}^7 y}{\mathrm{d}x^7} = \frac{21}{5}\frac{ut}{s} - \frac{84}{25}\frac{t^3}{s^2},$$

which is the ODE (20) from Theorem 5.2.

Using the prescription (8) we find that the conformal structure and the associated 3-form are represented by (9) and (12) with

$$\begin{aligned}
\theta^1 &= -2\Omega \sum_{\alpha=0}^{3} (p_2)^\alpha \mathrm{d}q_\alpha, \quad \theta^7 = -2\Omega \sum_{\alpha=0}^{3} (p_1)^\alpha \mathrm{d}q_\alpha, \\
\theta^2 &= -\frac{\Omega}{2}(p_2 - p_1)^2 (p_3)^{3/2} \mathrm{d}p_2, \quad \theta^6 = \frac{\Omega}{2}(p_2 - p_1)^2 (p_3)^{3/2} \mathrm{d}p_1, \\
\theta^3 &= -\frac{\Omega}{15}(3\,\mathrm{d}q_0 + (2p_2 + p_1)\,\mathrm{d}q_1 + (2p_1 p_2 + (p_2)^2)\,\mathrm{d}q_2 + 3p_1(p_2)^2\,\mathrm{d}q_3), \\
\theta^5 &= -\frac{\Omega}{15}(3\,\mathrm{d}q_0 + (2p_1 + p_2)\,\mathrm{d}q_1 + (2p_1 p_2 + (p_1)^2)\,\mathrm{d}q_2 + 3p_2(p_1)^2\,\mathrm{d}q_3), \\
\theta^4 &= -\frac{3\Omega}{20}(p_2 - p_1)^2 \sqrt{p_3}\,\mathrm{d}(p_3(p_2 - p_1)),
\end{aligned} \tag{22}$$

where $\Omega = (p_1 - p_2)^{-12/5}(p_3)^{-9/10}$.

This conformal G_2 structure can be analytically continued to Riemannian signature: Setting $p_2 = p$, $p_1 = \bar{p}$ where $p \in \mathbb{C}$ and keeping $(q_0, q_1, q_2, q_3, p_3)$ real gives purely imaginary θ^4 and

$$\theta^7 = \overline{\theta^1}, \quad \theta^6 = -\overline{\theta^2}, \quad \theta^3 = \overline{\theta^5}.$$

The corresponding conformal structure is positive definite and the 3-form ϕ is real. It gives rise to a closed (in a sense of decomposition (14)) G_2 structure as $\mathrm{d}\phi = 0$, $\mathrm{d} * \phi = -\tau_2 \wedge \phi$ in agreement with Theorem 5.2. This theorem also implies that up to diffeomorphisms this is the only closed G_2 structure arising from ODEs.

EXAMPLE 6.2 Consider a rational curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bidegree $(1, k)$

$$y = \frac{r_0 + r_1 x + \cdots + r_k x^k}{s_0 + s_1 x + \cdots + s_k x^k}.$$

It has self-intersection number $2k$, and the enumerator of the perturbed curve (section of a normal bundle) δy defines the conformal structure (7) with

$$\theta^{i+1} = \binom{2k}{i}^{-1} \sum_{\alpha+\beta=i} (r_\alpha\,\mathrm{d}s_\beta - s_\beta\,\mathrm{d}r_\alpha), \quad i = 0, \ldots, 2k.$$

This conformal structure is defined on a hypersurface where the resultant of the denominator and enumerator in y has a non-zero fixed value. Alternatively we can fix the ambiguity by choosing affine coordinates, say $r_k = 1$. Now restrict to the seven-dimensional case $k = 3$. This also gives $\phi \wedge d\phi = 0$. The corresponding seventh-order ODE is

$$\frac{\mathrm{d}^7 y}{\mathrm{d}x^7} = \frac{P}{Q},$$

where

$$
\begin{aligned}
P &= 420q^2u^2 + 2520qst^2 - 1680qrut - 2100qs^2u - 504pt^3 \\
&\quad + 1680r^2t^2 - 6300trs^2 + 840tups + 2625s^4 - 280u^2rp + 2800ur^2s, \\
Q &= 360q^2t - 1200rqs - 240rtp + 800r^3 + 300s^2p.
\end{aligned}
$$

This example has a six-dimensional group of point symmetries, given by the Möbius transformations of x and y.

EXAMPLE 6.3 We can construct less trivial conformal structures and the associated 3-forms by generalizing the last example, and taking a double covering of a neighbourhood of a non-singular curve of bidegree (1, 6) branched along a fixed curve. Consider a (1, 6) curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$

$$
y = \frac{R(x)}{S(x)}, \tag{23}
$$

where

$$
S = s_0 + s_1x + \cdots + s_6x^6, \quad R = r_0 + r_1x + \cdots + r_6x^6.
$$

This curve has normal bundle $\mathcal{O}(12)$, and is parametrized by $\mathbb{CP}^{13}$ minus a hypersurface where both polynomials have a common factor. We take the branch locus to be the (1, 6) curve

$$
y = x^6.
$$

The curves in a covering space we are constructing project to those curves (23) which meet the branch locus in seven points to second-order. Thus

$$
x^6S(x) - R(x) = (t_0 + t_1x + \cdots + t_6x^6)^2.
$$

This gives 13 conditions on 20 coefficients (s, r, t), leaving the seven-dimensional moduli space of curves.

EXAMPLE 6.4 In [**10**] it was shown that the moduli space of solutions to the ODE

$$
\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} = \left(\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right)^{(n+1)/n}
$$

admits the GL(2, ℝ) structure. Consider a solution curve $x \to (x, y(x))$ and its perturbation δy

$$
\begin{aligned}
y &= t_1 + t_2x + \cdots + t_nx^{n-1} - \frac{n^n}{(n-1)!}\ln(x + t_{n+1}), \\
\delta y &= \frac{1}{x + t_{n+1}}\left(\left(-\frac{n^n}{(n-1)!}\delta t_{n+1} + t_{n+1}\delta t_1\right) + \sum_{i=1}^{n-1}(\delta t_i + t_{n+1}\delta t_{i+1})x^i + \delta t_n x^n\right).
\end{aligned}
$$

The enumerator of the polynomial δy defines a conformal structure (7) with

$$\theta^1 = -\frac{n^n}{(n-1)!}\,\mathrm{d}t_{n+1} + t_{n+1}\,\mathrm{d}t_1, \quad \theta^{i+1} = \binom{2k}{i}^{-1}(\mathrm{d}t_i + t_{n+1}\,\mathrm{d}t_{i+1}), \quad \theta^{2k+1} = \mathrm{d}t_{2k},$$

where $i = 1, \ldots, 2k-1$. We can now specify $2k = 6$ and construct the 3-form. We find that $\phi \wedge \mathrm{d}\phi = 0$ so that $\lambda = 0$ but there is no conformal scale which makes ϕ closed.

7. Construction of the Cartan connection

We shall now describe the $\mathrm{GL}(2, \mathbb{R})$ and conformal G_2^{split} structures arising from seventh-order ODEs by constructing a $\mathfrak{gl}(2, \mathbb{R})$-valued linear connection on M. The basic object in this description is the torsion, which contains lowest order invariants of the $\mathrm{GL}(2, \mathbb{R})$ geometry, identifies the Fernandez–Gray types of the associated conformal G_2^{split} geometry, and expresses these quantities by contact invariants of the underlying ODE. This approach will give us a better handle on the various torsion components. Our aim is to express these components in terms of invariants of the seventh-order ODE and eventually prove Theorems 5.1 and 5.2. Our treatment of Cartan's connection follows closely that of [13].

We shall use an equivalent form of Definition 1 and regard a $\mathrm{GL}(2, \mathbb{R})$ geometry on a manifold M as a reduction of the frame bundle FM to a $\mathrm{GL}(2, \mathbb{R})$-sub-bundle, where $\mathrm{GL}(2, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$ acts irreducibly in each tangent space [13]. We shall focus on the case $n = 6$ where

$$\mathrm{GL}(2, \mathbb{R}) \subset \mathbb{R}^+ \times G_2^{\mathrm{split}} \subset \mathbb{R}^+ \times \mathrm{SO}(3, 4)$$

holds (see Proposition 4.1). The central role will be played by the six-jet space J^6 and its description via the Tanaka–Morimoto theory, [18, 25], which is a special version of Cartan's method of equivalence. We shall first construct a $\mathfrak{gl}(2, \mathbb{R}) \oplus_{\cdot} \mathbb{R}^7$-valued Cartan connection Ω on a bundle over J^6 and then re-interpret Ω from the point of view of the $\mathrm{GL}(2, \mathbb{R})$ structure. The conditions for the existence of the geometry appear to be certain linear conditions for the curvature of Ω. If they are satisfied, then the $\mathfrak{gl}(2, \mathbb{R})$-part of Ω is the desired linear connection on M.

7.1. *Jet space*

Let us consider the space J^6 of six-jets of functions from $\mathbb{R}$ to $\mathbb{R}$. It is an eight-dimensional real manifold, locally parametrized by $(x, y, y_1, \ldots, y_6)$, and such that each curve $x \mapsto (x, f(x))$ in the xy-space has a unique lift to a curve in J^6 given by $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$ in the above coordinate system. This gives a distinguished family of all curves lifted from the xy-space in J^6. One may encode this family in a coordinate-free language of distributions. Let us fix a point $w \in J^6$ and consider all lifted curves through w. The linear span of their tangent vectors at w is a two-dimensional subspace C_w in $T_w J^6$. The collection $C = \bigcup_w C_w$ is by definition *the contact distribution* on J^6. It is generated by two vector fields

$$\mathfrak{D} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + \ldots + y_6\partial_{y_5} \quad \text{and} \quad \partial_{y_6}.$$

Given the distribution C we define

$$\partial C = [C, C], \quad \partial^2 C = [\partial C, C], \ldots, \partial^5 C = [\partial^4 C, C], \quad \partial^6 C = TJ^6.$$

The distributions constitute a filtration, that is

$$C \subset \partial C \subset \ldots \subset \partial^5 C \subset \partial^6 C = TJ^6 \tag{24}$$

and

$$[\partial^i C, \partial^j C] = \partial^{i+j+1} C. \tag{25}$$

The diffeomorphisms of J^6 which preserve C are called contact transformations. The well-known Lie–Bäcklund theorem states that all contact transformations of J^6 are uniquely defined by the contact transformations of J^1 and have the following form

$$x \longmapsto \bar{x}(x, y, y_1), \quad y \longmapsto \bar{y}(x, y, y_1), \quad y_1 \longmapsto \bar{y}_1(x, y, y_1), \tag{26}$$

and for higher-order jet coordinates

$$y_{k+1} \longmapsto \frac{\mathfrak{D}\bar{y}_k}{\mathfrak{D}\bar{x}}, \quad k = 1, 2, \ldots, n.$$

The functions $\bar{x}$, $\bar{y}$ and $\bar{y}_1$ in (26) are not arbitrary but subject to the condition

$$\bar{y}_1 = \frac{\mathfrak{D}\bar{y}}{\mathfrak{D}\bar{x}}.$$

The contact transformation preserves the whole filtration (24).

Now consider the seventh-order ODE (18). Any solution $y = f(x)$ of the equation is uniquely defined by a choice of $f(x_0), f'(x_0), \ldots, f^{(6)}(x_0)$ at some x_0. Since this choice of initial data is equivalent to a choice of a point in J^6 there exists exactly one lifted curve $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$ through any point of J^6. Therefore the solutions form a one-dimensional foliation in J^6. The corresponding tangent distribution is spanned by

$$\mathcal{D} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + \ldots + y_6\partial_{y_5} + F\partial_{y_6}.$$

An important consequence of this is that $J^6 \to M$ is locally a line bundle, where M is the solution space of the seventh-order ODE (18).

Definition 7.1 The contact geometry of seventh-order ODEs is the jet space J^6 equipped with

(i) the filtration $C \subset \ldots \subset \partial^5 C \subset \partial^6 C = TJ^6$;
(ii) the foliation by the solutions, tangent to the field $\mathcal{D}$.

One may associate to the contact geometry of the ODEs a sub-bundle

$$\tilde{G} \longrightarrow \tilde{P} \longrightarrow J^6 \tag{27}$$

of the frame bundle FJ^6: the structure group $\tilde{G}$ is the lower triangular group preserving the filtration and the 1-distribution span$\{\mathcal{D}\} \subset C$ tangent to solutions.

7.2. *Cartan connection*

The main object we use in the construction is Cartan connection defined here as in [15].

DEFINITION 7.2 Let M be a manifold of dimension n, G be a Lie group, H be a closed subgroup of G with $\dim G/H = n$ and $H \to P \xrightarrow{\pi} M$ be a principal bundle. A Cartan connection of type (G, H) on P is a 1-form Ω with values in the Lie algebra $\mathfrak{g}$ of G satisfying the following conditions:

(i) $\Omega_u : T_u P \to \mathfrak{g}$ for every $u \in P$ is an isomorphism of vector spaces;
(ii) $\Omega(A^*) = A$ for every $A \in \mathfrak{h}$ and the corresponding fundamental field A^*;
(iii) $R_h^* \Omega = \mathrm{Ad}(h^{-1})\Omega$ for every $h \in H$.

The curvature of a Cartan connection is a $\mathfrak{g}$-valued 2-form on P defined by

$$K(X, Y) = \mathrm{d}\Omega(X, Y) + \tfrac{1}{2}[\Omega(X), \Omega(Y)].$$

If Ω is given in a matrix representation, then

$$K = \mathrm{d}\Omega + \Omega \wedge \Omega. \tag{28}$$

The curvature is *horizontal*, that is it vanishes on each vertical vector field:

$$K(X, \cdot) = 0 \quad \text{if } \pi_*(X) = 0. \tag{29}$$

Horizontality of the curvature is locally equivalent to the property (iii) in Definition 7.2. Cartan connections with vanishing curvature are called flat.

We are now in a position to describe the construction of $\mathrm{GL}(2, \mathbb{R})$ geometry on the solution space. We start from the bundle $\tilde{P}$ of the contact geometry of ODEs. We are interested in invariants of this geometry. The filtration is preserved by the contact transformations but the foliation of solutions is not, and generates the contact invariants of the underlying ODE. However, the situation further is complicated by the fact that the object generating the invariants—a Cartan connection—exists on a sub-bundle $P \subset \tilde{P}$ rather than $\tilde{P}$ itself. Using the Tanaka–Morimoto theory we shall construct the sub-bundle $H \to P \to J^6$ together with a Cartan connection Ω of type $(\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^7, H)$, where H is isomorphic to the group of triangular 2×2 matrices. The curvature K of Ω contains all the local information about the contact geometry of the ODEs. The contact invariants are either components of K or certain combinations of their derivatives of sufficiently high order.

The jet space J^6 is a bundle over the solution space M and $P \to M$ is also a principal bundle with the structure group $\mathrm{GL}(2, \mathbb{R})$. That Ω generates the $\mathrm{GL}(2, \mathbb{R})$ geometry on M only if certain conditions (which we will determine) hold. First of all, we ask whether Ω (which is a Cartan connection on $P \to J^6$) satisfies the conditions for the Cartan connection of $P \to M$. It holds if and only if

$$K(X, \cdot) = 0 \qquad \text{for all } X \text{ vertical with respect to } P \longrightarrow M.$$

This condition is not satisfied automatically but only holds for the ODEs with vanishing Wünschmann invariants (Appendix A).

The Cartan connection Ω on $P \to M$ is of type $(\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^7, \mathrm{GL}(2, \mathbb{R}))$. It naturally decomposes into the $\mathbb{R}^7$-part and the $\mathfrak{gl}(2, \mathbb{R})$-part. The former behaves like a canonical form θ on a principal bundle and turns P into a sub-bundle of the frame bundle FM. The latter is a linear $\mathfrak{gl}(2,\mathbb{R})$-valued connection Γ on P. Together θ and Γ define a $GL(2,\mathbb{R})$ geometry on M. The torsion T and curvature of Γ contain the information about local invariants of the geometry, which are in turn expressed by contact invariants of the underlying ODE, since Ω also describes the contact geometry of the ODEs.

EXAMPLE 7.3 For the trivial equation $y^{(7)} = 0$, all the objects may be immediately constructed by means of the symmetry group. The full group of contact symmetries is $GL(2,\mathbb{R}) \ltimes \mathbb{R}^7$. Its action on J^6 is transitive and turns it into a homogeneous space $GL(2,\mathbb{R}) \ltimes \mathbb{R}^7/H$, where H is isomorphic to the group of triangular 2×2 matrices. Thus we have the bundle $H \to P \to J^6$ and $P = GL(2,\mathbb{R}) \ltimes \mathbb{R}^7$ locally. The connection Ω, flat in this case, is given by the Maurer–Cartan form on P.

8. The Tanaka–Morimoto theory

We turn to detailed description of the construction. First of all, we briefly describe the general pattern, next we apply it to our case. The references for this subsection are [**8**, **17**, **18**, **24**, **25**]. The contact geometry of ODEs contains the filtration (24) which is encoded by the graded tangent bundle $\mathrm{gr}\, TJ^6$, denoted here by gr for short. Its fibre over $w \in J^6$ is $\mathrm{gr}(w) = \bigoplus_{i=1}^{7} \mathrm{gr}_{-i}(w)$, where

$$\mathrm{gr}_{-1}(w) = C_w, \quad \mathrm{gr}_{-2}(w) = \partial C_w/C_w, \ldots, \mathrm{gr}_{-6}(w) = \partial^5 C_w/\partial^4 C_w, \quad \mathrm{gr}_{-7}(w) = T_w J^6/\partial^5 C_w.$$

The relation (25) implies that $\mathrm{gr}(w)$ carries the structure of a nilpotent graded Lie algebra, that is

$$[\mathrm{gr}_{-i}(w), \mathrm{gr}_{-j}(w)] \subset \mathrm{gr}_{-i-j}(w), \quad \text{and } \mathrm{gr}(w) \text{ is generated by } \mathrm{gr}_{-1}(w).$$

Let

$$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \ldots \oplus \mathfrak{g}_{-7},$$

where $\mathrm{gr}_{-i} \cong \mathfrak{g}_{-i}$. We have $\dim \mathfrak{m} = \dim \mathrm{gr}(w) = \dim T_w J^6 = 8$ and

$$[\mathfrak{g}_{-i}, \mathfrak{g}_{-j}] \subset \mathfrak{g}_{-i-j}.$$

The additional piece of structure—the distribution $\mathrm{span}\{\mathcal{D}\}$—is encoded in the following manner. One defines a weighted frame z_w at $w \in J^6$ to be an isomorphism of graded Lie algebras $z_w : \mathfrak{m} \to \mathrm{gr}(w)$. The bundle of weighted frames $\mathcal{R}J^6$ is a principal bundle over J^6 with the structure group $G_0(\mathfrak{m})$ being the group of all grading preserving algebra automorphisms of $\mathfrak{m}$.

The vector $\mathcal{D}_w$ at any w belongs to C_w and is complementary to the one-dimensional subspace of C_w which is vertical with respect to $J^6 \to J^5$ and spanned by ∂_{y_6}. At the level of $\mathfrak{m}$ it is reflected by a decomposition of $\mathfrak{g}_{-1}$ into two one-dimensional subspaces. These subspaces, call them D and V for short, are then encoded by reducing $\mathcal{R}J^6$ to a G_0-sub-bundle, where G_0 is the two-dimensional subgroup of $G_0(\mathfrak{m})$ preserving the decomposition $\mathfrak{g}_{-1} = D \oplus V$.

However, the G_0-sub-bundle still is not the bundle P where the connection Ω exists. In order to construct P one needs to prolong the G_0-bundle. The procedure of prolongation is quite involved and the reader is referred to the original paper [**24**]. The underlying idea is, however, simple. One aims to extend the G_0-bundle so that it is large enough to contain all the symmetries in the most symmetric homogeneous case. From the example of the trivial ODE we know that the total space P must be 11-dimensional, so one dimension is lacking. After the prolongation one obtains the desired $H \to P \to J^6$ with the structural group H being a product of G_0 and the one-dimensional prolongation, isomorphic to the group of triangular 2×2 matrices.

At the algebraic level the filtration is encoded by $\mathfrak{m}$ and the full (prolonged) structural group H is encoded by its algebra $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Since commutators $[\mathfrak{m}, \mathfrak{h}]$ are known from the construction, we obtain a graded algebra

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}_{-7} \oplus \mathfrak{g}_{-6} \oplus \ldots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{gl}(2, \mathbb{R}) \oplus_{.} \mathbb{R}^7. \tag{30}$$

The Cartan connection Ω takes values in this algebra.

The next step is constructing the form Ω using the normality conditions of Tanaka and Morimoto. The normality conditions, which are certain linear constraints for the curvature, were originally introduced by Cartan in the context of conformal and projective geometries. The purpose was fixing ambiguity in the choice of Cartan connections and providing canonical connections for these geometries in a sense analogous to the Levi-Civita connection in Riemannian geometry. Later, these conditions were generalized to the case of the filtered manifolds. We discuss them below.

The connection 1-form at $p \in P$ is a vector space isomorphism $\Omega_p : T_pP \to \mathfrak{g}$. We define $\mathcal{V}_p = \Omega_p^{-1}(\mathfrak{h})$ and $\mathcal{H}_p = \Omega_p^{-1}(\mathfrak{m})$, hence $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$. The curvature K_p is then characterized by a tensor $\kappa_p \in \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ given by

$$\kappa_p(A, B) = K_p(\Omega_p^{-1}(A), \Omega_p^{-1}(B)), \quad A, B \in \mathfrak{m}. \tag{31}$$

In the space $\mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ let us define $\mathrm{Hom}^1(\wedge^2\mathfrak{m}, \mathfrak{g})$ to be the space of all $\alpha \in \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ fulfilling

$$\alpha(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}_{i+j+1} \oplus \ldots \oplus \mathfrak{g}_k \quad \text{for } i, j < 0.$$

The algebra $\mathfrak{g}$ is equipped with the following complex

$$\cdots \xrightarrow{\partial} \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g}) \xrightarrow{\partial} \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g}) \xrightarrow{\partial} \cdots$$

with $\partial\colon\ \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g})$ given by

$$\begin{aligned}(\partial^*\alpha)(A_1\wedge \ldots \wedge A_{q+1}) = &\sum_i (-1)^{i+1}[A_i, \alpha(A_1\wedge \ldots \wedge \hat{A}_i\wedge \ldots \wedge A_{q+1})] \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([A_i, A_j]\wedge A_1 \ldots \wedge \hat{A}_i\wedge \ldots \wedge \hat{A}_j\wedge \ldots \wedge A_{q+1}),\end{aligned}$$

where $\alpha \in \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g})$ and $A_1, \ldots A_{q+1} \in \mathfrak{m}$.

Consider a positive definite scalar product $(\cdot, \cdot)$ in $\mathfrak{g}$ satisfying the following three conditions:

(i) $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for $i \neq j$;
(ii) there exists a mapping $\tau : \mathfrak{h} \to \mathfrak{g}$ such that

$$\begin{aligned}&\tau(\mathfrak{g}_i) \subset \mathfrak{g}_{-i} \quad \text{for } i \geq 0, \\ &([A, X], Y) = (X, [\tau(A), Y]) \quad \text{for } X, Y \in \mathfrak{g},\ A \in \mathfrak{h};\end{aligned} \tag{32}$$

(iii) there exists a mapping $\tau_0 : G_0 \to G_0$ such that

$$(aX, Y) = (X, \tau_0(a)Y) \quad \text{for } X, Y \in \mathfrak{g},\ a \in G_0.$$

This product extends to $\mathrm{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ through

$$(\alpha, \beta) = \frac{1}{q!} \sum_{i_1,\ldots,i_q} (\alpha(v_{i_1} \wedge \ldots \wedge v_{i_q}), \beta(v_{i_1} \wedge \ldots \wedge v_{i_q})),$$

where $\alpha, \beta \in \mathrm{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g})$ and (v_i) is any orthonormal basis of $\mathfrak{g}$. Given ∂ and $(\cdot, \cdot)$ the formal adjoint operator

$$\cdots \xrightarrow{\partial^*} \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g}) \xrightarrow{\partial^*} \mathrm{Hom}(\wedge^q \mathfrak{m}, \mathfrak{g}) \xrightarrow{\partial^*} \cdots$$

is defined by

$$(\partial^* \alpha, \beta) = (\alpha, \partial\beta).$$

A normal connection is defined as follows.

DEFINITION 8.1 A Cartan connection Ω is normal if κ given by (31) satisfies the following conditions:

(i) $\kappa \in \mathrm{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g})$;
(ii) $\partial^* \kappa = 0$.

By a general result of Morimoto [**18**, Theorem 2.3 and Proposition 2.10], given an inner product satisfying the three properties (32) one can construct the normal Cartan connection which preserves the contact equivalence of the underlying ODEs.

9. Application to seventh-order ODEs

Define seven 1-forms on J^6 by

$$\omega^i = \mathrm{d}y_{i-1} - y_i \,\mathrm{d}x, \quad \omega^7 = \mathrm{d}y_6 - F\,\mathrm{d}x, \quad i = 1, \ldots, 6. \tag{33}$$

These forms encode the geometry of an ODE, and in particular C is annihilated by the ideal $\mathrm{span}\{\omega^1, \omega^2, \ldots, \omega^6\}$ and the foliation by solutions is annihilated by $\mathrm{span}\{\omega^1, \omega^2, \ldots, \omega^7\}$. On $\tilde{P}$ there is the fundamental $\mathbb{R}^8$-valued 1-form, whose components are denoted by $\theta^1, \ldots, \theta^7$ and Γ_+. (The notation Γ_+ instead of θ^8 will be useful later on.) One may introduce a coordinate system $(x, y, y_1, \ldots, y_6, u_1, u_2, \ldots, u_{36})$ compatible with the local trivialization $\tilde{P} \cong J^6 \times \tilde{G}$ and such that locally

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \\ \theta^7 \\ \Gamma_+ \end{pmatrix} = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 & 0 & 0 & 0 & 0 \\ u_7 & u_8 & u_9 & u_{10} & 0 & 0 & 0 & 0 \\ u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & 0 & 0 & 0 \\ u_{16} & u_{17} & u_{18} & u_{19} & u_{20} & u_{21} & 0 & 0 \\ u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} & u_{28} & 0 \\ u_{29} & u_{30} & u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & u_{36} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \\ \mathrm{d}x \end{pmatrix}. \tag{34}$$

The structural group $\tilde{G}$ is the group of the lower triangular matrices as above.

We choose a representation of $\mathfrak{gl}(2,\mathbb{R})\oplus_{\cdot}\mathbb{R}^7$ and write down Ω in the following matrix form

$$\Omega=\begin{pmatrix}-6\Gamma_0-6\Gamma_1 & 6\Gamma_+ & 0 & 0 & 0 & 0 & 0 & \theta^1\\ \Gamma_- & -4\Gamma_0-6\Gamma_1 & 5\Gamma_+ & 0 & 0 & 0 & 0 & \theta^2\\ 0 & 2\Gamma_- & -2\Gamma_0-6\Gamma_1 & 4\Gamma_+ & 0 & 0 & 0 & \theta^3\\ 0 & 0 & 3\Gamma_- & -6\Gamma_1 & 3\Gamma_+ & 0 & 0 & \theta^4\\ 0 & 0 & 0 & 4\Gamma_- & 2\Gamma_0-6\Gamma_1 & 2\Gamma_+ & 0 & \theta^5\\ 0 & 0 & 0 & 0 & 5\Gamma_- & 4\Gamma_0-6\Gamma_1 & \Gamma_+ & \theta^6\\ 0 & 0 & 0 & 0 & 0 & 6\Gamma_- & 6\Gamma_0-6\Gamma_1 & \theta^7\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{35}$$

Here $\theta^1,\dots,\theta^7,\Gamma_+,\Gamma_0,\Gamma_1$ and Γ_- are 1-forms on P.

Starting from this representation we construct a basis (e_μ), $\mu=1,\dots,11$ of $\mathfrak{gl}(2,\mathbb{R})\oplus_{\cdot}\mathbb{R}^7$. To get the element e_1 we formally set $\theta^1=1$ and the remaining 1-forms equal to zero. All the remaining elements of the basis can be obtained in an analogous way, so that (35) may be written as

$$\Omega=\sum_{i=1}^{7}\theta^i e_i+\Gamma_+e_8+\Gamma_0e_9+\Gamma_1e_{10}+\Gamma_-e_{11}. \tag{36}$$

The basis satisfies

$$\begin{aligned}&\mathfrak{g}_{-7}=\mathrm{span}\{e_1\},\quad \mathfrak{g}_{-6}=\mathrm{span}\{e_2\},\quad \mathfrak{g}_{-5}=\mathrm{span}\{e_3\},\\ &\mathfrak{g}_{-4}=\mathrm{span}\{e_4\},\quad \mathfrak{g}_{-3}=\mathrm{span}\{e_5\},\quad \mathfrak{g}_{-2}=\mathrm{span}\{e_6\},\\ &\mathfrak{g}_{-1}=\mathrm{span}\{e_7,e_8\},\quad \mathfrak{g}_{0}=\mathrm{span}\{e_9,e_{10}\},\quad \mathfrak{g}_{1}=\mathrm{span}\{e_{11}\},\end{aligned}$$

and, moreover,

$$\mathfrak{gl}(2,\mathbb{R})=\mathrm{span}\{e_8,\dots,e_{11}\},\quad \mathbb{R}^7=\mathrm{span}\{e_1,\dots,e_7\}.$$

To construct P and Ω we need to:

(i) find a scalar product satisfying the conditions (32);
(ii) find formulae of $P\hookrightarrow\tilde{P}$ by expressing $u_4,\dots,u_{36}$ as certain functions of $u_1,u_2,u_3,x,y,y_1,\dots,y_6$; then $(u_1,u_2,u_3,x,y,y_1,\dots,y_6)$ is a local coordinate system in P and the forms $\theta^1,\dots,\theta^7,\Gamma_+$ of (35) are given by the pull-back of (34);
(iii) find formulae for Γ_-,Γ_0 and Γ_1.

We choose a scalar product on $\mathfrak{g}$ so that the basis $(e_1,\dots,e_{11})$ is orthogonal and

$$\begin{aligned}&(e_1,e_1)=1,\quad (e_2,e_2)=6,\quad (e_3,e_3)=15,\\ &(e_4,e_4)=20,\quad (e_5,e_5)=15,\quad (e_6,e_6)=6,\\ &(e_7,e_7)=1,\quad (e_8,e_8)=1,\quad (e_9,e_9)=2,\\ &(e_{10},e_{10})=1,\quad (e_{11},e_{11})=1.\end{aligned}$$

The product satisfies the conditions (32) if we set $\tau_0=\mathrm{id}$, $\tau(e_9)=e_9$, $\tau(e_{10})=e_{10}$ and $\tau(e_{11})=e_8$.

Both (ii) and (iii) are obtained from the horizontality condition (29) and the normality conditions of Definition 8.1 with the scalar product as above. The 1-forms Γ_0, Γ_1 and Γ_+ on P are a priori arbitrary

$$\Gamma_A = \sum_{j=1}^{3} a^j{}_A \mathrm{d}u_j + \sum_{i=1}^{7} b^i{}_A \theta^i + b^+_A \Gamma_+ \quad A = -, 0, 1.$$

The functions a and b are arbitrary but sufficiently smooth on P, so they depend on the jet coordinates and u_1, u_2, u_3.

The curvature (28) becomes

$$K = \sum_{\mu=1}^{11} \sum_{j=1}^{7} K^{\mu}_{8j} \Gamma_+ \wedge \theta^j \otimes e_\mu + \frac{1}{2} \sum_{\mu=1}^{11} \sum_{i,j=1}^{7} K^{\mu}_{ij} \theta^i \wedge \theta^j \otimes e_\mu. \tag{37}$$

and $K^\rho_{\mu\nu} = -K^\rho_{\nu\mu}$, The terms proportional to Γ_0, Γ_1 and Γ_- must be absent since K is horizontal. This produces a set of first-order differential equations for the functions a, which may be determined without ambiguity giving

$$\begin{aligned}
\Gamma_0 =& \frac{1}{2}\frac{\mathrm{d}u_1}{u_1} - \frac{1}{2}\frac{\mathrm{d}u_3}{u_3} + \sum_i b^i{}_0 \theta^i + b^+_0 \Gamma_+, \\
\Gamma_1 =& -\frac{1}{3}\frac{\mathrm{d}u_1}{u_1} + \frac{1}{2}\frac{\mathrm{d}u_3}{u_3} + \sum_i b^i{}_1 \theta^i + b^+_1 \Gamma_+, \\
\Gamma_- =& \frac{\mathrm{d}u_2}{u_1} + \frac{u_2 \mathrm{d}u_3}{u_1 u_3} + \sum_i b^i{}_- \theta^i + b^+_- \Gamma_+.
\end{aligned} \tag{38}$$

The tensor κ is equal to

$$\kappa = \frac{1}{2} \sum_{\mu=1}^{11} \sum_{i,j=1}^{8} K^{\mu}_{ij} e^i \wedge e^j \otimes e_\mu.$$

The condition $\kappa \in \mathrm{Hom}^1(\wedge^2 \mathfrak{m}, \mathfrak{g})$ is equivalent to vanishing of the following components of K.

$$\begin{matrix}
K^{1,2,3,4,5}_{67} & K^{1,2,3,4}_{57} & K^{1,2,3}_{47} \\
K^{1,2}_{37} & K^{1}_{27} & K^{1,2,3}_{56} \\
K^{1,2}_{46} & K^{1}_{36} & K^{1}_{45} \\
K^{1,2,3,4,5,6}_{87} & K^{1,2,3,4,5}_{86} & K^{1,2,3,4}_{85} \\
K^{1,2,3}_{84} & K^{1,2}_{83} & K^{1}_{82},
\end{matrix} \tag{39}$$

where $K^{1,2}_{37}$ is an abbreviation for K^1_{37}, K^2_{37} and so on.

In order to evaluate the condition $\partial^* \kappa = 0$ we introduce the notation $(e_\mu, e_\mu) = p_{\mu\mu}$ (no summation) and $[e_\mu, e_\nu] = c^\rho_{\mu\nu} e_\rho$. The explicit form of $\partial^* \kappa = 0$ is

$$4 \sum_{\nu=1}^{11} \sum_{j=1}^{8} \frac{p_{\nu\nu}}{p_{ii} p_{jj}} K^{\nu}_{ij} c^{\nu}{}_{j\mu} + \sum_{j,k=1}^{8} \frac{p_{\mu\mu}}{p_{jj} p_{kk}} K^{\mu}_{jk} c^{i}{}_{jk} = 0, \tag{40}$$

where $\mu = 1, \ldots, 11$ and $i = 1, \ldots, 8$.

We compute K via (33), (35) and (38). The conditions (37) and (39) become a set of easy algebraic and differential equations for the functions $u_4, \dots, u_{36}$ and b. By solving these equations we obtain $u_4, \dots, u_{36}$ and b as rational functions of u_1, u_2, u_3 with coefficients given by arbitrary functions of the jet coordinates. After these substitutions the normality condition (40) becomes a set of algebraic and differential equations on the coefficients. The equations, although complicated, are overdetermined and may be solved without integration. It is enough to perform usual algebraic elimination of the functions, provided it is done in an appropriate order. The elimination also assures us that the solution—the Cartan connection—is unique. We have therefore proved

PROPOSITION 9.1 *Given a seventh-order* ODE $y^{(7)} = F(x, y, y', \dots, y^{(6)})$ *one can construct*:

(i) a principal fibre bundle $H \to P \to J^6$, *where* $H = \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R})$;
(ii) a Cartan connection Ω *on* P *of type* $(\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^7, H)$.

Two ODE*s*

$$y^{(7)} = F(x, y, y', \dots, y^{(6)})$$

and

$$\bar{y}^{(7)} = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(6)})$$

are locally contact equivalent if and only if there exists a local bundle diffeomorphism $\Phi \colon \bar{P} \to P$ *such that* $\Phi^*\Omega = \bar{\Omega}$. *The connection is given by* (35), *where*

$$\begin{aligned}
\theta^1 &= u_1\omega^1,\\
\theta^2 &= u_2\omega^1 + u_3\omega^2,\\
\theta^3 &= \frac{u_2^2}{u_1}\omega^1 + 2\frac{u_2u_3}{u_1}\omega^2 + \frac{u_3^2}{u_1}\left(\left(\frac{3}{14}(\mathcal{D}F)_6 - \frac{12}{35}F_5 - \frac{13}{49}F_6^2\right)\omega^1 - \frac{2}{35}F_6\omega^2 + \frac{6}{5}\omega^3\right),\\
\theta^4 &= \dots .
\end{aligned}$$

The explicit formulae for $\theta^i,\ i = 4, 5, 7,$ *and* $\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1$ *are omitted since they are complicated and unilluminating.*

10. GL(2, $\mathbb{R}$) geometry from Cartan connection

The manifold P is endowed with two structures of a principal bundle: $H \to P \to J^6$ given by construction, and $\mathrm{GL}(2, \mathbb{R}) \to P \to M$ over the solution space which is generated by the connection Ω. Let X_μ, $\mu = 1, \dots, 11$, denote the frame dual to the coframe $(\theta^2, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^7, \Gamma_+, \Gamma_0, \Gamma_1, \Gamma_-)$ of (35). The curvature K written in the form

$$\mathrm{d}\Omega = -\Omega \wedge \Omega + K, \tag{41}$$

and split into scalar-valued equations reads

$$
\begin{aligned}
\mathrm{d}\theta^1 &= 6(\Gamma_1+\Gamma_0)\wedge\theta^1 - 6\Gamma_+\wedge\theta^2 + \tfrac{1}{2}K^1{}_{ij}\theta^i\wedge\theta^j,\\
\mathrm{d}\theta^2 &= -\Gamma_-\wedge\theta^1 + (6\Gamma_1+4\Gamma_0)\wedge\theta^2 - 5\Gamma_+\wedge\theta^3 + \tfrac{1}{2}K^2{}_{ij}\theta^i\wedge\theta^j,\\
\mathrm{d}\theta^3 &= -2\Gamma_-\wedge\theta^2 + (6\Gamma_1+2\Gamma_0)\wedge\theta^3 - 4\Gamma_+\wedge\theta^4 + \tfrac{1}{2}K^3{}_{ij}\theta^i\wedge\theta^j\\
&\quad + K^3{}_{18}\theta^1\wedge\Gamma_+,\\
\mathrm{d}\theta^4 &= -3\Gamma_-\wedge\theta^3 + 6\Gamma_1\wedge\theta^4 - 3\Gamma_+\wedge\theta^5 + \tfrac{1}{2}K^4{}_{ij}\theta^i\wedge\theta^j\\
&\quad + (K^4{}_{18}\theta^1 + K^4{}_{28}\theta^2)\wedge\Gamma_+,\\
\mathrm{d}\theta^5 &= -4\Gamma_-\wedge\theta^4 + (6\Gamma_1-2\Gamma_0)\wedge\theta^5 - 2\Gamma_+\wedge\theta^6 + \tfrac{1}{2}K^5{}_{ij}\theta^i\wedge\theta^j\\
&\quad + (K^5{}_{18}\theta^1 + K^5{}_{28}\theta^2 + K^5{}_{38}\theta^3)\wedge\Gamma_+,\\
\mathrm{d}\theta^6 &= -5\Gamma_-\wedge\theta^5 + (6\Gamma_1-4\Gamma_0)\wedge\theta^6 - \Gamma_+\wedge\theta^7 + \tfrac{1}{2}K^6{}_{ij}\theta^i\wedge\theta^j\\
&\quad + (K^6{}_{18}\theta^1 + K^6{}_{28}\theta^2 + K^6{}_{38}\theta^3 + K^6{}_{48}\theta^4)\wedge\Gamma_+,\\
\mathrm{d}\theta^7 &= -6\Gamma_-\wedge\theta^6 + (6\Gamma_1-6\Gamma_0)\wedge\theta^7 + \tfrac{1}{2}K^7{}_{ij}\theta^i\wedge\theta^j\\
&\quad + (K^7{}_{18}\theta^1 + K^7{}_{28}\theta^2 + K^7{}_{38}\theta^3 + K^7{}_{48}\theta^4 + K^7{}_{58}\theta^5)\wedge\Gamma_+,\\
\mathrm{d}\Gamma_+ &= 2\Gamma_0\wedge\Gamma_+ + \tfrac{1}{2}K^8{}_{ij}\theta^i\wedge\theta^j + K^8{}_{i8}\theta^i\wedge\Gamma_+,\\
\mathrm{d}\Gamma_0 &= \Gamma_+\wedge\Gamma_- + \tfrac{1}{2}K^9{}_{ij}\theta^i\wedge\theta^j + K^9{}_{i8}\theta^i\wedge\Gamma_+,\\
\mathrm{d}\Gamma_1 &= \tfrac{1}{2}K^{10}{}_{ij}\theta^i\wedge\theta^j + K^{10}{}_{i8}\theta^i\wedge\Gamma_+,\\
\mathrm{d}\Gamma_- &= -2\Gamma_0\wedge\Gamma_- + \tfrac{1}{2}K^{11}{}_{ij}\theta^i\wedge\theta^j + K^{11}{}_{i8}\theta^i\wedge\Gamma_+.
\end{aligned}
\tag{42}
$$

Since J^6 is a bundle over M then so is P and Theorem 9.1 together with equation (33) guarantees that the fibres of the projection $P \to M$ are annihilated by the simple ideal $\mathrm{span}\{\theta^1, \dots, \theta^7\}$. The relation (42) implies that this ideal is closed

$$\mathrm{d}\theta^i\wedge\theta^1\wedge\dots\wedge\theta^7 = 0 \quad \text{for } i = 1, \dots, 7.$$

It is annihilated by an integrable distribution $\mathrm{span}\{X_8, X_9, X_{10}, X_{11}\}$ and the maximal integral leaves of this distribution are locally the fibres of the projection $P \to M$. Moreover, by (42) the commutation relations of the vector fields are isomorphic to the commutation relations of the algebra $\mathfrak{gl}(2, \mathbb{R})$. This allows us to *define* an action of $GL(2, \mathbb{R})$ on P by defining X_8, X_9, X_{10}, X_{11} to be the associated fundamental vector fields.

10.1. *Existence of* GL(2, ℝ) *geometry*

Does the bundle $GL(2, \mathbb{R}) \to P \to M$ define a $GL(2, \mathbb{R})$ geometry on M? The answer to this question is positive only if P may be identified with a sub-bundle of the frame bundle FM. However, this may only be done if the original ODE satisfies additional conditions. An object that turns P into a sub-bundle of FM is the canonical $\mathbb{R}^7$-valued 1-form. The $\mathbb{R}^7$-part of Ω (the 1-forms $\theta^1, \dots, \theta^7$ arranged into a column) is the natural candidate for it here, but one must still check whether the canonical 1-form has the property $R_g^*\theta = g^{-1}\theta$ under the actions of $\mathrm{GL}(2,\mathbb{R})$ in P and $\mathbb{R}^7$. In the language of the curvature K this is equivalent to the horizontality with respect to the projection $P \to M$. Since K is already horizontal with respect to $P \to J^6$ we must only impose $K(X_8, \cdot) = 0$ which amounts to

$$K^{\mu}{}_{\nu 8} = 0, \quad \mu, \nu = 1, \dots, 11.$$

Due to algebraic and differential relations among the curvature components this condition may be further reduced to

$$K^3{}_{18} = 0, \quad K^4{}_{18} = 0, \quad K^5{}_{18} = 0, \quad K^6{}_{18} = 0, \quad K^7{}_{18} = 0, \tag{43}$$

where

$$K^{\alpha+2}_{18} = \sum_{\beta=1}^{\alpha} c^{\alpha}{}_{\beta} W_{\beta}, \quad \alpha = 1, \dots, 5.$$

The expressions $W_1, W_2, \dots, W_5$ are the Wünschmann conditions discussed in Section 1 and given by (Appendix A), and $c^{\alpha}{}_{\beta}$ are rational functions of u_1, u_2, u_3. The condition (43) is therefore equivalent to the vanishing of W_α. Simultaneous vanishing of these expressions is a property of a seventh-order ODE invariant under contact transformations. It is also equivalent to the conditions for trivial linearizations obtained in [**6**, **7**].

From now on we restrict our considerations to those ODEs which satisfy all five conditions in Appendix A. Then the curvature contains no $\Gamma_+ \wedge \theta^i$ terms, and the equations (42) may be written as the structural equations for a $\mathfrak{gl}(2,\mathbb{R})$-connection. We have proven

THEOREM 10.1 *Consider a seventh-order* ODE *satisfying the conditions $W_\alpha = 0$, $\alpha = 1, \dots, 5$. Then its solution space M is equipped with a $\mathfrak{gl}(2,\mathbb{R})$ geometry. Let*

$$\Gamma = \begin{pmatrix} -6\Gamma_0 - 6\Gamma_1 & 6\Gamma_+ & 0 & 0 & 0 & 0 & 0 \\ \Gamma_- & -4\Gamma_0 - 6\Gamma_1 & 5\Gamma_+ & 0 & 0 & 0 & 0 \\ 0 & 2\Gamma_- & -2\Gamma_0 - 6\Gamma_1 & 4\Gamma_+ & 0 & 0 & 0 \\ 0 & 0 & 3\Gamma_- & -6\Gamma_1 & 3\Gamma_+ & 0 & 0 \\ 0 & 0 & 0 & 4\Gamma_- & 2\Gamma_0 - 6\Gamma_1 & 2\Gamma_+ & 0 \\ 0 & 0 & 0 & 0 & 5\Gamma_- & 4\Gamma_0 - 6\Gamma_1 & \Gamma_+ \\ 0 & 0 & 0 & 0 & 0 & 6\Gamma_- & 6\Gamma_0 - 6\Gamma_1 \end{pmatrix}. \tag{44}$$

be the $\mathfrak{gl}(2,\mathbb{R})$-part of the Cartan connection Ω of Proposition 9.1 *and $\theta = (\theta^i)$ be its $\mathbb{R}^7$ part. Then Γ is a $\mathfrak{gl}(2,\mathbb{R})$ linear connection on P compatible with the* $\mathrm{GL}(2,\mathbb{R})$ *geometry and the equations* (28) *and* (42) *read*

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = \tfrac{1}{2} T^i{}_{kl} \theta^k \wedge \theta^l, \quad i, j = 1, \dots, 7, \tag{45}$$

$$\mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \tfrac{1}{2} R^i{}_{jlm} \theta^l \wedge \theta^m, \tag{46}$$

where T and R are the torsion and curvature of Γ, *respectively.*

We will construct two tensor fields on M preserved by the GL(2, ℝ) geometry: the conformal metric g and the conformal 3-form ϕ. This is done as follows. The action of $\mathfrak{gl}(2,\mathbb{R})$ on $\mathbb{R}^7$ is given by the matrix representation (44), in particular it defines two of the conformal classes of tensors represented by $g \in S^2\mathbb{R}^{7*}$ and $\phi \in \Lambda^3\mathbb{R}^{7*}$. Next we transport these tensors to T^*P. The connection Ω gives the identification $e^i \leftrightarrow \theta^i$, $i = 1, \ldots, 7$, where (e^i) is dual of the basis (e_i) of (35) and (36). By this identification we get the tensor fields on P:

$$g = \theta^1\theta^7 - 6\theta^2\theta^6 + 15\theta^3\theta^5 - 10(\theta^4)^2$$

and

$$\phi = 3\theta^2\wedge\theta^3\wedge\theta^7 - 6\theta^2\wedge\theta^4\wedge\theta^6 - \theta^1\wedge\theta^4\wedge\theta^7 + 3\theta^1\wedge\theta^5\wedge\theta^6 + 15\theta^3\wedge\theta^4\wedge\theta^5. \tag{47}$$

Finally, we project these fields to conformal fields on M. This projection is well defined because g and ϕ satisfy two conditions: (i) the vertical directions of $P \to M$ are degenerate for g and ϕ, and (ii) the vertical directions are conformal symmetries of g and ϕ, that is

$$\mathcal{L}_{X_8}g = 0, \quad \mathcal{L}_{X_9}g = 0, \quad \mathcal{L}_{X_{10}}g = 12g, \quad \mathcal{L}_{X_{11}}g = 0,$$
$$\mathcal{L}_{X_8}\phi = 0, \quad \mathcal{L}_{X_9}\phi = 0, \quad \mathcal{L}_{X_{10}}\phi = 18\phi, \quad \mathcal{L}_{X_{11}}\phi = 0.$$

It is worth noting that $\mathcal{L}_{X_8}g = \mathcal{L}_{X_8}\phi = 0$ are equivalent to conditions listed in Appendix A. The conformal fields on M will be also denoted by g and ϕ—on solutions to the ODE they coincide with (9) and (12), respectively.

The following fact is an immediate consequence of Theorem 10.1.

Proposition 10.2 *Let ∇ denote the covariant derivative on M associated to Γ. We have*

$$\nabla_X g = -A(X)g,$$
$$\nabla_X\phi = -\tfrac{3}{2}A(X)\phi,$$

where the 1-form A is proportional to the trace of the connection matrix:

$$A = \frac{2}{7}\sum_j \Gamma^j{}_j = \sum_{i,j}\langle\nabla_i X_j, \xi^j\rangle\xi^i,$$

for any frame (X_i) and the dual coframe (ξ^i) such that $g = g_{ij}\xi^i \otimes \xi^j$ with constant g_{ij}.

Of course, g and ϕ do not reduce GL(7, ℝ) to GL(2, ℝ), since their conformal stabilizers are $CO(3,4)$ and $\mathbb{R}^+ \times G_2^{\mathrm{split}}$, respectively. The object whose conformal stabilizer is precisely the irreducible GL(2, ℝ) is a certain totally symmetric 4-tensor Υ_{ijkl}, which is however irrelevant in our approach.

10.2. *Torsion*

In this section we consider only those ODEs which admit the GL(2, ℝ) geometry on the solution space. First we shall characterize the torsion T of Γ. Let V^k denote the k-dimensional irreducible representation of $GL(2, \mathbb{R})$ as before. Torsion of any $\mathfrak{gl}(2, \mathbb{R})$-connection at $p \in P$ belongs to the representation $\Lambda^2 V^{7*} \otimes V^7$ which decomposes as

$$\Lambda^2 V^{7*} \otimes V^7 = V^1 \oplus V^3 \oplus 3V^5 \oplus 3V^7 \oplus 3V^9 \oplus 2V^{11} \oplus 2V^{13} \oplus V^{15} \oplus V^{17}.$$

PROPOSITION 10.3 *The only non-vanishing components of the torsion T of the connection Γ in Theorem 10.1 are in the one-dimensional, the three-dimensional, and a fixed five-dimensional representation in the above decomposition.*

$$T = T_1 + T_3 + T_5.$$

Explicit form of T in (45) is given in Appendix B, where λ spans T_1; a_1, a_2, a_3 span T_3, and b_1, b_2, b_3, b_4, b_5 span T_5.

Proof. To prove the formula of Appendix B we use Proposition 9.1 to explicitly calculate (45). Next we check that T only occupies the irreducible representations as above.

We are now ready to prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1 Using (47), (45) and (14) we calculate λ, τ_2, τ_3 and Θ in terms of the torsion coefficients and the forms θ^i. We find that $\tau_2 = 0$ if $T_3 = 0$, $\tau_3 = 0$ if $T_5 = 0$, and also $\Theta = 24\Gamma_1$. Next we calculate explicitly λ, a_i and b_i using formulae for Ω given in Theorem 9.1. Since the components T_3 and T_5 lie in irreducible representations they vanish if and only if any of the components a_i or b_i vanishes. In the theorem we gave the simplest ones.

Proof of Theorem 5.2 In order to prove this result we need to extensively use the Bianchi identities. First, we suppose that $\tau_3 = 0$ which is equivalent to vanishing of $b_1, b_2, \ldots, b_5$. Then from $d^2\theta^i = 0$ we find that either (i) $a_i = 0$ (equivalently $\tau_2 = 0$) or (ii) $\lambda = 0$.

Suppose (i). Then the torsion is reduced to λ and it makes all the curvature except the Ricci scalar vanish. In particular the Lee form $\Theta = 24\Gamma_1$ is closed. Therefore there exists a conformal gauge in which locally $\Theta = 0$ and $\lambda = \text{const}$, and which defines a 10-dimensional sub-bundle P' of P. Equations (45) and (46) pulled-back to P' become the structural equations of $SO(3, 2)$ while the integrable distribution on P' annihilated by $\theta^1, \ldots, \theta^7$ defines the action of $SO(2, 1)$ on P', which is vertical with respect to $P' \to M$. This also means that the maximal symmetry group of an underlying ODE is $SO(3, 2)$. We find the ODE of point 2 by integration of the conditions from Appendix A and the conditions of Theorem 5.1.

Suppose (ii). Lengthy but straightforward calculations show that the condition $\lambda = 0$ specifies curvature in equation (46); all torsion and curvature coefficients and their coframe derivatives are polynomials of a_1, a_2 and a_3, which span T_3. Again, we have $d\Theta = 0$. Since T_3 belongs to the three-dimensional representation V^3 of $GL(2, \mathbb{R})$ we may classify it by the orbit it sweeps out. The component T_3 is a tensor field on P, which is a $\mathfrak{gl}(2, \mathbb{R})$ bundle over M. If we fix $x \in M$ and sweep out the fibre P_x then T_3 at points $p \in P_x$ sweeps a $GL(2, \mathbb{R})$-orbit in V^3. These orbits are labelled by the sign of $\langle \cdot, \cdot \rangle$, the conformal product in V^3 preserved by $GL(2, \mathbb{R})$. The case $\langle T_3, T_3 \rangle = 0$ is forbidden by the Bianchi identities. The only remaining possibilities are $\langle T_3, T_3 \rangle > 0$ and $\langle T_3, T_3 \rangle < 0$. The ODE (20) generates both cases in two disjoint areas of M, depending on the sign of $\langle T_3, T_3 \rangle = \text{const} \cdot (5y^{(6)}y^{(4)} - 6(y^{(5)})^2)$.

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A. Appendix

The five Wünschmann conditions for the seventh-order ODE are as follows:

$$
\begin{aligned}
W_1 &= 245\mathcal{D}^2F_6 - 245\mathcal{D}F_5 + 98F_4 - 210\mathcal{D}F_6F_6 + 70F_5F_6 + 20F_6^3,\\
W_2 &= 6860\mathcal{D}^2F_5 - 10976\mathcal{D}F_4 + 6615(\mathcal{D}F_6)^2 + 6860F_3 - 8330\mathcal{D}F_6F_5\\
&\quad + 1715F_5^2 - 1960\mathcal{D}F_5F_6 + 1568F_4F_6 - 1890\mathcal{D}F_6F_6^2 + 1190F_5F_6^2 + 135F_6^4,\\
W_3 &= 9604\mathcal{D}^2F_4 - 24010\mathcal{D}F_3 + 15435\mathcal{D}F_5\mathcal{D}F_6 + 24010F_2 - 14749\mathcal{D}F_6F_4\\
&\quad - 5145\mathcal{D}F_5F_5 + 4459F_4F_5 - 2744\mathcal{D}F_4F_6 + 6615(\mathcal{D}F_6)^2F_6 + 3430F_3F_6\\
&\quad - 6615\mathcal{D}F_6F_5F_6 + 1470F_5^2F_6 - 2205\mathcal{D}F_5F_6^2 + 2107F_4F_6^2\\
&\quad - 1890\mathcal{D}F_6F_6^3 + 945F_5F_6^3 + 135F_6^5,\\
W_4 &= 336140\mathcal{D}^2F_3 - 1344560\mathcal{D}F_2 + 180075(\mathcal{D}F_5)^2 + 432180\mathcal{D}F_4\mathcal{D}F_6\\
&\quad + 2352980F_1 - 624260\mathcal{D}F_6F_3 - 216090\mathcal{D}F_5F_4 + 64827F_4^2\\
&\quad - 144060\mathcal{D}F_4F_5 + 154350(\mathcal{D}F_6)^2F_5 + 192080F_3F_5 - 102900\mathcal{D}F_6F_5^2\\
&\quad + 17150F_5^3 - 96040\mathcal{D}F_3F_6 + 308700\mathcal{D}F_5\mathcal{D}F_6F_6 + 192080F_2F_6\\
&\quad - 246960\mathcal{D}F_6F_4F_6 - 154350\mathcal{D}F_5F_5F_6 + 113190F_4F_5F_6 - 61740\mathcal{D}F_4F_6^2\\
&\quad + 132300(\mathcal{D}F_6)^2F_6^2 + 89180F_3F_6^2 - 176400\mathcal{D}F_6F_5F_6^2 + 47775F_5^2F_6^2\\
&\quad - 44100\mathcal{D}F_5F_6^3 + 35280F_4F_6^3 - 37800\mathcal{D}F_6F_6^4 + 22050F_5F_6^4 + 2700F_6^6,
\end{aligned}
$$

$$
\begin{aligned}
W_5 = {} & 2352980\mathcal{D}^2F_2 - 16470860\mathcal{D}F_1 + 1512630\mathcal{D}F_4\mathcal{D}F_5 + 2268945\mathcal{D}F_3\mathcal{D}F_6 \\
& - 5126135\mathcal{D}F_6F_2 - 1512630\mathcal{D}F_5F_3 - 907578\mathcal{D}F_4F_4 + 648270(\mathcal{D}F_6)^2F_4 \\
& + 907578F_3F_4 - 756315\mathcal{D}F_3F_5 + 1080450\mathcal{D}F_5\mathcal{D}F_6F_5 + 1596665F_2F_5 \\
& - 1080450\mathcal{D}F_6F_4F_5 - 360150\mathcal{D}F_5F_5^2 + 288120F_4F_5^2 - 672280\mathcal{D}F_2F_6 \\
& + 540225(\mathcal{D}F_5)^2F_6 + 1296540\mathcal{D}F_4\mathcal{D}F_6F_6 + 2352980F_1F_6 \\
& - 1620675\mathcal{D}F_6F_3F_6 - 864360\mathcal{D}F_5F_4F_6 + 324135F_4^2F_6 - 648270\mathcal{D}F_4F_5F_6 \\
& + 926100(\mathcal{D}F_6)^2F_5F_6 + 756315F_3F_5F_6 - 771750\mathcal{D}F_6F_5^2F_6 + 154350F_5^3F_6 \\
& - 324135\mathcal{D}F_3F_6^2 + 926100\mathcal{D}F_5\mathcal{D}F_6F_6^2 + 732305F_2F_6^2 - 926100\mathcal{D}F_6F_4F_6^2 \\
& - 617400\mathcal{D}F_5F_5F_6^2 + 524790F_4F_5F_6^2 - 185220\mathcal{D}F_4F_6^3 + 396900(\mathcal{D}F_6)^2F_6^3 \\
& + 231525F_3F_6^3 - 661500\mathcal{D}F_6F_5F_6^3 + 209475F_5^2F_6^3 - 132300\mathcal{D}F_5F_6^4 \\
& + 119070F_4F_6^4 - 113400\mathcal{D}F_6F_6^5 + 75600F_5F_6^5 + 8100F_6^7 + 65883440F_0.
\end{aligned}
$$

B. Appendix

The torsion components in Proposition 10.3 are as follows:

$$
\begin{aligned}
T^1 = {} & \frac{55}{18}b_1\theta^1\wedge\theta^2 + \frac{55}{9}b_4\theta^1\wedge\theta^3 + \left(\frac{55}{18}b_3 - \frac{10}{3}\lambda - 3a_3\right)\theta^1\wedge\theta^4 \\
& + \left(-\frac{55}{9}b_5 + \frac{3}{2}a_2\right)\theta^1\wedge\theta^5 - \frac{77}{36}b_2\theta^1\wedge\theta^6 + \left(-\frac{55}{2}b_3 + 10\lambda + 9a_3\right)\theta^2\wedge\theta^3 \\
& + \left(\frac{55}{3}b_5 - 3a_2\right)\theta^2\wedge\theta^4 + \frac{55}{12}b_2\theta^2\wedge\theta^5, \\
T^2 = {} & \frac{55}{36}b_1\theta^1\wedge\theta^3 + \left(\frac{275}{54}b_4 + \frac{1}{2}a_1\right)\theta^1\wedge\theta^4 + \left(-\frac{55}{36}b_3 - \frac{5}{3}\lambda - a_3\right)\theta^1\wedge\theta^5 \\
& + \left(-\frac{11}{18}b_5 + \frac{1}{2}a_2\right)\theta^1\wedge\theta^6 - \frac{77}{216}b_2\theta^1\wedge\theta^7 + \left(-\frac{55}{18}b_4 - \frac{3}{2}a_1\right)\theta^2\wedge\theta^3 \\
& + \left(-\frac{55}{9}b_3 + 2a_3 + \frac{10}{3}\lambda\right)\theta^2\wedge\theta^4 - \frac{11}{18}b_2\theta^2\wedge\theta^6 + \left(\frac{275}{18}b_5 - \frac{5}{2}a_2\right)\theta^3\wedge\theta^4 \\
& + \frac{275}{72}b_2\theta^3\wedge\theta^5, \\
T^3 = {} & \frac{11}{54}b_1\theta^1\wedge\theta^4 + \left(\frac{22}{9}b_4 + \frac{1}{2}a_1\right)\theta^1\wedge\theta^5 + \left(-\frac{44}{45}b_3 - \frac{1}{5}a_3 - \frac{2}{3}\lambda\right)\theta^1\wedge\theta^6 \\
& + \left(\frac{22}{135}b_5 + \frac{1}{10}a_2\right)\theta^1\wedge\theta^7 + \frac{22}{9}b_1\theta^2\wedge\theta^3 + \left(\frac{11}{9}b_4 - a_1\right)\theta^2\wedge\theta^4 \\
& + \left(-\frac{11}{45}b_5 + \frac{3}{5}a_2\right)\theta^2\wedge\theta^6 - \frac{11}{20}b_2\theta^2\wedge\theta^7 + \left(\frac{55}{18}b_3 + \frac{10}{3}\lambda + a_3\right)\theta^3\wedge\theta^4 \\
& + \left(\frac{55}{9}b_5 - \frac{3}{2}a_2\right)\theta^3\wedge\theta^5 + \frac{11}{12}b_2\theta^3\wedge\theta^6 + \frac{55}{18}b_2\theta^4\wedge\theta^5 - \frac{11}{2}b_3\theta^2\wedge\theta^5,
\end{aligned}
$$

$$
\begin{aligned}
T^4 = & -\frac{11}{24}b_1\theta^1\wedge\theta^5 + \left(\frac{11}{15}b_4 + \frac{3}{10}a_1\right)\theta^1\wedge\theta^6 + \left(-\frac{11}{90}b_3 - \frac{1}{6}\lambda\right)\theta^1\wedge\theta^7 \\
& + \frac{11}{6}b_4\theta^2\wedge\theta^5 + \left(-\frac{22}{5}b_3 - \lambda\right)\theta^2\wedge\theta^6 + \left(\frac{11}{15}b_5 + \frac{3}{10}a_2\right)\theta^2\wedge\theta^7 \\
& + \left(\frac{55}{18}b_4 - \frac{3}{2}a_1\right)\theta^3\wedge\theta^4 + \frac{5}{2}\lambda\theta^3\wedge\theta^5 + \frac{11}{6}b_5\theta^3\wedge\theta^6 - \frac{11}{24}b_2\theta^3\wedge\theta^7 \\
& + \left(\frac{55}{18}b_5 - \frac{3}{2}a_2\right)\theta^4\wedge\theta^5 + \frac{22}{9}b_2\theta^4\wedge\theta^6 + \frac{22}{9}b_1\theta^2\wedge\theta^4, \\
T^5 = & -\frac{11}{20}b_1\theta^1\wedge\theta^6 + \left(\frac{22}{135}b_4 + \frac{1}{10}a_1\right)\theta^1\wedge\theta^7 + \frac{11}{12}b_1\theta^2\wedge\theta^5 \\
& + \left(-\frac{11}{45}b_4 + \frac{3}{5}a_1\right)\theta^2\wedge\theta^6 + \left(-\frac{44}{45}b_3 - \frac{2}{3}\lambda + \frac{1}{5}a_3\right)\theta^2\wedge\theta^7 \\
& + \frac{55}{18}b_1\theta^3\wedge\theta^4 + \left(\frac{55}{9}b_4 - \frac{3}{2}a_1\right)\theta^3\wedge\theta^5 - \frac{11}{2}b_3\theta^3\wedge\theta^6 \\
& + \left(\frac{22}{9}b_5 + \frac{1}{2}a_2\right)\theta^3\wedge\theta^7 + \left(\frac{55}{18}b_3 + \frac{10}{3}\lambda - a_3\right)\theta^4\wedge\theta^5 \\
& + \left(\frac{11}{9}b_5 - a_2\right)\theta^4\wedge\theta^6 + \frac{11}{54}b_2\theta^4\wedge\theta^7 + \frac{22}{9}b_2\theta^5\wedge\theta^6, \\
T^6 = & -\frac{77}{216}b_1\theta^1\wedge\theta^7 - \frac{11}{18}b_1\theta^2\wedge\theta^6 + \left(-\frac{11}{18}b_4 + \frac{1}{2}a_1\right)\theta^2\wedge\theta^7 \\
& + \left(-\frac{55}{36}b_3 - \frac{5}{3}\lambda + a_3\right)\theta^3\wedge\theta^7 + \left(\frac{275}{18}b_4 - \frac{5}{2}a_1\right)\theta^4\wedge\theta^5 \\
& + \left(-\frac{55}{9}b_3 + \frac{10}{3}\lambda - 2a_3\right)\theta^4\wedge\theta^6 + \left(\frac{275}{54}b_5 + \frac{1}{2}a_2\right)\theta^4\wedge\theta^7 \\
& + \left(-\frac{55}{18}b_5 - \frac{3}{2}a_2\right)\theta^5\wedge\theta^6 + \frac{55}{36}b_2\theta^5\wedge\theta^7 + \frac{275}{72}b_1\theta^3\wedge\theta^5, \\
T^7 = & -\frac{77}{36}b_1\theta^2\wedge\theta^7 + \frac{55}{12}b_1\theta^3\wedge\theta^6 + \left(-\frac{55}{9}b_4 + \frac{3}{2}a_1\right)\theta^3\wedge\theta^7 \\
& + \left(\frac{55}{3}b_4 - 3a_1\right)\theta^4\wedge\theta^6 + \left(\frac{55}{18}b_3 - \frac{10}{3}\lambda + 3a_3\right)\theta^4\wedge\theta^7 \\
& + \left(-\frac{55}{2}b_3 + 10\lambda - 9a_3\right)\theta^5\wedge\theta^6 + \frac{55}{9}b_5\theta^5\wedge\theta^7 + \frac{55}{18}b_2\theta^6\wedge\theta^7.
\end{aligned}
$$