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2011 Class. Quantum Grav. 28 025007

(http://iopscience.iop.org/0264-9381/28/2/025007)

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Class. Quantum Grav. **28** (2011) 025007 (16pp) doi:10.1088/0264-9381/28/2/025007

Cosmological Einstein–Maxwell instantons and Euclidean supersymmetry: anti-self-dual solutions

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Received 3 July 2010, in final form 8 November 2010
Published 22 December 2010
Online at stacks.iop.org/CQG/28/025007

Abstract
We classify super-symmetric solutions of the minimal $N = 2$ gauged Euclidean supergravity in four dimensions. The solutions with an anti-self-dual Maxwell field give rise to anti-self-dual Einstein metrics given in terms of solutions to the $SU(\infty)$ Toda equation and more general three-dimensional Einstein–Weyl structures. Euclidean Kastor–Traschen metrics are also characterized by the existence of a certain supercovariantly constant spinor.

PACS numbers: 04.65.+e, 04.40.Nr

1. Introduction

The bosonic sector of $N = 2$ supergravity (SUGRA) in four dimensions coincides with the Einstein–Maxwell theory. In [28], all solutions which admit a supercovariantly constant spinor have been found.

In this work, we shall classify supersymmetric solutions of Euclidean Einstein–Maxwell equations with a nonzero cosmological constant. It will be shown that the field equations in various branches of our classification reduce to the Einstein–Weyl (EW) system in three dimensions [7, 18, 19] which is integrable by twistor construction. Some of the Euclidean solutions arise from analytic continuations of real Lorentzian solutions—for example the Euclidean analogs of Kastor–Traschen metrics [20] belong to this class—while others do not have Lorentzian counterparts. In particular, all solutions with an anti-self-dual (ASD) Maxwell field belong to the latter class. It turns out (proposition 2.1) that the anti-self-duality of the Maxwell field implies the conformal anti-self-duality of the Weyl tensor. In this paper, we shall focus on constructing all solutions belonging to this ASD class. The non-ASD solutions will be constructed in [8]. Some of these have a Lorentzian counterpart [3, 5, 14–16, 24].

In the second part of this section, we shall discuss the Euclidean Einstein–Maxwell theory and explain the origin of various sign choices in the Euclidean signature. In section 2, we shall use the two-component spinor calculus to classify all supersymmetric solutions. The Killing spinor equations (2.7) contain a continuous parameter, and we shall show that the Killing spinor gives rise to a Killing vector only for one special value of this parameter. In this symmetric case, the metric is given in terms of the solutions to the $SU(\infty)$ Toda equation (proposition 2.2). For all other values of the parameter, the solutions do not in general admit an isometry. They do however admit a conformal retraction (propositions 2.3 and 2.4). In section 3, we shall characterize the Euclidean Kastor–Traschen solutions by the existence of a supercovariantly constant spinor with certain additional properties (proposition 3.1). The solutions constructed in this section are not ASD.

There are several motivations for studying Euclidean gauged SUGRA solutions. From the differential geometric perspective, the supersymmetric solutions constructed in propositions 2.3 and 2.4 provide examples of ASD Einstein metrics. The point is that the energy–momentum tensor of the ASD Maxwell field vanishes and the Maxwell equations decouple from the Einstein equations. In Euclidean quantum gravity, instantons provide the semi-classical description of black hole creations and in the cosmological context this has been implemented in [17, 22, 23, 27]. Finally, the solutions of ungauged ($\Lambda = 0$) SUGRA can be used to construct the supersymmetric solutions of Lorentzian minimal SUGRA theories in five and higher dimensions [1, 2, 10]. It remains to be seen whether solutions to the gauged $D = 4$ Euclidean SUGRA admit such lifts.

1.1. Euclidean Einstein–Maxwell equations

Consider Lorentzian Einstein–Maxwell equations possibly with a non-vanishing cosmological constant:

$$G_{ab} + 6\Lambda g_{ab} = -T_{ab}, \qquad \mathrm{d}F = 0, \qquad d * F = 0, \tag{1.1}$$

where

$$T_{ab} = \tfrac{1}{2} g_{ab}|F|^2 - 2F_{ac}F_b{}^c$$

is the Maxwell energy–momentum tensor[6] and $|F|^2 = F_{cd}F^{cd}$. Swapping the electric and magnetic fields, i.e. replacing F by its Hodge dual $*F$, maps solutions to solutions as the Lorentzian Maxwell energy–momentum tensor is unchanged by this transformation. This can be easily seen in the two-component spinor notation [25] where

$$T_{ab} = 2\phi_{AB}\overline{\phi}_{A'B'}$$

and the duality transformation is $\overline{\phi}_{A'B'} \to \mathrm{i}\overline{\phi}_{A'B'}$ and $\phi_{AB} \to -\mathrm{i}\phi_{AB}$. This is no longer the case in the Riemannian signature where

$$T_{ab} = 2\phi_{AB}\tilde{\phi}_{A'B'}$$

and the spinors ϕ_{AB} and $\tilde{\phi}_{A'B'}$ are independent. The transformation $F \to *F$ entails to $\tilde{\phi}_{A'B'} \to \tilde{\phi}_{A'B'}$ and $\phi_{AB} \to -\phi_{AB}$, thus $T_{ab} \to -T_{ab}$. This duality transformation can be used to 'fix wrong signs' arising from the analytic continuation of a Lorentzian solution.

[6] Our conventions follow Penrose and Rindler [25]: $[\nabla_a, \nabla_b]V^d = R_{abc}{}^d V^c$, $R_{ab} = R_{acb}{}^c = 6\Lambda g_{ab} - 2\Phi_{ab}$, where Φ_{ab} is the traceless Ricci tensor. Using these conventions in the Riemannian settings implies that the hyperbolic space H^4 has $\Lambda > 0$ and the 4-sphere S^4 has $\Lambda < 0$.

As an example consider the Reissner–Nordström–de Sitter (RNdS) spacetime

$$g = -V(r)\,\mathrm{d}t^2 + V(r)^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \qquad A = -Q\frac{\mathrm{d}t}{r}, \tag{1.2}$$

where

$$V(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - 2\Lambda r^2,$$

$m \geqslant 0$ and Q are constants, and $F = \mathrm{d}A$. Now continue this analytically to the Riemannian signature setting $t = \mathrm{i}\tau$ and assuming that r lies between the middle roots of the quartic $r^2V(r) = 0$. We encounter an immediate problem as the potential A is now purely imaginary. There does not seem to be a universally accepted resolution of this problem, and the way in which one proceeds is dictated by an overall aim of the analytic continuation. According to Hawking and Ross [17] one should, at least in the quantum-mechanical context, accept that the electrically charged solution has an imaginary gauge potential. Alternatively, especially if our interest lies in classical solutions, one could replace A by $\mathrm{i}A$ which is real. However, this continuation changes the overall sign of T_{ab} and leads to a 'wrong' coupling between the gravitational and electromagnetic fields. The coupling can now be 'made right' by replacing $F \to *F$, resulting in the Maxwell field $F = -*\,\mathrm{d}(Qr^{-1}\,\mathrm{d}\tau)$.

In this discussion, we used 'wrong' and 'right' in inverted commas, as the energy of the Riemannian Maxwell field is not positive definite, and (unlike in the Lorentzian case) the positivity cannot be used to fix the relative sign between G_{ab} and T_{ab}. The cosmological constant in our example does not change under the analytic continuation. In section 3, we shall however see a different class of examples (Kastor–Traschen cosmological multi black holes [20]) where the analytic continuation leads to a real solution only if Λ changes sign: asymptotically de Sitter–Lorentizan solutions become asymptotically hyperbolic Riemannian solutions.

To avoid making the various sign choices, we shall simply look for real solutions of the Euler–Lagrange equations arising from the Lagrangian density

$$\mathcal{L} = \sqrt{g}(R + \gamma|F|^2 + \delta),$$

where γ and δ are the real constants. The cosmological constant can then be read off from δ and the sign of the Maxwell energy–momentum tensor can be adjusted if necessary replacing F by its Hodge dual as explained above.

2. Supersymmetric solutions with an ASD Maxwell field

Let the 2-form F be an ASD Maxwell field on a Riemannian four-manifold (M, g), i.e.

$$\mathrm{d}F = 0, \qquad *F = -F.$$

We shall make use of an isomorphism

$$\mathbb{C} \otimes TM \cong \mathbb{S} \otimes \mathbb{S}'$$

where the complex rank-2 vector bundles $\mathbb{S}$, $\mathbb{S}'$ (called spin-bundles) over M are equipped with parallel symplectic structures ε, ε' such that $g = \varepsilon \otimes \varepsilon'$. We use the standard convention [7, 25] in which spinor indices are capital letters, unprimed for sections of $\mathbb{S}$ and primed for sections of $\mathbb{S}'$. The two-component spinor formalism will be adapted to the Riemannian signature, where the spinor conjugation preserves the type of spinors. Thus, if $\alpha_A = (p, q)$ we can define $\hat{\alpha}_A = (\overline{q}, -\overline{p})$ so that $\hat{\hat{\alpha}}_A = -\alpha_A$. This Hermitian conjugation induces a positive inner product

$$\hat{\alpha}_A\alpha^A = \varepsilon_{AB}\hat{\alpha}^A\alpha^B = |p|^2 + |q|^2.$$

We define the inner product on the primed spinors in the same way. Here, ε_{AB} and $\varepsilon_{A'B'}$ are the covariantly constant symplectic forms with $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These are used to raise and lower spinor indices according to $\alpha_B = \varepsilon_{AB}\alpha^A$ and $\alpha^B = \varepsilon^{BA}\alpha_A$, and similarly for primed spinors.

The spinor decomposition of the Riemann tensor is

$$R_{abcd} = \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} + \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \Phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'B'}\varepsilon_{C'D'} - \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{A'D'}\varepsilon_{B'C'}), \quad (2.1)$$

where ψ_{ABCD} and $\psi_{A'B'C'D'}$ are ASD and SD Weyl spinors which are symmetric in their indices, $\Phi_{A'B'CD} = \Phi_{(A'B')(CD)}$ is the traceless Ricci spinor, and $\Lambda = R/24$ is the cosmological constant.

Making use of the isomorphism $\Lambda^2_- \cong \mathbb{S} \odot \mathbb{S}$, we can write $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ where the symmetric spinor $\phi_{AB} = \hat{\phi}_{AB}$ satisfies the ASD Maxwell equations

$$\nabla^A{}_{A'}\phi_{AB} = 0.$$

Consider the Killing spinor equations [13, 28]

$$\begin{aligned} &\nabla_{AA'}\alpha_B + c_0 A_a\alpha_B + (c_1\phi_{AB} + c_2\varepsilon_{AB})\beta_{A'} = 0, \\ &\nabla_{AA'}\beta_{B'} + c_3 A_a\beta_{B'} + c_4\varepsilon_{A'B'}\alpha_A = 0, \end{aligned} \quad (2.2)$$

where A_a is a real 1-form and $c_0, \ldots, c_4$ are some constant coefficients which we shall now determine. Differentiating (2.2) covariantly, commuting covariant derivatives and using the spinor Ricci identities

$$\nabla^A{}_{(A'}\nabla_{B')A}\alpha_B + \Phi_{A'B'AB}\alpha^A = 0, \quad (2.3)$$

$$\nabla^{A'}{}_{(A}\nabla_{B)A'}\beta_{B'} + \Phi_{A'B'AB}\beta^{A'} = 0, \quad (2.4)$$

$$\nabla^{A'}{}_{(A}\nabla_{B)A'}\alpha_C + \psi_{ABCD}\alpha^D - 2\Lambda\alpha_{(A}\varepsilon_{B)C} = 0, \quad (2.5)$$

$$\nabla^A{}_{(A'}\nabla_{B')A}\beta_{C'} + \psi_{A'B'C'D'}\beta^{D'} - 2\Lambda\beta_{(A'}\varepsilon_{B')C'} = 0, \quad (2.6)$$

leads to the compatibility conditions. Equations (2.3) give

$$\Phi_{ABA'B'} = 0, \qquad c_0\nabla^A{}_{(A'}A_{B')A} = 0, \qquad c_0 = c_3.$$

Equations (2.4) give $F = 2dA$, if $\nabla^A{}_{(A'}A_{B')A} = -\tilde{\phi}_{A'B'}$, $\nabla^{A'}{}_{(A}A_{B)A'} = -\phi_{AB}$ or

$$c_3 = -c_1c_4.$$

Equations (2.5) give

$$c_2c_4 = -\Lambda.$$

Finally, equations (2.6) give

$$\psi_{A'B'C'D'} = 0$$

so we deduce

Proposition 2.1. *A Riemannian four-manifold which admits a solution to the Killing spinor equations (2.2) with an ASD Maxwell field F is ASD and Einstein.*

The case $c_0 = c_1 = c_3 = 0$ leads to some non-trivial solutions in (2, 2) signature, but not in (4, 0) so we shall not consider it. If $c_0 \neq 0$, then we can redefine A_a, ϕ_{AB} and $\beta_{A'}$ by rescalings to get rid of some constants. Set $c_0 = -c\,\mathrm{e}^{\mathrm{i}\theta}$, where c and θ are real. The resulting equations are

$$\nabla_{AA'}\alpha_B = \mathrm{e}^{\mathrm{i}\theta} A_a \alpha_B + \left(\frac{\mathrm{e}^{\mathrm{i}\theta}}{\Lambda}\phi_{AB} - \varepsilon_{AB}\right)\beta_{A'}$$
$$\nabla_{AA'}\beta_{B'} = \mathrm{e}^{\mathrm{i}\theta} A_a \beta_{B'} + \Lambda \varepsilon_{A'B'}\alpha_A, \tag{2.7}$$

together with equations for spinor conjugates

$$\nabla_{AA'}\hat{\alpha}_B = \mathrm{e}^{-\mathrm{i}\theta} A_a \hat{\alpha}_B + \left(\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\Lambda}\phi_{AB} - \varepsilon_{AB}\right)\hat{\beta}_{A'}$$
$$\nabla_{AA'}\hat{\beta}_{B'} = \mathrm{e}^{-\mathrm{i}\theta} A_a \hat{\beta}_{B'} + \Lambda \varepsilon_{A'B'}\hat{\alpha}_A. \tag{2.8}$$

2.1. $\theta = \pi/2$ and $SU(\infty)$ Toda equation

Now we shall consider the case $\theta = \pi/2$ and show that the resulting metric is the most general ASD Einstein metric with symmetry, and can be found from solutions to the $SU(\infty)$ Toda equation.

Proposition 2.2. *Let the Riemannian four-manifold (M, g) admit a solution to the Killing spinor equations (2.7) with $\theta = \pi/2$ such that $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ is an ASD Maxwell field with $F = 2dA$. Then, g satisfies ASD Einstein equations with nonzero Λ. Moreover, g admits a Killing vector and the local coordinates (x, y, z, τ) can be chosen so that*

$$g = \frac{1}{z^2}(V(\mathrm{d}z^2 + \mathrm{e}^u(\mathrm{d}x^2 + \mathrm{d}y^2)) + V^{-1}(\mathrm{d}\tau + \omega)^2), \tag{2.9}$$

where $u = u(x, y, z)$ is a solution of the $SU(\infty)$ Toda equation

$$u_{xx} + u_{yy} + (\mathrm{e}^u)_{zz} = 0, \tag{2.10}$$

the function V is given by $-4\Lambda V = zu_z - 2$, and ω is a 1-form such that

$$\mathrm{d}\omega = -V_x\,\mathrm{d}y \wedge \mathrm{d}z - V_y\,\mathrm{d}z \wedge \mathrm{d}x - (V\,\mathrm{e}^u)_z\,\mathrm{d}x \wedge \mathrm{d}y. \tag{2.11}$$

We have already shown that g is ASD and Einstein. Once we establish the existence of a symmetry, we could refer to results of Tod [29] and Przanowski [26] to deduce the canonical form of the metric (2.9). In the proof given below, we shall however give a direct derivation of this form using the Killing spinor equations.

Proof. Define two real nonzero functions $U, \widetilde{U}$ by

$$U = (\varepsilon_{AB}\hat{\alpha}^A\alpha^B)^{-1}, \qquad \widetilde{U} = (\varepsilon_{A'B'}\hat{\beta}^{A'}\beta^{B'})^{-1}. \tag{2.12}$$

Consider a complex tetrad of 1-forms

$$K_a = \mathrm{i}(\hat{\alpha}_A\beta_{A'} + \alpha_A\hat{\beta}_{A'}), \qquad X_a = \hat{\alpha}_A\beta_{A'} - \alpha_A\hat{\beta}_{A'}, \qquad Z_a = \alpha_A\beta_{A'}. \tag{2.13}$$

The 1-forms $X = X_a\mathrm{e}^a$ and $K = K_a\mathrm{e}^a$ are real and the 1-form $Z = Z_a\mathrm{e}^a$ is complex. Using the Killing spinor equations (2.7) and their conjugations (2.8), we find

$$\nabla_a K_b = \varepsilon_{A'B'}\left(\Lambda(\hat{\alpha}_A\alpha_B + \alpha_A\hat{\alpha}_B) - \frac{\mathrm{i}}{\Lambda\widetilde{U}}\phi_{AB}\right) - \varepsilon_{AB}(\hat{\beta}_{A'}\beta_{B'} + \beta_{A'}\hat{\beta}_{B'}) \tag{2.14}$$

so that $\nabla_{(a}K_{b)} = 0$ and K is a Killing vector. Moreover, we find

$$\mathrm{d}X = 0, \qquad Z \wedge \mathrm{d}Z = 0$$

and deduce the existence of a local coordinate system $(\tau, \zeta, q, \overline{q})$ on M such that

$$K^a \nabla_a = \sqrt{2}\frac{\partial}{\partial \tau}, \qquad X = \sqrt{2}\,\mathrm{d}\zeta, \qquad Z = \frac{1}{\sqrt{2}} p\,\mathrm{d}q$$

for some complex-valued function $p = p(\zeta, q, \overline{q})$. Therefore, the 1-form dual to the Killing vector is $K = \Omega(\mathrm{d}\tau + \omega)$, where Ω and ω are a function and a 1-form on the space of orbits of K in M, respectively. Using

$$\varepsilon_{AB} = U(\hat{\alpha}_A\alpha_B - \hat{\alpha}_B\alpha_A), \qquad \varepsilon_{A'B'} = \widetilde{U}(\hat{\beta}_{A'}\beta_{B'} - \hat{\beta}_{B'}\beta_{A'}) \tag{2.15}$$

we find the metric to be

$$g = \varepsilon_{AB}\varepsilon_{A'B'}\,\mathrm{e}^{AA'}\,\mathrm{e}^{BB'} = U\widetilde{U}(\mathrm{d}\zeta^2 + |p|^2\,\mathrm{d}q\,\mathrm{d}\overline{q} + \Omega^2(\mathrm{d}\tau + \omega)^2).$$

Finally, using $K_aK^a = 2(U\widetilde{U})^{-1}$ and setting $q = x + \mathrm{i}y$, $|p|^2 = \mathrm{e}^{\phi}$ where $\phi = \phi(x, y, \zeta)$ is a real-valued function yields

$$g = U\widetilde{U}(\mathrm{e}^{\phi}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}\zeta^2) + \frac{1}{U\widetilde{U}}(\mathrm{d}\tau + \omega)^2.$$

Now we need to find equations for ϕ, U, $\widetilde{U}$ and ω. Using the Killing spinor equations (2.7) gives

$$\nabla_a\left(\frac{1}{\widetilde{U}}\right) = \Lambda X_a, \qquad \nabla_a\left(\frac{1}{U}\right) = -\frac{1}{\Lambda}\phi_A{}^C K_{CA'} + X_a. \tag{2.16}$$

Therefore, $\widetilde{U} = (\sqrt{2}\Lambda\zeta)^{-1}$, where we absorbed the integration constant into the definition of ζ. Defining a coordinate $z = \zeta^{-1}$ and setting

$$U = \sqrt{2}\Lambda z V, \qquad \phi = u + 4\log\zeta$$

where $V = V(x, y, z)$, $u = u(x, y, z)$ yields the final form of the metric (2.9). We now use (2.16) to find

$$\phi_{AB} = \frac{2\Lambda}{|K|^2}K_B^{A'}\nabla_{AA'}\left(\frac{1}{U} - \frac{\Lambda}{\widetilde{U}}\right),$$

and substitute this to (2.14). This yields $-4\Lambda V = zu_z - 2$, where u satisfies the $SU(\infty)$ Toda equation (2.10), and (2.11) holds. □

Note that the rescaled metric $\hat{g} = z^2 g$ is of the form given by the LeBrun ansatz [21] because V satisfies the linearized $SU(\infty)$ Toda equation. Therefore, $\hat{g}$ is Kähler with a vanishing Ricci scalar. Scalar-flat Kähler metrics are also the solutions to Einstein–Maxwell equations in the Riemannian signature [11], where the SD and ASD parts of the Maxwell field are given by the Kähler form Ω and (half of) the Ricci form ρ respectively:

$$F = \Omega + \frac{\rho}{2}.$$

Note that $\Omega \in \Lambda^2_+$ and $\rho \in \Lambda^2_-$. Thus, there exist two conformally related metrics: one non-supersymmetric $\hat{g}$ which solves Euclidean ungauged SUGRA equations (Einstein–Maxwell with $\Lambda = 0$), and one supersymmetric g which solves the gauged SUGRA (Einstein–Maxwell with $\Lambda \neq 0$).

2.2. $\theta = 0$ and the hyperCR equation

The ASD Einstein metrics corresponding to $\theta \neq \pi/2$ in (2.7) do not in general admit a continuous symmetry. In this subsection, we shall find a general local form of the metric in the case when $\theta = 0$.

Proposition 2.3. *Let the Riemannian four-manifold (M, g) admit a solution to the Killing spinor equations (2.7) with $\theta = 0$ such that $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ is an ASD Maxwell field with $F = 2dA$. Then, g satisfies ASD Einstein equations with $\Lambda > 0$. Moreover, a local coordinate ψ can be chosen so that*

$$g = \frac{\Lambda}{8}\sinh{(2\psi)}^2 h + \frac{2}{\Lambda}(\mathrm{d}\psi - \coth{(\psi)}\omega)^2, \qquad F = 2\,\mathrm{d}(\coth{(\psi)}^2\omega) \tag{2.17}$$

where $h = \boldsymbol{e}_1^2 + \boldsymbol{e}_2^2 + \boldsymbol{e}_3^2$ is a 3-metric, the ψ-independent 1-forms $(\boldsymbol{e}_i, \omega)$ satisfy $\partial_\psi \lrcorner\, \boldsymbol{e}_i = \partial_\psi \lrcorner\, \omega = 0$,

$$\begin{aligned} \mathrm{d}\boldsymbol{e}_1 &= -2\omega\wedge\boldsymbol{e}_1 - \Lambda\boldsymbol{e}_2\wedge\boldsymbol{e}_3,\\ \mathrm{d}\boldsymbol{e}_2 &= -2\omega\wedge\boldsymbol{e}_2 - \Lambda\boldsymbol{e}_3\wedge\boldsymbol{e}_1,\\ \mathrm{d}\boldsymbol{e}_3 &= -2\omega\wedge\boldsymbol{e}_3 - \Lambda\boldsymbol{e}_1\wedge\boldsymbol{e}_2,\end{aligned} \tag{2.18}$$

and

$$\mathrm{d}\omega = \Lambda *_h \omega, \tag{2.19}$$

where $*_h$ is the Hodge operator of h.

Proof. The ASD Einstein equations follow from the integrability conditions as we have already explained.

The gauge freedom

$$\alpha_A \to \mathrm{e}^f\alpha_A, \qquad \beta_{A'} \to \mathrm{e}^f\beta_{A'}, \qquad A \to A - \mathrm{d}f, \qquad \text{where} \quad f : M \longrightarrow \mathbb{R}$$

can be used to set $\widetilde{U} = 1$. Consider a tetrad (2.13), so that with our gauge choice

$$g_{ab} = \frac{U}{2}(4Z_{(a}\overline{Z}_{b)} + K_aK_b + X_aX_b)$$

and $X_aX^a = K_aK^a = 2Z_a\overline{Z}^a = 2U^{-1}$ and all other inner products vanish.

The condition $\mathrm{d}(\widetilde{U}^{-1}) = 0$ implies

$$A_a = \frac{\Lambda}{2}X_a.$$

We also find

$$X^a\nabla_a(U^{-1}) = 2U^{-1}(\Lambda U^{-1} - 1),$$

so that if τ is a local coordinate for which $X^a\nabla_a = \partial/\partial\tau$, then

$$U = \Lambda(1 + \mathrm{e}^{2\tau}c^2), \tag{2.20}$$

where c is a local function on M independent of τ. Now we use the Killing spinor equations (2.7) and (2.15) to find

$$\begin{aligned} \mathrm{d}Z &= (-UX + \mathrm{i}(U - \Lambda)K)\wedge Z\\ \mathrm{d}K &= -UX\wedge K + 2\mathrm{i}(U-\Lambda)Z\wedge\overline{Z},\end{aligned} \tag{2.21}$$

so regarding $X = \partial/\partial\tau$ as a vector field

$$\mathcal{L}_X Z = -2Z, \qquad \mathcal{L}_X K = -2K$$

where $\mathcal{L}_X$ denotes the Lie derivative along the vector field $X = X^a \nabla_a$. Therefore, we can set

$$Z = e^{-2\tau} \widetilde{Z}, \qquad K = e^{-2\tau} \widetilde{K}$$

where $\widetilde{Z}$ and $\widetilde{K}$ are 1-forms which Lie derive along X. The 1-form X_a is given by $X = 2U^{-1}(d\tau + \Omega)$, where Ω is a 1-form which in general can depend on τ. We now have to consider the following two cases.

(1) $U = \Lambda$, which corresponds to vanishing of the function c in (2.20). Now,

$$d\widetilde{Z} = -2\Omega \wedge \widetilde{Z}, \qquad d\widetilde{K} = -2\Omega \wedge \widetilde{K},$$

so that Ω is independent of τ. Taking the exterior derivatives of these equations gives the integrability condition $d\Omega = 0$. Therefore, locally there exist τ-independent real-valued functions (ϕ, x, y, z) such that

$$\Omega = d\phi, \qquad \widetilde{Z} = \tfrac{1}{2} e^{-2\phi}(dx + i\, dy), \qquad K = e^{-2\phi} dz.$$

Finally, setting $\tilde{\tau} = \tau + \phi$ gives the hyperbolic metric

$$g = \frac{\Lambda}{2} e^{-4\tilde{\tau}}(dx^2 + dy^2 + dz^2) + \frac{2}{\Lambda} d\tilde{\tau}^2 \tag{2.22}$$

and the vanishing Maxwell field $F = 0$. This metric has $\Lambda > 0$ which is consistent with our curvature conventions.

(2) Now assume $U \neq \Lambda$. Equations (2.21) imply

$$\begin{aligned} d\widetilde{Z} &= (-2\Omega + i\Lambda c^2 \widetilde{K}) \wedge \widetilde{Z} \\ d\widetilde{K} &= -2\Omega \wedge \widetilde{K} + 2i\Lambda c^2 \widetilde{Z} \wedge \overline{\widetilde{Z}}. \end{aligned} \tag{2.23}$$

We can redefine coordinates to set $c = 1$. To see it put

$$\widetilde{Z} = \frac{c^{-2}}{2}(\mathbf{e}_1 + i\mathbf{e}_2), \qquad \widetilde{K} = c^{-2}\mathbf{e}_3, \qquad \tilde{\tau} = \tau - \log c, \qquad \omega = \Omega + d\log c. \tag{2.24}$$

Then, the metric is given by

$$g = \Lambda(1 + e^{2\tilde{\tau}}) \left(\frac{1}{2} e^{-4\tilde{\tau}} h + \frac{2}{\Lambda^2 (1 + e^{2\tilde{\tau}})^2} (d\tilde{\tau} + \omega)^2 \right),$$

where $h = \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2$. Substituting (2.24) into (2.23) gives the system (2.18) for the 1-forms $\mathbf{e}_i$.The Maxwell field is given by

$$F = d\left(\frac{2\omega}{1 + e^{2\tilde{\tau}}} \right)$$

and the anti-self-duality condition $F = -*F$ yields (2.19). This is also the integrability condition for (2.18). Setting $\psi = -\text{arctanh}(\sqrt{1 + e^{2\tilde{\tau}}})$ yields the form of the metric and the Maxwell field given in the statement of the proposition. □

Remarks

- Making an analytic continuation $\psi \to i\psi$ and changing the sign of Λ lead to an ASD Einstein metric given in terms of trigonometric (rather than hyperbolic) functions. Setting $\Lambda = -4$ yields

$$g = \tfrac{1}{4} \sin^2(2\psi) h + \tfrac{1}{4}(d\psi + \cot\psi\, \omega)^2.$$

- A three-manifold admitting a system of 1-forms $(\mathbf{e}_i, \omega)$ satisfying equations (2.18) and (2.19) admits a hyperCR EW structure [12]. There is a well-known construction [19] which associates ASD conformal structures with symmetry to any EW structure. Proposition 2.3 reveals another connection between the hyperCR EW structures and ASD four-manifolds, where the Einstein metric in an ASD conformal class does not admit a symmetry.
- In [9], it was shown how to reduce the hyperCR conditions (2.18) and (2.19) to a single second-order integrable PDE (which therefore plays a role analogous to the $SU(\infty)$ Toda equation) for one function of three variables.

The metric (2.17) degenerates at $\psi = 0$ but this degeneracy can be absorbed into a conformal factor as

$$g = \sinh(2\psi)^2 \hat{g}, \qquad \text{where} \quad \hat{g} = \frac{\Lambda}{8}h + \frac{2}{\Lambda}(\mathrm{d}\chi + \mathrm{e}^{-2\chi}\sinh(2\chi)\omega)^2$$

and $\chi = -\mathrm{arctanh}(\mathrm{e}^{2\psi})$. The conformal structure will therefore be regular if the pair (h, ω) is. An example is provided by the Berger sphere, where[7]

$$h = (\sigma_1)^2 + (\sigma_2)^2 + \Lambda^2(\sigma_3)^2, \qquad \omega = \tfrac{1}{2}\Lambda\sqrt{1-\Lambda^2}\sigma_3.$$

Here, $0 < \Lambda \leqslant 1$ and σ_i are the left-invariant 1-forms on S^3 satisfying

$$\mathrm{d}\sigma_1 = \sigma_2 \wedge \sigma_3, \qquad \mathrm{d}\sigma_2 = \sigma_3 \wedge \sigma_1, \qquad \mathrm{d}\sigma_3 = \sigma_1 \wedge \sigma_2.$$

2.3. *General θ and interpolating EW equations*

Finally, we shall analyze the Killing spinor equations (2.7) and (2.8), where the parameter θ is allowed to take arbitrary values. Similar to the cases $\theta = \pi/2$ and $\theta = 0$, we shall find that the spacetime admits a local fibration over a three-dimensional manifold with an EW structure. The relevant EW structure has arisen in [9] as the most general symmetry reductions of ASD Ricci-flat equations by a conformal Killing vector. It contains both the $SU(\infty)$ and hyperCR equations as special cases. The class of ASD Einstein metrics characterized in the following proposition does not in general admit an isometry (or a conformal isometry) unless $\theta = \pi/2$. Instead, it will be shown to admit an ASD conformal retraction in a sense of [18] and [4]. (In [4], it is referred to a conformal submersion. The metrics from proposition 2.3 belong to the class described in theorem IX in this reference. The metrics characterized by the proposition given below appear to be new.)

Proposition 2.4. *Let (M, g) be a Riemannian four-manifold which admits a solution of the Killing spinor equations (2.7) and (2.8) such that the 2-form $F = 2\,\mathrm{d}A$ is ASD. Then, g is ASD and Einstein with $\Lambda \neq 0$ and locally is of the form*

$$g = \frac{2}{\Lambda}\left(\frac{\mathrm{e}^{2\cos\theta\tau}}{1+\mathrm{e}^{2\cos\theta\tau}}\right)\left(\mathrm{d}\tau - \omega + \frac{1}{2}\Lambda\tan\theta\,\mathrm{e}^{2\cos\theta\tau}\,\boldsymbol{e}_3\right)^2 + \frac{\Lambda}{2}\mathrm{e}^{4\cos\theta\tau}(1+\mathrm{e}^{-2\cos\theta\tau})h \tag{2.25}$$

where $h = \boldsymbol{e}_1^2 + \boldsymbol{e}_2^2 + \boldsymbol{e}_3^2$ is a 3-metric, the $\tau-$ independent 1-forms $(\boldsymbol{e}_i, \omega)$ satisfy $\partial_\tau \lrcorner\, \boldsymbol{e}_i = \partial_\tau \lrcorner\, \omega = 0$, and

$$\begin{aligned} \mathrm{d}\boldsymbol{e}_3 &= -2\cos\theta\omega \wedge \boldsymbol{e}_3 - \Lambda\cos\theta\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \\ \mathrm{d}(\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2) &= (-2\mathrm{e}^{-\mathrm{i}\theta}\omega - \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\Lambda\boldsymbol{e}_3) \wedge (\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2). \end{aligned} \tag{2.26}$$

Proof. To establish this result we shall use the same strategy as in the proof of proposition 2.3. The calculations are further complicated by the presence of θ, but the main steps are as before:

[7] Equations (2.18) hold, but not in the 'obvious' frame $\mathbf{e}_1 = \sigma_1$, $\mathbf{e}_2 = \sigma_2$, $\mathbf{e}_3 = \Lambda\sigma_3$. See [6] for relevant formulae.

use the gauge freedom to set $\widetilde{U}$ to a constant, explore the Killing spinor equations to solve for the Maxwell potential A and use the Frobenius theorem to construct a triad of 1-forms out of the Killing spinors defining a conformal structure on a three-manifold.

Using (2.7) and (2.8), we find

$$\nabla_a U^{-1} = 2U^{-1}\cos\theta A_a + \alpha_A\hat{\beta}_{A'} - \hat{\alpha}_A\beta_{A'} + \frac{1}{\Lambda}\phi_{AB}(\mathrm{e}^{-\mathrm{i}\theta}\alpha^B\hat{\beta}_{A'} - \mathrm{e}^{\mathrm{i}\theta}\hat{\alpha}^B\beta_{A'}),$$
$$\nabla_a\widetilde{U}^{-1} = 2\widetilde{U}^{-1}\cos\theta A_a + \Lambda(\alpha_A\hat{\beta}_{A'} - \hat{\alpha}_A\beta_{A'}),$$

where U and $\widetilde{U}$ are defined by (2.12). Use the gauge freedom in scalings of the spinors to set $\widetilde{U}$ to a constant. This gives an expression for A. Set

$$T^a = \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\theta}\alpha^A\hat{\beta}^{A'} - \mathrm{e}^{\mathrm{i}\theta}\hat{\alpha}^A\beta^{A'}),$$

and define a real 1-form W by $\alpha_A\hat{\beta}_{A'} = \sqrt{2}^{-1}\mathrm{e}^{\mathrm{i}\theta}(T_a + \mathrm{i}W_a)$, so that

$$g(W,W) = g(T,T) = (U\widetilde{U})^{-1}.$$

We also define two real 1-forms $\mathbf{e}_1$ and $\mathbf{e}_2$ by $Z = f\sqrt{2}^{-1}(\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2)$, where $Z^a = \alpha^A\beta^{A'}$ and f is some function. Now introduce a local coordinate τ such that $T^a\nabla_a = \partial/\partial\tau$ and so

$$T = (U\widetilde{U})^{-1}(\mathrm{d}\tau + \alpha) \tag{2.27}$$

for some 1-form α which in general depends on τ. Calculating $T^a\nabla_a U^{-1}$ yields

$$\frac{\partial}{\partial\tau}U^{-1} = \sqrt{2}\cos\theta U^{-1}(\widetilde{U}^{-1} - \Lambda U^{-1}).$$

There are two cases to consider. If $U = \lambda\widetilde{U} = \text{const}$, then $\phi_{AB} = 0$, so $dA = 0$ and without loss of generality we can set $A = 0$ in some gauge. Using an argument analogous to the one leading to (2.22), we find that g is a hyperbolic metric with constant scalar curvature (this has $\Lambda > 0$ in our conventions). Otherwise, we have

$$\tau = \frac{1}{\mu}\left(\ln\frac{\widetilde{U}}{(U - \Lambda\widetilde{U})}\right) + \ln(c), \qquad \text{where} \quad \mu = \sqrt{2}\widetilde{U}^{-1}\cos\theta, \qquad c = \text{const}.$$

The Killing spinor equations give

$$\mathrm{d}Z = 2\,\mathrm{e}^{\mathrm{i}\theta}A\wedge Z - 2UZ\wedge\bar{Y} - 2\Lambda\widetilde{U}Z\wedge Y,$$

where $Y_a = \alpha_A\hat{\beta}_{A'}$. Set $Z = f(\tau)Z_0$, where $\dot{f}/f = \sqrt{2}\exp(-\mathrm{i}\theta)\widetilde{U}^{-1}$ so that

$$\mathrm{d}Z_0 = \left(\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\theta}\widetilde{U}^{-1}\alpha + \mathrm{i}\left(\frac{\sqrt{2}\Lambda\widetilde{U}}{\cos\theta} - \sqrt{2}\mathrm{e}^{-\mathrm{i}\theta}U\right)W\right)\wedge Z_0.$$

We also find

$$\mathrm{d}T + \mathrm{i}\mathrm{d}W = \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\theta}(\mathrm{i}U\;T\wedge W + (U - \Lambda\widetilde{U})|f|^2 Z_0\wedge\bar{Z}_0)$$

and

$$\mathrm{d}W = \sqrt{2}\cos\theta(UT\wedge W - (U - \Lambda\widetilde{U})|f|^2\mathbf{e}_1\wedge\mathbf{e}_2).$$

Defining a 1-form $\mathbf{e}_3$ by $W = g(\tau)\mathbf{e}_3$, where $g = g(0)\exp(\mu\tau)$ and substituting this into the expression for $\mathrm{d}W$ yields

$$\mathrm{d}\mathbf{e}_3 = \sqrt{2}\widetilde{U}^{-1}\cos\theta\alpha\wedge\mathbf{e}_3 - \sqrt{2}\beta\cos\theta\mathbf{e}_1\wedge\mathbf{e}_2, \tag{2.28}$$

where $\beta = \widetilde{U}|f(0)|^2/(cg(0))$ is a constant. Similarly, the expression for $\mathrm{d}Z_0$ yields

$$\mathrm{d}(\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2) = \left(\sqrt{2}\mathrm{e}^{-\mathrm{i}\theta}\widetilde{U}^{-1}\alpha + \mathrm{i}\left(\frac{2\Lambda\widetilde{U}}{\cos\theta} - 2\mathrm{e}^{-\mathrm{i}\theta}U\right)g\mathbf{e}_3\right)\wedge(\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2). \tag{2.29}$$

We now have to establish the dependence of α on τ. The Killing spinor equations yield

$$\nabla_{(a}T_{b)} = 2\cos\theta A_{(a}T_{b)} + \sqrt{2}^{-1}(\widetilde{U}^{-1} + \Lambda U^{-1})\cos\theta g_{ab}.$$

Therefore, $\mathcal{L}_T h = \theta\ h$, where h_{ab} is the part of g_{ab} orthogonal to T^a and the last equality is valid modulo T. Thus, T_a is a conformal retraction. Moreover, this retraction is ASD in the sense of [4] as dA is ASD. We further find

$$\mathcal{L}_T T_a = \sqrt{2}\cos\theta(\widetilde{U}^{-1} - \Lambda U^{-1})T_a - \sqrt{2}U^{-1}\sin\theta\, W_a.$$

Finally, using (2.27) gives

$$\alpha = -\omega + \Lambda\widetilde{U}^2 g(0)\tan\theta\, \mathrm{e}^{\mu\tau}\,\mathbf{e}_3,$$

where ω is some τ-independent 1-form orthogonal to $\partial/\partial\tau$. To obtain equations (2.26) in the proposition, we substitute this expression into (2.28) and (2.29), and make the following choices for the so far unspecified constants

$$\widetilde{U} = \frac{1}{\sqrt{2}}, \qquad g(0) = c\Lambda, \qquad \beta = \sqrt{2}^{-1}\Lambda$$

which are consistent if we also choose $(f(0))^2 = (g(0))^2$. Note that c can also be chosen arbitrarily by adding a constant to τ. To obtain the formulae in the proposition we set $c = \Lambda^{-1}$. The metric g is given by

$$g = U\widetilde{U}(T^2 + Z^2 + |f|^2((\mathbf{e}_1)^2 + (\mathbf{e}_2)^2)),$$

where

$$U = \frac{\widetilde{U} + c\widetilde{U}\Lambda\,\mathrm{e}^{\mu\tau}}{c\,\mathrm{e}^{\mu\tau}}.$$

This, with our choice of constants, gives (2.25). □

Remark. A three-dimensional EW structure consists of a conformal structure $[h]$ and a torsion-free connection D such that

$$D_i h_{jk} = \nu_i h_{jk}, \qquad R_{ij} + \tfrac{1}{2}\nabla_{(i}\nu_{j)} + \tfrac{1}{4}\nu_i\nu_j = \mathcal{W}h_{jk}, \qquad i, j, k = 1, 2, 3,$$

where ν is a 1-form, ∇_i and R_{ij} are respectively the Levi-Civita connection and the Ricci tensor of $h \in [h]$, and $\mathcal{W}$ is a function which can be read off by taking a trace of both sides of the latter equation. In [9], it was shown that

$$h = (\mathbf{e}_1)^2 + (\mathbf{e}_2)^2 + (\mathbf{e}_3)^2, \qquad \nu = -4\omega\cos\theta - 4\Lambda\sin\theta\mathbf{e}_3, \qquad \Lambda = \text{const}$$

satisfy the EW equations if the triad $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ satisfies (2.26). Moreover, the EW structure arising this way is the most general symmetry reduction of hyper-Käher metric in four dimensions by a conformal symmetry.

3. Euclidean Kastor–Traschen solutions

In this section, we shall drop the ASD condition on the Maxwell field so that

$$F_{ab} = \phi_{AB}\varepsilon_{A'B'} + \tilde{\phi}_{A'B'}\varepsilon_{AB}.$$

Thus, the Killing spinor equations (2.2) are replaced by

$$\nabla_{AA'}\alpha_B + c_0 A_a\alpha_B + (c_1\phi_{AB} + c_2\varepsilon_{AB})\beta_{A'} = 0,$$
$$\nabla_{AA'}\beta_{B'} + c_3 A_a\beta_{B'} + (c_5\tilde{\phi}_{A'B'} + c_4\varepsilon_{A'B'})\alpha_A = 0,$$

where an additional term involving $\tilde{\phi}_{A'B'}$ is present. We now impose the integrability conditions (2.3)–(2.6) and proceed as before, but use the Einstein–Maxwell condition

$$\Phi_{ABA'B'} = 2\phi_{AB}\tilde{\phi}_{A'B'}.$$

We find that

$$c_0 = c_3 = \frac{1}{L}, \qquad c_2 = \frac{1}{2L}c_1, \qquad c_5 = -\frac{2}{c_1}, \qquad c_4 = -\frac{1}{Lc_1},$$

where

$$\Lambda = \frac{1}{2L^2}$$

is a cosmological constant (which at this stage can be positive or negative if L is real or imaginary respectively). A constant rescaling of $\beta_{A'}$ can be used to set c_1 to any given nonzero constant. To achieve a symmetric form of the equations, we replace $\beta_{A'}$ by $\sqrt{2}\beta_{A'}/c_1$ which results in $c_1 = \sqrt{2}$. The final form of the Killing spinor condition is

$$\begin{aligned} &\nabla_{AA'}\alpha_B + \frac{1}{L}A_a\alpha_B + \sqrt{2}\left(\phi_{AB} + \frac{1}{2L}\varepsilon_{AB}\right)\beta_{A'} = 0, \\ &\nabla_{AA'}\beta_{B'} + \frac{1}{L}A_a\beta_{B'} - \sqrt{2}\left(\tilde{\phi}_{A'B'} + \frac{1}{2L}\varepsilon_{A'B'}\right)\alpha_A = 0. \end{aligned} \tag{3.1}$$

We conclude that the non-ASD case is more 'rigid' than the ASD one. The Killing spinor equations (2.7) with an ASD Maxwell field contain one essential parameter θ. If the Maxwell field is not ASD (or SD), the integrability conditions fix all the parameters in terms of the cosmological constant. Further analysis depends on the sign of the cosmological constant. If $\Lambda < 0$, the resulting metric admits a Killing vector $K_a = \mathrm{i}(\hat{\alpha}_A\beta_{A'} + \alpha_A\hat{\beta}_{A'})$ and is given by a Riemannian analog of the Caldarelli–Klemm solution [3, 5]. If $\Lambda > 0$, the symmetry is not present in general, and the metric is a Riemannian version of the solutions obtained in [16, 24].

In the proposition given below we shall characterize Riemannian Kastor–Traschen solutions [20] as those where the ratio of norms of the spinors α_A and $\beta_{A'}$ is a constant. We define two real nonzero functions U and $\widetilde{U}$ by

$$U = (\varepsilon_{AB}\hat{\alpha}^A\alpha^B)^{-1}, \qquad \widetilde{U} = (\varepsilon_{A'B'}\hat{\beta}^{A'}\beta^{B'})^{-1}.$$

as before. The gauge transformations $\alpha \to \mathrm{e}^f\alpha$ and $\beta \to \mathrm{e}^f\beta$, where $f : M \longrightarrow \mathbb{R}$, result in

$$U \to \mathrm{e}^{-2f}U, \qquad \widetilde{U} \to \mathrm{e}^{-2f}\widetilde{U}$$

so that the ratio $U/\widetilde{U}$ is gauge invariant. In the rest of this section, we shall assume $L = l \in \mathbb{R}$ and the cosmological constant is positive.

Proposition 3.1. *Let the Riemannian four-manifold (M, g) admit a solution to the Killing spinor equations (3.1) with $\Lambda > 0$ such that the gauge invariant condition*

$$\frac{U}{\widetilde{U}} = \mathrm{const}$$

holds. Then, (M, g) is Einstein–Maxwell with[8] *$F = 2\,\mathrm{d}A$ and the local coordinates (x, y, z, T) can be chosen so that*

$$g = \left(\mathcal{U} + \frac{1}{l}T\right)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \left(\mathcal{U} + \frac{1}{l}T\right)^{-2}\mathrm{d}T^2, \qquad F = 2\,d\left(\frac{\mathrm{d}T}{\mathcal{U} + l^{-1}T}\right) \tag{3.2}$$

[8] The sign of the energy–momentum tensor in this example is opposite to the one in (1.1). This sign can be changed if desired as explained in the introduction by using the Maxwell field $F = 2 * \mathrm{d}((\mathcal{U} + l^{-1}T)^{-1}\mathrm{d}T)$.

where $\mathcal{U} = \mathcal{U}(x, y, z)$ satisfies the Laplace equation on $\mathbb{R}^3$:

$$\frac{\partial^2\mathcal{U}}{\partial x^2} + \frac{\partial^2\mathcal{U}}{\partial y^2} + \frac{\partial^2\mathcal{U}}{\partial z^2} = 0.$$

Proof. The Einstein–Maxwell equations with $\Lambda > 0$ follow from the integrability conditions for (3.1). To find the local form of the metric, first choose a gauge

$$U\widetilde{U} = 1.$$

Set $L = l \in \mathbb{R}$. The Killing spinor equations and their conjugates can be used to find

$$\nabla_a(U^{-1}) = -\frac{2}{l}A_aU^{-1} - \sqrt{2}\phi_A^B X_{BA'} - \frac{1}{l\sqrt{2}}X_a = 0,$$
$$\nabla_a(\widetilde{U}^{-1}) = -\frac{2}{l}A_a\widetilde{U}^{-1} - \sqrt{2}\tilde{\phi}_{A'}^{B'} X_{AB'} - \frac{1}{l\sqrt{2}}X_a = 0.$$

These equations imply

$$U = \widetilde{U} = 1, \qquad A_aX^a = -\frac{1}{\sqrt{2}}, \qquad \tilde{\phi}_{A'}^{B'}X_{AB'} = \phi_A^B X_{BA'}.$$

The expression for A is found to be

$$A_a = -\frac{l}{2\sqrt{2}}\left(E_a + \frac{1}{l}X_a\right), \tag{3.3}$$

where $E_a = 2\phi_A{}^B X_{BA'} = X^b F_{ab}$. We also find

$$F = E \wedge X. \tag{3.4}$$

Further application of the Killing spinor equations (3.1) gives

$$\begin{aligned} \mathrm{d}Z &= (-l^{-1}\sqrt{2}X - 2l^{-1}A) \wedge Z, \\ \mathrm{d}K &= (-l^{-1}\sqrt{2}X - 2l^{-1}A) \wedge K, \\ \mathrm{d}X &= -2l^{-1}A \wedge X - \sqrt{2}F, \end{aligned} \tag{3.5}$$

and consequently

$$\mathcal{L}_X Z = -\frac{\sqrt{2}}{l}Z, \qquad \mathcal{L}_X K = -\frac{\sqrt{2}}{l}K, \qquad X \wedge \mathrm{d}X = 0.$$

The integrability conditions for the first two equations in (3.5) come down to $\mathrm{d}(X + \sqrt{2}A) = 0$, so that locally

$$X + \sqrt{2}A = \mathrm{d}\gamma \tag{3.6}$$

for some function γ. Let τ be a local coordinate such that $X^a\nabla_a = \partial/\partial\tau$. We find that $X(\gamma) = 1$, so $\gamma = \tau + \tilde{\gamma}$, where $\tilde{\gamma}$ is a function which does not depend on τ. The 1-form dual to $X = \frac{\partial}{\partial\tau}$ is $X = 2(\mathrm{d}\tau + \Omega)$ for some 1-form Ω. We can fix the value of the constant $\tilde{\gamma}$ reabsorbing $\mathrm{d}\tilde{\gamma}$ into the definition of Ω. Equations (3.5) now imply the existence of real local coordinates (x, y, z) such that

$$K = \frac{1}{\sqrt{2}}\mathrm{e}^{-\sqrt{2}\tau/l}\,\mathrm{d}z, \qquad Z = \frac{1}{2\sqrt{2}}\mathrm{e}^{-\sqrt{2}\tau/l}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y)$$

and combining (3.3) with $X + \sqrt{2}A = \mathrm{d}\tau$ yields $\Omega = lE/2$ so that the metric is

$$g = \mathrm{e}^{-2\sqrt{2}\tau/l}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + 2\left(\mathrm{d}\tau + l\frac{E}{2}\right)^2.$$

Using (3.6) and (3.4), we calculate $\mathrm{d}X = l\,\mathrm{d}E = -(\sqrt{2})^{-1}F$ and

$$\mathcal{L}_X E = -\frac{1}{l\sqrt{2}} X \lrcorner F = -\frac{1}{l\sqrt{2}} X \lrcorner (E \wedge X)$$
$$= \frac{\sqrt{2}}{l} E$$

as $X \cdot X = 2$. Therefore, $E = \mathrm{e}^{\sqrt{2}\tau/l}\omega$, where ω is a 1-form independent on τ. The condition $X \wedge \mathrm{d}X = 0$ implies that $\mathrm{d}\omega = 0$ so that locally $\omega = \mathrm{d}\phi$, where $\phi = \phi(x, y, z)$ is some function. Using (3.6), we find

$$F = 2\,\mathrm{e}^{\sqrt{2}\tau/l}\,\mathrm{d}\phi \wedge \mathrm{d}\tau.$$

Moreover,

$$*F = \frac{4\sqrt{2}}{l} *_3 \mathrm{d}\phi,$$

where $*_3$ is the Hodge operator of the flat 3-metric. Therefore, the Maxwell equation $d*F = 0$ implies that ϕ is harmonic on $\mathbb{R}^3$ and

$$g = \mathrm{e}^{-2\sqrt{2}\tau/l}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + 2\left(\mathrm{d}\tau + \frac{l}{2}\,\mathrm{e}^{\sqrt{2}\tau/l}\,\mathrm{d}\phi\right)^2.$$

To put the metric and the Maxwell field in the form (3.2) set

$$T = l\frac{\phi}{\sqrt{2}} - l\mathrm{e}^{-\sqrt{2}\tau/l}, \qquad \mathcal{U} = -\frac{\phi}{\sqrt{2}}.$$

□

Solution (3.2) can be obtained as an analytic continuation of the Kastor–Traschen cosmological black holes [20]. This continuation requires the sign of the cosmological constant to change.

Example. Setting $\mathcal{U} = 0$ in (3.2) gives the hyperbolic space. Consider $\mathcal{U} = m/R$, where m is a constant, and R is the radial coordinate on $\mathbb{R}^3$ so that the metric becomes

$$g = \left(\frac{m}{R} + \frac{T}{l}\right)^2\left(\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \left(\frac{m}{R} + \frac{T}{l}\right)^{-2}\mathrm{d}T^2. \qquad (3.7)$$

This metric admits an isometry $(R, T) \to (c^{-1}R, cT)$ generated by the Killing vector

$$\mathcal{K} = T\frac{\partial}{\partial T} - R\frac{\partial}{\partial R}.$$

Introduce the coordinates (s, r) adapted to this isometry by

$$R = \mathrm{e}^{-s/l}, \qquad T = l(r - m)\,\mathrm{e}^{s/l}$$

so that $\mathcal{K}(s) = 0$ and $\mathcal{K}(r) = 1$. Let $\psi = \psi(s, r)$ be a function such that

$$\mathrm{d}\psi = \mathrm{d}s + \frac{l(r - m)}{Vr^2}\,\mathrm{d}r,$$

where

$$V = \frac{r^2}{l^2} + \left(1 - \frac{m}{r}\right)^2.$$

The metric then takes the form

$$g = V\mathrm{d}\psi^2 + V^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \qquad (3.8)$$

It closely resembles the analytic continuation of the RNdS metric (1.2) described in the introduction in the extremal case $|Q| = m$. The difference between these two solutions lies in the sign of the cosmological constant. The extremal RNdS instanton with $\Lambda < 0$ has been named the lukewarm instanton in [27]. The conical singularities in the metric are not present as the black hole and the cosmological horizons have the same Hawking temperatures, i.e. $V(r_1) = V(r_2) = 0$ at these horizons and $|V'(r_1)| = |V'(r_2)|$. This instanton has been interpreted [22, 27] as describing a pair creation of non-extreme black holes in thermal equilibrium.

In our case $\Lambda > 0$. At $r \to \infty$, the metric approaches the constant curvature hyperbolic space. The limit $r \to 0$ is singular. This reflects the fact that the metric (3.7) is an analytic continuation of the Lorentzian RNdS spacetime, where the singularity is not hidden inside a horizon.

Example. If $\Lambda = 0$ and

$$\mathcal{U} = c + \sum_{m=1}^{N} \frac{a_m}{|\mathbf{x} - \mathbf{x}_m|}, \qquad a_1, \ldots, a_N, c = \text{const},$$

then (3.2) becomes the Majumdar–Papapetrou Einstein–Maxwell multi instanton [10]. The metric is asymptotically locally Robinson–Bertotti if $c = 0$, or asymptotically flat if $c \neq 0$ and T is periodic. In [10], it was shown how these instantons can be lifted to regular solitonic solutions to $\mathcal{N} = 2$ minimal five-dimensional SUGRA. It remains to be seen whether the solutions (3.2) with nonzero Λ can also be uplifted to higher dimensions.

4. Conclusions

We have classified supersymmetric solutions of the minimal $N = 2$ gauged Euclidean SUGRA in four dimensions, under the additional assumptions that the Maxwell field is ASD. The resulting metrics are Einstein, have ASD Weyl curvature and are given in terms of solutions to three-dimensional EW equations. We have also found one class of examples corresponding to non-ASD Maxwell field. These examples are Euclidean analogs of Kastor–Traschen cosmological metrics. The solutions constructed in this paper provide new examples of Einstein metrics in four dimensions. It remains to be seen whether they can be used to describe cosmological black hole creations and in the context of Euclidean quantum gravity.

Acknowledgments

MD, JG and PT thank the American University of Beirut for hospitality when some of this work was carried over. The work of WS is supported in part by the National Science Foundation under grant no PHY-0903134. JG is supported by EPSRC grant EP/F069774/1. The authors thank David Calderbank for the interesting comments on the manuscript.

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