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Multidimensional integrable systems and deformations of Lie algebra homomorphisms

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We use deformations of Lie algebra homomorphisms to construct deformations of dispersionless integrable systems arising as symmetry reductions of anti-self-dual Yang-Mills equations with a gauge group $\text{Diff}(S^1)$. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2777008]

I. INTRODUCTION

A dispersionless limit of partial differential equations (PDEs) is taken by rescaling the independent variables $X^a \to X^a/\varepsilon$ and taking the limit $\varepsilon \to 0$. This is a delicate procedure, as the limit of the solutions of a given PDE does not usually correspond to solutions of the limiting dispersionless equation. Moreover, inequivalent PDEs may have the same dispersionless limit, so the problem

- Recover the original PDE from its dispersionless limit

is, of course, ill posed. Some progress can nevertheless be made if the dispersionless equation is integrable, and one insists that its dispersive analog is also integrable. In the next section, we shall explain how dispersionless limits of solitonic PDEs are equivalent to the Wentzel-Kramers-Brillouin (WKB) quasiclassical approximation of the associated linear problems. This suggests that the reconstruction of the dispersive solitonic system should involve a quantization of some kind.

Such a quantization procedure has been developed in the seminal work of Kupershmidt.[15] This procedure is based on the Moyal product and works well if the Lie algebra underlying the dispersionless linear problem is the algebra $\text{sdiff}(\Sigma^2)$ of divergence-free vector fields on a two-surface Σ^2. This is the case for the dispersionless Kadomtsev-Petviashvili (dKP) and $\text{SU}(\infty)$ Toda equations in $2+1$ dimensions. Similar progress can also be made in higher dimensions and, indeed, one of us has constructed integrable deformations of Plebanski's first heavenly equation[29] by replacing the underlying Poisson bracket with the Moyal bracket.

The idea of deforming integrable systems while retaining the integrability of the resulting equation has now been studied from a number of different points of view:

- Takasaki studied properties of the deformed heavenly equations and described how solutions may be described in terms of a Riemann-Hilbert splitting in a Moyal algebra valued loop

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group.[33] Extensions of this led to Moyal-KP hierarchies[34] and deformations of the self-dual Yang-Mills equations.[32] The deformed Riemann-Hilbert procedure was recently fully developed by Formanski and Przanowski.[9,10]

- Nekrasov and Schwarz introduced instantons on noncommutative space-time.[22] This led to the development of noncommutative soliton equations. These may be viewed as a deformation of the standard, commutative, soliton equations. Many of these may be studied as reductions of the noncommutative self-dual Yang-Mills equations.[12,16]
- Associated with any Frobenius manifold is a hierarchy of integrable equations of hydrodynamic type. Integrable deformations of these equations arise naturally when one studies the genus expansion in the corresponding topological quantum field theories.[3]

In the present paper, deformations of multidimensional integrable systems are based on the algebra[1] Diff(Σ), the Lie algebra of vector fields on Σ, where $\Sigma \cong S^1$ or $\mathbb{R}$. It turns out, however, that this algebra admits no nontrivial deformations.[17] However, an alternative method of deforming these integrable systems may be developed. This method is based on the approach of Ovsienko and Rogers[24] where a homomorphism from Diff(Σ) to the Poisson algebra on $T^*\Sigma$ can be used to construct nontrivial deformations. We shall use this idea to construct integrable deformations of various equations associated with the algebra Diff(Σ).

It should be pointed out that the original, undeformed, equations have a natural interpretation in terms of the twistor theory, via the nonlinear graviton construction and its variants. It would seem desirable to develop a "deformed" version of the twistor theory that would encode solutions of the deformed equations as some sort of deformed holomorphic conditions. This idea was what was behind the paper,[31] but the problem remains open (though see Ref. 20 for some ideas on how Nijenhuis structures may be deformed).

II. DISPERSIONLESS LIMIT IN 2+1 DIMENSIONS

Certain dispersionless integrable systems can arise from solitonic systems in a following way. Let

$$A\left(\frac{\partial}{\partial X}\right) = \frac{\partial^n}{\partial X^n} + a_1(X^a)\frac{\partial^{n-1}}{\partial X^{n-1}} + \cdots + a_n(X^a),$$

$$B\left(\frac{\partial}{\partial X}\right) = \frac{\partial^m}{\partial X^m} + b_1(X^a)\frac{\partial^{m-1}}{\partial X^{m-1}} + \cdots + b_m(X^a)$$

be differential operators on $\mathbb{R}$ with coefficients depending on local coordinates $X^a=(X,Y,T)$ on $\mathbb{R}^3$. The overdetermined linear system,

$$\Psi_Y = A\left(\frac{\partial}{\partial X}\right)\Psi, \quad \Psi_T = B\left(\frac{\partial}{\partial X}\right)\Psi,$$

admits a solution $\Psi(X,Y,T)$ on a neighborhood of initial point (X,Y_0,T_0) for arbitrary initial data $\Psi(X,Y_0,T_0)=f(X)$ if and only if the integrability conditions $\Psi_{YT}=\Psi_{TY}$ or

$$A_T - B_Y + [A,B] = 0 \tag{2.1}$$

are satisfied. Nonlinear system (2.1) for $a_1,\ldots,a_n,b_1,\ldots,b_m$ can be solved by the inverse scattering transform (IST). Integrable systems which admit a Lax representation [Eq. (2.1)] will be referred to as solitonic or dispersive.

The dispersionless limit[37] is obtained by substituting

[1]In the remainder of this paper, the superscript, denoting the dimension of the manifold, will be dropped.

$$\frac{\partial}{\partial X^a} = \varepsilon \frac{\partial}{\partial x^a}, \quad \Psi(X^a) = \exp(\psi(x^a/\varepsilon))$$

and taking the limit $\varepsilon \to 0$. In the limit, the commutators of differential operators are replaced by the Poisson brackets of their symbols according to the relation

$$\frac{\partial^k}{\partial X^k}\Psi \to (\psi_x)^k\Psi, \quad [A,B] \to \frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial \lambda} = \{A,B\}, \quad \lambda = \psi_x,$$

where A,B are polynomials in λ, with coefficients depending on $x^a=(x,y,t)$. The dispersionless limit of system (2.1) is

$$A_t - B_y + \{A,B\} = 0. \tag{2.2}$$

Nonlinear differential equations of the form of Eq. (2.2) are called dispersionless integrable systems. One motivation for studying dispersionless integrable systems is their role in constructing partition functions in topological field theories.[14]

A natural approach to solving Eq. (2.2) would be an attempt to take a quasiclassical limit of the IST which linearizes Eq. (2.1). This does not yield the expected result, as the quasiclassical limit of the Lax representation for Eq. (2.1) is the system of Hamilton-Jacobi equations

$$\psi_y = A(\psi_x, x^a), \quad \psi_t = B(\psi_x, x^a),$$

with "two times" t and y, and the initial value problem for Eq. (2.2) would require a reconstruction of a potential from the asymptotic form of the Hamiltonians. This classical inverse scattering problem is so far open.

There are alternative methods of solving Eq. (2.2).[13,8,35,6] In particular, the minitwistor approach of Ref. 6 works as follows. System (2.2) is equivalent to the integrability $[L,M]=0$ of a two-dimensional distribution of vector fields

$$L = \partial_t - B_\lambda \partial_x + B_x \partial_\lambda, \quad M = \partial_y - A_\lambda \partial_x + A_x \partial_\lambda, \tag{2.3}$$

on $\mathbb{R}^3 \times \mathbb{RP}^1$. Assume that L,M are real analytic and complexify $\mathbb{R}^3$ to $\mathbb{C}^3$. The minitwistor space Z is the two complex dimensional quotient manifold

$$Z = \mathbb{C}^3 \times \mathbb{CP}^1/(L,M), \quad \lambda \in \mathbb{CP}^1, \quad x^a \in \mathbb{C}^3.$$

That is to say that the local coordinates on Z lift to functions on $\mathbb{C}^3 \times \mathbb{CP}^1$ constant along L,M.

The minitwistor space is equipped with a three parameter family of certain rational curves. All solutions to Eq. (2.2) can in principle be reconstructed from a complex structure of the minitwistor space.

In fact, the twistor approach outlined above is capable of solving a wider class of equations. We shall therefore generalize the notion of the dispersionless integrable systems by allowing distributions of vector fields more general than Eq. (2.3). The derivatives $A_\lambda, A_x, B_\lambda, B_x$ of the symbols (A,B) of operators can be replaced by independent polynomials A_1, A_2, B_1, B_2 in λ with coefficients depending on (x,y,t),

$$L = \partial_t - B_1 \partial_x + B_2 \partial_\lambda, \quad M = \partial_y - A_1 \partial_x + A_2 \partial_\lambda. \tag{2.4}$$

If A_1, B_1 are linear in λ and A_2, B_2 are at most cubic in λ, then the rational curves in Z have normal bundle $\mathcal{O}(2)$ (the line bundle over $\mathbb{CP}^1$ with transition functions λ^{-2} from the set $\lambda \neq \infty$ to $\lambda \neq 0$, i.e., Chern class 2) and the three-dimensional moduli space of such curves in Z can be parametrized by (x,y,t). Allowing polynomials of higher degrees would lead to hierarchies of dispersionless equations. We take the integrability of this generalized distribution [Eq. (2.4)] as our definition of the dispersionless integrable system. The definition is intrinsic in a sense that it does not refer to an underlying dispersive equation.

III. Diff(S^1) DISPERSIONLESS INTEGRABLE SYSTEMS

In this section, two integrable systems associated with the gauge group Diff(S^1) will be given. The first has been extensively studied in Refs. 26, 8, 4, 19, 7, 18, and 25 so only a new gauge-theoretic description will be given—the reader is referred to these earlier papers for more details. The second system, which arises from a Nahm-type system, is new and this system is discussed in more detail.

A. A (2+1) dimensional dispersionless integrable system

An example of a dispersionless system which is integrable in the sense of the outlined twistor correspondence is given by the following distribution:

$$L = \partial_t - w\partial_x - \lambda\partial_y, \quad M = \partial_y + u\partial_x - \lambda\partial_x. \tag{3.1}$$

A linear combination of this distribution leads to a special case of Eq. (2.4) with $A_2 = B_2 = 0$. Its integrability leads to the pair of quasilinear PDEs,

$$u_t + w_y + uw_x - wu_x = 0, \quad u_y + w_x = 0, \tag{3.2}$$

for two real functions $u = u(x,y,t)$, $w = w(x,y,t)$. This system of equations has recently been studied in Refs. 26, 8, 4, 19, 7, 18, and 25 in connection with the Einstein-Weyl geometry, hydrodynamic chains, and symmetry reductions of anti-self-dual Yang-Mills equations. From the twistor point of view, Eq. (3.2) is invariantly characterized[4] by requiring that the minitwistor space Z fibers holomorphically over $\mathbb{CP}^1$. The second equation can be used to introduce a potential H such that $u = H_x$, $w = -H_y$. The first equation then gives

$$H_{xt} - H_{yy} + H_y H_{xx} - H_x H_{xy} = 0. \tag{3.3}$$

System (3.2) arises as a symmetry reduction of the anti-self-dual Yang-Mills equations in signature (2,2) with the infinite-dimensional gauge group Diff(Σ) and two commuting translational symmetries exactly one of which is null.[7] This combined with the embedding of $SU(1,1) \subset \text{Diff}(\Sigma)$ gives rise to explicit solutions to Eq. (3.2) in terms of solutions to the nonlinear Schrödinger equation and the Korteweg–de Vries equation.[7]

The Lie algebra of the group of diffeomorphisms Diff(Σ), where $\Sigma = S^1$ or $\mathbb{R}$, is isomorphic to the infinite-dimensional Lie algebra of functions on Σ with the Wronskian,

$$\langle f, g \rangle := f g_x - f_x g, \tag{3.4}$$

as the Lie bracket, where $f, g \in C^\infty(\Sigma)$ and x is a local coordinate on Σ. An alternative gauge-theoretic interpretation can be given to Eq. (3.2). Observe that the first equation in Eq. (3.2) can be interpreted as the flatness of a gauge connection on $\mathbb{R}^2$, where the gauge group is Diff(Σ). Indeed, choose local coordinates (t,y) on $\mathbb{R}^2$ and consider $\mathcal{A} \in \Lambda^1(\mathbb{R}^2) \otimes \mathbb{C}^\infty(\Sigma)$ of the general form

$$\mathcal{A} = -w\mathrm{d}t + u\mathrm{d}y,$$

where $u, w : \mathbb{R}^2 \to C^\infty(\Sigma)$ depend on (x,y,t).

The flatness of this connection yields

$$\mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (u_t + w_y + \langle u, w \rangle)\mathrm{d}t \wedge \mathrm{d}y = 0,$$

as claimed. Therefore, the connection is a pure gauge and can be written as $\mathcal{A} = g^{-1}\mathrm{d}g$, where $g = g(x,y,t) \in \text{Map}(\mathbb{R}^2, \text{Diff}(\Sigma))$ and

$$w = -g^{-1}g_t, \quad u = g^{-1}g_y.$$

The second equation in Eq. (3.2) yields the following system:

$$(g^{-1}g_y)_y - (g^{-1}g_t)_x = 0, \tag{3.5}$$

where $g = \exp(\mathcal{A})$ is a finite diffeomorphism of Σ and terms like $g^{-1}g_t$ should be understood as

$$g^{-1}g_t = \mathcal{A}_t - \langle \mathcal{A}, \mathcal{A}_t \rangle + \frac{1}{2}\langle \mathcal{A}, \langle \mathcal{A}, \mathcal{A}_t \rangle\rangle + \cdots .$$

B. A (3+1) dimensional dispersionless integrable system

In this section, we shall present another example of an integrable system associated with the Lie algebra of Diff(S^1). We shall first write it as a Nahm system,

$$\dot{\mathbf{e}}_i = \frac{1}{2}\varepsilon_{ijk}[\mathbf{e}_j, \mathbf{e}_k], \quad i = 1,2,3, \tag{3.6}$$

where $\mathbf{e}_i$ are vector fields on an open set in $\mathbb{R}^4$ given by

$$\mathbf{e}_i = \frac{\partial}{\partial y^i} - N_i(x, y^j)\frac{\partial}{\partial x},$$

and (x, y^j) are local coordinates. Rewrite Eq. (3.6) as

$$\partial_x N_i + \varepsilon_{ijk}\partial_j N_k - \frac{1}{2}\varepsilon_{ijk}\langle N_j, N_k \rangle = 0. \tag{3.7}$$

We shall now discuss the origin and possible applications of Eq. (3.7).

(1) Any solution to Eq. (3.7) defines a hyper-Hermitian conformal structure represented by the metric

$$g = \mathbf{n}^2 + \delta_{ij}\mathrm{d}y^i \mathrm{d}y^j, \tag{3.8}$$

where

$$\mathbf{n} = \mathrm{d}x + N_i \mathrm{d}y^i.$$

The three complex structures $\mathbf{I}_i$, $i=1,2,3$ satisfying the algebra of quaternions,

$$\mathbf{I}_i\mathbf{I}_j = -\delta_{ij}\mathbf{1} + \varepsilon_{ijk}\mathbf{I}_k,$$

are given by

$$\mathbf{I}_i(\mathbf{n}) = \mathrm{d}y^i.$$

These formulas together with the algebraic relations satisfied by $\mathbf{I}_j$ determine the complex structures uniquely, e.g.,

$$\mathbf{I}_i(\mathrm{d}y^j) = -\delta_{ij}(\mathbf{n}) + \varepsilon_{ijk}\mathrm{d}y^k.$$

One way to impose integrability of the complex structures is to use the explicit form of the complex structures on the basis $(\mathrm{d}y^1, \mathrm{d}y^2, \mathrm{d}y^3, \mathbf{n})$ and demand that the space $\Lambda^{(1,0)}$ is closed under exterior differentiation. We begin by defining a basis of self-dual two forms,

$$\mathbf{\Sigma}^i = \mathbf{n} \wedge \mathrm{d}y^i + \frac{1}{2}\varepsilon^{ijk}\mathrm{d}y^j \wedge \mathrm{d}y^k.$$

The integrability of the complex structures is then equivalent to the anti-self-duality of the two-form $\mathrm{d}\mathbf{n}$,

$$\mathbf{\Sigma}^i \wedge \mathrm{d}\mathbf{n} = 0. \tag{3.9}$$

This condition is equivalent to Eq. (3.7).

A dual formulation leads to the Lax pair of vector fields, which is a special form of the hyper-Hermitian Lax pair.[5,11,2] To see it, set $\mathbf{e}_4=\partial_x$ and define complex vector fields

$$\mathbf{w} = \mathbf{e}_1 - i\mathbf{e}_2, \quad \mathbf{z} = \mathbf{e}_3 - i\mathbf{e}_4.$$

System (3.7) is equivalent to the commutativity of the Lax pair,

$$[\mathbf{w} - \lambda\bar{\mathbf{z}}, \mathbf{z} + \lambda\bar{\mathbf{w}}] = 0, \tag{3.10}$$

for all values of the parameter λ.

(2) System (3.7) arises as a symmetry reduction of the anti-self-dual Yang-Mills equations on $\mathbb{R}^4$ with the infinite-dimensional gauge group $\mathrm{Diff}(S^1)$ and one translational symmetry. In fact, any such symmetry reduction is gauge equivalent to (3.7). To see it, consider the flat metric on $\mathbb{R}^4$, which in double null coordinates $w=y^1+iy^2$, $z=y^3+iy^4$ takes the form

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}w\mathrm{d}\bar{w},$$

and choose the volume element $\mathrm{d}w\wedge\mathrm{d}\bar{w}\wedge\mathrm{d}z\wedge\mathrm{d}\bar{z}$. Let $A\in T^*\mathbb{R}^4\otimes\mathfrak{g}$ be a connection one-form and let F be its curvature two form. Here, $\mathfrak{g}$ is the Lie algebra of some (possibly infinite dimensional) gauge group G. In a local trivialization, $A=A_\mu\mathrm{d}y^\mu$ and $F=(1/2)F_{\mu\nu}\mathrm{d}y^\mu\wedge\mathrm{d}y^\nu$, where $F_{\mu\nu}=[D_\mu,D_\nu]$ takes its values in $\mathfrak{g}$. Here, $D_\mu=\partial_\mu+A_\mu$ is the covariant derivative. The connection is defined up to gauge transformations $A\to b^{-1}Ab-b^{-1}\mathrm{d}b$, where $b\in\mathrm{Map}(\mathbb{R}^4,G)$. The anti-self-dual Yang-Mills (ASDYM) equations on A_μ are $F=-*F$ or

$$F_{wz}=0, \quad F_{w\bar{w}}+F_{z\bar{z}}=0, \quad F_{\bar{w}\bar{z}}=0.$$

These equations are equivalent to the commutativity of the Lax pair,

$$L = D_w - \lambda D_{\bar{z}}, \quad M = D_z + \lambda D_{\bar{w}}, \tag{3.11}$$

for every value of the parameter λ.

We shall require that the connection possesses a symmetry which in our coordinates is given by $\partial/\partial y^4$. Choose a gauge such that the Higgs field A_4 is a constant in $\mathfrak{g}$. Now choose $G=\mathrm{Diff}(S^1)$, so that the components of the one-form A become vector fields on S^1. We can choose a local coordinate x on S^1 such that $A_4=\partial_x$ and $A_i=-N_i\partial_x$, where $N_i=N_i(x,y^j)$ are smooth functions on $\mathbb{R}^4$. The Lax pair [Eq. (3.11)] is identical to [Eq. (3.10)] and the ASDYM equations reduce to the first order PDEs [Eq. (3.7)].

(3) Example: An ansatz $\mathbf{N}(x,y^j)=f(x)\mathbf{A}(y^j)$, where $\mathbf{N}=(N_1,N_2,N_3)^T$, reduces Eq. (3.7) to a pair of linear equations

$$\dot{f} = cf, \quad c\mathbf{A} + \nabla\wedge\mathbf{A} = 0,$$

where c is a constant. If $c=0$, then $\mathbf{N}$ may be absorbed into a redefinition of the coordinate x in the metric [Eq. (3.8)]. Therefore, we assume that $c\neq 0$. We set $c=1$ by rescaling y^j and solve for $f=\exp(x)$, reabsorbing another constant of integration into $\mathbf{A}$. Now define a new coordinate $\rho=\exp(-x)$. Rescaling the metric [Eq. (3.8)] yields

$$\hat{g} = \rho\mathrm{d}\mathbf{y}^2 + \rho^{-1}(\mathrm{d}\rho - \mathbf{A}\cdot\mathrm{d}\mathbf{y})^2. \tag{3.12}$$

This metric is hyper-Hermitian if and only if the vector $\mathbf{A}(y^j)$ satisfies the Beltrami equation

$$\mathbf{A} + \nabla\wedge\mathbf{A} = 0. \tag{3.13}$$

This is a slight improvement of the result of Ref. 36 where it is shown that Eq. (3.12) is ASD if and only iff Eq. (3.13) holds.

The Beltrami equation implies that $\mathbf{A}$ is divergence-free and satisfies $\triangle\mathbf{A}+\mathbf{A}=0$, where $\triangle=\nabla^2$ is the scalar Laplacian on $\mathbb{R}^3$ acting on components of $\mathbf{A}$. Existence of solutions of Eq. (3.13), at least in the analytic case, can be proved by an application of the Cartan-Kähler theorem [c.f. Example 3.7 in Chap. III of Ref. 1].

(4) System (3.7) can be put in the hydrodynamic form

$$\partial_x\mathbf{N}=\mathbf{M}\,\mathrm{curl}\,\mathbf{N},$$

where

$$\mathbf{M}=-\begin{pmatrix} 1 & N_3 & -N_2 \\ -N_3 & 1 & N_1 \\ N_2 & -N_1 & 1 \end{pmatrix}^{-1}.$$

A different analytic continuation of Eq. (3.7) can be obtained at the level of the hyper-Hermitian geometry. This comes down to looking for conformal structures [Eq. (3.8)] of signature $++\,--$. To achieve this, we regard y_1,y_2,N_1,N_2 as imaginary and define

$$Y_1=iy_1,\quad Y_2=iy_2,\quad Y_3=y_3,\quad \mathcal{N}_1=iN_1,\quad \mathcal{N}_2=iN_2,\quad \mathcal{N}_3=N_3.$$

The desired system for $\mathcal{N}_i=\mathcal{N}_i(Y^j,x)$ arises from Eq. (3.7).

Clearly, there are many further properties of these dispersionless systems that may be studied. We now turn our attention to the construction of nontrivial integrable deformations of Eqs. (3.2) and (3.7).

IV. DISPERSIVE DEFORMATIONS

Given a dispersionless integrable system, it is natural to ask whether it arises as a limit of some dispersive (or solitonic) system. One would expect the reconstruction of a solitonic system to involve a quantization of some kind because taking a dispersionless limit of Eq. (2.1) was equivalent to a quasiclassical limit of the wave function $\Psi(X^a)$. This is indeed the case, and the paradigm example is provided by the connection between the Kadomtsev-Petviashvili (KP) equation and its dispersionless limit dKP. One can reconstruct KP from dKP by expressing the latter in the form of Eq. (2.2) and replacing the Poisson brackets by the Moyal bracket.[15,35,30,31,27] The infinite series involved in a Moyal product truncates in this case because the symbols A and B are polynomials in momentum λ. The deformation parameter can then be set to 1 and can be removed from the construction. It seems, however, that this beautiful example is rather exceptional and that the reconstruction of dispersive systems (if at all possible) is in general nonunique and can lead to systems which involve a formal power series.

Generalizing the definition of the dispersionless systems to non-Hamiltonian distributions (L,M) such as Eq. (3.1) makes things even worse, as the Poisson bracket is not present, and the connection with known quantization procedures of a classical phase space has been lost. It could be argued that Eq. (3.3) should be regarded as its own deformation as it admits a dual (classical and quantum) description. It is a solitonic system [Eq. (2.1)] with

$$A=H_X\frac{\partial}{\partial X},\quad B=H_Y\frac{\partial}{\partial X}$$

or a dispersionless limit [Eq. (2.2)] with $A=\lambda H_x, B=\lambda H_y$.

One attempt to find a dispersive analog of Eq. (3.2) would be to use the centrally extended Virasoro algebra in place of diff(Σ). Recall that such a procedure has been used to produce dispersive systems from dispersionless systems in a different context. Namely,[23] one can view the periodic Monge equation $u_t=uu_x$ as the equation for affinely parametrized geodesics with respect to the right-invariant metric on Diff(S^1) constructed from the L^2 inner product on the Lie algebra. Going to the central extension, one finds that affinely parametrized geodesics on the Virasoro-Bott group correspond to solutions of the Korteweg de Vries (KdV) equation (for a recent review of such constructions, see Ref. 21). In the current situation, we view a general element of the extended algebra as a pair

$$(f,a) := f(x)\frac{\mathrm{d}}{\mathrm{d}x} - iac,$$

where $a \in \mathbb{R}$ does not depend on x and c is a constant. Assuming that x is a periodic variable, the modified commutation relation is

$$[f\partial_x, g\partial_x]_c = \langle f,g\rangle\partial_x + \frac{ic}{48\pi}\int (f_{xxx}g - fg_{xxx})\mathrm{d}x,$$

and we see that the central term is a function of (t,y) only. Applying this procedure to Eq. (3.3) with

$$H(x,y,t) = \sum_k h_k(y,t)L^k, \tag{4.1}$$

where L^k are generators of the centrally extended Virasoro algebra satisfying

$$[L^k, L^m] = (k-m)L^{k+m} + \frac{c}{12}k(k^2-1)\,\delta_{k,-m}, \tag{4.2}$$

would modify only one equation in the infinite chain of PDEs for the functions h_k.

In the remaining part of this paper, we shall present a construction[2] which leads to nontrivial dispersive analogs of Eq. (3.2), or its equivalent form Eq. (3.3), and Eq. (3.7).

A. Deforming Lie algebra homomorphisms

To find a nontrivial deformation, one would wish to deform the Lie algebra of vector fields on Σ, but this algebra is known not to admit any nontrivial deformations.[17]

We shall choose a different route[24] and deform the standard homomorphism between diff(Σ) and the Poisson algebra on $T^*\Sigma$, the point being that the homomorphisms between Lie algebras can admit nontrivial deformations even if one of the algebras is rigid. A deformation of Eq. (3.3) is achieved in two steps, each introducing a parameter. In the first step, we shall deform the embedding of diff(Σ) into the Lie algebra of volume-preserving vector fields on $T^*\Sigma$. This introduces the first parameter μ. The second step is a deformation quantization of the first one. The Poisson algebra on $T^*\Sigma$ is the quasiclassical limit of the Lie algebra of pseudodifferential operators on Σ, so (working at the level of symbols) one quantizes the deformed homomorphism by using the deformed associative product of symbols of pseudodifferential operators rather than a pointwise commutative product of functions. This introduces the second parameter ε. In what follows, we shall be interested in the polynomial deformations rather than the formal ones.

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[2]An alternative approach which we have not explored would be to consider a quantum deformation of the Virasoro algebra as in Ref. 28 and its free boson realization,

$$[T_m, T_n] = -\sum_{l=1}^{\infty} f_l(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-s}(s^n - s^{-m})\,\delta_{m+n,0}, \tag{4.3}$$

where $s = qt^{-1}$ and coefficients f_l are given by

$$f(z) = \sum_{l=0}^{\infty} f_l z^l = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+s^n}z^n\right).$$

The ordinary Virasoro algebra [Eq. (4.2)] is recovered as $a \to 1$. Applying Eq. (4.3) to Eq. (4.1) would lead to a q-deformed analog of Eq. (3.3)

The standard embedding $\pi:\mathrm{vect}(\Sigma)\to C^{\infty}(T^{*}\Sigma)$ is given by contracting a vector field $X_f=f(x)\partial_x$ with a canonical one-form Θ on $T^{*}\Sigma$. In our case, $T^{*}\Sigma=\mathbb{R}\times\Sigma$ and the Lie algebras $C^{\infty}(S^1)$ (with the Wronskian bracket) and $\mathrm{vect}(S^1)$ (with the Lie bracket) are isomorphic so we can regard π as defined on $C^{\infty}(\Sigma)$.

If λ is a local coordinate on the fibers of $T^{*}\Sigma$ and $\Theta=\lambda dx$, the map π is explicitly given by

$$(\pi(f))(\lambda,x) \coloneqq \lambda f(x).$$

It is a Lie algebra homomorphism as

$$\{\pi(f),\pi(g)\}=\pi(\langle f,g\rangle).$$

Given $\mu\in\mathbb{R}$, define[24]

$$(\pi_\mu(f))(\lambda,x)=(\pi(f))(\lambda,x+\mu/\lambda)=\lambda f(x+\mu/\lambda)=\lambda\left(f(x)+f'(x)\frac{\mu}{\lambda}+\frac{1}{2!}f''(x)\left(\frac{\mu}{\lambda}\right)^2+\cdots\right).$$

Note that $\{\pi_\mu(f),\pi_\mu(g)\}=\pi_\mu(\langle f,g\rangle)$, so that π_μ is also a Lie algebra homomorphism between $\mathrm{diff}(\Sigma)$ and $\mathrm{sdiff}(T^{*}\Sigma)$.

The next step is motivated by the canonical quantization $\lambda\to\partial/\partial x$. For any functions $F,G\in C^{\infty}(T^{*}\Sigma)$ which are also allowed to depend on a parameter μ, define the Kupershmidt-Manin product

$$F\star G=\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\frac{\partial^k F}{\partial\lambda^k}\frac{\partial^k G}{\partial x^k} \tag{4.4}$$

(this is equivalent, under an ϵ-valued change of variable to the Moyal product) and set

$$\{F,G\}_{\varepsilon}=\frac{1}{\epsilon}(F\star G-G\star F). \tag{4.5}$$

The Poisson bracket is recovered in the limiting procedure

$$\lim_{\epsilon\to 0}\{F,G\}_{\epsilon}=\{F,G\},$$

but the deformed bracket is equal to the Poisson bracket for all ϵ if F,G are linear in λ. This is why the first deformation parameter μ is needed.

We are now ready to propose the dispersive analog of the dispersionless equations Eqs. (3.3) and (3.7).

(1) Let $\hat{H}(\lambda,x,y,t;\mu,\epsilon)=\pi_\mu(H)$ take values in an algebra of formal power series in ϵ with an associative product defined by Eq. (4.4). The deformed analog of Eq. (3.3) is

$$\hat{H}_{xt}-\hat{H}_{yy}-\{\hat{H}_x,\hat{H}_y\}_{\epsilon}=0. \tag{4.6}$$

(2) Let $\hat{N}_i(\lambda,x^i;\mu,\epsilon)=\pi_\mu(N_i)$ take values in an algebra of formal power series in ϵ with an associative product defined by Eq. (4.4). The deformed analog of Eq. (3.7) is

$$\partial_x\hat{N}_i+\varepsilon_{ijk}\partial_j\hat{N}_k-\frac{1}{2}\varepsilon_{ijk}\{\hat{N}_j,\hat{N}_k\}_{\epsilon}=0. \tag{4.7}$$

Given a solution $\hat{H}$ of Eq. (4.6) such that $(\mu\partial_\mu-\lambda\partial_\lambda)(\lambda^{-1}\hat{H})=0$ and $\lambda^{-1}\hat{H}$ is smooth in (μ/λ), we can construct $H(x,y,t)$ satisfying Eq. (3.3) by taking any of the two limits $\mu\to 0$, $\epsilon\to 0$, and similar remarks hold for Eq. (4.7). Conversely, formal power series (in the deformation parameters) solution may be constructed from a solution to the original, undeformed, equation in an analogous manner to the way developed in Ref. 29 The extent to which such formal series converge in a suitable space of functions is as yet, however, unclear.

These deformed equations formally retain their integrability; the various manipulations hold at the level of the Lax pair as well as at the level of the equations themselves. However, as remarked in the Introduction, a direct twistor theory correspondence for these equations is lacking, though one should be able to adopt the methods developed by Takasaki[33] and Formanski-Przanowski[9,10] to study the geometry of the corresponding Riemann-Hilbert problem (in some suitable Moyal algebra valued loop group).

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