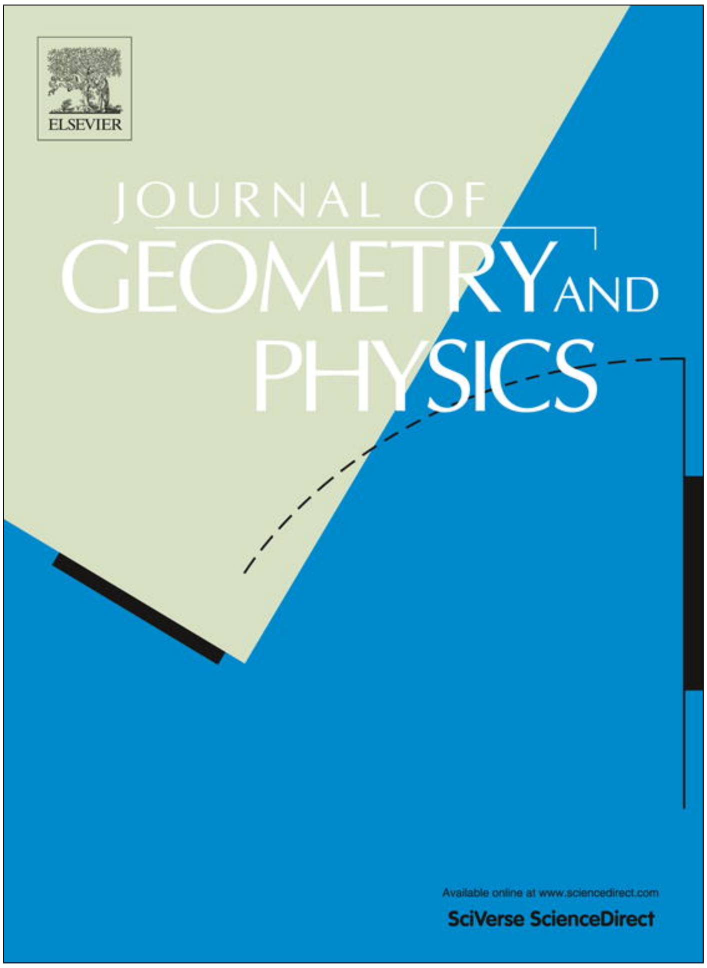
ELSEVIER
JOURNAL OF
GEOMETRY AND
PHYSICS
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Journal of Geometry and Physics 62 (2012) 1747–1759

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

$SU(2)$ solutions to self-duality equations in eight dimensions

Maciej Dunajski *, Moritz Hoegner

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

ARTICLE INFO

Article history:
Received 6 January 2012
Accepted 31 March 2012
Available online 9 April 2012

Dedicated to Jerzy Lukierski on the occasion of his 75th birthday

Keywords:
Instantons
Self-duality equations
Spin(7)

ABSTRACT

We consider the octonionic self-duality equations on eight-dimensional manifolds of the form $M_8 = M_4 \times \mathbb{R}^4$, where M_4 is a hyper-Kähler four-manifold. We construct explicit solutions to these equations and their symmetry reductions to the non-abelian Seiberg–Witten equations on M_4 in the case when the gauge group is $SU(2)$. These solutions are singular for flat and Eguchi–Hanson backgrounds. For $M_4 = \mathbb{R} \times \mathcal{G}$ with a cohomogeneity one hyper-Kähler metric, where $\mathcal{G}$ is a nilpotent (Bianchi II) Lie group, we find a solution which is singular only on a single-sided domain wall. This gives rise to a regular solution of the non-abelian Seiberg-Witten equations on a four-dimensional nilpotent Lie group which carries a regular conformally hyper-Kähler metric.

1. Introduction

Gauge theory in dimensions higher than four has been investigated in both theoretical physics [1–5] and pure mathematics [6,7] contexts. While the solutions to the full second order Yang–Mills equations seem to be out of reach, the first order higher dimensional analogues of four-dimensional self-duality equations admit some explicit solutions. Such equations can be written down on any n-dimensional Riemannian manifold M_n, once a closed differential form Ω of degree $(n-4)$ has been chosen. The generalised self-duality equations state that the curvature two-form of a Yang–Mills connection takes its values in one of the eigenspaces of the linear operator $T : \Lambda^2(M_n) \to \Lambda^2(M_n)$ given by $T(\mathbb{F}) = *(\Omega \wedge \mathbb{F})$. The full Yang–Mills equations are then implied by the Bianchi identity. If $n = 4$, and the zero-form $\Omega = 1$ is canonically given by the orientation, the eigenspaces of T are both two-dimensional, and are interchanged by reversing the orientation. In general the eigenspaces corresponding to different eigenvalues have different dimensions. For the construction to work, one of these eigenspaces must have dimension equal to $(n-1)(n-2)/2$, as only then does the number of equations match the number of unknowns modulo gauge.

Any Riemannian manifold with special holonomy $Hol \subset SO(n)$ admits a preferred parallel $(n-4)$-form, and the eigenspace conditions above can be equivalently stated as $\mathbb{F} \in \mathfrak{hol}$, where we have identified the Lie algebra $\mathfrak{hol}$ of the holonomy group with a subspace of $\Lambda^2(M_n) \cong \mathfrak{so}(n)$. One of the most interesting cases corresponds to eight-dimensional manifolds with holonomy $Spin(7)$. The only currently known explicit solution on $M_8 = \mathbb{R}^8$ with its flat metric has a gauge group $Spin(7)$. The aim of this paper is to construct explicit solutions to the system

$$*_8(\mathbb{F} \wedge \Omega) = -\mathbb{F},$$

with gauge group $SU(2)$. This will be achieved by exploiting the embedding $SU(2) \times SU(2) \subset Spin(7)$. This holonomy reduction allows a canonical symmetry reduction to the Yang–Mills–Higgs system in four dimensions — a non-abelian analogue of the Seiberg–Witten equations involving four Higgs fields [6,4,8]. The explicit $SU(2)$ solutions arise from a t'Hooft-like ansatz which turns out to be consistent despite a vast overdeterminancy of the equations. The resulting solutions on

* Corresponding author.
E-mail addresses: m.dunajski@damtp.cam.ac.uk (M. Dunajski), M.Hoegner@damtp.cam.ac.uk (M. Hoegner).

0393-0440/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2012.03.013

$\mathbb{R}^8$ fall into two classes, both of which are singular along a hypersurface. To overcome this, and to evade Derrick's theorem prohibiting finite action solutions in dimensions higher than four we shall consider the case of curved backgrounds of the form $M_8 = M_4 \times \mathbb{R}^4$, where M_4 is hyper-Kähler. The gauge fields on the Eguchi–Hanson gravitational instanton are still singular, but if M_4 is taken to be a Bianchi II gravitational instanton representing a domain wall [9], then the Yang–Mills curvature is regular away from the wall. This gives rise to a regular solution of the non-abelian Seiberg–Witten equation on a four-dimensional nilpotent Lie group $\mathcal{H}$ which carries a regular conformally hyper-Kähler metric.

Theorem 1.1. *Let $\mathcal{H}$ be the simply-connected Lie group whose left-invariant one-forms satisfy the Maurer–Cartan relations*

$$d\sigma_0 = 2\sigma_0 \wedge \sigma_3 - \sigma_1 \wedge \sigma_2, \qquad d\sigma_1 = \sigma_1 \wedge \sigma_3, \qquad d\sigma_2 = \sigma_2 \wedge \sigma_3, \qquad d\sigma_3 = 0.$$

- *The left-invariant metric $\hat{g} = {\sigma_0}^2 + {\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2$ on $\mathcal{H}$ is regular and conformally hyper-Kähler.*
- *The $\mathfrak{su}(2)$-valued one-forms*

$$A = \frac{3}{4}(\sigma_2 \otimes T_1 - \sigma_1 \otimes T_2 + \sigma_0 \otimes T_3), \qquad \Phi = -\frac{\sqrt{21}}{3}A$$

with $[T_1, T_2] = T_3,\ [T_3, T_1] = T_2,\ [T_2, T_3] = T_1$ satisfy

$$F_+ = \frac{1}{2}[\Phi, \Phi]_+, \qquad (D\Phi)_- = 0, \qquad D *_4 \Phi = 0,$$

where $D = d + [A, \ldots]$, $F = dA + A \wedge A$, and $\pm$ denote self-dual $(+)$ and anti-self-dual $(-)$ parts with respect to $\hat{g}$.

Finally we should mention that there are other candidates for 'self-duality' equations in higher dimensions. One possibility in dimension eight, exploited by Polchinski in the context of heterotic string theory [10], is to consider the system $*\mathbb{F}\wedge\mathbb{F} = \pm\mathbb{F}\wedge\mathbb{F}$. These equations are conformally invariant, and thus the finite action solutions compactify $\mathbb{R}^8$ to the eight-dimensional sphere, but unlike the system (2.2) considered in this paper they do not imply the Yang–Mills equations.

2. Self-duality in eight dimensions

Let (M_8, g_8) be an eight-dimensional oriented Riemannian manifold. The 21-dimensional Lie group $Spin(7)$ is a subgroup of $SO(8)$ preserving a self-dual four-form Ω. Set $e^{\mu\nu\rho\sigma} = e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\sigma$. There exists an orthonormal frame in which the metric and the four-form are given by

$$\begin{aligned} g_8 &= (e^0)^2 + (e^1)^2 + \cdots + (e^7)^2, \\ \Omega &= e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\ &\quad - e^{1247} - e^{1256} - e^{1346} + e^{1357} + e^{2345} + e^{2367} + e^{4567}. \end{aligned} \tag{2.1}$$

Let $T : \Lambda^2(M_8) \to \Lambda^2(M_8)$ be a self-adjoint operator given by

$$\omega \to *_8(\Omega \wedge \omega),$$

where $*_8$ is the Hodge operator of g_8 corresponding to the orientation $\Omega \wedge \Omega$. The 28-dimensional space of two-forms in eight dimensions splits into $\Lambda^2_{21} \oplus \Lambda^2_+$, where Λ^2_{21} and Λ^2_+ are eigenspaces of T with eigenvalues -1 and 3 respectively. The 21-dimensional space Λ^2_{21} can be identified with the Lie algebra $\mathfrak{spin}(7) \subset \mathfrak{so}(8) \cong \Lambda^2(M_8)$.

Let $\mathbb{A}$ be a one-form on $\mathbb{R}^8$ with values in a Lie algebra $\mathfrak{g}$ of a gauge group G. The $Spin(7)$ self-duality condition states that the curvature two form

$$\mathbb{F} = d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}]$$

takes its values in Λ^2_{21}. This leads to a system of seven first order equations

$$*_8(\mathbb{F} \wedge \Omega) = -\mathbb{F}, \tag{2.2}$$

explicitly given by

$$\begin{aligned} &\mathbb{F}_{01} + \mathbb{F}_{23} + \mathbb{F}_{45} + \mathbb{F}_{67} = 0, \\ &\mathbb{F}_{02} - \mathbb{F}_{13} + \mathbb{F}_{46} - \mathbb{F}_{57} = 0, \\ &\mathbb{F}_{03} + \mathbb{F}_{12} - \mathbb{F}_{47} - \mathbb{F}_{56} = 0, \\ &\mathbb{F}_{05} + \mathbb{F}_{14} + \mathbb{F}_{27} + \mathbb{F}_{36} = 0, \\ &\mathbb{F}_{06} - \mathbb{F}_{17} + \mathbb{F}_{24} - \mathbb{F}_{35} = 0, \\ &\mathbb{F}_{07} + \mathbb{F}_{16} - \mathbb{F}_{25} - \mathbb{F}_{34} = 0, \\ &\mathbb{F}_{04} - \mathbb{F}_{15} - \mathbb{F}_{26} + \mathbb{F}_{37} = 0. \end{aligned}$$

This is a determined system of PDEs as one of the eight components of $\mathbb{A}$ can be set to zero by a gauge transformation

$$\mathbb{A} \longrightarrow \rho\mathbb{A}\rho^{-1} - d\rho\ \rho^{-1}, \quad \text{where } \rho \in \text{Map}(M_8, G).$$

Eq. (2.2) were first investigated in [1], and some solutions were found in [11,2] for the gauge group $Spin(7)$. If $\mathbb{A}$ is a solution to (2.2), then it is a Yang–Mills connection because

$$D *_8 \mathbb{F} = -D\mathbb{F} \wedge \Omega = 0, \quad \text{where } D = d + [\mathbb{A}, \ldots]$$

by the Bianchi identities.[1]

2.1. Non-abelian Seiberg–Witten equations

2.1.1. Holonomy reduction

Eq. (2.2) are valid on curved eight-dimensional Riemannian manifolds with holonomy equal to, or contained in $Spin(7)$, as such manifolds are characterised by the existence of a parallel four-form given by (2.1). We shall consider the special case of product manifolds [13]

$$M_8 = M_4 \times \widetilde{M}_4, \qquad g_8 = g_4 + \tilde{g}_4, \tag{2.3}$$

where M_4 and $\widetilde{M}_4$ are hyper-Kähler manifolds. Let ${\psi_i}^{\pm}$ span the spaces $\Lambda^2_+(M_4)$ and $\Lambda^2_-(M_4)$ of self-dual and anti-self-dual two-forms respectively. Thus

$$g_4 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2, \quad \text{and} \quad {\psi_i}^{\pm} = e^0 \wedge e^i \pm \frac{1}{2}{\varepsilon^i}_{jk} e^j \wedge e^k, \tag{2.4}$$

where $i, j, \ldots = 1, 2, 3$ with analogous expressions for $\tilde{g}_4$. The $Spin(7)$ four-form (2.1) is then given by

$$\Omega = \text{vol} + \widetilde{\text{vol}} + \sum_{i,j=1}^{3} \eta_{ij} {\psi_i}^{+} \wedge {\tilde{\psi}_j}^{+},$$

where $\eta = \text{diag}(1, 1, -1)$ and vol, $\widetilde{\text{vol}}$ are volume forms on M_4 and $\widetilde{M}_4$ respectively. The self-dual four-form Ω is closed as a consequence of the closure of ψ_i and $\tilde{\psi}_i$ which can always be achieved by a choice of the orthonormal frame on hyper-Kähler manifolds.

2.1.2. Symmetry reduction

We shall now consider the self-duality equations (2.2) for a $\mathfrak{g}$-valued connection $\mathbb{A}$ over an eight-manifold M_8 of the form (2.3), where M_4 is an arbitrary hyper-Kähler four-manifold, and $\widetilde{M}_4 = \widetilde{\mathbb{R}}^4$ is flat. We shall look for solutions $\mathbb{A}$ that admit a four-dimensional symmetry group generated by the translations on $\widetilde{\mathbb{R}}^4$. If x^μ are local coordinates of M_8, then we denote the coordinates of M_4 by x^a and those of $\widetilde{\mathbb{R}}^4$ by $\tilde{x}^a$. The Greek indices run from 0 to 7 as Latin indices run from 0 to 3. We choose a frame e^μ in (2.1),/ where e^μ $(\mu = 0, \ \ldots \ , 3)$ is a frame (2.4) on M_4 in which ψ_i are closed and $e^\mu = d\tilde{x}^{\mu-4}$ $(\mu = 4, \ \ldots \ , 7)$. We can then write

$$\begin{aligned}\mathbb{A} &= \sum_{\mu=0}^{7} \mathbb{A}_\mu(x^b) e^\mu \\ &= \sum_{a=0}^{3} A_a(x^b)e^a + \Phi_0(x^b)e^4 - \Phi_1(x^b)e^5 - \Phi_2(x^b)e^6 + \Phi_3(x^b)e^7 \\ &= A + \Phi'\end{aligned} \tag{2.5}$$

where we have re-labelled coefficients and consequently defined A, A_a, Φ' and Φ_a. Thus A is a $\mathfrak{g}$-connection on M_4. Let F denote the curvature of A, and let $F_\pm$ be the SD and ASD parts of F with respect to the Hodge operator $*_4$ of g_4. Furthermore, we introduce the following notation: Let $\Phi = \Phi_a e^a$ be a $\mathfrak{g}$-valued one-form and let ∇_a be four vector fields dual to e^a, i.e. $\nabla_a \lrcorner\, e^b = \delta^b_a$. Set $D = e^a \otimes \nabla_a + [A, \cdot]$, and $D_a\Phi_b = \partial_a\Phi_b + [A_a, \Phi_b]$. Thus $D\Phi = D_{[a}\Phi_{b]} e^a \wedge e^b$ captures the antisymmetric part of $D_a\Phi_b$. Note that A, F, Φ and $D\Phi$ are $\mathfrak{su}(2)$-valued forms over M_4. We are thus splitting up the connection and curvature in various pieces. Note that $\Phi' \neq \Phi_a e^a$ due to the choice of indices and signs in (2.5).

---

[1] The Derrick scaling argument (see e.g. [12]) shows there are no nontrivial finite action solutions to the pure Yang–Mills equations on $\mathbb{R}^8$. This obstruction can be overcome if some dimensions are compactified. If (M_8, g_8) is a compact manifold with holonomy $Spin(7)$, then the YM connections which satisfy (2.2) are absolute minima of the Yang–Mills functional

$$E(\mathbb{A}) = \frac{1}{4\pi}\int_{M_8} |\mathbb{F}|^2 \, \text{vol}_{M_8}.$$

To see this write $\mathbb{F} = \mathbb{F}_+ + \mathbb{F}_-$, where $\mathbb{F}_+ \in \Lambda^2_+$, $\mathbb{F}_- \in \Lambda^2_{21}$, and verify that

$$\mathbb{F} \wedge *_8 \mathbb{F} = \mathbb{F}_+ \wedge *_8 \mathbb{F}_+ + \Omega \wedge \mathbb{F} \wedge \mathbb{F}.$$

The integral of the trace of the second term on the RHS is independent of $\mathbb{A}$.

Now we shall investigate Eq. (2.2) on the chosen product background M_8. Invoking translational symmetry along $\tilde{\mathbb{R}}^4$ as explained, we find the following.

Proposition 2.1. *For a connection of the form (2.5) Eq. (2.2) reduce to the following system of equations for the differential forms A and Φ over M_4:*

$$F_+ - \frac{1}{2}[\Phi, \Phi]_+ = 0 \tag{2.6}$$

$$[D\Phi]_- = 0 \tag{2.7}$$

$$D *_4 \Phi = 0, \tag{2.8}$$

where the $\pm$ denote the SD $(+)$ or ASD $(-)$ part with respect to the Hodge operator $*_4$.

Proof. This reduction has been performed before [4,6,14,8], but in a slightly different context.[2] We shall present a proof adapted to our setup. One obtains these equations by inserting the explicit expression for $\mathbb{A} = A + \Phi'$ and the definition of the curvature, $\mathbb{F} = d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}]$ into the system (2.2). For the curvature, we find

$$\begin{aligned}\mathbb{F} &= d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \\ &= dA + d\Phi' + \frac{1}{2}[A, A] + [A, \Phi'] + \frac{1}{2}[\Phi', \Phi'] \\ &= F + D\Phi' + \frac{1}{2}[\Phi', \Phi'].\end{aligned}$$

In the expression $\mathbb{F} = \frac{1}{2}\mathbb{F}_{\mu\nu} e^\mu \wedge e^\nu$, the two-form F accounts for coefficients $\mathbb{F}_{\mu\nu}$ with both indices in the range $0 \leq \mu, \nu \leq 3$, the term $\frac{1}{2}[\Phi', \Phi']$ for those coefficients $\mathbb{F}_{\mu\nu}$ with indices in the range $4 \leq \mu, \nu \leq 7$ and $D\Phi'$ for coefficients with one index each. This allows us to translate the components $\mathbb{F}_{\mu\nu}$, e.g.

$$\mathbb{F}_{01} = F_{01}, \qquad \mathbb{F}_{25} = (D\Phi')_{25} = -D_2\Phi_1, \qquad \mathbb{F}_{67} = \frac{1}{2}[\Phi', \Phi']_{67} = -\frac{1}{2}[\Phi_2, \Phi_3].$$

The sign and index changes are a result of the labelling of the components of Φ'. Applying this to the system (2.2), we find

$$\begin{aligned}&F_{01} + F_{23} - \frac{1}{2}[\Phi_0, \Phi_1] - \frac{1}{2}[\Phi_2, \Phi_3] = 0,\\ &F_{02} - F_{13} - \frac{1}{2}[\Phi_0, \Phi_2] + \frac{1}{2}[\Phi_1, \Phi_3] = 0,\\ &F_{03} + F_{12} - \frac{1}{2}[\Phi_0, \Phi_3] - \frac{1}{2}[\Phi_1, \Phi_2] = 0,\\ &-D_0\Phi_1 + D_1\Phi_0 + D_2\Phi_3 - D_3\Phi_2 = 0,\\ &-D_0\Phi_2 - D_1\Phi_3 + D_2\Phi_0 + D_3\Phi_1 = 0,\\ &D_0\Phi_3 - D_1\Phi_2 + D_2\Phi_1 - D_3\Phi_0 = 0,\\ &D_0\Phi_0 + D_1\Phi_1 + D_2\Phi_2 + D_3\Phi_3 = 0.\end{aligned}$$

This is exactly the system (2.6) with all components written out. □

The resulting system is a set of equations for a connection A and four non-abelian Higgs fields Φ_a over M_4. In particular they can be regarded as a non-abelian version [4,6,15,14,8] of the equations found by Seiberg and Witten [16]. We will call (2.6) the non-abelian Seiberg–Witten equations.

3. Ansatz for $SU(2)$ solutions

To find explicit solutions to (2.6) and (2.2) with the gauge group $SU(2)$ we shall proceed with an analogy to the t'Hooft ansatz for the self-dual Yang–Mills equations on $\mathbb{R}^4$.

Let T_i, $(i = 1, 2, 3)$ denote a basis of $\mathfrak{su}(2)$ with commutation relations $[T_i, T_j] = \epsilon_{ijk}T_k$ and $T_iT^i := T_iT_j\delta^{ij} = -\frac{3}{4}\mathbb{1}_2$. We can then define two $\mathfrak{su}(2)$-valued two-forms σ and $\tilde{\sigma}$ such that $*_4\,\sigma = \sigma$ and $*_4\,\tilde{\sigma} = -\tilde{\sigma}$ by

$$\sigma = \frac{1}{2}\sigma_{ab}e^a \wedge e^b = \sum_i T_i\ {\psi_i}^+, \qquad \tilde{\sigma} = \frac{1}{2}\tilde{\sigma}_{ab}e^a \wedge e^b = \sum_i T_i\ {\psi_i}^-, \tag{3.9}$$

[2] In the approach of [8] M_8 is the total space of the spinor bundle over M_4 and Eqs. (2.7) and (2.8) are combined into the non-abelian Dirac equation.

where $\psi_i^{\pm}$ are given by (2.4). Thus the forms σ_{ab} select the three-dimensional space of SD two-forms $\Lambda^2_+(M_4)$ from the six-dimensional space $\Lambda^2(M_4)$ and project it onto the three-dimensional subspace $\mathfrak{su}(2)$ of $\mathfrak{so}(4)$. An analogous isomorphism between $\Lambda^2_-(M_4)$ and another copy of $\mathfrak{su}(2)$ is provided by $\tilde{\sigma}$. The following identities hold

$$\tilde{\sigma}_{ab}\sigma^{ab} = 0, \qquad \sigma_{ab}\sigma^{b}{}_{c} = \frac{3}{4}\mathbb{1}_2\delta_{ac} + \sigma_{ac}, \qquad \sigma_{ab}\sigma^{ab} = -3\mathbb{1}_2. \tag{3.10}$$

We now return to Eq. (2.6) and make the following ansatz for the $\mathfrak{su}(2)$-valued one-forms A and Φ,

$$A = *_4(\sigma \wedge dG) = \sigma_{ab}\nabla^b G e^a, \qquad \Phi = *_4(\sigma \wedge dH) = \sigma_{ab}\nabla^b H e^a, \tag{3.11}$$

where $G, H : M_4 \to \mathbb{R}$ are functions on M_4 and ∇_a are the vector fields dual to e^a. Let $\Box = *d*d + d*d*$ be the Laplacian and ∇ be the gradient on M_4, and let $d(e^a) = C^a{}_{bc}e^b \wedge e^c$. The following proposition will be proved in the Appendix.

Proposition 3.1. *The non-abelian Seiberg–Witten equations (2.6) are satisfied by the ansatz (3.11) if and only if G and H satisfy the following system of coupled partial differential equations:*

$$\Box G + |\nabla G|^2 - |\nabla H|^2 = 0, \tag{3.12}$$

$$(\epsilon_{ea}{}^{bc}C^a{}_{bc}\sigma^{ed} - \sigma^{ab}C^d{}_{ab})\nabla_d G = 0, \tag{3.13}$$

$$\tilde{\sigma}_{ac}\sigma^{c}{}_{b}(\nabla^a\nabla^b H - 2\nabla^a G\nabla^b H) = 0, \tag{3.14}$$

$$\sigma_{ab}(\nabla^a\nabla^b H - 2\nabla^a G\nabla^b H) = 0. \tag{3.15}$$

Note that Eq. (3.15) is equivalent to the anti-self-duality of the antisymmetric part of

$$\nabla^a\nabla^b H - 2\nabla^a H\nabla^b G.$$

A similar interpretation of Eq. (3.14) is given by the following.

Lemma 3.2. *Let Σ_{ab} be an arbitrary tensor. Then*

$$\tilde{\sigma}^{ab}\sigma^{c}{}_{b}\Sigma_{ac} = 0 \Leftrightarrow \Sigma_{(ac)} = \frac{1}{4}\Sigma_b{}^b\delta_{ac}. \tag{3.16}$$

Proof. Starting from the left-hand side we first define a two-form $(\Sigma\sigma) = \sigma^c{}_{[b}\Sigma_{a]c}\, e^a \wedge e^b$. Therefore

$$\tilde{\sigma}^{ab}\sigma^{c}{}_{b}\Sigma_{ac} = \tilde{\sigma}^{ab}\sigma^{c}{}_{[b}\Sigma_{a]c} = *[\tilde{\sigma} \wedge (\Sigma\sigma)] = 0,$$

and so $(\Sigma\sigma)$ is self-dual, i.e.

$$(\Sigma\sigma)_{01} = (\Sigma\sigma)_{23}, \qquad (\Sigma\sigma)_{02} = -(\Sigma\sigma)_{13}, \qquad (\Sigma\sigma)_{03} = (\Sigma\sigma)_{12}. \tag{3.17}$$

Using the definition (3.9) of σ_{ab} in terms of the generators of $\mathfrak{su}(2)$ this is equivalent to a system of nine linear equations for the components of Σ_{ac}: six of them set off-diagonal terms to zero, three more equate the four diagonal terms of Σ_{ac}. Solving this system is straightforward: the only solution is $\Sigma_{(ac)} = \Sigma\delta_{ac}$ for some scalar function Σ. □

Thus Eqs. (3.14) and (3.15) together imply that $\nabla^a\nabla^b H - 2\nabla^a H\nabla^b G$ is the sum of a (symmetric) pure-trace term and an (anti-symmetric) ASD term. To continue with the analysis of (3.12) we need to distinguish between flat and curved background spaces.

3.1. Flat background

Our first choice for M_4 is the flat space $\mathbb{R}^4$ with $e^a = dx^a$ for Cartesian coordinates x^a. Since the one-forms e^a are closed we have $C^a{}_{bc} = 0$ and the dual vector fields $\nabla_a = \partial_a$ commute. This implies that (3.13) is identically satisfied. Eq. (3.15) implies that the simple two-form $dG \wedge dH$ is ASD. Therefore this form is equal to zero, since there are no real simple ASD two-forms in Euclidean signature and thus H and G are functionally dependent. Therefore we can set $H = H(G)$. Thus the tensor $\Sigma_{ab} = \partial_a\partial_b H - 2\partial_a H\partial_b G$ is symmetric. Next, we turn our attention to (3.14). Applying Lemma 3.2 we deduce that Σ_{ab} is pure trace. Defining a one-form $f = \exp(-2G)dH$ we find that

$$\partial_a f_c = \Sigma e^{-2G}\delta_{ac} \tag{3.18}$$

for some Σ. Equating the off-diagonal components of (3.18) to zero shows that f_c depends on x^c only, and the remaining four equations yield $dH = e^{2G}dw$, where

$$w = \frac{1}{2}\gamma x_a x^a + \kappa_a x^a,$$

for some constants γ, κ_a. Thus G also depends only on w and, defining $g(w) = \exp G(w)$, Eq. (3.12) yields

$$g''(2\gamma w + \kappa^2) + 4\gamma g' - g^5(2\gamma w + \kappa^2) = 0. \tag{3.19}$$

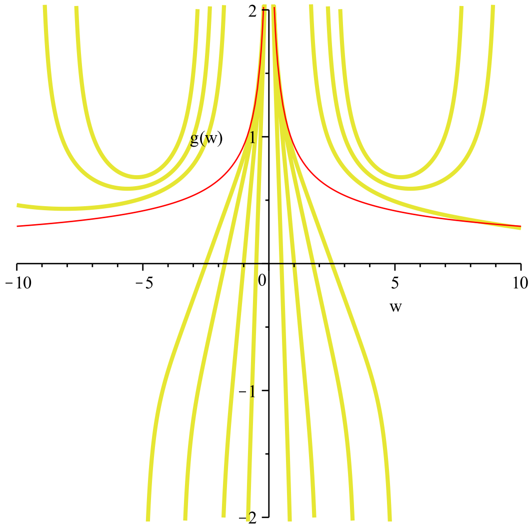

Fig. 1. Numerical plot of solutions to $g'' = g^5$.

There are two cases to consider

- Assume that $\gamma = 0$, in which case

$$g' = \pm\sqrt{\frac{1}{3}g^6 + \gamma_1}. \tag{3.20}$$

To obtain an explicit solution we set the constant $\gamma_1 = 0$. Using the translational invariance of (2.6) we can always put $w = x^3$. Reabsorbing the constant of integration and rescaling yields

$$G = -\frac{1}{2}\ln|x^3|, \qquad H = \frac{\sqrt{3}}{2}\ln|x^3|. \tag{3.21}$$

Using these functions in the ansatz (3.11) for the pair (A, Φ) will give rise to a curvature $\mathbb{F}$ such that (2.2) holds. Note however that the connection is singular along a hyperplane in $\mathbb{R}^4$ and thus $\mathbb{A}$ is also singular along a hyperplane in $\mathbb{R}^8$ because of the translational symmetry. The curvature for this solution is singular along a hyperplane with normal κ_a, and blows up like $|x^3|^{-2}$, thus the solution is singular. A numerical plot of solutions of (3.20) for different γ_1 is displayed in Fig. 1. Since the equation is autonomous, one can obtain the general solution by translating any curve in the x^3 direction. The red line corresponds to (3.21). Note that all other curves have two vertical asymptotes and do not extend to the whole range of x^3.

- We will now present a second, radially symmetric solution. If $\gamma \neq 0$ we translate the independent variable by $w \to w - \frac{\kappa^2}{2\gamma}$, then (3.19) is

$$g''w + 2g' - g^5 w = 0. \tag{3.22}$$

Figs. 2 and 3 contain the numerical plots of two one-parameter families of solutions. An explicit analytical solution is given by

$$g(w) = \frac{1}{\sqrt{\frac{1}{3}w^2 - 1}}.$$

If we define the radial coordinate $r := |\sqrt{\frac{\gamma}{2\sqrt{3}}}(x_a + \frac{\kappa_a}{\gamma})|$, then $w = \sqrt{3}r^2$ and

$$G(r) = -\frac{1}{2}\ln(r^4 - 1), \qquad H(r) = \frac{\sqrt{3}}{2}\ln\left[\frac{r^2 - 1}{r^2 + 1}\right]. \tag{3.23}$$

The pair (A, Φ) in (3.11) is singular on the sphere $r = 1$ in $\mathbb{R}^4$. In $\mathbb{R}^8$ this corresponds to cylinders of a hypersurface type. The curvature is given by

$$\mathbb{F} = \frac{K^i{}_{\mu\nu}T_i}{(r^4 - 1)^2}\, e^\mu \wedge e^\nu,$$

where $K^i{}_{\mu\nu}$ are quadratic polynomials in r^2. The numerical results suggest that there are no regular solutions to (3.22) and most solution curves do not even extend to the full range of r.

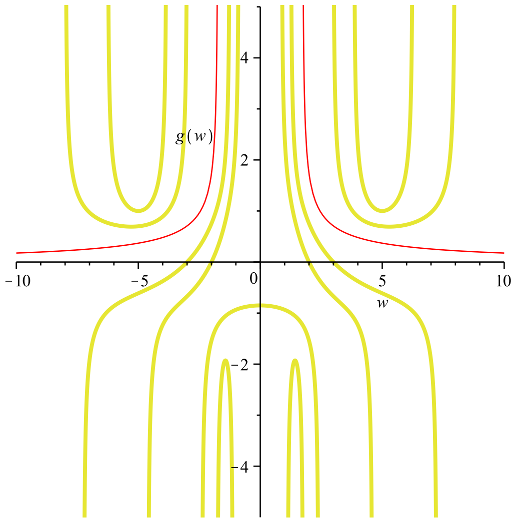

Fig. 2. Solutions of ODE (3.22) I.

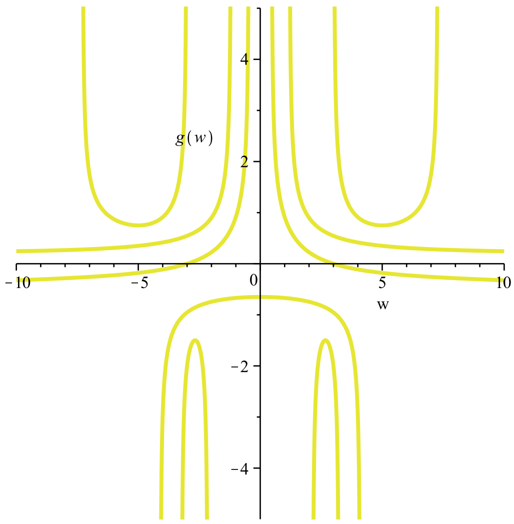

Fig. 3. Solutions of ODE (3.22) II.

This concludes the process of solving the initial system of coupled partial differential equations (3.12). We have shown that the most general solution to this system is given by two functions of one variable, G and H with $w := \frac{1}{2}\gamma x_a x^a + \kappa_a x^a$, which are determined by an ordinary differential equation. We presented two classes of solutions in closed form.

3.2. Curved backgrounds

The solutions we have found in the last subsection have extended singularities resulting in an unbounded curvature and infinite action. While we could argue that the former is an artefact resulting from the form of our ansatz, there is no hope to cure the latter. The existence of the finite action solutions to pure Yang–Mills theory on $\mathbb{R}^8$ or to Yang–Mills–Higgs theory on $\mathbb{R}^4$ is ruled out by the Derrick scaling argument [12].

To evade Derrick's argument we shall now look at curved hyper-Kähler manifolds M_4 in place of $\mathbb{R}^4$. The one-forms e^a in the orthonormal frame (2.4) are no longer closed and the vector fields ∇_a do not commute, as $C_{ab}^c \neq 0$. Eqs. (3.14) and (3.15) imply that $\nabla_a\nabla_b H - 2\nabla_a G\nabla_b H$ is a sum of a pure-trace term and an ASD term, but examining the integrability conditions shows that the trace term vanishes unless the metric g_4 is flat. Thus

$$\nabla_a H = \delta_a e^{2G}, \tag{3.24}$$

where δ_a are some constants of integration. We shall analyse two specific examples of M_4. The first class of solutions on the Eguchi–Hanson manifold generalises the spherically symmetric solutions (3.23), which were singular at $r = 1$. In the Eguchi–Hanson case the parameter in the metric can be chosen so that $r = 1$ does not belong to the manifold. The second class of solutions on the domain wall backgrounds generalises the solutions (3.21).

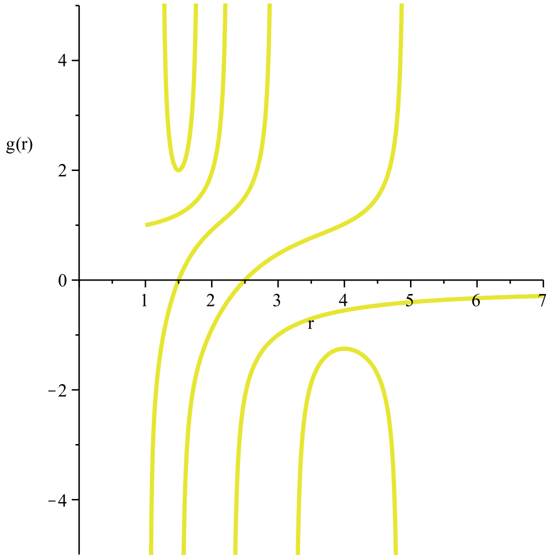

Fig. 4. Solutions of ODE (3.27) I.

3.2.1. *Eguchi–Hanson background*

Consider (M_4, g_4) to be the Eguchi–Hanson manifold [17], with the metric

$$g_4 = \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2\left(1 - \frac{a^4}{r^4}\right)\sigma_3^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2).$$

Here σ_i, $i = 1, 2, 3$ are the left-invariant one-forms on $SU(2)$

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\phi), \qquad \sigma_3 = d\psi + \cos\theta d\phi$$

and to obtain the regular metric we take the ranges

$$r > a, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi, \qquad 0 \le \psi \le 2\pi. \tag{3.25}$$

Choose an orthonormal frame

$$e^0 = \frac{1}{\sqrt{1 - \frac{a^4}{r^4}}}dr, \qquad e^1 = \frac{r}{2}\sqrt{1 - \frac{a^4}{r^4}}\sigma_3, \qquad e^2 = \frac{r}{2}\sigma_2, \qquad e^3 = \frac{r}{2}\sigma_1. \tag{3.26}$$

Computing the exterior derivatives $d(e^a)$ explicitly we can evaluate (3.13) and find that it is trivially zero. Furthermore, we know that Eqs. (3.14) and (3.15) are equivalent to (3.24). The integrability conditions $d^2H = 0$ imply

$$df = 2f \wedge dG, \quad \text{where } f = \delta_a e^a.$$

The condition $dG \neq 0$ implies $\delta_i = 0$. Then

$$f = \frac{\delta_0 dr}{\sqrt{1 - \frac{a^4}{r^4}}},$$

and $df = 0$. Thus $f \wedge dr = dH \wedge dr = dH \wedge dG = 0$ and consequently H and G depend on r only and satisfy the following relation:

$$\frac{dH}{dr} = \frac{\delta_0 e^{2G}}{\sqrt{1 - \frac{a^4}{r^4}}}.$$

Using this in Eq. (3.12) and substituting $g := \frac{e^G}{\sqrt{\delta_0}}$ yields

$$\left(1 - \frac{a^4}{r^4}\right)g'' + \frac{1}{r}\left(3 + \frac{a^4}{r^4}\right)g' - g^5 = 0. \tag{3.27}$$

The numerical results (Figs. 4 and 5, where $a = 1$) indicate that yet again there are no regular functions among the solutions. Analysing the limits $r \to a$ and $r \to \infty$ we find that the solution curves either blow up for $r \to a$ or, if they intersect with the line $r = a$ in the (r, g) plane, they will satisfy $g' = (a/4)g^5$. For the second limit (3.27) tends to $g'' = g^5$ which we have investigated in the previous section. Thus the behaviour for $r \to \infty$ is determined by Fig. 1. In the flat limit $a \to 0$, in which the Eguchi–Hanson manifold becomes $\mathbb{R}^4$, Eq. (3.27) does not reduce to the one we found for the ansatz over $\mathbb{R}^4$. This is to be expected, since the frame e^a we are working with will not reduce to an integrable coordinate frame even in the flat limit.

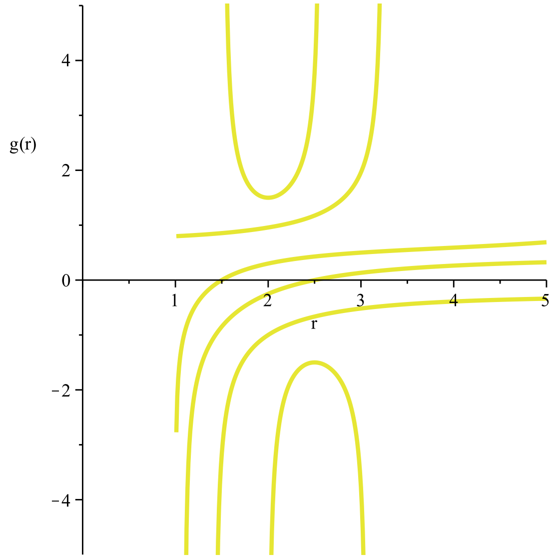

Fig. 5. Solutions of ODE (3.27) II.

3.2.2. Non-abelian Seiberg–Witten equations on Bianchi II domain wall

In this subsection we shall prove Theorem 1.1. Consider the Gibbons–Hawking [18] class of hyper-Kähler metrics characterised by the existence of a tri-holomorphic isometry. The metric is given by

$$g_4 = V((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + V^{-1}(dx^0 + \alpha)^2. \tag{3.28}$$

The function V and the one-form $\alpha = \alpha_i dx^i$ depend on x^j and satisfy

$$*_3\, dV = -d\alpha,$$

where $*_3$ is the Hodge operator on $\mathbb{R}^3$. Thus the function V is harmonic.

Choose the orthonormal frame

$$e^0 = \frac{1}{\sqrt{V}}(dx^0 + \alpha), \qquad e^i = \sqrt{V} dx^i,$$

and the dual vector fields ∇_0 and ∇_i. In comparison to the Eguchi–Hanson background, for the Gibbons–Hawking case Eq. (3.13) is no longer trivially satisfied. It only holds if $dG \wedge dV = 0$. Thus, in particular $\nabla_0 G = 0$. Eqs. (3.14) and (3.15) are equivalent to (3.24). The integrability conditions force $\delta_0 = 0$. Setting $w = \delta_i x^i$, we can determine H from the relation $dH = \sqrt{V}e^{2G}dw$. Thus H and $\sqrt{V}e^{2G}$ are functions of w only. We claim that $\sqrt{V}e^{2G} \neq C$ for any constant[3] C. Therefore $dV \wedge dw = dG \wedge dw = 0$, since $dV \wedge dG = 0$, and we must have $V := V(w)$, $G := G(w)$. Furthermore $V(w)$ is harmonic, so the potential must be linear in w, i.e. without loss of generality

$$V = x^3, \qquad \alpha = x^2 dx^1.$$

The resulting metric admits a Bianchi II (also called *Nil*) group of isometries generated by the vector fields

$$X_0 = \frac{\partial}{\partial x^0}, \qquad X_1 = \frac{\partial}{\partial x^1}, \qquad X_2 = \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^0}$$

with the Heisenberg Lie algebra structure

$$[X_0, X_1] = 0, \qquad [X_0, X_2] = 0, \qquad [X_2, X_1] = X_0.$$

There is also a homothety generated by

$$D = 2x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3},$$

3 Suppose the opposite. Using $V = C^2 e^{-4G}$ in (3.12) we find $\partial_i \partial^i G + \partial_i G \partial^i G = C^2 \delta_i \delta^i$. The Laplace equation on V implies $\partial_i \partial^i G = 4\partial_i G \partial^i G$, and

$$\partial_i \partial^i G = 4c^2, \qquad \partial_i G\, \partial^i G = c^2, \quad \text{where } c := \frac{C^2 \delta_i \delta^i}{\sqrt{5}}.$$

Differentiation of the first relation reveals that all derivatives of G are harmonic. Two partial differentiations of the second relation and contracting the indices then yields $|\partial_i \partial_j G|^2 = 0$. This implies $c = 0$ and thus $\partial_i G = 0$, which rules out this special case.

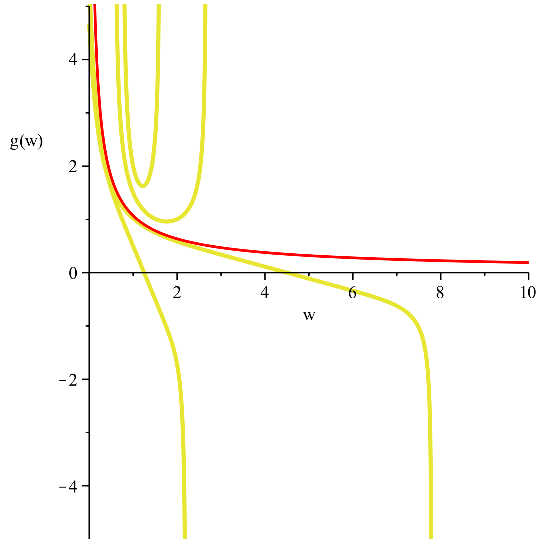

Fig. 6. Solutions of ODE (3.2) I.

such that

$$\mathcal{L}_D g_4 = 3g_4.$$

The conformally rescaled metric $\hat{g} = (x^3)^{-3} g_4$ admits D as a proper Killing vector. Thus $\{X_0, X_1, X_2\}$ span the Bianchi II algebra of isometries of $\hat{g}$ and $\{X_0, X_1, D\}$ span the Bianchi V group of isometries of $\hat{g}$. Setting $x^3 = \exp(\rho)$ puts g_4 in the form

$$g_4 = e^{3\rho}(d\rho^2 + e^{-2\rho}((dx^1)^2 + (dx^2)^2) + e^{-4\rho}(dx^0 + x^2 dx^1)^2).$$

This metric is singular at $\rho \to \pm\infty$ but we claim that this singularity is only present in an overall conformal factor, and g_4 is a conformal rescaling of a regular homogeneous metric on a four-dimensional Lie group with the underlying manifold $\mathcal{H} = Nil \times \mathbb{R}^+$ generated by the right-invariant vector fields $\{X_0, X_1, X_2, D\}$. To see it, set

$$\sigma_0 = e^{-2\rho}(dx^0 + x^2 dx^1), \qquad \sigma_1 = e^{-\rho} dx^1, \qquad \sigma_2 = e^{-\rho} dx^2, \qquad \sigma_3 = d\rho.$$

Then

$$g_4 = e^{3\rho}\hat{g} \quad \text{where } \hat{g} = {\sigma_0}^2 + {\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2, \tag{3.29}$$

and the left-invariant one-forms satisfy

$$d\sigma_0 = 2\sigma_0 \wedge \sigma_3 - \sigma_1 \wedge \sigma_2, \qquad d\sigma_1 = \sigma_1 \wedge \sigma_3, \qquad d\sigma_2 = \sigma_2 \wedge \sigma_3, \qquad d\sigma_3 = 0. \tag{3.30}$$

Thus the metric $\hat{g}$ is regular.

In [9] the singularity of g_4 at $\rho = -\infty$ has been interpreted as a single side domain wall in the space–time

$$M_4 \times \mathbb{R}^{p-3,1}$$

with its product metric. This domain wall is a p-brane: either a nine-brane of 11D supergravity if $p = 6$ or a three-brane of the $(4+1)$-dimensional space–time $g_4 - dt^2$. In all cases the direction ρ is transverse to the wall. In the approach of [9] the regions $x^3 > 0$ and $x^3 < 0$ are identified. In this reference it is argued that (M_4, g_4) with such identification is the approximate form of a regular metric constructed in [19] on a complement of a smooth cubic curve in $\mathbb{CP}^2$.

Using this linear potential $V = w = x^3$ in (3.12) and setting $g(w) := e^{G(w)}$ yields

$$g'' - wg^5 = 0.$$

This equation changes its character as w changes from positive to negative sign; we find infinitely many singularities for $G(w)$ for $w < 0$. We thus focus on the region $w > 0$, which is in agreement with the identification of these two regions proposed by Gibbons and Rychenkova [9]. Numerical plots for solutions of this equation are given in Figs. 6 and 7. One explicit solution is given by

$$g(w) = \pm\frac{1}{2}\sqrt[4]{21}\, w^{-\frac{3}{4}}. \tag{3.31}$$

If we choose $w = x^3$, the curvature for this solution blows up like $(x^3)^{-3}$. This is singular only on the domain wall.

Explicitly, the solution (3.31) gives

$$G = -\frac{3}{4}\rho + \frac{1}{4}\ln 21 - \ln 2, \qquad H = -\frac{\sqrt{21}}{3} G$$

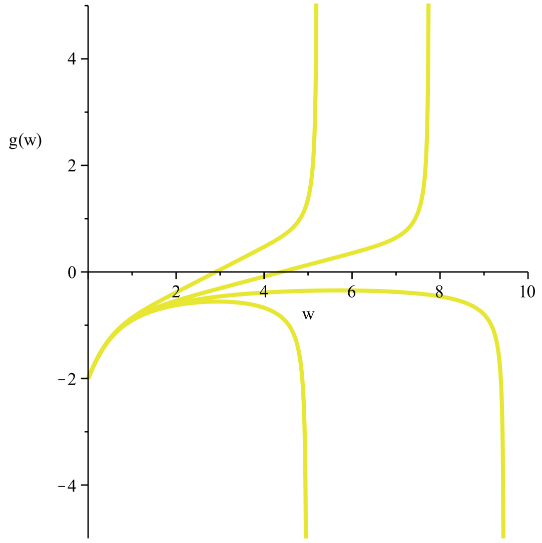

Fig. 7. Solutions of ODE (3.2) II.

and

$$A = \frac{3}{4}(\sigma_2 \otimes T_1 - \sigma_1 \otimes T_2 + \sigma_0 \otimes T_3), \qquad \Phi = -\frac{\sqrt{21}}{3} A, \tag{3.32}$$

$$F = \left(\frac{9}{16}\sigma_0 \wedge \sigma_1 + \frac{3}{4}\sigma_2 \wedge \sigma_3\right) \otimes T_1 + \left(\frac{9}{16}\sigma_0 \wedge \sigma_2 - \frac{3}{4}\sigma_1 \wedge \sigma_3\right) \otimes T_2 + \left(\frac{3}{2}\sigma_0 \wedge \sigma_3 - \frac{3}{16}\sigma_1 \wedge \sigma_2\right) \otimes T_3.$$

We claim that (A, Φ) is a regular solution to the non-abelian Seiberg–Witten equations on the Lie group corresponding to the Lie algebra (3.30) with its left-invariant metric $\hat{g}$ given by (3.29). To justify this claim, we need to consider the invariance of the non-abelian Seiberg–Witten equations under the conformal rescalings of the underlying metric. The first two Eqs. (2.6) and (2.7) are clearly invariant, which follows from the conformal invariance of the Hodge operator acting on two-forms in four dimensions. The third Eq. (2.8) is not invariant in general, but it still holds in our case with g_4 replaced by $\hat{g}_4$, as the conformal factor depends only on ρ and $d\rho \wedge *_4 \Phi = 0$ for the Higgs fields (3.32). We should stress that this solution does not lift to a solution of Yang–Mills equations in eight dimensions, as the product metric $\hat{g}_4 + \tilde{g}_4$ on $\mathcal{H} \times \widetilde{R}^4$ is not $Spin(7)$.

4. Conclusions and outlook

In this paper we have used the identification of $\mathbb{R}^8$ with $\mathbb{R}^4 \times \mathbb{R}^4$, or the curved analogue when one of the $\mathbb{R}^4$ factors is replaced by a hyper-Kähler four-manifold (M_4, g_4) to construct explicit solutions of the 'self-duality' equations in eight dimensions with a gauge group $SU(2)$. The solutions all admit four-dimensional symmetry group along the $\mathbb{R}^4$ factor, and thus they give rise to solutions of the non-abelian Seiberg–Witten equations on M_4.

We have analysed three cases, where M_4 is $\mathbb{R}^4$ with the flat metric, the Eguchi–Hanson gravitational instanton, and finally the cohomogeneity one hyper-Kähler metric with Bianchi II group acting isometrically with three-dimensional orbits. In this last case the singularity of the gauge field is regular on a conformally rescaled four-manifold. Alternatively, the singularity is present only on a domain wall in the space–time with the metric $g_4 - dt^2$.

The symmetry reduction to four dimensions was based on the holonomy reduction $SU(2) \times SU(2) \subset Spin(7)$. An analogous reduction from $\mathbb{R}^8$ with split signature metrics may provide a source of Lorentz invariant gauged solitons in $3+1$ dimensions. Moreover, there are other special realisations of $Spin(7)$ in terms of Lie groups G_2, $SU(3)$ and $SU(4)$. Each realisation leads to some symmetry reduction [20,21], and picks a preferred gauge group, where the ansatz analogous to (3.11) can be made.

Witten [22] considered a complex-valued connection $\mathcal{A} = A + i\Phi$ on bundles over four-manifolds of the form $M_4 = \mathbb{R} \times M_3$ with the product metric $g_4 = dw^2 + g_3$, where (M_3, g_3) is a three-dimensional Riemannian manifold. He showed that the gradient flow equation

$$\frac{d\mathcal{A}}{dw} = - *_3 \frac{\delta \mathcal{I}}{\delta \bar{\mathcal{A}}}$$

for the holomorphic Chern–Simons functional $\mathcal{I}$ yields Eqs. (2.6) and (2.7). In this setup neither A nor Φ have a dw component.

The example (3.21) fits into this framework: g_3 is the flat metric on $\mathbb{R}^3$, and the corresponding ODE is the reduction of the gradient flow equations. In all other examples in our paper the underlying four-manifold is also of the form $M_4 = \mathbb{R} \times M_3$, where M_3 is a three-dimensional Lie group with left-invariant one-forms σ_i. Moreover in all cases there exists a gauge such that neither A nor Φ have components in the $\mathbb{R}$ direction orthogonal to the group orbits. However the Riemannian metric $g_4 = dw^2 + h_{ij}(w)\sigma_i\sigma_j$ on M_4 is not a product metric unless h_{ij} does not depend on w. It remains to be seen whether the gradient flow formulation of the non-abelian Seiberg–Witten equations can be achieved in this more general setup.

Acknowledgements

We thank Gary Gibbons, Hermann Nicolai and Martin Wolf for useful discussions.

Appendix

Proof of Proposition 3.1. Rewrite Eq. (2.6) using the two-forms σ and $\tilde{\sigma}$:

$$*\left[\sigma \wedge \left(F - \frac{1}{2}[\Phi, \Phi]\right)\right] = \sigma^{ab}(F_{ab} - \Phi_a \wedge \Phi_b) = 0, \tag{A.1}$$

$$*(\tilde{\sigma} \wedge [D\Phi]) = -\tilde{\sigma}^{ab} D_a \Phi_b = 0, \tag{A.2}$$

$$D^a \Phi_a = 0. \tag{A.3}$$

Now, substituting (3.11) and using (3.10) in Eq. (A.1) yields

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^{ab}\left(F_{ab} - \frac{1}{2}[\Phi_a, \Phi_b]\right) \\ &= \frac{3}{4}\nabla_a\nabla^a G + \sigma_{ac}\nabla^a\nabla^c G + \sigma_{cd}\nabla^d G\sigma^{ab} d(e^c)_{ab} + \frac{3}{4}|\nabla G|^2 - \frac{3}{4}|\nabla H|^2. \end{aligned}$$

The term $\sigma_{cd}\nabla^d G\sigma^{ab} d(e^c)_{ab}$ decomposes as

$$\sigma_{cd}\nabla^d G\sigma^{ab} d(e^c)_{ab} = \frac{1}{4}[C^a{}_{da} + \epsilon_{da}{}^{bc} C^a{}_{bc}]\nabla^d G\, \mathbb{1}_2 + \epsilon_{ea}{}^{bc} C^a{}_{bc}\nabla^d G\sigma^e{}_d.$$

The closure condition $d\sigma = 0$ yields $\sigma_{a[b}C^a{}_{cd]} = 0$, which is a system of 12 linear equations. These equations imply the four relations $\epsilon_{da}{}^{bc}C^a{}_{bc} = 2C^a{}_{da}$. Then the identity-valued part of (A.1) becomes

$$\frac{3}{4}\nabla_a\nabla^a G + \frac{3}{4}C^a{}_{ba}\nabla^b G + \frac{3}{4}|\nabla G|^2 - \frac{3}{4}|\nabla H|^2 = 0.$$

The first two terms of these combine to give $\Box G$, as can be seen by computing

$$\begin{aligned} \Box G &= *d*dG = *d\left(\frac{1}{3!}\epsilon_{abcd}\nabla_a G e^b \wedge e^c \wedge e^d\right) \\ &= *(\nabla_a\nabla^a G + C^b{}_{ab}\nabla^a G) = (\nabla_a\nabla^a + C^b{}_{ab}\nabla^a)G. \end{aligned}$$

The other components of (A.1) are given by[4]

$$(\epsilon_{ea}{}^{bc}C^a{}_{bc}\sigma^{ed} - \sigma^{ab}C^d{}_{ab})\nabla_d G = 0.$$

We now move to Eq. (A.2),

$$\begin{aligned} \tilde{\sigma}_{ab}(D^a\Phi^b) &= \tilde{\sigma}_{ab}(\nabla^a\Phi^b + A^a\Phi^b - \Phi^b A^a) \\ &= \tilde{\sigma}_{ab}\sigma^{bc}\nabla^a\nabla_c H + 2\tilde{\sigma}_{ab}\sigma^{ad}\sigma^{bc}\nabla_{(c}G\nabla_{d)}H \\ &= \tilde{\sigma}_{ab}\sigma^b{}_c(\nabla^a\nabla^c H - 2\nabla^a H\nabla^c G). \end{aligned}$$

---

4 Using the spinor decomposition [12]

$$C^a{}_{bc} = \varepsilon^{A'}{}_{B'}\Gamma^A{}_{BCC'} + \varepsilon^A{}_B\Gamma^{A'}{}_{B'CC'}$$

with the anti-self-duality conditions $d\sigma = 0$ equivalent to $\Gamma^{A'}{}_{B'CC'} = 0$ gives

$$\Gamma^{AB}{}_{AC'}\sigma^{C'B'}\nabla_{BB'}G = 0,$$

where $\sigma^{A'B'} = \sigma^{(A'B')}$ and $\sigma^{ab} = \sigma^{A'B'}\varepsilon^{AB}$. Thus the three-dimensional distribution $\Gamma^{AB}{}_{A(C'}\nabla_{B')B}$ is integrable and G is in its kernel.

Here we had to explicitly evaluate and symmetrise a product of three σ-matrices to obtain the last line. And finally, for Eq. (A.3) we obtain

$$\begin{aligned} D_a\Phi^a &= (\nabla_a\Phi^a + [A_a, \Phi^a]) \\ &= \nabla_a(\sigma^{ab}\nabla_b H) + \sigma_{ab}\sigma^a{}_c\nabla^b G\nabla^c H - \sigma_{ac}\sigma^a{}_b\nabla^b G\nabla^c H \\ &= \sigma_{ab}(\nabla^a\nabla^b H - 2\nabla^a G\nabla^b H) = 0. \quad \square \end{aligned}$$

References

[1] E. Corrigan, C. Devchand, D.B. Fairlie, J. Nuyts, First order equations for gauge fields in spaces of dimension greater than four, Nucl. Phys. B214 (1983) 452.
[2] S. Fubini, H. Nicolai, The octontionic instanton, Phys. Lett. B155 (1985) 369.
[3] J.A. Strominger Harvey, A. Strominger, Octonionic superstring solitons, Phys. Rev. Lett. 66 (1991) 549.
[4] L. Baulieu, H. Kanno, I.M. Singer, Special quantum field theories in eight and other dimensions, Commun. Math. Phys. 194 (1998) 149–175.
[5] B.A. Bernevig, J. Hu, N. Toumbas, S.C. Zhang, Eight-dimensional quantum hall effect and octonions, Phys. Rev. Lett. 91 (2003) 236803.
[6] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in: Huggett, et al. (Eds.), The Geometric Universe, 1996, pp. 31–47, OUP.
[7] G. Tian, Gauge theory and calibrated geometry. I, Ann. of Math. 151 (2000) 193–268.
[8] A. Haydys, Gauge theory, calibrated geometry and harmonic spinors, arXiv.org:math.DG/0902.3738, 2009.
[9] G.W. Gibbons, P. Rychenkova, Single-sided domain walls in M-theory, J. Geom. Phys. 32 (2000) 311–340.
[10] J. Polchinski, Open heterotic strings, JHEP, 2006, 0609:082.
[11] D.B. Fairlie, J. Nuyts, Spherically symmetric solutions of gauge theories in eight dimensions, J. Phys. A17 (1984) 2867–2872.
[12] M. Dunajski, Solitons, Instantons & Twistors, in: Oxford Graduate Texts in Mathematics, vol. 19, Oxford University Press, 2009.
[13] D. Joyce, Compact Manifolds with Special Holonomy, Oxford University Press, 2000.
[14] S. Detournay, D. Klemm, C. Pedroli, Generalized instantons in $N = 4$ super Yang–Mills theory and spinorial geometry, J. High Energy Phys. 030 (2009).
[15] A.D. Popov, A.G. Sergeev, M. Wolf, Seiberg–Witten monopole equations on noncommutative R^4, J. Math. Phys. 44 (2003) 4527–4554.
[16] N. Seiberg, E. Witten, Monopole condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory, Nucl. Phys. B426 (1994) 19–52.
[17] T. Eguchi, A.J. Hanson, Asymptotically flat self-dual solutions to Euclidean gravity, Phys. Lett. B74 (1978) 249–251.
[18] G.W. Gibbons, S.W. Hawking, Gravitational multi-instantons, Phys. Lett. B78 (1978) 430.
[19] R. Kobayashi, Kaehler metric and moduli spaces, Adv. Stud. Pure Math. 18-II (1990) 137. (1990).
[20] D. Harland, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Yang–Mills flows on nearly Kaehler manifolds and G_2-instantons, Commun. Math. Phys. 300 (2010) 185–204.
[21] K.P. Gemmer, O. Lechtenfeld, C. Nolle, A.D. Popov, Yang–Mills instantons on cones and sine-cones over nearly Khler manifolds, arXiv:1108.3951, 2011.
[22] E. Witten, Analytic continuation of Chern–Simons theory, arXiv:1001.2933, 2010.