Jumps, folds and singularities of Kodaira moduli spaces

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Abstract

For any integer $k$ we construct an explicit example of a twistor space which contains a one-parameter family of jumping rational curves, where the normal bundle changes from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(k) \oplus \mathcal{O}(2 - k)$. For $k > 3$ the resulting anti-self-dual Ricci-flat manifold is a Zariski cone in the space of holomorphic sections of $\mathcal{O}(k)$. In the case $k = 2$ we recover the canonical example of Hitchin’s folded hyper-Kähler manifold, where the jumping lines form a three-parameter family. We show that in this case there exist normalisable solutions to the Schrödinger equation which extend through the fold.

1. Introduction

The non-linear graviton twistor construction of Penrose [18] gives a one-to-one correspondence between holomorphic anti-self-dual (ASD) Ricci-flat metrics on complex four-manifolds $M_C$, and complex threefolds $Z$ with a family of rational curves. The points in $M_C$ correspond to holomorphic sections of $Z \rightarrow \mathbb{CP}^1$ characterised by their normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, where $\mathcal{O}(k)$ is a line bundle over $\mathbb{CP}^1$ with Chern class $k$.

If the normal bundles of rational curves corresponding to points on a surface $S \subset M_C$ (of co-dimension one or more) change, then the metric becomes singular on $S$. There are other geometric structures, most notably the self-dual two-forms spanning $\Lambda^2$, which nevertheless remain regular on $S$. In the case of a single jump to $\mathcal{O}(2) \oplus \mathcal{O}$ this results in a folded hyper-Kähler structure in the sense of Hitchin [12]. Examples of such structures, and the underlying existence theorems are known [2], and some applications in theoretical physics have recently emerged [17].

The aim of this paper is to construct an explicit example of a twistor space where the normal bundle jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(k) \oplus \mathcal{O}(2 - k)$ for any integer $k > 2$. This jump occurs on a curve $\gamma \subset M_C$, and the corresponding metric on $M_C$ can be constructed explicitly. It admits a tri-holomorphic Killing vector, and so away from the jump it can be put in the standard Gibbons–Hawking form [10] by a coordinate transformation. This transformation removes the region of $M_C$ where the jump occurs. The resulting metric on $M_C \setminus \gamma$ is still singular on the surface $S$ where the Gibbons–Hawking harmonic function vanishes. The normal bundles of twistor lines corresponding to the points on $S$ jump to $\mathcal{O}(2) \oplus \mathcal{O}$.

In the next section we will set up the twistor correspondence, where the non-deformed twistor space $Z$ is an affine line bundle over the total space of $\mathcal{O}(k)$, for any $k$. In Theorem 3.1 we find Kodaira deformations preserving this affine bundle, and leading to a four-manifold $M_C$ arising as a Zariski cone in the $(k + 1)$-dimensional space of holomorphic sections $H^0(\mathbb{CP}^1, \mathcal{O}(k))$. In Sections 3.1 and 3.2 we give expressions for the metric in cases where $k = 3$ and $k = 4$, and show that the Gibbons–Hawking potential

$$V(X, Y, Z) = \frac{1}{2(k - 1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} (\lambda^2 (X - Y) + 2\lambda Z + (X + Y))^{-1/2}|_{\lambda=0},$$

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on $\mathbb{C}^3$ or $\mathbb{R}^{1,2}$ corresponds to the general $k$. In Section 4 we give an explicit coordinate transformation between a real form of the Sparling–Tod solution and the Eguchi–Hanson gravitational instanton and its dyonic limit.

In Section 5 it will be shown how a cascade of intermediate jumps

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathcal{O}(2) \oplus \mathcal{O} \rightarrow \cdots \rightarrow \mathcal{O}(k) \oplus \mathcal{O}(2-k)$$

arises on surfaces on $M_C$ with various co-dimensions. In Section 6 we will put the construction in the framework of the generalised Legendre transform [1, 7, 14, 15], and show how the Zariski cone $M_C$ arises as the zero–locus of a zero-rest-mass field corresponding to a cohomology class in $H^1(\mathcal{O}(k), \mathcal{O}(2-k))$.

Finally in Section 7 we will come back to the case $k = 2$, where the metric arising from Theorem 3.1 admits a Riemannian real slice $(M, g)$, which is the canonical model of a folded hyper-Kähler structure

$$g = Z(dX^2 + dY^2 + dZ^2) + Z^{-1}\left(dt + \frac{1}{2}XdY - \frac{1}{2}YdX\right)^2, \quad (1.1)$$

where $S \subset M$ given by $Z = 0$ is the fold. Answering a question of Manton, we will show that despite the blow-up in the metric there exist normalisable solutions to the Schrödinger equation which extend through the fold.

2. Twistor spaces as affine bundles

We will start off by reviewing the twistor correspondence [5, 13, 18]. Let $M_C$ be a complex four-manifold with a holomorphic orientation vol, and a holomorphic Ricci-flat metric $g$ such that the Weyl tensor is ASD. The anti-self-duality is the Frobenius integrability condition for the existence of a three-parameter family of self–dual totally null surfaces ($\alpha$–surfaces) in $M_C$, and the twistor space $Z$ is the three-dimensional complex manifold with the $\alpha$–surfaces as points. This leads to a double fibration picture

$$M_C \leftarrow \mathcal{F} \xrightarrow{\rho} Z,$$

where $\mathcal{F} \subset M_C \times Z$ is the space of incident pairs $(p, \xi)$ such that $p \in M_C$ lies on an $\alpha$-surface $\xi \subset M_C$. A point in $M_C$ corresponds to a projective line $L_p \cong \mathbb{C}P^1$ in $Z$ which consists of all $\alpha$-surfaces through $p$. A conformal structure $[g]$ on $M_C$ is encoded in the algebraic geometry of curves in $Z$: two points in $M_C$ are null-separated iff the corresponding curves in $Z$ intersect in one point.

There are two additional structures on $Z$ resulting from the existence of a Ricci-flat metric $g \in [g]$, and the canonical isomorphism

$$TM_C = S \otimes S',$$

where $S$ and $S'$ are two rank-two complex symplectic vector bundles over $M_C$.

- The Levi-Civita connection of $g$ gives a flat spin connection$^\dagger$ on $S$. Thus there exists a two-dimensional space of parallel sections of $S$. This, together with the isomorphism $\Lambda^2_+ = S \otimes S$ and a natural identification $\mathcal{F} = \mathbb{P}(S)$, gives a holomorphic projection

$$\mu : Z \rightarrow \mathbb{C}P^1, \quad (2.1)$$

such that the points in $M_C$ are holomorphic sections of $\mu$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

$^\dagger$To avoid the repeated usage of primed spinor indices in Section 6 we depart from the usual twistor conventions, and swap the roles of primed and unprimed indices.
The parallel basis \((\Sigma^{00}, \Sigma^{01}, \Sigma^{11})\) of \(A^2\) gives rise to a symplectic two-form \(\Sigma\) on the fibres of (2.1) which takes values in the line bundle \(O(2)\). The pull-back of \(\Sigma\) from \(Z\) to \(\mathbb{P}(S)\) is
\[
\rho^*(\Sigma) = \Sigma^{11} - 2\lambda \Sigma^{01} + \lambda^2 \Sigma^{00},
\]
where \(\lambda\) is an affine coordinate on the fibres of \(\mathbb{P}(S)\).

The two–form \(\Sigma\) fixes a volume form \(\text{vol}\) on \(M_C\): the condition \(\Sigma \wedge \Sigma = 0\) gives
\[
\text{vol} = \Sigma^{00} \wedge \Sigma^{11} = -2\Sigma^{01} \wedge \Sigma^{01}.
\]

### 2.1. Jumping lines

Let \([\pi_0, \pi_1]\) be homogeneous coordinates on \(\mathbb{CP}^1\). Cover \(\mathbb{CP}^1\) with two open sets
\[
U = \{[\pi] \in \mathbb{CP}^1, \pi_1 \neq 0\}, \quad \tilde{U} = \{[\pi] \in \mathbb{CP}^1, \pi_0 \neq 0\}
\]
and set \(\lambda = \pi_0/\pi_1\) on \(U \cap \tilde{U}\). The Birkhoff–Grothendieck theorem states that any rank-two holomorphic vector bundle over \(\mathbb{CP}^1\) is isomorphic to a direct sum of line bundles \(O(p) \oplus O(q)\) for some integers \(p, q\). Moreover the transition matrix \(F : \mathbb{C}^* \to GL(2, \mathbb{C})\) of this bundle can be written as
\[
F = \tilde{H} \text{diag}(\lambda^{-p}, \lambda^{-q}) H^{-1},
\]
where \(H : U \to GL(2, \mathbb{C})\) and \(\tilde{H} : \tilde{U} \to GL(2, \mathbb{C})\) are holomorphic.

Let \(Z_0 \longrightarrow \mathbb{CP}^1\) be a one-parameter family of rank-two vector bundles determined by the patching matrix
\[
F_b = \begin{pmatrix} \lambda^{k-2} & b\lambda^{-1} \\ 0 & \lambda^{-k} \end{pmatrix},
\]
where \(b\) is a constant and \(k\) is a positive integer. If \(b = 0\) then \(F_0 = \text{diag}(\lambda^{k-2}, \lambda^{-k})\) is the patching matrix for \(Z_0 = O(2-k) \oplus O(k)\) with \(H\) and \(\tilde{H}\) in (2.3) both equal to the identity matrix. If \(b \neq 0\) then
\[
F_b = \begin{pmatrix} 0 & b \\ -b^{-1} & \lambda^{1-k} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^{-1}\lambda^{-k} & 1 \end{pmatrix}^{-1}
\]
which is of the form (2.3). Thus \(Z_b = O(1) \oplus O(1)\) if \(b \neq 0\). This is the twistor space with the holomorphic sections of \(Z_b \to \mathbb{CP}^1\) parametrised by points in \(M_C = \mathbb{C}^4\) with the flat metric\(^\dagger\).

### 2.2. Twistor space as an affine bundle over \(O(k)\)

Let \((Q, \lambda)\) and \((\tilde{Q}, \tilde{\lambda} = \lambda^{-1})\) be coordinates on the pre-images of \(U\) and \(\tilde{U}\) in the total space of the line bundle \(O(k) \to \mathbb{CP}^1\). On the pre-image of \(U \cap \tilde{U}\) in \(Z_b\) we have
\[
\tilde{Q} = \lambda^{-k}Q, \quad \tilde{\tau} = \lambda^{-2}\tau + b\lambda^{-1}Q,
\]
where \(\tau\) and \(\tilde{\tau}\) are coordinates on the fibres of \(Z_b\) over \(U\) and \(\tilde{U}\), respectively. Restricting the inhomogeneous coordinates to a section of \(O(k) \to \mathbb{CP}^1\)
\[
Q = x_k\lambda^k + \cdots + x_1\lambda + x_0
\]
and performing the splitting of (2.4) gives
\[
\tilde{\tau} - b\lambda^{-1}x_0 - bx_1 = \lambda^{-2}\tau + b\lambda^{-1}x_k + \cdots + b\lambda x_2.
\]
\(^\dagger\)If \(k = 2\), and \(b\) is interpreted as the inverse of the speed of light, then the jumping from \(Z_b\) to \(Z_0\) is the Newtonian limit of the twistor correspondence [6].
The LHS and RHS of (2.6) are holomorphic on $\tilde{U}$ and $U$, respectively. Both sides of this relation are sections of a line bundle with a negative Chern class, so should vanish by the Liouville theorem. Thus
\[ \tau = -b(\lambda x_k - x_{k-1} - \lambda^{-1}x_{k-2} - \cdots - \lambda^{k-3}x_2). \] (2.7)

This will be holomorphic in $\lambda$ on $U$ if $(k-3)$ conditions
\[ x_{k-2} = x_{k-3} = \cdots = x_2 = 0 \] (2.8)
hold. These conditions arise only if $k > 3$. They define a holomorphic four-dimensional subspace $M_C$ in the $(k+1)$-dimensional space of holomorphic sections of $O(k)$.

The algebraic geometry of holomorphic sections of $Z_b \to \mathbb{CP}^1$ determines a conformal structure on $M_C$: two points in $M_C$ are null separated iff the corresponding sections intersect at one point in $Z_b$. Infinitesimally, a vector in $T_pM_C$ is null if the corresponding section of $N(L_p)$ vanishes at one point. This condition is equivalent to the existence of the unique solution $\lambda = \lambda_0$ to a simultaneous system
\[ \delta Q = 0, \quad \delta \tau = 0, \] (2.9)
where $Q$ and $\tau$ are given by (2.5) and (2.7). These conditions give
\[ \lambda^k \delta x_k + \cdots + \lambda \delta x_1 + \delta x_0 = 0, \quad \lambda \delta x_k + \delta x_{k-1} = 0. \]

Imposing (2.8) and using the second equation replaces the first equation by $\lambda \delta x_1 + \delta x_0 = 0$. Eliminating $\lambda$ between the two equations in (2.9) gives the quadratic conformal structure
\[ [g] = \delta x_k \delta x_0 - \delta x_{k-1} \delta x_1 \] (2.10)
which is flat.

3. Conformal structures on Zariski cones from Kodaira deformations

Consider an affine line bundle $Z \to O(k)$, with underlying translation bundle given by $O(2-k)$. Such bundles are classified by elements of $H^1(O(k), O(2-k))$ and we choose a cohomology representative which leads to the patching relations
\[ \tilde{Q} = \lambda^{-k}Q, \quad \tilde{\tau} = \lambda^{-2} \tau + a \lambda^{-2}Q^2, \quad \text{where } a = \text{const.} \] (3.1)

Restricting this to holomorphic sections (2.5) of $Z \to \mathbb{CP}^1$ and splitting gives
\[ \tilde{\tau} - a \lambda^{-2}(x_0^2 + 2\lambda x_0 x_1 + \lambda^2(2x_0 x_2 + x_1^2)) \]
\[ = \lambda^{k-2} \tau + a\lambda(2x_0 x_3 + 2x_1 x_2) + \cdots + a\lambda^{2k-2}x_k. \] (3.2)

Therefore $\tau$ is holomorphic in $\lambda$ if $(k-3)$ quadratic conditions
\[ x_0 x_3 + x_1 x_2 = 0, \quad x_0 x_4 + x_1 x_3 + \frac{x_2^2}{2} = 0, \ldots, x_0 x_{k-1} + x_1 x_{k-2} + \cdots \] (3.3)
hold. These constraints put no restrictions on $x_k$, and we can assume that
\[ t = x_0, \quad z = x_1, \quad y = x_2, \quad x = x_k \]
are coordinates on an open set in $M_C$, and that the remaining coordinates $(x_3, \ldots, x_{k-1})$ have been expressed as functions of $(y, z, t)$. To compute the ASD conformal structure $[g]$ on $M_C$ we follow the procedure leading to (2.10), except that to simplify the computations the condition (2.9) is replaced by the equivalent condition $\delta \tilde{Q} = 0, \delta \tilde{\tau} = 0$ (the resulting conformal structure does not depend on the choice of the open set) and pull the differentials $\delta x_i$ in $\delta \tilde{Q}$ back to $M_C$. To eliminate $\lambda$ we take the resultant $\text{Res}$ of the quadratic $\lambda^2 \delta \tilde{\tau}$ and the polynomial of degree
space \( k \) given by \( \lambda^k \delta \tilde{Q} \). The resultant is a section of \( \text{Sym}^{(k+2)}(T^* M_C) \) which factorises as \([g](\delta x_0)^k\), where \([g] \in \text{Sym}^2(T^* M_C)\) is the conformal structure given by

\[
[g] = \text{Res} \left( x_0 \delta x_0 + \lambda (x_0 \delta x_1 + x_1 \delta x_0) + \lambda^2 (x_0 \delta x_2 + x_2 \delta x_0 + x_1 \delta x_1),
\right)
\]

\[
\delta x_0 + \lambda \delta x_1 + \cdots + \lambda^k \delta x_k \right).
\]  

(3.4)

Here \( \delta x_0, \ldots, \delta x_{k-1} \) are the pull-backs from \( H^0(\mathbb{CP}^1, O(k)) \) to \( M_C \) defined by the relations (3.3). The deformation (3.1) preserves the fibration \( \mu : Z \to \mathbb{CP}^1 \), and the fibres of \( \mu \) are equipped with an \( O(2) \)-valued symplectic form

\[
\Sigma = \lambda^{-2} dQ \wedge d\tau = d\tilde{Q} \wedge d\tilde{\tau}.
\]

Therefore there exists a Ricci-flat metric \( g \subset [g] \) in the ASD conformal class (3.4).

\[ \text{Theorem 3.1.} \quad \text{Let} \quad (M_C, g) \quad \text{be an ASD Ricci-flat manifold corresponding to the twistor space with the patching relations (3.1). There exists a curve} \quad \gamma \subset M_C \quad \text{such that all points on} \quad \gamma \quad \text{correspond to rational curves} \quad Z \quad \text{where the normal bundle jumps from} \quad O(1) \oplus O(1) \quad \text{to} \quad O(k) \oplus O(2-k), \quad \text{and such that} \quad \gamma \quad \text{is preserved by a tri-holomorphic Killing vector.} \]

\[ \text{Proof.} \quad \text{We will first prove that} \quad (M_C, g) \quad \text{admits a triholomorphic Killing vector field. The coefficients of the conformal structure (3.4) do not depend on} \quad x \equiv x_k, \quad \text{so} \quad K = \partial/\partial x \quad \text{is a conformal Killing vector. Conformal Killing vectors in} \quad M_C \quad \text{generate one-parameter groups of transformations of} \quad M_C \quad \text{which map} \alpha\text{–surfaces to} \alpha\text{–surfaces. Thus (as the points in} \quad Z \quad \text{are} \alpha\text{–surfaces in} \quad M_C \quad \text{) conformal Killing vectors correspond to global holomorphic vector fields on} \quad Z. \text{Consider the holomorphic vector field} \quad K \quad \text{in} \quad Z \quad \text{corresponding to the conformal Killing vector} \quad K = \partial/\partial x. \text{We will compute this vector field on the open set} \quad \tilde{U} \]

\[ K = \frac{\partial \tilde{Q}}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \tilde{\tau}}{\partial x} \frac{\partial}{\partial \tilde{\tau}} = \frac{\partial}{\partial \tilde{Q}}, \]

where we have used (3.2). Therefore

\[ L_K \Sigma = 0, \quad L_K \lambda = 0, \]

and so \( K \) preserves the symplectic two-form on the fibres of \( Z \to \mathbb{CP}^1 \), as well as the fibration itself. The first condition implies that \( K \) is a Killing vector of the Ricci-flat metric singled out by \( \Sigma \) in the conformal structure \([g] \). The second condition means that \( K \) is tri-holomorphic (it acts trivially on the basis of parallel self-dual two-forms).

Now consider the normal bundle \( N(L_p) \) to a curve \( L_p \subset Z \) corresponding to a point \( p \in M_C \). For a generic \( p \), the bundle \( N(L_p) \) is biholomorphic to \( O(1) \oplus O(1) \). Its patching matrix is given by

\[ F_N = \begin{pmatrix} \frac{\partial \tilde{x}}{\partial \tilde{r}} & \frac{\partial \tilde{y}}{\partial \tilde{r}} \\ \frac{\partial \tilde{r}}{\partial \tilde{Q}} & \frac{\partial \tilde{Q}}{\partial \tilde{Q}} \end{pmatrix} = \begin{pmatrix} \lambda^{k-2} & 2a\lambda^{-2}Q \\ 0 & \lambda^{-k} \end{pmatrix}. \]

To investigate the non-generic points introduce the splitting matrices

\[ H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} 1 & \tilde{h} \\ 0 & 1 \end{pmatrix}. \]
which are invertible and holomorphic in $U$ and $\tilde{U}$, respectively. Now
\[
\tilde{H}F_N H^{-1} = \begin{pmatrix} \lambda^{k-2} & 2a\lambda^{-2}Q - h\lambda^{k-2} + \tilde{h}\lambda^{-k} \\ 0 & \lambda^{-k} \end{pmatrix}.
\]
Thus the normal bundle jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(2-k) \oplus \mathcal{O}(k)$ at points of $M_C$ where
\[
2a\lambda^{-2}Q - h\lambda^{k-2} + \tilde{h}\lambda^{-k} = 0.
\]
The functions $h$ and $\tilde{h}$ are holomorphic in $\lambda$ and $\lambda^{-1}$, respectively. Therefore the equality can be satisfied by some choice of $h$ and $\tilde{h}$ only if
\[
x_{k-1} = x_{k-2} = \cdots = x_0 = 0.
\]
These conditions imply the conditions (3.3), and leave the coordinate $x_k$ unspecified. Thus there is a one–parameter family $\gamma \subset M_C$ of jumping lines in $Z$, and $K$ is tangent to $\gamma$. □

In addition to $K$, the Ricci–flat metric arising from Theorem 3.1 admits a homothetic conformal Killing vector $x\partial_x + y\partial_y + z\partial_z + t\partial_t$. In what follows we will work out the metrics and their Gibbons–Hawking forms in detail when $k = 3$ and $k = 4$. We note that in these two cases the metric admits another Killing vector, where the coordinates scale with different weights.

3.1. Ricci–flat metric with $k = 3$

Parametrise the sections of $\mathcal{O}(3) \to \mathbb{C}P^1$ by
\[
Q = t + \lambda z + \lambda^2 y + \lambda^3 x.
\]
In this case the ASD conformal structure (3.4) is
\[
g = \Omega^2 \left( t^3 dx^2 + 2t^2 z dx dy + t(yt + z^2) dx dz + z(3yt - z^2) dx dt \\
+ tz^2 dy^2 + z(yt + z^2) dy dz + y(yt + z^2) dy dt + yz^2 dz^2 + 2y^2 z dz dt + y^3 dt^2 \right).
\]
We find that the choice
\[
\Omega^2 = \frac{2a^3}{z^2 - yt}
\]
makes the resulting metric Ricci-flat. If the coordinates $(x,y,z,t)$ are chosen to be real, then $g$ is real, and has neutral signature. The basis of self-dual parallel two-forms is
\[
\Sigma^{00} = 2dx \wedge (tdy + ydt + zdz), \\
\Sigma^{01} = tdx \wedge dz + zdx \wedge dt + zdy \wedge dz + ydy \wedge dt, \\
\Sigma^{11} = 2(tdx + ydz + zdy) \wedge dt.
\]
These forms are Lie derived by the Killing vector $K = \partial/\partial x$. This Killing vector is triholomorphic and therefore the metric $g$ can be cast in the Gibbons–Hawking form
\[
g = V h_{flat} + V^{-1}(dT + A)^2, \quad \text{where} \quad T \equiv x,
\]
where $h_{flat}$ is a flat metric on $\mathbb{R}^{1,2}$, and $V$ and $A$ are, respectively, a function, and a one–form on $\mathbb{R}^{1,2}$ which satisfy the Abelian monopole equation
\[
dV = *dA,
\]
where $*$ is the Hodge endomorphism of $h_{\text{flat}}$. The function $V$ can be read-off directly from $g$, and is given by

$$V = g(K, K)^{-1} = \frac{ty - z^2}{2a^4b^3}.$$ 

To construct the flat coordinates for the metric

$$h_{\text{flat}} = V^{-1}(g - VK \otimes K)$$

we first compute the six generators of its group of isometries. We then select a three-dimensional abelian subalgebra $(X_1, X_2, X_3)$ generated by translations. The corresponding one forms $h_{\text{flat}}(X_i, \cdot)$ are exact differentials of the flat coordinates $(Y, Z, T)$, where

$$y = \frac{X^2 - Y^2 - Z^2}{2(X + Y)^{3/2}}, \quad z = \frac{Z}{(X + Y)^{1/2}}, \quad t = (X + Y)^{1/2}. \quad (3.8)$$

Now

$$h_{\text{flat}} = a^6(dX^2 - dY^2 - dZ^2), \quad V = \frac{X^2 - Y^2 - 3Z^2}{4a^3(X + Y)^{5/2}} \quad (3.9)$$

and $V$ satisfies the wave equation on $\mathbb{R}^{1,2}$.

- Instead of using the patching relation (3.1) we could have started with

$$\tilde{\tau} = \lambda \tau + b \lambda^{-1} Q + a \lambda^{-2} Q^2,$$

which allows the limit $a \rightarrow 0$ corresponding to the patching (2.4), and resulting in a flat conformal structure. The corresponding metric and the conformal factor arise from $g$ and $\Omega$ given by (3.5) and (3.6) when one makes a replacement

$$z \rightarrow z + \frac{b}{2a}.$$ 

Therefore, if $a \neq 0$ then $b$ can be set to zero by translating $z$.

- The metric (3.5) admits a second Killing vector

$$K_2 = 5x \partial_x + 2y \partial_y - z \partial_z - 4t \partial_t$$

which is not tri-holomorphic. It Lie derives $\Sigma^{01}$, but rotates $\Sigma^{00}$ and $\Sigma^{11}$. Thus the space of orbits of $K_2$ in $M_C$ admits a Toda Einstein–Weyl structure [3, 24].

- There exists a combination of self-dual two-forms which is degenerate when the harmonic function $V$ in (3.5) vanishes, which is the surface $ty - z^2 = 0$ in $M_C$. All three forms vanish on the line $t = y = z = 0$ in $M_C$. The normal bundle of the twistor curves corresponding to this line jumps from $O(1) \oplus O(1)$ to $O(3) \oplus O(-1)$.

### 3.2. Ricci-flat metric with $k = 4$

Parametrise the sections of $O(4) \rightarrow \mathbb{C}P^1$ by

$$Q = t + \lambda z + \lambda^2 y + \lambda^3 w + \lambda^4 x. \quad (3.10)$$

In this case the splitting (2.6) is possible if

$$\phi \equiv tw + zy = 0. \quad (3.11)$$

Computing the resultant (3.4) leads to the conformal structure

$$g = \Omega^2 \left( t^6 dx^2 - t^3 z(ty + z^2) dx dz + 2t^4 (ty - z^2) dx dy + t^2 (2t^2 y^2 - 5tyz^2 + z^4) dx dt 
- 2tyz^2(ty - z^2) dz^2 - tz(t^2 y^2 - z^4) dx dy - yz(t^2 y^2 - 6tyz^2 + z^4) dz dt 
+ t^2(ty - z^2)^2 dy^2 + 2t^2 y^2(ty + z^2) dy dt + y^2(t^2 y^2 + 4tyz^2 - z^4) dt^2 \right). \quad (3.12)$$
The conformal factor making this metric Ricci flat is

\[ \Omega^2 = \frac{2a^4}{t^2 z(3ty - z^2)}. \]

The triholomorphic Killing vector \( \partial/\partial x \) Lie derives the covariantly constant basis of self-dual two–forms

\[
\Sigma_{00} = 2dx \wedge (tdy + ydt + zdz),
\]

\[
\Sigma_{01} = tdx \wedge dz + zdx \wedge dt - \frac{yt - z^2}{t} dz \wedge dy - \frac{y(yt + z^2)}{t^2} dz \wedge dt - \frac{2yz}{t} dy \wedge dt,
\]

\[
\Sigma_{11} = dt \wedge \left( \frac{4yz}{t} dz - 2tx dx - 2yt - z^2 t dy - \frac{y(t + z^2)}{t^2} t dz - \frac{2yz}{t} dt \right),
\] (3.13)

The coordinate transformation (3.8) brings the metric \( g \) to Gibbons–Hawking form (3.7), where

\[ V = \frac{z(3ty - z^2)}{2a^4 t^4} = \frac{Z(3X^2 - 3Y^2 - 5Z^2)}{4a^4(X + Y)^{7/2}}, \] (3.14)

and

\[ h_{flat} = a^8(dX^2 - dY^2 - dZ^2). \]

The metric \( g \) admits a homothety, as well as a second non-triholomorphic Killing vector \( K_2 = t\partial_t - y\partial_y - 3x\partial_x \).

3.3. Gibbons–Hawking potential for general \( k \)

There exists a map from an affine bundle over \( \mathcal{O}(k) \) with holomorphic charts \( (Q, \tau) \) and \( (\tilde{Q}, \tilde{\tau}) \) to an affine bundle over \( \mathcal{O}(2) \) with charts \( (q, p) \) and \( (\tilde{q}, \tilde{p}) \) such that the latter admits a four-parameter family of section only if the patching for the former satisfies some additional conditions. The explicit transformation is given by

\[ q = \lambda^k \tau + Q^2, \quad p = \sum_{n=1}^{\infty} \frac{(2n)!}{(1-2n)(n)!^2 4^n} \lambda^{(n-1)k} \tau^n (\lambda^k \tau + Q^2)^{1/2 - n} \]

on \( U \), and

\[ \tilde{q} = \tilde{\tau}, \quad \tilde{p} = \tilde{Q}. \]

This map is well defined only if some sections are removed from the \( \mathcal{O}(k) \) twistor space. This corresponds to removing the region from \( M_C \) corresponding to the ‘big jump’. In the case of (3.1) we find

\[ \tilde{q} = \lambda^{-2} q, \quad \tilde{p} = p + s(q, \lambda), \quad \text{where} \quad s = \lambda^{-k} \sqrt{q}. \]

The element of \( H^1(\mathcal{O}(2), \mathcal{O}(-2)) \) corresponding to the Gibbons–Hawking function is \( \partial s/\partial q \). Parametrising the sections of \( \mathcal{O}(2) \) by

\[ q = \lambda^2(X - Y) + 2\lambda Z + (X + Y) \]

and taking the contour enclosing \( \lambda = 0 \) in the twistor integral formula, leads to

\[ V(X, Y, Z) = \frac{1}{2(k-1)!} \partial^{k-1} \frac{\partial^{k-1} (\lambda^2(X - Y) + 2\lambda Z + (X + Y))^{1/2}}{\partial \lambda^{k-1}}|_{\lambda=0}, \]

which for \( k = 3 \) and \( k = 4 \) agrees with (3.9) and (3.14).
3.4. \( k = 2 \) and \( k = 1 \)

For completeness we will analyse the remaining cases \( k = 2 \) and \( k = 1 \) with the constant \( a \) set to 1. If \( k = 2 \), both sides of (3.2) are homogeneous of degree 0, and thus are equal to some \( x_{-1} \), so that

\[
\tilde{\tau} = x_{-1} + 2\tilde{x}_0 x_1 + \tilde{\lambda}^2 x_0^2, \quad \tilde{Q} = x_2 + \tilde{\lambda} x_1 + \tilde{\lambda}^2 x_0,
\]

where in \( \tilde{\tau} \) we have absorbed \((2x_0 x_2 + x_1^2)\) into \( x_{-1} \). Now compute the resultant (3.4), and set

\[
x_{-1} = -2iT + \frac{1}{2}(X^2 + Y^2) - Z^2, \quad x_0 = \frac{1}{\sqrt{2}}(X + iY), \quad x_1 = \sqrt{2}iZ, \quad x_2 = \frac{1}{\sqrt{2}}(X - iY).
\]

This yields the folded hyper-Kähler metric (1.1).

Finally if \( k = 1 \) then both sides of (3.2) are homogeneous of degree 1, and thus give rise to two coordinates on \( M_C \). The resulting metric is flat. In this case the twistor space \( Z \) fibres over \( O(1) \), and all metrics (corresponding to arbitrary patching) fall into the classification of [9].

4. From Sparling–Tod to Eguchi–Hanson

The holomorphic Sparling–Tod metric [20, 21]

\[
g = 4dudv - 4dxdy - 8\rho \Delta^{-3}(udv - xdy)^2, \quad \triangle \equiv uv - xy \tag{4.1}
\]

is ASD, Ricci–flat, and of Petrov-Penrose type \( D_{II} \). In [4] a twistor–theoretic argument was used to show that there exists a Riemannian real section of (4.1) which is equivalent to the Eguchi–Hanson gravitational instanton. The coordinate transformation below makes this explicit, by putting (4.1) in the ALE \( A_2 \) Gibbons–Hawking form.

We find that the four-dimensional isometry group of (4.1) contains \( SL(2, \mathbb{C}) \) which acts tri-holomorphically. Let us consider a pencil of tri-holomorphic Killing vectors given by

\[
K = \frac{b}{2}(v \partial_y + x \partial_u) - \frac{c}{2}(y \partial_x + u \partial_v),
\]

where \((b, c)\) are constants not both zero. A parallel basis of \( \Lambda^2_+ \) is Lie-derived along \( K \), and the corresponding moment maps are the flat coordinates on \( \mathbb{R}^3 \) in the Gibbons–Hawking form. They are given by

\[
Z = i(bxv + cyu), \quad X + iY = \sqrt{2}\rho(bx^2 + cu^2)\triangle^{-2} + \frac{\sqrt{2}}{2}(bx^2 + cu^2),
\]

\[
X - iY = \sqrt{2}(cy^2 + bu^2).
\]

The metric (4.1) takes the form

\[
g = V(dX^2 + dY^2 + dZ^2) + V^{-1}(dT + A)^2,
\]

where \( V \) is the harmonic function on \( \mathbb{R}^3 \) given by

\[
V = \frac{\Delta^3}{2\rho Z^2 + bc\Delta^4}
\]

and

\[
\Delta^2 = \frac{2bc\rho - R^2 + \epsilon\sqrt{(R^2 - 2bc\rho)^2 + 8bc\rho Z^2}}{2bc}, \quad R^2 \equiv X^2 + Y^2 + Z^2, \tag{4.2}
\]

where \( \epsilon = \pm 1 \). With the help of some algebra this simplifies to

\[
V = \frac{1}{2\sqrt{-bc}} \left( \frac{1}{|R + a|} - \epsilon \frac{1}{|R - a|} \right),
\]
where $R = (X, Y, Z)$, $a = (0, 0, \sqrt{-2bc \rho})$. If $b, c$ are real and such that $bc < 0$ then $V$ with $\epsilon = -1$ gives the positive-definite Eguchi–Hanson gravitational instanton. If $\epsilon = 1$ then $g$ is still positive-definite, but not complete. It is an example of a folded hyper-Kähler metric [12]. The points on the hypersurface $V = 0$ in $M$ correspond to lines in $Z$ where the normal bundle jumps to $O \oplus O(2)$. Moreover, the limit when $b$ or $c$ tends to zero gives a dyon.

5. Multi-jumps

The metrics resulting from Theorem 3.1 admit a tri-holomorphic Killing vector, and thus can locally be put in the holomorphic Gibbons–Hawking form [10], which depends on a solution $V$ to a holomorphic Laplace equation on $\mathbb{C}^3$. The hypersurface corresponding to $V = 0$ is singular, and can be characterised by the jumping phenomenon.

**Proposition 5.1.** Let $(M, g)$ be a Gibbons–Hawking metric

$$g = V(dX^2 - dY^2 - dZ^2) + V^{-1}(dT + A)^2,$$

where $dV = *_3dA$ and let $S = \{p \in M, V(p) = 0\}$. The points of $S$ correspond to rational curves in $Z$ with normal bundle $O(2) \oplus O$.

**Proof.** The twistor space of a Gibbons–Hawking manifold is an affine line bundle over the total space of $O(2)$ with transition functions

$$\tilde{\tau} = \tau + f(Q, \lambda), \quad \tilde{Q} = \lambda^{-2}Q,$$

where $f \in H^1(O(2), O)$. Restricting the cohomology class $f$ to a section of $O(2)$

$$Q = \lambda^2(X - Y) + 2\lambda Z + (X + Y)$$

(5.1)

gives rise to the harmonic function $V$ by

$$V(p) = \frac{1}{2\pi i} \oint_{\Gamma \subset L_p} \frac{\partial f}{\partial Q} d\lambda.$$ 

The normal bundle to $L_p$ is the restriction to (5.1) of

$$\begin{pmatrix} 1 & \frac{\partial f}{\partial Q} \\ 0 & \lambda^{-2} \end{pmatrix},$$

and then expanding

$$f(Q, \lambda) = \sum_{-\infty}^{\infty} a_k \lambda^k, \quad \text{with} \quad a_k = a_k(X, Y, Z).$$

Split the sum into two:

$$\tilde{h} = -\sum_{-\infty}^{0} a_k \lambda^k, \quad h = \sum_{0}^{\infty} a_k \lambda^k, \quad \text{so that} \quad f = h - \tilde{h}$$

and there is freedom to add a function of $(X, Y, Z)$ to both of $h, \tilde{h}$. Note that, after restricting,

$$\frac{\partial f}{\partial Z} = \frac{\partial f}{\partial Q} \frac{\partial Q}{\partial Z} = 2\lambda \frac{\partial f}{\partial Q}.$$
so that the transition matrix for the normal bundle is

\[ F := \begin{pmatrix} 1 & 1 \\
2\lambda \frac{\partial f}{\partial Z} & \lambda^{-2} 
\end{pmatrix} . \]

This is equivalent to

\[ F \to \begin{pmatrix} 1 & \tilde{p} \\
0 & 1 \end{pmatrix} F \begin{pmatrix} 1 & p \\
0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\
2\lambda \frac{\partial f}{\partial Z} + p + \tilde{p}\lambda^{-2} & \lambda^{-2} \end{pmatrix} , \]

where we choose \( \tilde{p}, p \) to remove from \( \frac{1}{2\lambda} \frac{\partial f}{\partial Z} \) all non-negative powers of \( \lambda \) and all negative powers less than or equal to \(-2\). All that remains is \( \frac{1}{2\lambda} \frac{\partial f}{\partial Z} \), and \( \frac{\partial f}{\partial Z} \) is equal to a multiple of \( V \). Where \( V \neq 0 \), we know that this is the transition matrix for \( O(1) \oplus O(1) \) but clearly where \( V = 0 \) it is the transition matrix for \( O(2) \oplus O \).

We conclude that the metrics arising from Theorem 3.1 have at least two jumps: the ‘small jump’ from \( O(1) \oplus O(1) \) to \( O(2) \oplus O \) on a surface \( S \) corresponding to the zero set of \( g(\partial_s, \partial_v)^{-1} \), and a big jump to \( O(k) \oplus O(2 - k) \) on a curve \( \gamma \). The argument below demonstrates that many intermediate jumps can arise.

Consider the twistor space of Theorem 3.1, with the moduli space of rational curves \( M_C \) given by the Zariski cone (3.3). The constraints defining \( M_C \) take the form

\[ x_0 x_n + x_1 x_{n-1} + \cdots = 0, \quad \text{for } 3 \leq n \leq k - 1. \]

For even \( n = 2m \) the last term in the constraint is \( x_m^2/2 \) and there will be a constraint like this for

\[ 1 \leq m \leq (k - 1)/2 \quad \text{for odd } k \quad \text{or} \quad 1 \leq m \leq k/2 - 1 \quad \text{for even } k. \]

We will be interested in solutions of the constraints for which all but one \( x_n \) are zero (for \( n < k \)) and the constraints will not be satisfied for \( n \) below a threshold. Thus we have a range of allowed \( n \), namely \((1 + k)/2 \leq n \leq k - 1\) for odd \( n \) or \( k/2 \leq n \leq k - 1\) for even \( n \). For \( n \) in these ranges the constraints are satisfied with \( x_n \neq 0 \) and \( x_i = 0 \) for all other \( i \) in the range \( 0 \leq i \leq k - 1 \). With any one of these solutions of the constraints, multiply \( F \) on the right with

\[ H^{-1} = \begin{pmatrix} 1 & -2ax_k \\
0 & 1 \end{pmatrix} \]

(5.2)

to remove \( x_k \)-term from \( Q \), leaving

\[ F = \begin{pmatrix} \lambda^{k-2} & 2a \sum_i x_i \lambda^{i-2} \\
0 & \lambda^{-k} \end{pmatrix} . \]

Now consider the product

\[ \tilde{H} \begin{pmatrix} \lambda^{i-2} & 0 \\
0 & \lambda^{-i} \end{pmatrix} H^{-1} = \begin{pmatrix} \alpha & 0 \\
\lambda^{2-i-k} & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{i-2} & 0 \\
0 & \lambda^{-i} \end{pmatrix} \begin{pmatrix} \alpha^{-1} \lambda^{k-i} & 1 \\
-1 & 0 \end{pmatrix} , \]

when \( \tilde{H} \) and \( H \), respectively, have only negative or only positive powers of \( \lambda \) and this product is \( F \) given the choice \( \alpha = 2ax_i \). Thus the normal bundle has jumped to \( O(2 - i) \oplus O(i) \) and there is an example like this for each \( i \) in the allowed range.
We also always have the case $x_0 \neq 0$, other $x_n$ zero when
\[
\begin{pmatrix}
\alpha & 0 \\
\lambda^{2-k} & -\alpha^{-1}
\end{pmatrix}
\begin{pmatrix}
\lambda^{-2} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha^{-1} & \lambda^k \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix}
\lambda^{k-2} & \alpha \lambda^{-2} \\
0 & \lambda^{-k}
\end{pmatrix} = F
\]
with $\alpha = 2ax_0$, so the jump to $O \oplus O(2)$ is always present.

We will see that all jumps are present if $k = 4$. There is enough here to prove this also for $k = 5$ and $k = 6$ but there is a gap at $k = 7$: the above constructions do not give an example of a curve with normal bundle $O(-1) \oplus O(3)$ but everything else up to $O(-5) \oplus O(7)$ occurs.

5.1. Jump cascade with $k = 4$

Restrict the transition function (3.1) with $k = 4$ to the line (3.10). The normal bundle is $O(1) \oplus O(1)$ away from $V = 0$, where $V$ is given by (3.14), and jumps to $O(-2) \oplus O(4)$ at $y = z = w = t = 0$. We have to look at the zero-set of $V$. Multiply $F$ on the right with (5.2) leaving
\[
F = \begin{pmatrix}
\lambda^2 & 2a(w\lambda + y + z + \frac{t}{\lambda^2}) \\
0 & \lambda^{-4}
\end{pmatrix}.
\]

There are six loci to investigate all of which have $V = 0$.

$S_1$. $w = y = z = t = 0$ when we know it jumps to $O(-2) \oplus O(4)$.

$S_2$. $w = y = z = 0$ but $t \neq 0$. Note that
\[
H^{-1} = \begin{pmatrix}
1/\beta & 0 \\
\lambda^4 & \beta
\end{pmatrix}, \quad \tilde{H} = \begin{pmatrix}
0 & 1 \\
-1 & \frac{1}{\beta \lambda^2}
\end{pmatrix}
\]
give
\[
\tilde{H} \begin{pmatrix}
1 & 0 \\
0 & \lambda^{-2}
\end{pmatrix} H^{-1} = \begin{pmatrix}
\lambda^2 & \beta \lambda^{-2} \\
0 & \lambda^{-4}
\end{pmatrix}
\]
which with $\beta = 2at$ shows $S_2$ is $O \oplus O(2)$.

$S_3$. $y = z = t = 0$ but $w \neq 0$. Take
\[
H^{-1} = \begin{pmatrix}
\lambda/\beta & 1 \\
-1 & 0
\end{pmatrix}, \quad \tilde{H} = \begin{pmatrix}
\beta & 0 \\
\lambda^{-5} & \frac{1}{\beta}
\end{pmatrix}
\]
which give
\[
\tilde{H} \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-3}
\end{pmatrix} H^{-1} = \begin{pmatrix}
\lambda^2 & \beta \lambda \\
0 & \lambda^{-4}
\end{pmatrix}
\]
which with $\beta = 2bw$ shows $S_3$ is $O(-1) \oplus O(3)$.

$S_4$. $z = t = 0$ but $y \neq 0$ (any $w$). Take
\[
H^{-1} = \begin{pmatrix}
\lambda^2 & \alpha + \beta \lambda \\
\beta \lambda & -\frac{1}{\alpha} \beta^2 + \frac{1}{\alpha \lambda^4}
\end{pmatrix}, \quad \tilde{H} = \begin{pmatrix}
1 & 0 \\
-\frac{\beta}{\alpha^2 \beta^3} + \frac{1}{\alpha \lambda^4} & 1
\end{pmatrix}
\]
which give
\[
\tilde{H} \begin{pmatrix}
1 & 0 \\
0 & \lambda^{-2}
\end{pmatrix} H^{-1} = \begin{pmatrix}
\lambda^2 & \alpha + \beta \lambda \\
0 & \lambda^{-4}
\end{pmatrix}
\]
so with $\alpha = 2ay, \beta = 2az$ this shows that $S_4$ is $\mathcal{O} \oplus \mathcal{O}(2)$.

$S_5$. $z = w = 0$ with $yt \neq 0$. Consider

$$H^{-1} = \begin{pmatrix} \frac{\lambda^2}{\alpha} + \frac{\beta}{\alpha^2} & 0 & -1 \\ -6pt & 1 & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} -\frac{\alpha}{\lambda^2} + \frac{\beta}{\alpha^2} & 0 \\ -1 & \frac{1}{\lambda^4} - \frac{1}{\alpha^2} + \frac{\beta}{\alpha^2} \lambda^2 \end{pmatrix}$$

so that

$$\tilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \frac{\lambda^2}{\alpha} + \frac{\beta}{\alpha^2} & 0 \\ \frac{1}{\lambda^4} - \frac{1}{\alpha^2} + \frac{\beta}{\alpha^2} \lambda^2 & 0 \end{pmatrix}.$$ 

With $\alpha = 2ay, \beta = 2at$ and $yt \neq 0$ this shows that $S_5$ is $\mathcal{O} \oplus \mathcal{O}(2)$.

$S_6$. $z^2 = 3ty$ with $yzt \neq 0$. Introduce $\chi := t \lambda^2 + z \lambda + y + w \lambda$ then set $t = \beta, z = 3\alpha t$. We have $3yt - z^2 = 0 = wt + yz$ so that $y = 3\alpha^2 \beta$ and $w = -9\alpha^3 \beta$ whence

$$\chi = \frac{\beta}{\lambda^2} (1 + 3\alpha \lambda + 3\alpha^2 \lambda^2 - 9\alpha^3 \lambda^3)$$

and this is the top-right entry in $F$. Consider

$$H^{-1} = \begin{pmatrix} 1 - 3\alpha \lambda + 6\alpha^2 \lambda^2 & 9\alpha^4 \beta (5 - 6\alpha \lambda) \\ \lambda^4 \beta^{-1} & \frac{\lambda^2 f}{\beta} \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} 0 & \beta \\ -1 & \frac{1}{\beta} - 3\frac{\alpha}{\lambda} + 6\alpha^2 \end{pmatrix}.$$ 

Then

$$\tilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \frac{\lambda^2}{\beta} & \chi \\ 0 & \lambda^{-4} \end{pmatrix}$$

so $S_6$ is also $\mathcal{O} \oplus \mathcal{O}(2)$.

We conclude that there are curves with normal bundle $(1, 1), (0, 2), (-1, 3)$ and $(-2, 4)$, that is, all possibilities up to the maximum jump occur.

6. **Generalised Legendre transform and self-dual two-forms**

There is another route directly from the cohomology class defining the affine line bundle $Z \to \mathcal{O}(k)$ to the ASD Ricci-flat metric directly without the need to use resultants as in (3.4). This follows [7, Theorem 4.4], and gives a version of the generalised Legendre transform [1, 14].

Affine line bundles over $\mathcal{O}(k)$ are classified by elements $[f]$ of $H^1(\mathcal{O}(k), \mathcal{O}(2 - k))$ as

$$\tilde{\tau} = \tau + f(Q, \lambda), \quad \tilde{Q} = \lambda^{-k} Q.$$ 

Any such cohomology class gives rise to $k - 3$ constraints

$$\phi_{A_1 \cdots A_{k-4}} := \frac{1}{2 \pi i} \oint_{\mathcal{C}} \pi_{A_1} \cdots \pi_{A_{k-4}} f(Q, \pi_A) \pi \cdot d\pi = 0,$$

which trace out a holomorphic four-manifold $M_C$ in a $(k + 1)$-dimensional space of holomorphic sections of $\mathcal{O}(k) \to \mathbb{C}P^1$. The ASD Ricci-flat metric on $M_C$ is determined by a basis of self-dual
two-forms \( \{ \Sigma^{00}, \Sigma^{01}, \Sigma^{11} \} \) which are pull-backs from \( \mathbb{C}^{k+1} \) to \( M \) of two-forms
\[
\Sigma^{AB} = \frac{1}{8} \psi^{AB} B_1 \cdots B_{k-3} C_1 \cdots C_{k-3} dx_{PQR} B_1 \cdots B_{k-3} \wedge dx_P \wedge \ldots \wedge dx_{C_1} \cdots C_{k-3} \tag{6.3}
\]
\[
+ \frac{3}{2} \psi_{B_1 \cdots B_{k-2} C_1 \cdots C_{k-2}} dx_P B_1 \cdots B_{k-2} (A \wedge dx^C) C_2 \cdots C_{k-2} R P,
\]
where
\[
\psi_{A_1 \ldots A_{2k-4}} = \frac{1}{2\pi i} \int_{\Sigma} \pi_{A_1} \cdots \pi_{A_{2k-4}} \frac{\partial f}{\partial Q} \pi \cdot d\pi
\]
is a zero-rest-mass field determined by \( [f] \), and \( Q \) in this formula is regarded as the coordinate on the fibres of \( \mathcal{O}(k) \rightarrow \mathbb{CP}^1 \) which is homogeneous of degree \( k \).

If \( k = 3 \) then there are no constraints to be imposed, and \( \psi_{AB} \) is a self-dual Maxwell field originally constructed in [23].

6.1. Example with \( k = 4 \)

The manifold \( M_\mathcal{C} \) is a surface \( \phi = 0 \) given by (6.2) in the five-dimensional space \( \mathcal{N} \) of holomorphic sections of the fibration \( \mathcal{O}(4) \rightarrow \mathbb{CP}^1 \). The function \( \phi \) satisfies the overdetermined system of linear PDEs \( \partial^4_{BCDEFGA} = 0 \) where \( \psi_{ABCD} \) is given by (6.4). Explicitly
\[
\phi_{yt} - \phi_{yz} = 0, \quad \phi_{tw} - \phi_{yz} = 0, \quad \phi_{tx} - \phi_{wz} = 0, \quad \phi_{wz} - \phi_{yy} = 0, \quad \phi_{xz} - \phi_{wy} = 0, \quad \phi_{xy} - \phi_{ww} = 0.
\]

Consider the cohomology class represented by \( f = Q^2 \lambda^{-k} \), and take \( k = 4 \). Comparing \( Q = x^{ABCD} \pi_A \pi_B \pi_C \pi_D \) with (3.10) gives
\[
t = x^{1111}, \quad z = 4x^{1110}, \quad y = 6x^{1100}, \quad w = 4x^{1000}, \quad x = x^{0000}.
\]
Evaluating the residue at the pole \( \lambda = 0 \) in (6.2) yields the constraint
\[
\phi = tw + zy = 0
\]
in agreement with (3.11). The spin-2 field (6.4) is
\[
\psi_{0000} = 0, \quad \psi_{0001} = t, \quad \psi_{0011} = z, \quad \psi_{0111} = y, \quad \psi_{1111} = w
\]
which gives the self-dual two-forms
\[
\Sigma^{00} = z \left( 2dx \wedge dz + \frac{1}{2} dw \wedge dy \right) + y \left( 2dx \wedge dt - \frac{1}{2} dz \wedge dw \right) + w \left( \frac{1}{2} dw \wedge dt \right) + t(2dx \wedge dy),
\]
\[
\Sigma^{01} = z \left( dx \wedge dt - dz \wedge dw \right) + y \left( dw \wedge dt + \frac{1}{2} dy \wedge dz \right)
\]
\[
+ w \left( \frac{1}{2} dy \wedge dt \right) + t \left( dz \wedge dw + \frac{1}{2} dw \wedge dy \right),
\]
\[
\Sigma^{11} = z \left( 2dw \wedge dt + \frac{1}{2} dy \wedge dz \right) + y \left( 2dy \wedge dt \right) + w \left( \frac{3}{2} dz \wedge dt \right) + t \left( 2dx \wedge dt - \frac{1}{2} dz \wedge dw \right).
\]
The pull-back of these two-forms to the cone (3.11) agrees with expressions (3.13).

The jump cascade discussed in Section 5.1 can be now understood in the framework of the generalised Legendre transform presented in [8, 16]. Using the Kodaira isomorphism
\[
T^* p \mathcal{N} \cong H^0(L_p, \mathcal{O}(4)) = \text{Sym}^4 (\mathbb{C}^2)
\]
\footnote{This formula corrects (4.27) in [7].}
we can identify the gradient $d\phi$ with a binary quartic
\[
d\phi \rightarrow Q(d\phi) = \alpha s^4 + 4\beta s^3 + 6\gamma s^2 + 4\delta s + \epsilon
\]
\[
= \phi_x s^4 - 4\phi_w s^3 + 6\phi_y s^2 - 4\phi_z s + \phi_t.
\]
Binary quartics admit two classical invariants
\[
I = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \quad \text{and} \quad J = \det \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \end{pmatrix}.
\]
(6.5)

If $\phi$ is given by (3.11) then
\[
I = 3z^2 - 4ty, \quad J = z^3 - 2tzw + t^2w.
\]
The points in $M_C$ where $d\phi = 0$ correspond to twistor curves with normal bundle $O(-2) \oplus O(4)$. The points where $d\phi \neq 0$, but $I = J = 0$ correspond to twistor curves with normal bundle $O \oplus O(2)$. Finally the generic points have $I \neq 0$ and $J \neq 0$. Such points correspond to twistor curves with the normal bundle $O \oplus O(2)$.

6.2. A Riemannian example

The Riemannian reality conditions require $k = 2n$ to be even. The real sections satisfy
\[
\overline{Q(\lambda)} = (-1)^n \lambda^{-2n} Q(-1/\lambda),
\]
which in the case $k = 4$ implies that
\[
Q = t + \lambda z + \lambda^2 y - \lambda^3 \overline{\tau} + \lambda^4 t
\]
with $t, z$ complex and $y$ real. The surface (3.11) becomes $t\overline{\tau} + zy = 0$ which is of co-dimension two in the space of real sections of $O(4)$. Thus the metric (3.12) does not admit a Riemannian slice.

To construct a Riemannian metric which admits a jump to $O(-2) \oplus O(4)$ consider a twistor space defined by the patching relation
\[
\tilde{Q} = \lambda^{-4} Q, \quad \tilde{\tau} = \lambda^2 \tau + s(Q, \lambda), \quad \text{where} \quad s = 3Q^2(1 - \lambda^{-6}).
\]
(6.6)
The metric can be computed as in (3) using the resultant (3.4), and constructing a conformal factor which makes the metric Ricci-flat. We will instead perform the Legendre transform of [14] which leads directly to a Kähler potential for the metric. To make contact with the notation and formalism of [14] define $G(Q, \lambda)$ by
\[
\frac{\partial G}{\partial Q} = \frac{s}{\lambda^2}, \quad \text{so that} \quad G = \frac{Q^3}{\lambda^2}(1 - \lambda^{-6}),
\]
and set
\[
F = \frac{1}{2\pi i} \oint_{\Gamma \subset \mathbb{CP}^1} \frac{1}{\lambda^2} G(t + \lambda z + \lambda^2 y - \lambda^3 \overline{\tau} + \lambda^4 t, \lambda) d\lambda
\]
\[
= 6ytz + 6y\overline{t}z + z^3 + \overline{\tau}^3 - 3zt^2 - 3\overline{t}^2,
\]
where the contour $\Gamma$ encloses $\lambda = 0$. The real four-manifold $M$ is defined as the surface
\[
\phi := \frac{\partial F}{\partial y} = 6(tz + \overline{t}z)
\]
\[
= 0
\]
\[\text{To make contact with (6.1) divide the expression (6.6) for } \tilde{\tau} \text{ by } \lambda^2, \text{ and set } f = s/\lambda^2.\]
in the space of real sections of $O(4)$. Using the splitting method in the proof of Theorem 3.1, or equivalently computing the $I$ and $J$ invariants (6.5) we find that the points in $M$ where $t = z = 0$ correspond to curves with normal bundle $O(-2) \oplus O(4)$. This is a curve parametrised by $y$.

Now perform the Legendre transform
\[ u := \frac{\partial F}{\partial z} = 6yt + 3z^2 - 3t^2 \]
and eliminate the coordinates $(z, \overline{z}, y)$ using $(t, \overline{t}, u, \overline{u})$ as holomorphic and anti-holomorphic coordinates on $M$. The Kähler potential is
\[
\Omega(t, \overline{t}, u, \overline{u}) = F - uz - \overline{u}\overline{z} = -2(z^3 + \overline{z}^3)
\]
\[
= -2i(t^3 - \overline{t}^3)R^3, \quad \text{where} \quad R^2 = -1 - \frac{\overline{mt} - u\overline{t}}{3(t^3 - \overline{t}^3)} \in \mathbb{R}^+.
\]
The Kähler potential satisfies the first heavenly equation [19]
\[
\Omega_t \Omega_u \Omega_v - \Omega_t \Omega_u \Omega_v = 1
\]
and the resulting metric on $M$
\[
g = \Omega_{u\overline{u}}udu + \Omega_{u\overline{t}}udu + \Omega_{v\overline{u}}dtd\overline{u} + \Omega_{v\overline{t}}dtd\overline{t}
\]
is hyper-Kähler. The line of jumping points in $M$ has been blown down to a point $u = t = 0$ by the Legendre transform.

7. Schrödinger equation on folded hyper-Kähler manifolds

In this Section we will demonstrate that the Schrödinger equation on a canonical folded hyper-Kähler manifold (corresponding to $k = 2$ in Theorem 3.1)
\[
g = Z\left(dX^2 + dY^2 + dZ^2\right) + \frac{1}{Z} \left(dt + \frac{1}{2}XdY - \frac{1}{2}YdX\right)^2
\]
admits normalisable solutions which extend to both sides of the fold $Z = 0$ where the metric degenerates.

The time-independent Schrödinger equation
\[
\frac{1}{\sqrt{|g|}} \partial_a \left( \sqrt{|g|} g^{ab} \partial_b \phi \right) = E\phi
\]
takes the form
\[
\frac{1}{Z} \left( \frac{1}{4}(X^2 + Y^2) + Z^2 \right) \partial_T \partial_\phi - \frac{X}{Z} \partial_Y \partial_T \phi + \frac{Y}{Z} \partial_X \partial_T \phi + \frac{1}{Z} \delta^{ij} \partial_i \partial_j \phi = E\phi.
\]  \hspace{1cm} (7.1)

We will take the coordinate $T$ to be periodic, and consider solutions of the form
\[
\phi(T, X, Y, Z) = e^{isT}\varphi(X, Y, Z)
\]
for $s$ a non-zero integer. The Schrödinger equation (7.1) becomes
\[
-\frac{s^2}{Z} \left( \frac{1}{4}(X^2 + Y^2) + Z^2 \right) \varphi - \frac{isX}{Z} \partial_Y \varphi + \frac{isY}{Z} \partial_X \varphi + \frac{1}{Z} \delta^{ij} \partial_i \partial_j \varphi = E\varphi,
\]
which separates as \( \varphi = G(X,Y)F(Z) \) into

\[
\frac{d^2 F}{dz^2} - (s^2Z^2 + EZ + \kappa)F = 0 \tag{7.2}
\]

and

\[
- \frac{1}{4}s^2(X^2 + Y^2) - \frac{isX}{G}\partial_Y G + \frac{isY}{G}\partial_x G + \frac{1}{G}(\partial^2_X + \partial^2_Y)G + \kappa = 0. \tag{7.3}
\]

If \( s = 0 \) then the first equation becomes the Airy equations and one can show that non-normalisable solutions exist on both sides of the fold. The second equation describes a free particle on a plane, and no bound states exist in this case either.

Let us therefore assume that \( s \neq 0 \), and consider the equation for (7.2) for \( F(Z) \), which has the form of the Schrödinger equation describing a displaced harmonic oscillator. This is readily solved to give

\[
F(Z) = H_\gamma \left( \sqrt{s} \left( Z + \frac{E}{2s^2} \right) \right) \exp \left\{ -\frac{1}{2} s \left( Z + \frac{E}{2s^2} \right)^2 \right\},
\]

where \( H_\gamma(\xi) \) solves the Hermite equation

\[
\frac{d^2 H}{d\xi^2} - 2\xi \frac{dH}{d\xi} + 2\gamma H = 0, \quad \text{with} \quad \gamma = \frac{1}{2s} \left( \frac{E^2}{4s^2} - (\kappa + s) \right).
\]

If \( \gamma \) is a non-negative integer then \( H_\gamma \) is a Hermite polynomial and thus \( F(Z) \) is clearly normalisable for \( s > 0 \) (even with the folded background’s factor of \( \sqrt{|g|} = Z \)) due to the exponential fall-off at large \( Z \). If, however, \( \gamma \) fails to be a non-negative integer then \( H_\gamma \) is more complicated, being most readily expressed as a series expansion. In this case normalisability is less clear, so let us restrict ourselves to the case where \( \gamma \) is a non-negative integer.

Let us now proceed to consider the \( G(X,Y) \) equation (7.3). This has the form of the Schrödinger equation describing motion in a constant magnetic field. In the usual manner let us then define the canonical (Hermitian) momenta

\[
\Pi_X = -i\partial_X + \frac{1}{2}sY \quad \Pi_Y = -i\partial_Y - \frac{1}{2}sX
\]

and ladder operators

\[
a = \Pi_X + i\Pi_Y \quad a^\dagger = \Pi_X - i\Pi_Y.
\]

The \( G(X,Y) \) equation is then

\[
(a^\dagger a + s - \kappa)G = 0
\]

and we can construct some solutions (choosing \( \kappa = s \)) by solving \( aG_0(X,Y) = 0 \), and then applying copies of \( a^\dagger \) to \( G_0 \). For example, one solution is

\[
G(X,Y) \propto \exp \left\{ -\frac{1}{4}s(X^2 + Y^2) \right\},
\]

and thus we conclude that there do exist normalisable solutions. One class of normalisable solutions is

\[
\phi = H_\gamma \left( \sqrt{s} \left( Z + \frac{E}{2s^2} \right) \right) \exp \left\{ -\frac{1}{2} s \left( Z + \frac{E}{2s^2} \right)^2 \right\} \exp \left\{ -\frac{1}{4} s(X^2 + Y^2) \right\} \exp \{isT\}
\]

with \( s \) a positive non-zero integer and \( E \) chosen such that

\[
\gamma = \frac{E^2}{8s^3} - 1
\]

is a positive integer.
Another example of a metric which admits a three-parameter family of jumping lines, and yet there exists normalisable solutions to the Schrödinger equation is the Taub-NUT space with negative mass [11].

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