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Jumps, folds and singularities of Kodaira moduli spaces

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Abstract

For any integer k we construct an explicit example of a twistor space which contains a one-parameter family of jumping rational curves, where the normal bundle changes from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$. For $k > 3$ the resulting anti-self-dual Ricci-flat manifold is a Zariski cone in the space of holomorphic sections of $\mathcal{O}(k)$. In the case $k = 2$ we recover the canonical example of Hitchin's folded hyper-Kähler manifold, where the jumping lines form a three-parameter family. We show that in this case there exist normalisable solutions to the Schrödinger equation which extend through the fold.

1. *Introduction*

The non-linear graviton twistor construction of Penrose [**18**] gives a one-to-one correspondence between holomorphic anti-self-dual (ASD) Ricci-flat metrics on complex four-manifolds $M_{\mathbb{C}}$, and complex threefolds $\mathcal{Z}$ with a family of rational curves. The points in $M_{\mathbb{C}}$ correspond to holomorphic sections of $\mathcal{Z} \longrightarrow \mathbb{CP}^1$ characterised by their normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, where $\mathcal{O}(k)$ is a line bundle over $\mathbb{CP}^1$ with Chern class k.

If the normal bundles of rational curves corresponding to points on a surface $S \subset M_{\mathbb{C}}$ (of co-dimension one or more) change, then the metric becomes singular on S. There are other geometric structures, most notably the self-dual two-forms spanning Λ^2_+, which nevertheless remain regular on S. In the case of a single jump to $\mathcal{O}(2) \oplus \mathcal{O}$ this results in a folded hyper-Kähler structure in the sense of Hitchin [**12**]. Examples of such structures, and the underlying existence theorems are known [**2**], and some applications in theoretical physics have recently emerged [**17**].

The aim of this paper is to construct an explicit example of a twistor space where the normal bundle jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$ for any integer $k > 2$. This jump occurs on a curve $\gamma \subset M_{\mathbb{C}}$, and the corresponding metric on $M_{\mathbb{C}}$ can be constructed explicitly. It admits a tri-holomorphic Killing vector, and so away from the jump it can be put in the standard Gibbons–Hawking form [**10**] by a coordinate transformation. This transformation removes the region of $M_{\mathbb{C}}$ where the jump occurs. The resulting metric on $M_{\mathbb{C}} \setminus \gamma$ is still singular on the surface S where the Gibbons–Hawking harmonic function vanishes. The normal bundles of twistor lines corresponding to the points on S jump to $\mathcal{O}(2) \oplus \mathcal{O}$.

In the next section we will set up the twistor correspondence, where the non-deformed twistor space $\mathcal{Z}$ is an affine line bundle over the total space of $\mathcal{O}(k)$, for any k. In Theorem 3.1 we find Kodaira deformations preserving this affine bundle, and leading to a four-manifold $M_{\mathbb{C}}$ arising as a Zariski cone in the $(k+1)$-dimensional space of holomorphic sections $H^0(\mathbb{CP}^1, \mathcal{O}(k))$. In Sections 3.1 and 3.2 we give expressions for the metric in cases where $k = 3$ and $k = 4$, and show that the Gibbons–Hawking potential

$$V(X,Y,Z) = \frac{1}{2(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}(\lambda^2(X-Y) + 2\lambda Z + (X+Y))^{-1/2}|_{\lambda=0},$$

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on $\mathbb{C}^3$ or $\mathbb{R}^{1,2}$ corresponds to the general k. In Section 4 we give an explicit coordinate transformation between a real form of the Sparling–Tod solution and the Eguchi–Hanson gravitational instanton and its dyonic limit.

In Section 5 it will be shown how a cascade of intermediate jumps

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \oplus \mathcal{O} \to \cdots \to \mathcal{O}(k) \oplus \mathcal{O}(2-k)$$

arises on surfaces on $M_{\mathbb{C}}$ with various co-dimensions. In Section 6 we will put the construction in the framework of the generalised Legendre transform [**1**, **7**, **14**], and show how the Zariski cone $M_{\mathbb{C}}$ arises as the zero–locus of a zero-rest-mass field corresponding to a cohomology class in $H^1(\mathcal{O}(k), \mathcal{O}(2-k))$.

Finally in Section 7 we will come back to the case $k = 2$, where the metric arising from Theorem 3.1 admits a Riemannian real slice (M, g), which is the canonical model of a folded hyper-Kähler structure

$$g = Z(dX^2 + dY^2 + dZ^2) + Z^{-1}\Big(dT + \frac{1}{2}XdY - \frac{1}{2}YdX\Big)^2, \tag{1.1}$$

where $S \subset M$ given by $Z = 0$ is the fold. Answering a question of Manton, we will show that despite the blow-up in the metric there exist normalisable solutions to the Schrödinger equation which extend through the fold.

2. *Twistor spaces as affine bundles*

We will start off by reviewing the twistor correspondence [**5**, **13**, **18**]. Let $M_{\mathbb{C}}$ be a complex four-manifold with a holomorphic orientation vol, and a holomorphic Ricci-flat metric g such that the Weyl tensor is ASD. The anti-self-duality is the Frobenius integrability condition for the existence of a three-parameter family of self–dual totally null surfaces (α-surfaces) in $M_{\mathbb{C}}$, and the twistor space $\mathcal{Z}$ is the three-dimensional complex manifold with the α–surfaces as points. This leads to a double fibration picture

$$M_{\mathbb{C}} \longleftarrow \mathcal{F} \stackrel{\rho}{\longrightarrow} \mathcal{Z},$$

where $\mathcal{F} \subset M_{\mathbb{C}} \times \mathcal{Z}$ is the space of incident pairs (p, ξ) such that $p \in M_{\mathbb{C}}$ lies on an α-surface $\xi \subset M_{\mathbb{C}}$. A point in $M_{\mathbb{C}}$ corresponds to a projective line $L_p \cong \mathbb{CP}^1$ in $\mathcal{Z}$ which consists of all α-surfaces through p. A conformal structure $[g]$ on $M_{\mathbb{C}}$ is encoded in the algebraic geometry of curves in $\mathcal{Z}$: two points in $M_{\mathbb{C}}$ are null-separated iff the corresponding curves in $\mathcal{Z}$ intersect in one point.

There are two additional structures on $\mathcal{Z}$ resulting from the existence of a Ricci-flat metric $g \in [g]$, and the canonical isomorphism

$$TM_{\mathbb{C}} = \mathbb{S} \otimes \mathbb{S}',$$

where $\mathbb{S}$ and $\mathbb{S}'$ are two rank-two complex symplectic vector bundles over $M_{\mathbb{C}}$.

- The Levi-Civita connection of g gives a flat spin connection[†] on $\mathbb{S}$. Thus there exists a two-dimensional space of parallel sections of $\mathbb{S}$. This, together with the isomorphism $\Lambda^2_+ \cong \mathbb{S} \odot \mathbb{S}$ and a natural identification $\mathcal{F} = \mathbb{P}(\mathbb{S})$, gives a holomorphic projection

$$\mu : \mathcal{Z} \longrightarrow \mathbb{CP}^1, \tag{2.1}$$

such that the points in $M_{\mathbb{C}}$ are holomorphic sections of μ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

[†]To avoid the repeated usage of primed spinor indices in Section 6 we depart from the usual twistor conventions, and swap the roles of primed and unprimed indices.

• The parallel basis $(\Sigma^{00}, \Sigma^{01}, \Sigma^{11})$ of Λ^2_+ gives rise to a symplectic two-form Σ on the fibres of (2.1) which takes values in the line bundle $\mathcal{O}(2)$. The pull-back of Σ from $\mathcal{Z}$ to $\mathbb{P}(\mathbb{S})$ is

$$\rho^*(\Sigma) = \Sigma^{11} - 2\lambda\Sigma^{01} + \lambda^2\Sigma^{00},$$

where λ is an affine coordinate on the fibres of $\mathbb{P}(\mathbb{S})$.

The two–form Σ fixes a volume form vol on $M_{\mathbb{C}}$: the condition $\Sigma \wedge \Sigma = 0$ gives

$$\text{vol} = \Sigma^{00} \wedge \Sigma^{11} = -2\Sigma^{01} \wedge \Sigma^{01}.$$

2.1. *Jumping lines*

Let $[\pi_0, \pi_1]$ be homogeneous coordinates on $\mathbb{CP}^1$. Cover $\mathbb{CP}^1$ with two open sets

$$U = \{[\pi] \in \mathbb{CP}^1, \pi_1 \neq 0\}, \quad \widetilde{U} = \{[\pi] \in \mathbb{CP}^1, \pi_0 \neq 0\} \tag{2.2}$$

and set $\lambda = \pi_0/\pi_1$ on $U \cap \widetilde{U}$. The Birkhoff–Grothendieck theorem states that any rank-two holomorphic vector bundle over $\mathbb{CP}^1$ is isomorphic to a direct sum of line bundles $\mathcal{O}(p) \oplus \mathcal{O}(q)$ for some integers p, q. Moreover the transition matrix $F : \mathbb{C}^* \to GL(2, \mathbb{C})$ of this bundle can be written as

$$F = \widetilde{H} \ \text{diag}(\lambda^{-p}, \lambda^{-q}) \ H^{-1}, \tag{2.3}$$

where $H : U \to GL(2, \mathbb{C})$ and $\widetilde{H} : \widetilde{U} \to GL(2, \mathbb{C})$ are holomorphic.

Let $\mathcal{Z}_b \longrightarrow \mathbb{CP}^1$ be a one-parameter family of rank-two vector bundles determined by the patching matrix

$$F_b = \begin{pmatrix} \lambda^{k-2} & b\lambda^{-1} \\ 0 & \lambda^{-k} \end{pmatrix},$$

where b is a constant and k is a positive integer. If $b = 0$ then $F_0 = \text{diag}(\lambda^{k-2}, \lambda^{-k})$ is the patching matrix for $\mathcal{Z}_0 = \mathcal{O}(2-k) \oplus \mathcal{O}(k)$ with H and $\widetilde{H}$ in (2.3) both equal to the identity matrix. If $b \neq 0$ then

$$F_b = \begin{pmatrix} 0 & b \\ -b^{-1} & \lambda^{1-k} \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^{-1}\lambda^{k-1} & 1 \end{pmatrix}^{-1}$$

which is of the form (2.3). Thus $\mathcal{Z}_b = \mathcal{O}(1) \oplus \mathcal{O}(1)$ if $b \neq 0$. This is the twistor space with the holomorphic sections of $\mathcal{Z}_b \to \mathbb{CP}^1$ parametrised by points in $M_{\mathbb{C}} = \mathbb{C}^4$ with the flat metric†.

2.2. *Twistor space as an affine bundle over $\mathcal{O}(k)$*

Let (Q, λ) and $(\widetilde{Q}, \tilde{\lambda} = \lambda^{-1})$ be coordinates on the pre-images of U and $\widetilde{U}$ in the total space of the line bundle $\mathcal{O}(k) \to \mathbb{CP}^1$. On the pre-image of $U \cap \widetilde{U}$ in $\mathcal{Z}_b$ we have

$$\widetilde{Q} = \lambda^{-k}Q, \quad \tilde{\tau} = \lambda^{k-2}\tau + b\lambda^{-1}Q, \tag{2.4}$$

where τ and $\tilde{\tau}$ are coordinates on the fibres of $\mathcal{Z}_b$ over U and $\widetilde{U}$, respectively. Restricting the inhomogeneous coordinates to a section of $\mathcal{O}(k) \to \mathbb{CP}^1$

$$Q = x_k\lambda^k + \cdots + x_1\lambda + x_0 \tag{2.5}$$

and performing the splitting of (2.4) gives

$$\tilde{\tau} - b\lambda^{-1}x_0 - bx_1 = \lambda^{k-2}\tau + b\lambda^{k-1}x_k + \cdots + b\lambda x_2. \tag{2.6}$$

†If $k = 2$, and b is interpreted as the inverse of the speed of light, then the jumping from $\mathcal{Z}_b$ to $\mathcal{Z}_0$ is the Newtonian limit of the twistor correspondence [6].

The LHS and RHS of (2.6) are holomorphic on $\widetilde{U}$ and U, respectively. Both sides of this relation are sections of a line bundle with a negative Chern class, so should vanish by the Liouville theorem. Thus

$$\tau = -b(\lambda x_k - x_{k-1} - \lambda^{-1}x_{k-2} - \cdots - \lambda^{k-3}x_2). \tag{2.7}$$

This will be holomorphic in λ on U if $(k-3)$ conditions

$$x_{k-2} = x_{k-3} = \cdots = x_2 = 0 \tag{2.8}$$

hold. These conditions arise only if $k > 3$. They define a holomorphic four-dimensional subspace $M_{\mathbb{C}}$ in the $(k+1)$-dimensional space of holomorphic sections of $\mathcal{O}(k)$.

The algebraic geometry of holomorphic sections of $\mathcal{Z}_b \to \mathbb{CP}^1$ determines a conformal structure on $M_{\mathbb{C}}$: two points in $M_{\mathbb{C}}$ are null separated iff the corresponding sections intersect at one point in $\mathcal{Z}_b$. Infinitesimally, a vector in $T_pM_{\mathbb{C}}$ is null if the corresponding section of $N(L_p)$ vanishes at one point. This condition is equivalent to the existence of the unique solution $\lambda = \lambda_0$ to a simultaneous system

$$\delta Q = 0, \quad \delta\tau = 0, \tag{2.9}$$

where Q and τ are given by (2.5) and (2.7). These conditions give

$$\lambda^k\delta x_k + \cdots + \lambda\delta x_1 + \delta x_0 = 0, \quad \lambda\delta x_k + \delta x_{k-1} = 0.$$

Imposing (2.8) and using the second equation replaces the first equation by $\lambda\delta x_1 + \delta x_0 = 0$. Eliminating λ between the two equations in (2.9) gives the quadratic conformal structure

$$[g] = \delta x_k\delta x_0 - \delta x_{k-1}\delta x_1 \tag{2.10}$$

which is flat.

3. *Conformal structures on Zariski cones from Kodaira deformations*

Consider an affine line bundle $\mathcal{Z} \to \mathcal{O}(k)$, with underlying translation bundle given by $\mathcal{O}(2-k)$. Such bundles are classified by elements of $H^1(\mathcal{O}(k), \mathcal{O}(2-k))$ and we choose a cohomology representative which leads to the patching relations

$$\widetilde{Q} = \lambda^{-k}Q, \quad \tilde{\tau} = \lambda^{k-2}\tau + a\lambda^{-2}Q^2, \quad \text{where } a = \text{const}. \tag{3.1}$$

Restricting this to holomorphic sections (2.5) of $\mathcal{Z} \to \mathbb{CP}^1$ and splitting gives

$$\begin{aligned}\tilde{\tau} - a\lambda^{-2}({x_0}^2 + 2\lambda x_0x_1 + \lambda^2(2x_0x_2 + {x_1}^2))\\ = \lambda^{k-2}\tau + a\lambda(2x_0x_3 + 2x_1x_2) + \cdots + a\lambda^{2k-2}{x_k}^2.\end{aligned} \tag{3.2}$$

Therefore τ is holomorphic in λ if $(k-3)$ quadratic conditions

$$x_0x_3 + x_1x_2 = 0, \quad x_0x_4 + x_1x_3 + \frac{{x_2}^2}{2} = 0, \ldots, x_0x_{k-1} + x_1x_{k-2} + \cdots \tag{3.3}$$

hold. These constraints put no restrictions on x_k, and we can assume that

$$t = x_0, \quad z = x_1, \quad y = x_2, \quad x = x_k$$

are coordinates on an open set in $M_{\mathbb{C}}$, and that the remaining coordinates $(x_3, \ldots, x_{k-1})$ have been expressed as functions of (y, z, t). To compute the ASD conformal structure $[g]$ on $M_{\mathbb{C}}$ we follow the procedure leading to (2.10), except that to simplify the computations the condition (2.9) is replaced by the equivalent condition $\delta\widetilde{Q} = 0, \delta\tilde{\tau} = 0$ (the resulting conformal structure does not depend on the choice of the open set) and pull the differentials δx_i in $\delta\widetilde{Q}$ back to $M_{\mathbb{C}}$. To eliminate λ we take the resultant Res of the quadratic $\lambda^2\delta\tilde{\tau}$ and the polynomial of degree

k given by $\lambda^k \delta\widetilde{Q}$. The resultant is a section of $\text{Sym}^{(k+2)}(T^*M_{\mathbb{C}})$ which factorises as $[g](\delta x_0)^k$, where $[g] \in \text{Sym}^2(T^*M_{\mathbb{C}})$ is the conformal structure given by

$$[g] = \text{Res}\Big(x_0\delta x_0 + \lambda(x_0\delta x_1 + x_1\delta x_0) + \lambda^2(x_0\delta x_2 + x_2\delta x_0 + x_1\delta x_1),$$
$$\delta x_0 + \lambda\delta x_1 + \cdots + \lambda^k\delta x_k\Big). \tag{3.4}$$

Here $\delta x_3, \ldots, \delta x_{k-1}$ are the pull-backs from $H^0(\mathbb{CP}^1, \mathcal{O}(k))$ to $M_{\mathbb{C}}$ defined by the relations (3.3). The deformation (3.1) preserves the fibration $\mu : \mathcal{Z} \to \mathbb{CP}^1$, and the fibres of μ are equipped with an $\mathcal{O}(2)$-valued symplectic form

$$\Sigma = \lambda^{-2} dQ \wedge d\tau = d\widetilde{Q} \wedge d\tilde{\tau}.$$

Therefore there exists a Ricci-flat metric $g \subset [g]$ in the ASD conformal class (3.4).

THEOREM 3.1. *Let $(M_{\mathbb{C}}, g)$ be an ASD Ricci–flat manifold corresponding to the twistor space with the patching relations (3.1). There exists a curve $\gamma \subset M_{\mathbb{C}}$ such that all points on γ correspond to rational curves $\mathcal{Z}$ where the normal bundle jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$, and such that γ is preserved by a tri–holomorphic Killing vector.*

Proof. We will first prove that $(M_{\mathbb{C}}, g)$ admits a triholomorphic Killing vector field. The coefficients of the conformal structure (3.4) do not depend on $x \equiv x_k$, so $K = \partial/\partial x$ is a conformal Killing vector. Conformal Killing vectors in $M_{\mathbb{C}}$ generate one-parameter groups of transformations of $M_{\mathbb{C}}$ which map α–surfaces to α–surfaces. Thus (as the points in $\mathcal{Z}$ are α–surfaces in $M_{\mathbb{C}}$) conformal Killing vectors correspond to global holomorphic vector fields on $\mathcal{Z}$. Consider the holomorphic vector field $\mathcal{K}$ in $\mathcal{Z}$ corresponding to the conformal Killing vector $K = \partial/\partial x$. We will compute this vector field on the open set $\widetilde{U}$

$$\mathcal{K} = \frac{\partial \widetilde{Q}}{\partial x}\frac{\partial}{\partial \widetilde{Q}} + \frac{\partial \tilde{\tau}}{\partial x}\frac{\partial}{\partial \tilde{\tau}}$$
$$= \frac{\partial}{\partial \widetilde{Q}},$$

where we have used (3.2). Therefore

$$\mathcal{L}_{\mathcal{K}}\Sigma = 0, \quad \mathcal{L}_{\mathcal{K}}\lambda = 0,$$

and so $\mathcal{K}$ preserves the symplectic two-form on the fibres of $\mathcal{Z} \to \mathbb{CP}^1$, as well as the fibration itself. The first condition implies that K is a Killing vector of the Ricci-flat metric singled out by Σ in the conformal structure $[g]$. The second condition means that K is tri-holomorphic (it acts trivially on the basis of parallel self-dual two-forms).

Now consider the normal bundle $N(L_p)$ to a curve $L_p \subset \mathcal{Z}$ corresponding to a point $p \in M_{\mathbb{C}}$. For a generic p, the bundle $N(L_p)$ is biholomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Its patching matrix is given by

$$F_N = \begin{pmatrix} \frac{\partial \tilde{\tau}}{\partial \tilde{\tau}} & \frac{\partial \tilde{\tau}}{\partial \widetilde{Q}} \\ \frac{\partial \widetilde{Q}}{\partial \tilde{\tau}} & \frac{\partial \widetilde{Q}}{\partial \widetilde{Q}} \end{pmatrix} = \begin{pmatrix} \lambda^{k-2} & 2a\lambda^{-2}Q \\ 0 & \lambda^{-k} \end{pmatrix}.$$

To investigate the non-generic points introduce the splitting matrices

$$H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 1 & \tilde{h} \\ 0 & 1 \end{pmatrix}$$

which are invertible and holomorphic in U and $\widetilde{U}$, respectively. Now

$$\widetilde{H}F_N H^{-1} = \begin{pmatrix} \lambda^{k-2} & 2a\lambda^{-2}Q - h\lambda^{k-2} + \tilde{h}\lambda^{-k} \\ 0 & \lambda^{-k} \end{pmatrix}.$$

Thus the normal bundle jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(2-k) \oplus \mathcal{O}(k)$ at points of $M_{\mathbb{C}}$ where

$$2a\lambda^{-2}Q - h\lambda^{k-2} + \tilde{h}\lambda^{-k} = 0.$$

The functions h and $\tilde{h}$ are holomorphic in λ and λ^{-1}, respectively. Therefore the equality can be satisfied by some choice of h and $\tilde{h}$ only if

$$x_{k-1} = x_{k-2} = \cdots = x_0 = 0.$$

These conditions imply the conditions (3.3), and leave the coordinate x_k unspecified. Thus there is a one–parameter family $\gamma \subset M_{\mathbb{C}}$ of jumping lines in $\mathcal{Z}$, and K is tangent to γ. □

In addition to K, the Ricci–flat metric arising from Theorem 3.1 admits a homothetic conformal Killing vector $x\partial_x + y\partial_y + z\partial_z + t\partial_t$. In what follows we will work out the metrics and their Gibbons–Hawking forms in detail when $k = 3$ and $k = 4$. We note that in these two cases the metric admits another Killing vector, where the coordinates scale with different weights.

3.1. *Ricci–flat metric with $k = 3$*

Parametrise the sections of $\mathcal{O}(3) \to \mathbb{CP}^1$ by

$$Q = t + \lambda z + \lambda^2 y + \lambda^3 x.$$

In this case the ASD conformal structure (3.4) is

$$g = \Omega^2 \Big(t^3 dx^2 + 2t^2 z dx dy + t(yt + z^2) dx dz + z(3yt - z^2) dx dt + tz^2 dy^2 + z(yt + z^2) dy dz + y(yt + z^2) dy dt + yz^2 dz^2 + 2y^2 z dz dt + y^3 dt^2 \Big). \quad (3.5)$$

We find that the choice

$$\Omega^2 = \frac{2a^3}{z^2 - yt} \quad (3.6)$$

makes the resulting metric Ricci-flat. If the coordinates (x, y, z, t) are chosen to be real, then g is real, and has neutral signature. The basis of self-dual parallel two-forms is

$$\Sigma^{00} = 2dx \wedge (tdy + ydt + zdz),$$
$$\Sigma^{01} = tdx \wedge dz + zdx \wedge dt + zdy \wedge dz + ydy \wedge dt,$$
$$\Sigma^{11} = 2(tdx + ydz + zdy) \wedge dt.$$

These forms are Lie derived by the Killing vector $K = \partial/\partial x$. This Killing vector is tri-holomorphic and therefore the metric g can be cast in the Gibbons–Hawking form

$$g = V h_{flat} + V^{-1}(dT + A)^2, \quad \text{where} \quad T \equiv x, \quad (3.7)$$

where h_{flat} is a flat metric on $\mathbb{R}^{1,2}$, and V and A are, respectively, a function, and a one–form on $\mathbb{R}^{1,2}$ which satisfy the Abelian monopole equation

$$dV = *dA,$$

where $*$ is the Hodge endomorphism of h_{flat}. The function V can be read-off directly from g, and is given by

$$V = g(K,K)^{-1} = \frac{ty - z^2}{2a^3t^3}.$$

To construct the flat coordinates for the metric

$$h_{flat} = V^{-1}(g - VK \otimes K)$$

we first compute the six generators of its group of isometries. We then select a three-dimensional abelian subalgebra (X_1, X_2, X_3) generated by translations. The corresponding one forms $h_{flat}(X_i, .)$ are exact differentials of the flat coordinates (Y, Z, T), where

$$y = \frac{X^2 - Y^2 - Z^2}{2(X+Y)^{3/2}}, \quad z = \frac{Z}{(X+Y)^{1/2}}, \quad t = (X+Y)^{1/2}. \tag{3.8}$$

Now

$$h_{flat} = a^6(dX^2 - dY^2 - dZ^2), \quad V = \frac{X^2 - Y^2 - 3Z^2}{4a^3(X+Y)^{5/2}} \tag{3.9}$$

and V satisfies the wave equation on $\mathbb{R}^{1,2}$.

- Instead of using the patching relation (3.1) we could have started with

$$\tilde{\tau} = \lambda\tau + b\lambda^{-1}Q + a\lambda^{-2}Q^2,$$

which allows the limit $a \to 0$ corresponding to the patching (2.4), and resulting in a flat conformal structure. The corresponding metric and the conformal factor arise from g and Ω given by (3.5) and (3.6) when one makes a replacement

$$z \longrightarrow z + \frac{b}{2a}.$$

Therefore, if $a \neq 0$ then b can be set to zero by translating z.

- The metric (3.5) admits a second Killing vector

$$K_2 = 5x\partial_x + 2y\partial_y - z\partial_z - 4t\partial_t$$

which is not tri-holomorphic. It Lie derives Σ^{01}, but rotates Σ^{00} and Σ^{11}. Thus the space of orbits of K_2 in $M_{\mathbb{C}}$ admits a Toda Einstein–Weyl structure [**3**, **24**].

- There exists a combination of self-dual two-forms which is degenerate when the harmonic function V in (3.5) vanishes, which is the surface $ty - z^2 = 0$ in $M_{\mathbb{C}}$. All three forms vanish on the line $t = y = z = 0$ in $M_{\mathbb{C}}$. The normal bundle of the twistor curves corresponding to this line jumps from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(3) \oplus \mathcal{O}(-1)$.

3.2. *Ricci-flat metric with $k = 4$*

Parametrise the sections of $\mathcal{O}(4) \to \mathbb{CP}^1$ by

$$Q = t + \lambda z + \lambda^2 y + \lambda^3 w + \lambda^4 x. \tag{3.10}$$

In this case the splitting (2.6) is possible if

$$\phi \equiv tw + zy = 0. \tag{3.11}$$

Computing the resultant (3.4) leads to the conformal structure

$$\begin{aligned} g = \Omega^2\Big(& t^6dx^2 - t^3z(ty+z^2)dxdz + 2t^4(ty-z^2)dxdy + t^2(2t^2y^2 - 5tyz^2 + z^4)dxdt \\ & -2tyz^2(ty-z^2)dz^2 - tz(t^2y^2 - z^4)dzdy - yz(t^2y^2 - 6tyz^2 + z^4)dzdt \\ & + t^2(ty-z^2)^2dy^2 + 2t^2y^2(ty+z^2)dydt + y^2(t^2y^2 + 4tyz^2 - z^4)dt^2\Big). \end{aligned} \tag{3.12}$$

The conformal factor making this metric Ricci flat is

$$\Omega^2 = \frac{2a^4}{t^2 z(3ty - z^2)}.$$

The triholomorphic Killing vector $\partial/\partial x$ Lie derives the covariantly constant basis of self-dual two–forms

$$\begin{aligned}
\Sigma^{00} &= 2dx \wedge (tdy + ydt + zdz), \\
\Sigma^{01} &= tdx \wedge dz + zdx \wedge dt - \frac{yt - z^2}{t} dz \wedge dy - \frac{y(yt + z^2)}{t^2} dz \wedge dt - \frac{2yz}{t} dy \wedge dt, \\
\Sigma^{11} &= dt \wedge \left(4\frac{yz}{t} dz - 2tdx - 2\frac{yt - z^2}{t} dy \right).
\end{aligned} \tag{3.13}$$

The coordinate transformation (3.8) brings the metric g to Gibbons–Hawking form (3.7), where

$$V = \frac{z(3ty - z^2)}{2a^4 t^4} = \frac{Z(3X^2 - 3Y^2 - 5Z^2)}{4a^4 (X + Y)^{7/2}}, \tag{3.14}$$

and

$$h_{flat} = a^8 (dX^2 - dY^2 - dZ^2).$$

The metric g admits a homothety, as well as a second non-triholomorphic Killing vector $K_2 = t\partial_t - y\partial_y - 3x\partial_x$.

3.3. *Gibbons–Hawking potential for general k*

There exists a map from an affine bundle over $\mathcal{O}(k)$ with holomorphic charts (Q, τ) and $(\widetilde{Q}, \tilde{\tau})$ to an affine bundle over $\mathcal{O}(2)$ with charts (q, p) and $(\tilde{q}, \tilde{p})$ such that the latter admits a four-parameter family of section only if the patching for the former satisfies some additional conditions. The explicit transformation is given by

$$q = \lambda^k \tau + Q^2, \quad p = \sum_{n=1}^{\infty} \frac{(2n)!}{(1 - 2n)(n!)^2 4^n} \lambda^{(n-1)k} \tau^n (\lambda^k \tau + Q^2)^{1/2 - n}$$

on U, and

$$\tilde{q} = \tilde{\tau}, \quad \tilde{p} = \widetilde{Q}.$$

on $\widetilde{U}$. This map is well defined only if some sections are removed from the $\mathcal{O}(k)$ twistor space. This corresponds to removing the region from $M_{\mathbb{C}}$ corresponding to the 'big jump'. In the case of (3.1) we find

$$\tilde{q} = \lambda^{-2} q, \quad \tilde{p} = p + s(q, \lambda), \quad \text{where} \quad s = \lambda^{-k} \sqrt{q}.$$

The element of $H^1(\mathcal{O}(2), \mathcal{O}(-2))$ corresponding to the Gibbons–Hawking function is $\partial s/\partial q$. Parametrising the sections of $\mathcal{O}(2)$ by

$$q = \lambda^2 (X - Y) + 2\lambda Z + (X + Y)$$

and taking the contour enclosing $\lambda = 0$ in the twistor integral formula, leads to

$$V(X, Y, Z) = \frac{1}{2(k-1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} (\lambda^2 (X - Y) + 2\lambda Z + (X + Y))^{-1/2} |_{\lambda = 0},$$

which for $k = 3$ and $k = 4$ agrees with (3.9) and (3.14).

3.4. $k = 2$ and $k = 1$

For completeness we will analyse the remaining cases $k = 2$ and $k = 1$ with the constant a set to 1. If $k = 2$, both sides of (3.2) are homogeneous of degree 0, and thus are equal to some x_{-1}, so that

$$\tilde{\tau} = x_{-1} + 2\tilde{\lambda} x_0 x_1 + \tilde{\lambda}^2 x_0^2, \quad \widetilde{Q} = x_2 + \tilde{\lambda} x_1 + \tilde{\lambda}^2 x_0,$$

where in $\tilde{\tau}$ we have absorbed $(2x_0x_2 + {x_1}^2)$ into x_{-1}. Now compute the resultant (3.4), and set

$$x_{-1} = -2iT + \frac{1}{2}(X^2 + Y^2) - Z^2, \quad x_0 = \frac{1}{\sqrt{2}}(X + iY), \quad x_1 = \sqrt{2} iZ, \quad x_2 = \frac{1}{\sqrt{2}}(X - iY).$$

This yields the folded hyper-Kähler metric (1.1).

Finally if $k = 1$ then both sides of (3.2) are homogeneous of degree 1, and thus give rise to two coordinates on $M_{\mathbb{C}}$. The resulting metric is flat. In this case the twistor space $\mathcal{Z}$ fibres over $\mathcal{O}(1)$, and all metrics (corresponding to arbitrary patching) fall into the classification of [**9**].

4. *From Sparling–Tod to Eguchi–Hanson*

The holomorphic Sparling–Tod metric [**20**, **21**]

$$g = 4dudv - 4dxdy - 8\rho \triangle^{-3}(udv - xdy)^2, \quad \triangle \equiv uv - xy \tag{4.1}$$

is ASD, Ricci–flat, and of Petrov-Penrose type D. In [**4**] a twistor–theoretic argument was used to show that there exists a Riemannian real section of (4.1) which is equivalent to the Eguchi–Hanson gravitational instanton. The coordinate transformation below makes this explicit, by putting (4.1) in the ALE A_2 Gibbons–Hawking form.

We find that the four-dimensional isometry group of (4.1) contains $SL(2, \mathbb{C})$ which acts tri-holomorphically. Let us consider a pencil of tri-holomorphic Killing vectors given by

$$K = \frac{b}{2}(v\partial_y + x\partial_u) - \frac{c}{2}(y\partial_v + u\partial_x),$$

where (b, c) are constants not both zero. A parallel basis of ${\Lambda^2}_+$ is Lie-derived along K, and the corresponding moment maps are the flat coordinates on $\mathbb{R}^3$ in the Gibbons–Hawking form. They are given by

$$Z = i(bxv + cyu), \quad X + iY = \sqrt{2}\rho(bx^2 + cu^2)\triangle^{-2} + \frac{\sqrt{2}}{2}(bx^2 + cu^2),$$

$$X - iY = \sqrt{2}(cy^2 + bv^2).$$

The metric (4.1) takes the form

$$g = V(dX^2 + dY^2 + dZ^2) + V^{-1}(dT + A)^2,$$

where V is the harmonic function on $\mathbb{R}^3$ given by

$$V = \frac{\triangle^3}{2\rho Z^2 + bc\triangle^4}$$

and

$$\triangle^2 = \frac{2bc\rho - R^2 + \epsilon\sqrt{(R^2 - 2bc\rho)^2 + 8bc\rho Z^2}}{2bc}, \quad R^2 \equiv X^2 + Y^2 + Z^2, \tag{4.2}$$

where $\epsilon = \pm 1$. With the help of some algebra this simplifies to

$$V = \frac{1}{2\sqrt{-bc}}\left(\frac{1}{|\mathbf{R} + \mathbf{a}|} - \epsilon\frac{1}{|\mathbf{R} - \mathbf{a}|}\right),$$

where $\mathbf{R} = (X, Y, Z), \mathbf{a} = (0, 0, \sqrt{-2bc\rho})$. If b, c are real and such that $bc < 0$ then V with $\epsilon = -1$ gives the positive-definite Eguchi–Hanson gravitational instanton. If $\epsilon = 1$ then g is still positive-definite, but not complete. It is an example of a folded hyper-Kähler metric [**12**]. The points on the hypersurface $V = 0$ in M correspond to lines in $\mathcal{Z}$ where the normal bundle jumps to $\mathcal{O} \oplus \mathcal{O}(2)$. Moreover, the limit when b or c tends to zero gives a dyon.

5. *Multi-jumps*

The metrics resulting from Theorem 3.1 admit a tri-holomorphic Killing vector, and thus can locally be put in the holomorphic Gibbons–Hawking form [**10**], which depends on a solution V to a holomorphic Laplace equation on $\mathbb{C}^3$. The hypersurface corresponding to $V = 0$ is singular, and can be characterised by the jumping phenomenon.

PROPOSITION 5.1. *Let $(M_{\mathbb{C}}, g)$ be a Gibbons–Hawking metric*

$$g = V(dX^2 - dY^2 - dZ^2) + V^{-1}(dT + A)^2, \quad \textit{where} \quad dV = *_3 dA$$

and let $S = \{p \in M_{\mathbb{C}}, V(p) = 0\}$. The points of S correspond to rational curves in $\mathcal{Z}$ with normal bundle $\mathcal{O}(2) \oplus \mathcal{O}$.

Proof. The twistor space of a Gibbons–Hawking manifold is an affine line bundle over the total space of $\mathcal{O}(2)$ with transition functions

$$\widetilde{\tau} = \tau + f(Q, \lambda), \quad \widetilde{Q} = \lambda^{-2} Q,$$

where $f \in H^1(\mathcal{O}(2), \mathcal{O})$. Restricting the cohomology class f to a section of $\mathcal{O}(2)$

$$Q = \lambda^2(X - Y) + 2\lambda Z + (X + Y) \tag{5.1}$$

gives rise to the harmonic function V by

$$V(p) = \frac{1}{2\pi i} \oint_{\Gamma \subset L_p} \frac{\partial f}{\partial Q} d\lambda.$$

The normal bundle to L_p is the restriction to (5.1) of

$$\begin{pmatrix} 1 & \dfrac{\partial f}{\partial Q} \\ 0 & \lambda^{-2} \end{pmatrix}$$

and then expanding

$$f(Q, \lambda) = \sum_{-\infty}^{\infty} a_k \lambda^k, \quad \text{with} \quad a_k = a_k(X, Y, Z).$$

Split the sum into two:

$$\tilde{h} = -\sum_{-\infty}^{-1} a_k \lambda^k, \quad h = \sum_{0}^{\infty} a_k \lambda^k, \quad \text{so that} \quad f = h - \tilde{h}$$

and there is freedom to add a function of (X, Y, Z) to both of $h, \tilde{h}$. Note that, after restricting,

$$\frac{\partial f}{\partial Z} = \frac{\partial f}{\partial Q} \frac{\partial Q}{\partial Z} = 2\lambda \frac{\partial f}{\partial Q}$$

so that the transition matrix for the normal bundle is

$$F := \begin{pmatrix} 1 & \dfrac{1}{2\lambda}\dfrac{\partial f}{\partial Z} \\ 0 & \lambda^{-2} \end{pmatrix}.$$

This is equivalent to

$$F \to \begin{pmatrix} 1 & \tilde{p} \\ 0 & 1 \end{pmatrix} F \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{1}{2\lambda}\dfrac{\partial f}{\partial Z} + p + \tilde{p}\lambda^{-2} \\ 0 & \lambda^{-2} \end{pmatrix},$$

where we choose $\tilde{p}, p$ to remove from $\frac{1}{2\lambda}\frac{\partial f}{\partial Z}$ all non-negative powers of λ and all negative powers less than or equal to -2. All that remains is $\frac{1}{2\lambda}\frac{\partial a_0}{\partial Z}$, and $\frac{\partial a_0}{\partial Z}$ is equal to a multiple of V. Where $V \neq 0$, we know that this is the transition matrix for $\mathcal{O}(1) \oplus \mathcal{O}(1)$ but clearly where $V = 0$ it is the transition matrix for $\mathcal{O}(2) \oplus \mathcal{O}$. □

We conclude that the metrics arising from Theorem 3.1 have at least two jumps: the 'small jump' from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to $\mathcal{O}(2) \oplus \mathcal{O}$ on a surface S corresponding to the zero set of $g(\partial_x, \partial_x)^{-1}$, and a big jump to $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$ on a curve γ. The argument below demonstrates that many intermediate jumps can arise.

Consider the twistor space of Theorem 3.1, with the moduli space of rational curves $M_{\mathbb{C}}$ given by the Zariski cone (3.3). The constraints defining $M_{\mathbb{C}}$ take the form

$$x_0 x_n + x_1 x_{n-1} + \cdots = 0, \quad \text{for } 3 \leqslant n \leqslant k-1.$$

For even $n = 2m$ the last term in the constraint is $x_m^2/2$ and there will be a constraint like this for

$$1 \leqslant m \leqslant (k-1)/2 \quad \text{for odd } k \quad \text{or} \quad 1 \leqslant m \leqslant k/2 - 1 \quad \text{for even } k.$$

We will be interested in solutions of the constraints for which all but one x_n are zero (for $n < k$) and the constraints will not be satisfied for n below a threshold. Thus we have a range of allowed n, namely $(1+k)/2 \leqslant n \leqslant k-1$ for odd n or $k/2 \leqslant n \leqslant k-1$ for n even. For n in these ranges the constraints are satisfied with $x_n \neq 0$ and $x_i = 0$ for all other i in the range $0 \leqslant i \leqslant k-1$. With any one of these solutions of the constraints, multiply F on the right with

$$H^{-1} = \begin{pmatrix} 1 & -2ax_k \\ 0 & 1 \end{pmatrix} \tag{5.2}$$

to remove x_k-term from Q, leaving

$$F = \begin{pmatrix} \lambda^{k-2} & 2a\displaystyle\sum_i x_i \lambda^{i-2} \\ 0 & \lambda^{-k} \end{pmatrix}.$$

Now consider the product

$$\widetilde{H} \begin{pmatrix} \lambda^{i-2} & 0 \\ 0 & \lambda^{-i} \end{pmatrix} H^{-1} = \begin{pmatrix} \alpha & 0 \\ \lambda^{2-i-k} & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{i-2} & 0 \\ 0 & \lambda^{-i} \end{pmatrix} \begin{pmatrix} \alpha^{-1}\lambda^{k-i} & 1 \\ -1 & 0 \end{pmatrix},$$

when $\widetilde{H}$ and H, respectively, have only negative or only positive powers of λ and this product is F given the choice $\alpha = 2ax_i$. Thus the normal bundle has jumped to $\mathcal{O}(2-i) \oplus \mathcal{O}(i)$ and there is an example like this for each i in the allowed range.

We also always have the case $x_0 \neq 0$, other x_n zero when

$$\begin{pmatrix} \alpha & 0 \\ \lambda^{2-k} & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} \lambda^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1}\lambda^k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^{k-2} & \alpha\lambda^{-2} \\ 0 & \lambda^{-k} \end{pmatrix} = F$$

with $\alpha = 2ax_0$, so the jump to $\mathcal{O} \oplus \mathcal{O}(2)$ is always present.

We will see that all jumps are present if $k = 4$. There is enough here to prove this also for $k = 5$ and $k = 6$ but there is a gap at $k = 7$: the above constructions do not give an example of a curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(3)$ but everything else up to $\mathcal{O}(-5) \oplus \mathcal{O}(7)$ occurs.

5.1. *Jump cascade with $k = 4$*

Restrict the transition function (3.1) with $k = 4$ to the line (3.10). The normal bundle is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ away from $V = 0$, where V is given by (3.14), and jumps to $\mathcal{O}(-2) \oplus \mathcal{O}(4)$ at $y = z = w = t = 0$. We have to look at the zero-set of V. Multiply F on the right with (5.2) leaving

$$F = \begin{pmatrix} \lambda^2 & 2a(w\lambda + y + \frac{z}{\lambda} + \frac{t}{\lambda^2}) \\ 0 & \lambda^{-4} \end{pmatrix}.$$

There are six loci to investigate all of which have $V = 0$.

S_1. $w = y = z = t = 0$ when we know it jumps to $\mathcal{O}(-2) \oplus \mathcal{O}(4)$.
S_2. $w = y = z = 0$ but $t \neq 0$. Note that

$$H^{-1} = \begin{pmatrix} 1/\beta & 0 \\ \lambda^4 & \beta \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{\beta\lambda^2} \end{pmatrix}$$

give

$$\widetilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \lambda^2 & \beta\lambda^{-2} \\ 0 & \lambda^{-4} \end{pmatrix}$$

which with $\beta = 2at$ shows S_2 is $\mathcal{O} \oplus \mathcal{O}(2)$.
S_3. $y = z = t = 0$ but $w \neq 0$. Take

$$H^{-1} = \begin{pmatrix} \lambda/\beta & 1 \\ -1 & 0 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} \beta & 0 \\ \lambda^{-5} & \frac{1}{\beta} \end{pmatrix}$$

which give

$$\widetilde{H} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-3} \end{pmatrix} H^{-1} = \begin{pmatrix} \lambda^2 & \beta\lambda \\ 0 & \lambda^{-4} \end{pmatrix}$$

which with $\beta = 2bw$ shows S_3 is $\mathcal{O}(-1) \oplus \mathcal{O}(3)$.
S_4. $z = t = 0$ but $y \neq 0$ (any w). Take

$$H^{-1} = \begin{pmatrix} \lambda^2 & \alpha + \beta\lambda \\ \frac{\beta\lambda}{\alpha^2} - \frac{1}{\alpha} & \frac{\beta^2}{\alpha^2} \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 1 & 0 \\ -\frac{\beta}{\alpha^2\lambda^3} + \frac{1}{\alpha\lambda^4} & 1 \end{pmatrix}$$

which give

$$\widetilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \lambda^2 & \alpha + \beta\lambda \\ 0 & \lambda^{-4} \end{pmatrix}$$

so with $\alpha = 2ay, \beta = 2az$ this shows that S_4 is $\mathcal{O} \oplus \mathcal{O}(2)$.
S_5. $z = w = 0$ with $yt \neq 0$. Consider

$$H^{-1} = \begin{pmatrix} -\dfrac{\lambda^2}{\alpha} + \dfrac{\beta}{\alpha^2} & -1 \\ -6pt]1 & 0 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} -\alpha - \dfrac{\beta}{\lambda^2} & \frac{\beta^2}{\alpha^2} \\ -\dfrac{1}{\lambda^4} & -\frac{1}{\alpha} + \frac{\beta}{\alpha^2\lambda^2} \end{pmatrix}$$

so that

$$\widetilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \lambda^2 & \alpha + \dfrac{\beta}{\lambda^2} \\ 0 & \lambda^{-4} \end{pmatrix}.$$

With $\alpha = 2ay, \beta = 2at$ and $yt \neq 0$ this shows that S_5 is $\mathcal{O} \oplus \mathcal{O}(2)$.
S_6. $z^2 = 3ty$ with $yzt \neq 0$. Introduce

$$\chi := \frac{t}{\lambda^2} + \frac{z}{\lambda} + y + w\lambda$$

then set $t = \beta, z = 3\alpha t$. We have $3yt - z^2 = 0 = wt + yz$ so that $y = 3\alpha^2\beta$ and $w = -9\alpha^3\beta$ whence

$$\chi = \frac{\beta}{\lambda^2}(1 + 3\alpha\lambda + 3\alpha^2\lambda^2 - 9\alpha^3\lambda^3)$$

and this is the top-right entry in F. Consider

$$H^{-1} = \begin{pmatrix} 1 - 3\alpha\lambda + 6\alpha^2\lambda^2 & 9\alpha^4\beta(5 - 6\alpha\lambda) \\ \dfrac{\lambda^4}{\beta} & \dfrac{\lambda^2 f}{\beta} \end{pmatrix}, \quad \widetilde{H} \begin{pmatrix} 0 & \beta \\ -\dfrac{1}{\beta} & \dfrac{1}{\lambda^2} - 3\dfrac{\alpha}{\lambda} + 6\alpha^2 \end{pmatrix}.$$

Then

$$\widetilde{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1} = \begin{pmatrix} \lambda^2 & \chi \\ 0 & \lambda^{-4} \end{pmatrix}$$

so S_6 is also $\mathcal{O} \oplus \mathcal{O}(2)$.

We conclude that there are curves with normal bundle $(1,1), (0,2), (-1,3)$ and $(-2,4)$, that is, all possibilities up to the maximum jump occur.

6. *Generalised Legendre transform and self-dual two-forms*

There is another route directly from the cohomology class defining the affine line bundle $\mathcal{Z} \longrightarrow \mathcal{O}(k)$ to the ASD Ricci-flat metric directly without the need to use resultants as in (3.4). This follows [**7**, Theorem 4.4], and gives a version of the generalised Legendre transform [**1**, **14**].

Affine line bundles over $\mathcal{O}(k)$ are classified by elements $[f]$ of $H^1(\mathcal{O}(k), \mathcal{O}(2-k))$ as

$$\tilde{\tau} = \tau + f(Q, \lambda), \quad \widetilde{Q} = \lambda^{-k} Q. \tag{6.1}$$

Any such cohomology class gives rise to $k - 3$ constraints

$$\phi_{A_1 \cdots A_{k-4}} := \frac{1}{2\pi i} \oint_\Gamma \pi_{A_1} \cdots \pi_{A_{k-4}} f(Q, \pi_A) \pi \cdot d\pi = 0 \tag{6.2}$$

which trace out a holomorphic four-manifold $M_{\mathbb{C}}$ in a $(k+1)$-dimensional space of holomorphic sections of $\mathcal{O}(k) \to \mathbb{CP}^1$. The ASD Ricci-flat metric on $M_{\mathbb{C}}$ is determined by a basis of self-dual two-forms $\{\Sigma^{00},\Sigma^{01},\Sigma^{11}\}$ which are pull-backs from $\mathbb{C}^{k+1}$ to M of two-forms[†]

$$\Sigma^{AB} = \frac{1}{8}\psi^{AB}{}_{B_1\cdots B_{k-3}C_1\cdots C_{k-3}}dx_{PQR}{}^{B_1\cdots B_{k-3}}\wedge dx^{PQRC_1\cdots C_{k-3}} + \frac{3}{2}\psi_{B_1\cdots B_{k-2}C_1\cdots C_{k-2}}dx_P{}^{B_1\cdots B_{k-2}\left(A\wedge dx^{C_1}\right)C_2\cdots C_{k-2}BP}, \tag{6.3}$$

where

$$\psi_{A_1\dots A_{2k-4}} = \frac{1}{2\pi i}\oint_\Gamma \pi_{A_1}\dots\pi_{A_{2k-4}}\frac{\partial f}{\partial Q}\pi\cdot d\pi \tag{6.4}$$

is a zero-rest-mass field determined by $[f]$, and Q in this formula is regarded as the coordinate on the fibres of $\mathcal{O}(k)\to\mathbb{CP}^1$ which is homogeneous of degree k.

If $k=3$ then there are no constraints to be imposed, and ψ_{AB} is a self-dual Maxwell field originally constructed in [**23**].

6.1. *Example with $k=4$*

The manifold $M_{\mathbb{C}}$ is a surface $\phi=0$ given by (6.2) in the five-dimensional space $\mathcal{N}$ of holomorphic sections of the fibration $\mathcal{O}(4)\to\mathbb{CP}^1$. The function ϕ satisfies the overdetermined system of linear PDEs $\partial^A{}_{BCD}\psi_{EFGA}=0$ where ψ_{ABCD} is given by (6.4). Explicitly

$$\phi_{yt}-\phi_{zz}=0,\quad \phi_{tw}-\phi_{yz}=0,\quad \phi_{tx}-\phi_{wz}=0,$$
$$\phi_{wz}-\phi_{yy}=0,\quad \phi_{xz}-\phi_{wy}=0,\quad \phi_{xy}-\phi_{ww}=0.$$

Consider the cohomology class represented by $f=Q^2\lambda^{-k}$, and take $k=4$. Comparing $Q=x^{ABCD}\pi_A\pi_B\pi_C\pi_D$ with (3.10) gives

$$t=x^{1111},\quad z=4x^{1110},\quad y=6x^{1100},\quad w=4x^{1000},\quad x=x^{0000}.$$

Evaluating the residue at the pole $\lambda=0$ in (6.2) yields the constraint

$$\phi=tw+zy=0$$

in agreement with (3.11). The spin-2 field (6.4) is

$$\psi_{0000}=0,\quad \psi_{0001}=t,\quad \psi_{0011}=z,\quad \psi_{0111}=y,\quad \psi_{1111}=w$$

which gives the self-dual two-forms

$$\Sigma^{00}=z\left(2dx\wedge dz+\frac{1}{2}dw\wedge dy\right)+y\left(2dx\wedge dt-\frac{1}{2}dz\wedge dw\right)+w\left(\frac{1}{2}dw\wedge dt\right)+t(2dx\wedge dy),$$

$$\Sigma^{01}=z\left(dx\wedge dt-dz\wedge dw\right)+y\left(dw\wedge dt+\frac{1}{2}dy\wedge dz\right)+w\left(\frac{1}{2}dy\wedge dt\right)+t\left(dx\wedge dz+\frac{1}{2}dw\wedge dy\right),$$

$$\Sigma^{11}=z\left(2dw\wedge dt+\frac{1}{2}dy\wedge dz\right)+y\left(2dy\wedge dt\right)+w\left(\frac{3}{2}dz\wedge dt\right)+t\left(2dx\wedge dt-\frac{1}{2}dz\wedge dw\right).$$

The pull-back of these two–forms to the cone (3.11) agrees with expressions (3.13).

The jump cascade discussed in Section 5.1 can be now understood in the framework of the generalised Legendre transform presented in [**8**, **16**]. Using the Kodaira isomorphism

$$T^*{}_p\mathcal{N}\cong H^0(L_p,\mathcal{O}(4))=\mathrm{Sym}^4(\mathbb{C}^2)$$

[†]This formula corrects (4.27) in [**7**].

we can identify the gradient $d\phi$ with a binary quartic

$$d\phi \to \mathcal{Q}(d\phi) = \alpha s^4 + 4\beta s^3 + 6\gamma s^2 + 4\delta s + \epsilon$$
$$= \phi_x s^4 - 4\phi_w s^3 + 6\phi_y s^2 - 4\phi_z s + \phi_t.$$

Binary quartics admit two classical invariants

$$\mathcal{I} = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \quad \text{and} \quad \mathcal{J} = \det \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \end{pmatrix}. \tag{6.5}$$

If ϕ is given by (3.11) then

$$\mathcal{I} = 3z^2 - 4ty, \quad \mathcal{J} = z^3 - 2tzw + t^2 w.$$

The points in $M_{\mathbb{C}}$ where $d\phi = 0$ correspond to twistor curves with normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(4)$. The points where $d\phi \neq 0$, but $\mathcal{I} = \mathcal{J} = 0$ correspond to twistor curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(3)$. The points where $\mathcal{I} \neq 0$ and $\mathcal{J} = 0$ correspond to curves with normal bundle $\mathcal{O} \oplus \mathcal{O}(2)$. Finally the generic points have $\mathcal{I} \neq 0, \mathcal{J} \neq 0$. Such points correspond to twistor curves with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

6.2. *A Riemannian example*

The Riemannian reality conditions require $k = 2n$ to be even. The real sections satisfy

$$\overline{Q(\lambda)} = (-1)^n \overline{\lambda}^{2n} Q(-1/\overline{\lambda}),$$

which in the case $k = 4$ implies that

$$Q = t + \lambda z + \lambda^2 y - \lambda^3 \overline{z} + \lambda^4 \overline{t}$$

with t, z complex and y real. The surface (3.11) becomes $t\overline{z} + zy = 0$ which is of co-dimension two in the space of real sections of $\mathcal{O}(4)$. Thus the metric (3.12) does not admit a Riemannian slice.

To construct a Riemannian metric which admits a jump to $\mathcal{O}(-2) \oplus \mathcal{O}(4)$ consider a twistor space defined by the patching relation†

$$\widetilde{Q} = \lambda^{-4} Q, \quad \tilde{\tau} = \lambda^2 \tau + s(Q, \lambda), \quad \text{where} \quad s = 3Q^2(1 - \lambda^{-6}). \tag{6.6}$$

The metric can be computed as in (3) using the resultant (3.4), and constructing a conformal factor which makes the metric Ricci–flat. We will instead perform the Legendre transform of [14] which leads directly to a Kähler potential for the metric. To make contact with the notation and formalism of [14] define $G(Q, \lambda)$ by

$$\frac{\partial G}{\partial Q} = \frac{s}{\lambda^2}, \quad \text{so that} \quad G = \frac{Q^3}{\lambda^2}(1 - \lambda^{-6}),$$

and set

$$F = \frac{1}{2\pi i} \oint_{\Gamma \subset \mathbb{CP}^1} \frac{1}{\lambda^2} G(t + \lambda z + \lambda^2 y - \lambda^3 \overline{z} + \lambda^4 \overline{t}, \lambda) d\lambda$$
$$= 6ytz + 6y\overline{tz} + z^3 + \overline{z}^3 - 3z\overline{t}^2 - 3\overline{z}t^2,$$

where the contour Γ encloses $\lambda = 0$. The real four-manifold M is defined as the surface

$$\phi := \frac{\partial F}{\partial y} = 6(tz + \overline{tz})$$
$$= 0$$

† To make contact with (6.1) divide the expression (6.6) for $\tilde{\tau}$ by λ^2, and set $f = s/\lambda^2$.

in the space of real sections of $\mathcal{O}(4)$. Using the splitting method in the proof of Theorem 3.1, or equivalently computing the $\mathcal{I}$ and $\mathcal{J}$ invariants (6.5) we find that the points in M where $t = z = 0$ correspond to curves with normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(4)$. This is a curve parametrised by y.

Now perform the Legendre transform

$$u := \frac{\partial F}{\partial z} = 6yt + 3z^2 - 3\bar{t}^2$$

and eliminate the coordinates $(z, \overline{z}, y)$ using $(t, \bar{t}, u, \overline{u})$ as holomorphic and anti-holomorphic coordinates on M. The Kähler potential is

$$\begin{aligned}\Omega(t, \bar{t}, u, \overline{u}) &= F - uz - \overline{uz}\\ &= -2(z^3 + \overline{z}^3)\\ &= -2i(t^3 - \bar{t}^3)R^3, \quad \text{where} \quad R^2 = -1 - \frac{\overline{u}t - u\bar{t}}{3(t^3 - \bar{t}^3)} \in \mathbb{R}^+.\end{aligned}$$

The Kähler potential satisfies the first heavenly equation [**19**]

$$\Omega_{t\bar{t}}\Omega_{u\overline{u}} - \Omega_{t\overline{u}}\Omega_{u\bar{t}} = 1$$

and the resulting metric on M

$$g = \Omega_{u\overline{u}} du d\overline{u} + \Omega_{u\bar{t}} du d\bar{t} + \Omega_{t\overline{u}} dt d\overline{u} + \Omega_{t\bar{t}} dt d\bar{t}$$

is hyper-Kähler. The line of jumping points in M has been blown down to a point $u = t = 0$ by the Legendre transform.

7. *Schrödinger equation on folded hyper-Kähler manifolds*

In this Section we will demonstrate that the Schrödinger equation on a canonical folded hyper-Kähler manifold (corresponding to $k = 2$ in Theorem 3.1)

$$g = Z\left(dX^2 + dY^2 + dZ^2\right) + \frac{1}{Z}\left(dT + \frac{1}{2}XdY - \frac{1}{2}YdX\right)^2$$

admits normalisable solutions which extend to both sides of the fold $Z = 0$ where the metric degenerates.

The time-independent Schrödinger equation

$$\frac{1}{\sqrt{|g|}}\partial_a\left(\sqrt{|g|}g^{ab}\partial_b\phi\right) = E\phi$$

takes the form

$$\frac{1}{Z}\left(\frac{1}{4}(X^2 + Y^2) + Z^2\right)\partial_T\partial_T\phi - \frac{X}{Z}\partial_Y\partial_T\phi + \frac{Y}{Z}\partial_X\partial_T\phi + \frac{1}{Z}\delta^{ij}\partial_i\partial_j\phi = E\phi. \tag{7.1}$$

We will take the coordinate T to be periodic, and consider solutions of the form

$$\phi(T, X, Y, Z) = e^{isT}\varphi(X, Y, Z)$$

for s a non-zero integer. The Schrödinger equation (7.1) becomes

$$-\frac{s^2}{Z}\left(\frac{1}{4}(X^2 + Y^2) + Z^2\right)\varphi - \frac{isX}{Z}\partial_Y\varphi + \frac{isY}{Z}\partial_X\varphi + \frac{1}{Z}\delta^{ij}\partial_i\partial_j\varphi = E\varphi,$$

which separates as $\varphi = G(X,Y)F(Z)$ into

$$\frac{d^2F}{dZ^2} - (s^2Z^2 + EZ + \kappa)F = 0 \tag{7.2}$$

and

$$-\frac{1}{4}s^2(X^2+Y^2) - \frac{isX}{G}\partial_Y G + \frac{isY}{G}\partial_X G + \frac{1}{G}(\partial_X^2 + \partial_Y^2)G + \kappa = 0. \tag{7.3}$$

If $s = 0$ then the first equation becomes the Airy equations and one can show that no-normalisable solutions exist on both sides of the fold. The second equation describes a free particle on a plane, and no bound states exist in this case either.

Let us therefore assume that $s \neq 0$, and consider the equation for (7.2) for $F(Z)$, which has the form of the Schrödinger equation describing a displaced harmonic oscillator. This is readily solved to give

$$F(Z) = H_\gamma\left(\sqrt{s}\left(Z + \frac{E}{2s^2}\right)\right)\exp\left\{-\frac{1}{2}s\left(Z+\frac{E}{2s^2}\right)^2\right\},$$

where $H_\gamma(\xi)$ solves the Hermite equation

$$\frac{d^2H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + 2\gamma H = 0, \quad \text{with} \quad \gamma = \frac{1}{2s}\left(\frac{E^2}{4s^2} - (\kappa + s)\right).$$

If γ is a non-negative integer then H_γ is a Hermite polynomial and thus $F(Z)$ is clearly normalisable for $s > 0$ (even with the folded background's factor of $\sqrt{|g|} = Z$) due to the exponential fall-off at large Z. If, however, γ fails to be a non-negative integer then H_γ is more complicated, being most readily expressed as a series expansion. In this case normalisability is less clear, so let us restrict ourselves to the case where γ is a non-negative integer.

Let us now proceed to consider the $G(X,Y)$ equation (7.3). This has the form of the Schrödinger equation describing motion in a constant magnetic field. In the usual manner let us then define the canonical (Hermitian) momenta

$$\Pi_X = -i\partial_X + \frac{1}{2}sY \qquad \Pi_Y = -i\partial_Y - \frac{1}{2}sX$$

and ladder operators

$$a = \Pi_X + i\Pi_Y \qquad a^\dagger = \Pi_X - i\Pi_Y.$$

The $G(X,Y)$ equation is then

$$(a^\dagger a + s - \kappa)G = 0$$

and we can construct some solutions (choosing $\kappa = s$) by solving $aG_0(X,Y) = 0$, and then applying copies of $a^\dagger$ to G_0. For example, one solution is

$$G(X,Y) \propto \exp\left\{-\frac{1}{4}s(X^2+Y^2)\right\},$$

and thus we conclude that there do exist normalisable solutions. One class of normalisable solutions is

$$\phi = H_\gamma\left(\sqrt{s}\left(Z+\frac{E}{2s^2}\right)\right)\exp\left\{-\frac{1}{2}s\left(Z+\frac{E}{2s^2}\right)^2\right\}\exp\left\{-\frac{1}{4}s(X^2+Y^2)\right\}\exp\{isT\}$$

with s a positive non-zero integer and E chosen such that

$$\gamma = \frac{E^2}{8s^3} - 1$$

is a positive integer.

Another example of a metric which admits a three-parameter family of jumping lines, and yet there exists normalisable solutions to the Schrödiner equation is the Taub-NUT space with negative mass [**11**].

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