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Strominger–Yau–Zaslow Geometry, Affine Spheres and Painlevé III

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Abstract: We give a gauge invariant characterisation of the elliptic affine sphere equation and the closely related Tzitzéica equation as reductions of real forms of $SL(3, \mathbb{C})$ anti–self–dual Yang–Mills equations by two translations, or equivalently as a special case of the Hitchin equation.

We use the Loftin–Yau–Zaslow construction to give an explicit expression for a six–real dimensional semi–flat Calabi–Yau metric in terms of a solution to the affine-sphere equation and show how a subclass of such metrics arises from 3rd Painlevé transcendents.

1. Introduction

Let X be a six real dimensional Calabi–Yau (CY) manifold - a complex Kähler three-fold with covariantly constant holomorphic three-form Ω. Any such manifold admits a Ricci flat Kähler metric with holonomy contained in $SU(3)$.

We shall consider a subclass of CY manifolds which are fibred over a real three dimensional manifold B, and the fibres are special Lagrangian tori T^3. This means that there exists a projection

$$\pi : X \longrightarrow B$$

such that the restrictions of the Kähler form ω and the real part of the holomorphic three-form $\text{Re}(\Omega)$ vanish on any fibre $\pi^{-1}(p) \cong T^3$ over a point $p \in B$.

The corresponding CY metric is called semi–flat if it is flat along the fibres. Consider the Kähler form $\omega = i\partial\overline{\partial}\phi$, where ϕ is the Kähler potential. A natural class of semi–flat CY manifolds are the T^3 invariant manifolds. In this case the potential ϕ can be chosen not to depend on the coordinates of the fibres of π. The Ricci–flat condition $\det\left(\frac{\partial^2\phi}{\partial z^j\partial\bar{z}^k}\right) = 1$ then reduces to the real Monge–Ampére equation

$$\det\left(\frac{\partial^2\phi}{\partial x^j\partial x^k}\right) = 1, \tag{1.1}$$

where x^j, $j = 1, 2, 3$, are local coordinates on B. The work of Cheng and Yau [6] shows that semi–flat CY metrics on compact complex three-fold are flat, so in what follows we allow CY manifolds to be non–compact, and some fibres of π to be singular.

The conjecture of Strominger, Yau and Zaslow (SYZ) [28] states that near the large complex structure limit both X and its mirror should be the fibrations over the moduli space of special Lagrangian tori. More precisely, SYZ consider the moduli space of special Lagrangian submanifolds admitting a unitary flat connection. They write down a metric on X and compute the metric on the moduli space. In the tree level contribution this metric is derived from the Born–Infeld action for the brane, assuming that the moduli parameters slowly vary in time and expanding the action up to second order in time derivatives. The metric on the moduli space Y arises from the kinetic term in the Born–Infeld action. This method is based on Manton's moduli space approximation [21] and was originally used by SYZ. The metric resulting on Y admits the T^3 action even if the original metric on X does not. The full agreement between Y and the mirror of X is therefore expected when instanton contribution from minimal area holomorphic discs whose boundaries wrap the tori are taken into account. These corrections are suppressed in the large complex structure limit.

One approach to a proof of the Strominger Yau Zaslow conjecture [28] would be to describe Ricci-flat metrics on Calabi-Yau manifolds near large complex structure limits. It is expected that in the large complex structure limit the base of the fibration $\pi : X \longrightarrow B$ admits an affine structure and a special metric of Hessian form. To test this conjecture Loftin, Yau and Zaslow (LYZ) [20] aimed to prove the existence of the metric of Hessian form[1]

$$g_B = \frac{\partial^2 \phi}{\partial x^j \partial x^k} dx^j \otimes dx^k, \tag{1.2}$$

where ϕ is homogeneous of degree 2 in x^j and satisfies (1.1). Given such a Hessian metric on B, the semi–flat Calabi–Yau metric g on TB and the corresponding Kähler form are given by

$$g = \phi_{jk}(dx^j \otimes dx^k + dy^j \otimes dy^k), \qquad \omega = \frac{i}{2}\phi_{jk} dz^j \wedge d\overline{z}^k, \tag{1.3}$$

where y^j are coordinates on the fibres of TB and $z^j = x^j + iy^j$.

LYZ constructed a candidate for such metric as a cone over the elliptic affine sphere metric with three singular points. One consequence of Mirror Conjecture is that the base metric g_B should have singularities in codimension two, and LYZ were interested in a local metric model near the trivalent vertex of a Y-shaped singularity. The monodromy of the resulting affine structure has not been calculated, so it is not yet clear that the metric coincides with the one predicted by Gross-Siebert [10] and Haase-Zharkov [12].

The LYZ construction of the metric comes down to looking for solutions of the definite affine sphere equation [27]

$$\psi_{z\bar{z}} + \frac{1}{2}e^{\psi} + |U|^2 e^{-2\psi} = 0, \qquad U_{\bar{z}} = 0, \tag{1.4}$$

[1] It follows from the work of Hitchin [13] that the natural Weil-Petersson metric on the space of special Lagrangian submanifolds has this form. More precisely, it is shown in [13] that the Kähler potentials of X and its mirror Y both satisfy the Monge-Ampére equation (1.1) and are related by a Legendre transform on the base. The fibres of the special Lagrangian fibration of Y are dual (by a Fourier transform) tori to the fibres of $\pi : X \longrightarrow B$.

where ψ and U are real and complex functions respectively on an open set in $\mathbb{C}$. LYZ set $U = z^{-2}$ to account for the singularity of the metric they considered. They then proved the existence of the radially symmetric solution ψ of (1.4) with a prescribed behaviour near the singularity $z = 0$, and established the existence of the global solution to the coordinate-independent version of (1.4) on S^2 minus three points.

In this paper, we study the integrability of Eq. (1.4). We show that the affine sphere equation and a closely related equation called the Tzitzéica equation arise as reductions of anti–self–dual Yang–Mills (ASDYM) system by two translations, and hence it admits a twistor interpretation. Moreover, the ODE characterising its radial solutions gives rise to an isomonodromy problem described by the Painlevé III ODE. The two-dimensional group of translations reduces the Euclidean ASDYM equations to the Hitchin equations [14] and Theorem 1.1 below gives an invariant characterisation of (1.4) as a special case of the $SU(2, 1)$ Hitchin equations.

Let A be an $\mathfrak{su}(2, 1)$ valued connection on a rank 3 complex vector bundle $E \to \mathbb{C}$ with the curvature $F_A = dA + A \wedge A$ and let Φ be a one-form with values in $\mathrm{adj}(E)$. Choose a local trivialisation of E and set

$$A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Q d\bar{z}, \quad D = d + A,$$

where $m^* := -\eta^{-1}\bar{m}^t\eta$ with $\eta = \mathrm{diag}(1, 1, -1)$, so that $\Phi^* = Q^* dz$.

Theorem 1.1. *The Hitchin equations*

$$F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \qquad D\Phi = 0 \tag{1.5}$$

hold with

$$A_z = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2}\psi_z & -U e^{-\psi} \\ 0 & 0 & \frac{1}{2}\psi_z \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6}$$

if the functions (ψ, U) satisfy the affine sphere equation (1.4).

Conversely, any solution to the $SU(2, 1)$ Hitchin equations such that

1. *Q has minimal polynomial t^2 and $Tr(QQ^*) \neq 0$,*
2. *$Tr\left((D_z Q^*)^2\right) = 0, \quad Tr\left((D_z Q^*)^2 (D_{\bar{z}} Q)^2\right) \neq 0$,*
3. *$Tr[(QQ^*)^4 - (Q^*Q)^2 (D_z Q^*)(D_{\bar{z}} Q) + Q^* Q (D_z Q^*) Q Q^* (D_{\bar{z}} Q)] = 0$*

is equivalent to (1.6) by gauge and coordinate transformations.

The connection between solutions to the affine sphere equation (1.4) and the Calabi–Yau metric (1.3) in six dimensions has not been made explicit in [20]. The Lax representation of (1.4) will be used to prove the following

Proposition 1.2. *Given a semi-flat Calabi–Yau metric (1.3), where $\phi(x)$ satisfies the Monge–Ampére equation (1.1), and $\phi(cx) = c^2\phi(x)$, where c is a non–zero constant, there exist complex coordinates $\{z, w, \xi\}$ such that the metric g and the Kähler form ω can be written as*

$$\begin{aligned} g &= e_1\bar{e}_1 + e_2\bar{e}_2 + e_3\bar{e}_3, \\ \omega &= \frac{i}{2}\left(e_1 \wedge \bar{e}_1 + e_2 \wedge \bar{e}_2 + e_3 \wedge \bar{e}_3\right), \end{aligned} \tag{1.7}$$

where

$$
\begin{aligned}
e_1 &= dw - \frac{i}{2}e^{\psi}(\bar{\xi}dz + \xi d\bar{z}),\\
e_2 &= \frac{e^{\psi/2}}{\sqrt{2}}\left((w + i\xi\psi_z)dz + i(d\xi + e^{-\psi}\bar{U}\bar{\xi}d\bar{z}) \right),\\
e_3 &= \frac{e^{\psi/2}}{\sqrt{2}}\left(i(d\bar{\xi} + e^{-\psi}U\xi dz) + (w + i\bar{\xi}\psi_{\bar{z}})d\bar{z} \right),
\end{aligned}
\tag{1.8}
$$

and $\psi(z,\bar{z})$, $U(z)$ *are real and complex functions respectively defined on an open set in* $\mathbb{C}$ *which satisfy the affine sphere equation* (1.4).

The Hitchin equations (1.5) are integrable as they arise from ASDYM and their solutions can be described by holomorphic twistor data. Therefore any ODE arising as reduction of (1.4) by another symmetry must be of Painlevé type in agreement with an integrable dogma [1,8,22].

If $U = z^n,\ n \in \mathbb{Z}$, Eq. (1.4) admits rotational symmetry

$$z \to e^{ic}z, \quad c \in \mathbb{R}. \tag{1.9}$$

Therefore one can consider the group invariant solutions ψ and look for the ODE characterising such reduction. For concreteness, let us consider $U = z^{-2}$ following LYZ.

Proposition 1.3. *Solutions to* (1.4) *with* $U = z^{-2}$ *invariant under a group of rotations* (1.9) *are of the form*

$$\psi(z,\bar{z}) = \log H(s) - 3\log(s), \qquad s = |z|^{1/2},$$

where H *satisfies*

$$H_{ss} = \frac{(H_s)^2}{H} - \frac{H_s}{s} - \frac{8H^2}{s} - \frac{16}{H},$$

which is the Painlevé III equation with parameters $(-8, 0, 0, -16)$.

In the next section we follow Leung [18] and review the semi–flat Calabi-Yau manifolds. Then, in Sect. 3 we summarise the results about affine spheres which are used in the LYZ construction [20]. In Sect. 4 we prove Theorem 1.1 and give a gauge invariant characterisation of the definite affine sphere equation and the closely related Tzitzéica equation as symmetry reductions of the anti–self–dual Yang–Mills equations. As a byproduct, in Sect. 5 we shall obtain a characterisation of a reduction of the Hitchin equations to the $\mathbb{Z}_3$ two dimensional Toda chain. In Sect. 6 we discuss other possible gauge inequivalent reductions of the ASDYM equations to the affine sphere equation and the Tzitzéica equation. In Sect. 7 we give a proof of Proposition 1.2 and recover the toric Calabi–Yau metric in terms of the solutions of the affine sphere equation. Finaly in Sect. 8 we establish Proposition 1.3 and demonstrate that the existence theorem for Hessian metrics with prescribed monodromy comes down to the study of the Painlevé III equation with special values of parameters, and obtain the corresponding 3×3 iso-monodromic Lax pair.

2. Semi–Flat Calabi–Yau Manifolds and the SYZ Conjecture

Let $z^j = x^j + iy^j$ be holomorphic coordinates on a Calabi–Yau three-fold X, and let $\phi(z^j, \bar{z}^j)$ be the Kähler potential such that $\omega = i\partial\overline{\partial}\phi$. The Ricci–flat condition for the corresponding Riemannian metric is

$$\Omega \wedge \overline{\Omega} = \omega^3,$$

where $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ is the holomorphic three-form on X.

Now let us consider the T^3 invariant case. Assume that the potential ϕ is invariant under translations in the imaginary directions y^j. In this case the Riemannian metric and the Kähler form are given by (1.3) where

$$\phi_{jk} := \frac{\partial^2\phi}{\partial x^j \partial x^k}$$

and the Ricci–flat condition reduces to the real Monge–Ampére equation (1.1) for $\phi = \phi(x^1, x^2, x^3)$.

We shall regard the x^j as local coordinates in an open set $B \subset \mathbb{R}^3$. The freedom in choosing the coordinates x^j without changing Eq. (1.1) is given by affine transformations $\mathbf{x} \to M\mathbf{x}+\mathbf{b}$, where $M \in SL(3, \mathbb{R})$, and $\mathbf{b}$ is a vector. The affine transformations induce the change in the potential $\phi \longrightarrow (\det M)^2\phi$, thus ϕ should be regarded as a section of the second power of the real determinant line bundle over B. Conversely, given a three real dimensional affine manifold B with a metric of Hessian type (1.2), where ϕ satisfies the Hessian condition (1.1) one can construct the Calabi–Yau metric on $X = TB$ by (1.3). We then compactify the fibres quotienting them by a lattice thus producing a T^3 invariant Calabi–Yau structure on the total space of a toric fibration $\pi : X \longrightarrow B$.

We are now ready to formulate the SYZ conjecture. If X, Y are mirror Calabi–Yau manifolds (see [11] for a discussion of what it means) then there exists a compact real three-manifold B such that

- $\pi : X \longrightarrow B, \quad \rho : Y \longrightarrow B$ are special Lagrangian fibrations by tori (the fibres can be singular at some points of B).
- The fibres of π and ρ are dual tori.

The second condition only makes sense for flat tori, therefore the conjecture holds in the *large complex structure limit*, where the volume of the fibres is small in comparison to the volume of the base space and the metric on the fibres is approximately flat.

To understand the large complex structure limit consider a one parameter family of complex structures $J(t)$ given by the holomorphic coordinates

$$z^j(t) = t^{-1}x^j + iy^j,$$

and the corresponding Calabi–Yau metrics rescaled by t^2

$$g(t) = \phi_{ij}(dx^j dx^k + t^2 dy^j dy^k).$$

Thus we get a one parameter family of special Lagrangian fibrations. In a limit $t \longrightarrow 0$ the Gromov–Hausdorff limit of metric $g(t)$ is the Hessian metric (1.2) on B, and the size of the fibres shrinks to zero. The SYZ conjecture predicts that such a limit exists for any Calabi–Yau metric on a (not necessarily T^3 symmetric) toric special Lagrangian fibration.

3. Affine Geometry and Hessian Metrics

The Hessian equation (1.1) is known not to be integrable, at least in the sense of the hydrodynamic reductions [9]. Its homogeneous solutions are however characterised by an integrable PDE. We shall carry over the homogeneity analysis for a general Hessian metric in $(n+1)$ dimensions, and then restrict our attention to $n = 2$ where there is a direct connection with the semi–flat CY manifolds on one side and integrability on the other.

The following proposition follows from combining results of Calabi [5] and Baues-Cortés [2] about parabolic and elliptic affine spheres. Here, we give a direct elementary proof not based on affine differential geometry. It has certain advantages as it exhibits explicit coordinate transformations between solutions to various forms of homogeneous Hessian equations.

Proposition 3.1. *Let $\phi = \phi(x^i)$ be a solution to the Hessian equation (1.1) on an open ball $B \subset \mathbb{R}^{n+1}$ such that $\phi(cx) = c^2\phi(x)$ for any non-zero constant c. Then there exists a local coordinate system $(p_1, \ldots, p_n, r)$ on B such that the metric (1.2) is*

$$g_B = dr^2 + r^2 \frac{1}{w}\left(\frac{\partial^2 w}{\partial p_\alpha \partial p_\beta}\right) dp_\alpha dp_\beta, \qquad \alpha, \beta = 1, \ldots, n, \tag{3.1}$$

where $w = w(p_\alpha)$ satisfies

$$\det\left(\frac{\partial^2 w}{\partial p_\alpha \partial p_\beta}\right) = \frac{1}{w^{n+2}}. \tag{3.2}$$

Proof. Consider the Hessian metric (1.2) with ϕ homogeneous of degree 2. Therefore $V = x^i\partial/\partial x^i$ is a homothety with $\mathcal{L}_V g_B = 2g_B$. Locally there exists a function $r : B \longrightarrow \mathbb{R}$ such that $V = r\partial/\partial r$ and

$$g_B = \gamma(dr + r\alpha)^2 + r^2 h,$$

where h, α, γ are a metric, a one–form and a function respectively on the space of orbits of V. The relation $\partial_i(x^j\phi_j) = 2\phi_i$ gives

$$g_B(V, \ldots) = x^i\phi_{ij}dx^j = d\phi.$$

Thus $d(\gamma(dr + r\alpha)) = 0$ and we can redefine r to set $\alpha = 0$ and $\gamma = 1$. We also note that $|V|^2 = x^i x^j \phi_{ij} = 2\phi$, and recognise g_B as a cone over h,

$$g_B = dr^2 + r^2 h, \quad \phi = \frac{r^2}{2}. \tag{3.3}$$

Now let us consider the surface $r = 1$ given by a graph in $\mathbb{R}^{n+1}$,

$$(\tilde{x}^1, \ldots, \tilde{x}^n) \longmapsto (\tilde{x}^1, \ldots, \tilde{x}^n, v(\tilde{x}^\alpha)),$$

where $\tilde{x}^\alpha$, $\alpha = 1, \ldots, n$, parametrise the surface. We shall show that its induced metric h is given by

$$h = \frac{\partial_\alpha \partial_\beta v}{\tilde{x}^\gamma \partial_\gamma v - v} d\tilde{x}^\alpha d\tilde{x}^\beta, \tag{3.4}$$

where $\partial_\alpha := \partial/\partial\tilde{x}^\alpha$. To prove it, restrict the function ϕ to the surface $r = 1$. This gives an identity $\phi(\tilde{x}^\alpha, v(\tilde{x}^\alpha)) = 1/2$. We differentiate this identity implicitly with respect to $\tilde{x}^\alpha$ and express the first and second derivatives of ϕ in terms of the derivatives of v,

$$\begin{aligned} 0 &= \partial_\alpha\phi + \partial_{n+1}\phi\,\partial_\alpha v, \\ 0 &= \partial_\alpha\partial_\beta\phi + \partial_\alpha\partial_{n+1}\phi\,\partial_\beta v + \partial_\beta\partial_{n+1}\phi\,\partial_\alpha v + \partial_{n+1}^2\phi\,\partial_\alpha v\partial_\beta v + \partial_{n+1}\phi\,\partial_\alpha\partial_\beta v, \\ 2\phi &= \tilde{x}^\alpha\partial_\alpha\phi + v\partial_{n+1}\phi = 1, \end{aligned}$$

where the last relation is just the homogeneity condition restricted to the hypersurface $\phi = 1/2$. Substituting all that to g_B gives (3.4).

Now if the function ϕ in the Hessian metric g_B satisfies the Hessian condition (1.1) then v satisfies

$$\det\frac{\partial^2 v}{\partial\tilde{x}^\alpha\partial\tilde{x}^\beta} = (\tilde{x}^\alpha\partial_\alpha v - v)^{n+2}. \tag{3.5}$$

To see it, let us write the coordinates x^i on $\mathbb{R}^{n+1}$ as $(x^1, \ldots, x^n, x^{n+1}) = (r\tilde{x}^1, \ldots, r\tilde{x}^n, rv(\tilde{x}^\alpha))$, that is, regard $\mathbb{R}^{n+1}$ as the cone over the $r = 1$ surface. Now consider the invariant volume element

$$\sqrt{|g_B|}\,dx^1\wedge\cdots\wedge dx^n\wedge dx^{n+1} = \sqrt{|\tilde{g}_B|}\,d\tilde{x}^1\wedge\cdots\wedge d\tilde{x}^n\wedge dr, \tag{3.6}$$

where $|g_B|$ is the absolute value of the determinant of Hessian metric (1.2) written in the coordinates x^i and $\tilde{g}_B$ is the same metric expressed in the basis $\{d\tilde{x}^\alpha, dr\}$. We contract both sides of (3.6) with V. On the LHS of (3.6) we use the form $V = x^i\partial/\partial x^i$ and on the RHS use $V = r\partial/\partial r$. We now set $r = 1$ and impose the Hessian equation (1.1), $\det g_B = \det\phi_{ij} = 1$. This yields

$$v - \tilde{x}^\alpha\partial_\alpha v = \sqrt{|\tilde{g}_B|}.$$

On the surface $r = 1$, one has $\det\tilde{g}_B = \det h$, where h is given by (3.4). Substituting this in the above formula and taking squares of both sides yields (3.5). Note[2] that we have taken $\det h > 0$ from the assumption that $\det g_B = \det\phi_{jk} = 1$.

To obtain the statement in the proposition, perform a Legendre transform

$$p_\alpha = \frac{\partial v}{\partial\tilde{x}^\alpha}, \quad w(p_\alpha) = \tilde{x}^\alpha\frac{\partial v}{\partial\tilde{x}^\alpha} - v, \quad \tilde{x}^\alpha = \frac{\partial w}{\partial p_\alpha}.$$

Using $dp_\alpha = \partial_\alpha\partial_\beta v\,d\tilde{x}^\beta$ yields

$$h = \frac{1}{w}\frac{\partial^2 w}{\partial p_\alpha\partial p_\beta}dp_\alpha dp_\beta \tag{3.7}$$

and

$$\frac{\partial^2 w}{\partial p_\alpha\partial p_\beta} = \left(\frac{\partial^2 v}{\partial\tilde{x}^\alpha\partial\tilde{x}^\beta}\right)^{-1},$$

which implies (3.1) and (3.2). □

[2] If we started with $\det\phi_{ij} = -1$, which implies $\det h < 0$, the analogous argument would lead to $\det\frac{\partial^2 v}{\partial\tilde{x}^\alpha\partial\tilde{x}^\beta} = -(\tilde{x}^\alpha\partial_\alpha v - v)^{n+2}$.

Now, let us consider a hypersurface Σ immersed in $\mathbb{R}^{n+1}$ with the flat metric $\delta_{jk}\, dx^j dx^k$, given by a graph

$$\mathbf{r} = (\tilde{x}^1, \ldots, \tilde{x}^n, v(\tilde{x}^1, \ldots, \tilde{x}^n)). \tag{3.8}$$

The first and second fundamental forms on Σ are given by

$$h_I = d\mathbf{r} \cdot d\mathbf{r} = (\delta_{\alpha\beta} + \partial_\alpha v \partial_\beta v) d\tilde{x}^\alpha d\tilde{x}^\beta,$$
$$h_{II} = -d\mathbf{r} \cdot d\mathbf{n} = \frac{1}{\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2}} \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} d\tilde{x}^\alpha d\tilde{x}^\beta,$$

where $\mathbf{n}$ is the unit normal to Σ. Tzitzéica [29,30] has studied surfaces Σ in $\mathbb{R}^3$ for which the ratio of the Gaussian curvature $\mathcal{K}$ to the fourth power of a distance from a tangent plane to some fixed point is a constant. If $\mathcal{K} \neq 0$, we can always rescale the coordinates to set this constant to $+1$ or -1 depending on the sign of the Gaussian curvature. We shall call this the Tzitzéica condition. The generalisation of the Tzitzéica condition to hypersurfaces in $\mathbb{R}^{n+1}$ is given by

$$\mathcal{K} = \pm \mathcal{D}^{n+2},$$

where $\mathcal{D} = \mathbf{r} \cdot \mathbf{n}$ is the same as the distance up to sign. In the adapted coordinates, $\mathcal{D}$ and the Gaussian curvature $\mathcal{K}$ are given by

$$\mathcal{D} = \frac{v - \tilde{x}^\alpha \partial_\alpha v}{\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2}},$$
$$\mathcal{K} = \frac{1}{(\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2})^{n+2}} \det \left(\frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} \right).$$

It follows that the Tzitzéica condition holds if and only if v satisfies

$$\det \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} = \pm (v - \tilde{x}^\alpha \partial_\alpha v)^{n+2}, \tag{3.9}$$

where plus and minus signs correspond to positive and negative Gaussian curvature respectively.

It is well known in affine differential geometry that an immersed hypersurface Σ in $\mathbb{R}^{n+1}$ is an affine hypersphere with the origin as its centre if and only if the Tzitzéica condition (3.9) holds [25]. It turns out that the metric (3.4), with v satisfying (3.5), is the same as the Blaschke metric (or affine metric) of a proper affine hypersphere. The Blaschke metric is conformally related to the second fundamental form, and is defined as follows. Let $\mathbf{N}$ denote the transversal vector field of the surface Σ such that the unit normal $\mathbf{n}$ is given by $\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$, i.e. $\mathbf{N} = \nabla(\tilde{x}^{n+1} - v(\tilde{x}^1, \ldots, \tilde{x}^n))$. Consider a bilinear form

$$\hat{h} = -d\mathbf{r} \cdot d\mathbf{N} = |\mathbf{N}|\, h_{II}.$$

The Blaschke metric is then given by

$$h := |\det \hat{h}|^{-\frac{1}{n+2}}\, \hat{h}. \tag{3.10}$$

Therefore, for the surface Σ given by the graph (3.8), we have

$$h = \left|\det \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}\right|^{-\frac{1}{n+2}} \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} d\tilde{x}^\alpha d\tilde{x}^\beta,$$

which coincides with the metric (3.4) if Eq. (3.5) holds.

In affine differential geometry, it is also known [5] that a Hessian metric (1.2) which satisfies $\det \phi_{ij} = 1$ is a parabolic (improper) affine hypersphere metric. We have demonstrated that Hessian equation (1.1) on ϕ implies (3.5) on v. Therefore, this is in agreement with a result of Baues and Cortéz [2] that a parabolic affine hypersphere metric which admits a homothety $\mathcal{L}_V g_B = 2g_B$ is the metric cone over a proper affine hypersphere.

Let us now restrict our attention to $n = 2$, and consider the metric h (3.4). For $n = 2$, $\det h > 0$ implies that h is a definite metric. In the context of the Calabi–Yau manifolds, the metric g_B is Riemannian, hence one is interested in positive–definite h. Baues and Cortés [2] have shown that in such case h is the Blaschke metric of a definite elliptic affine sphere, with affine mean curvature 1. Since h is positive definite we can adopt isothermal coordinates for the affine metric (which are asymptotic coordinates for the second fundamental form h_{II}) and write it as

$$h = e^\psi dz d\bar{z}, \tag{3.11}$$

for some real valued function $\psi = \psi(z, \bar{z})$. In this form, Simon and Wang [27] proved that the structure equations[3] of definite affine sphere imply that ψ necessarily satisfies Eq. (1.4),

$$\psi_{z\bar{z}} + \frac{1}{2}e^\psi + |U|^2 e^{-2\psi} = 0, \qquad U_{\bar{z}} = 0,$$

where $U dz^3$ is the holomorphic cubic differential.

Conversely, given a solution of (1.4) one can construct an affine sphere with $h = e^\psi dz d\bar{z}$ as its Blaschke metric. We should note here that if the holomorphic cubic

[3] The usual affine immersion in $\mathbb{R}^{n+1}$ only assumes a flat connection D and a parallel volume element on $\mathbb{R}^{n+1}$, but not an ambient metric. In particular, the structure equations of a Blaschke hypersurface immersion $f : (\Sigma, \nabla) \longrightarrow (\mathbb{R}^{n+1}, D)$ are given by

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi, \tag{3.12}$$

$$D_X \xi = -f_*(SX), \tag{3.13}$$

where ∇ is an affine connection on Σ, $X, Y \in T\Sigma$, ξ is a transversal vector field chosen uniquely up to sign to satisfy certain properties, called the affine normal field, and h is the Blaschke metric defined by (3.12). This definition turns out to be equivalent to (3.10) if one were to use the Euclidean metric on $\mathbb{R}^{n+1}$. The operator $S : T\Sigma \longrightarrow T\Sigma$ is called the affine shape operator and $H = \frac{1}{n}\mathrm{Tr}(S)$ the affine mean curvature. A proper affine sphere is defined to be a Blaschke hypersurface with $S = HI$, I being the identity metric. Another affine invariant quantity is a totally symmetric tensor called the cubic form $\hat{C}$ and is defined by

$$\hat{C}(X, Y, Z) = h(C(X, Y), Z),$$

where C is the difference tensor $C = \hat{\nabla} - \nabla$ and $\hat{\nabla}$ is the Levi-Civita connection of h. Consider h as in (3.11) and let C^i_{jk}, $i, j, k \in \{1, \bar{1}\}$ be the components of C in the basis $e^1 = dz, e^{\bar{1}} = d\bar{z}$. Then it can be shown that the only nonvanishing components of C are $C^{\bar{1}}_{11}$ and $C^1_{\bar{1}\bar{1}} = \overline{C^{\bar{1}}_{11}}$, and the function U in (1.4) is defined by $U = C^{\bar{1}}_{11} e^\psi$. It follows that the cubic form is $\hat{C} = U dz^3 + \bar{U} d\bar{z}^3$. See [5,19,25,27] for details.

differential $U(z)dz^3$ is non-zero, we can choose the isothermal coordinates such that $U = 1$. For example, defining $\xi = \xi(z)$ by $d\xi = 2^{-1/3}U^{1/3}dz$ transforms (1.4) into

$$\hat{\psi}_{\xi\bar{\xi}} + e^{\hat{\psi}} + e^{-2\hat{\psi}} = 0, \tag{3.14}$$

where

$$\hat{\psi} = \psi - \frac{1}{3}\log U - \frac{1}{3}\log \bar{U} - \frac{1}{3}\log 2.$$

We will make use of such coordinate transformation in Sect. 4.[4] Loftin, Yau and Zaslow [20] proved the existence of a semi–flat Calabi–Yau metric (1.3) with the base metric g_B as the metric cone over an elliptic affine sphere

$$g_B = \phi_{ij}dx^i dx^j = dr^2 + r^2 e^{\psi} dz d\bar{z}, \tag{3.15}$$

with the prescribed singularity, by proving the existence of a radially symmetric solution ψ of (1.4) for $U(z) = z^{-2}$ and the corresponding global solution on S^2 minus three points.

Motivated by this work, we are interested in the integrability of the definite affine sphere equation (1.4). The affine sphere equation is closely related to a well known integrable equation, namely the Tzitzéica equation

$$u_{xy} = e^u - e^{-2u}. \tag{3.16}$$

In the context of affine spheres, the Tzitzéica equation arises if $\det h < 0$. By writing the metric in isothermal coordinates as $h = 2e^u\ dxdy$ and considering the structure equations, Simon and Wang [27] also show that h is the Blaschke metric of the indefinite affine sphere (with negative affine mean curvature) if and only if u satisfies $u_{xy} = e^u - r(x)b(y)e^{-2u}$, where $r(x), b(y)$ are arbitrary non-vanishing functions of one variable, which can be normalised by rescaling the isothermal coordinates. Thus, we obtain

$$u_{xy} = e^u - \epsilon e^{-2u}, \tag{3.17}$$

where $\epsilon = \pm 1$. The equation with $\epsilon = 1$, (3.16), was first derived in [29,30] for the Tzitzéica surface in $\mathbb{R}^3$ with negative Gaussian curvature $\mathcal{K} = -\mathcal{D}^4$, where the indefinite second fundamental form is written in asymptotic coordinates as $h_{II} = 2e^u\mathcal{D}\ dxdy$.

The difference between the two equations (3.16) and (1.4) lies in the relative sign of the two exponential terms on the RHS. For the Tzitzéica equation $u = 0$ is a solution and other solutions may be constructed using Darboux and Bäcklund transformations, for example see [4]. The definite affine sphere equation does not seem to have such obvious solutions. However, Calabi [5] has shown that an elliptic affine hypersphere with complete Blaschke metric is an ellipsoid. This is in agreement with the fact that (1.4) admits solutions in term of elliptic functions, which can be found by making an ansatz $\psi(z, \bar{z}) = f(z + \bar{z})$ in (3.14).

[4] We note that the analytic continuation

$$\hat{\psi}_{\xi\bar{\xi}} + e^{\hat{\psi}} - e^{-2\hat{\psi}} = 0$$

of Eq. (3.14) was used by McIntosh [23] to describe minimal Lagrangian immersions in $\mathbb{CP}^2$ and special Lagrangian cones in $\mathbb{C}^3$.

4. Reduction of ASDYM

It was shown in [7] that the Tzitzéica equation (3.16) can be obtained from a special ansatz to the anti–self–dual Yang–Mills in $\mathbb{R}^{2,2}$ with gauge group $SL(3,\mathbb{R})$. In this section, we shall give a gauge and coordinate invariant characterisation of the Tzitzéica equation and the definite affine sphere equation as different real forms of a reduction of ASDYM on $\mathbb{C}^4$ with gauge group $SL(3,\mathbb{C})$, via the holomorphic Hitchin equations on $\mathbb{C}^2$.

4.1. Holomorphic Tzitzéica equation. Consider a holomorphic metric and volume element on $\mathbb{C}^4$,

$$ds^2 = 2(dz\, d\tilde{z} - dw\, d\tilde{w}), \quad \nu = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}.$$

Let $\mathcal{A} = A_z dz + A_w dw + A_{\tilde{z}} d\tilde{z} + A_{\tilde{w}} d\tilde{w}$ be a Lie algebra valued connection on a vector bundle $E \to \mathbb{C}^4$. The anti–self–dual Yang–Mills equations are given by

$$F_{zw} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad F_{\tilde{z}\tilde{w}} = 0.$$

These equations arise from a Lax pair

$$[D_z + \lambda D_{\tilde{w}}, D_w + \lambda D_{\tilde{z}}] = 0, \tag{4.1}$$

where $D_z = \partial_z + A_z$, etc, are covariant derivatives, $F_{z\tilde{z}} = [D_z, D_{\tilde{z}}]$, and (4.1) is required to hold for any value of the spectral parameter λ.

Choose a gauge group to be $SL(3,\mathbb{C})$ and assume that $\mathcal{A}$ is invariant under the action of two dimensional group of translations $\mathbb{C}^2$ such that the metric restricted to the planes spanned by the generators of the group is non-degenerate. Let X_1, X_2 be the generators of the group, then the Higgs fields

$$P = X_1 \lrcorner A, \quad Q = X_2 \lrcorner A$$

belong to the adjoint representation. We can always choose the coordinates so that the group is generated by the two null vectors $X_1 = \partial/\partial\tilde{w}$ and $X_2 = \partial/\partial w$. The ASDYM system reduces to the holomorphic form of the Hitchin equations [14]

$$D_z Q = 0, \tag{4.2a}$$
$$D_{\tilde{z}} P = 0, \tag{4.2b}$$
$$F_{z\tilde{z}} + [P, Q] = 0, \tag{4.2c}$$

where

$$F_{z\tilde{z}} = \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + [A_z, A_{\tilde{z}}]$$

is a curvature of a holomorphic connection $A = A_z dz + A_{\tilde{z}} d\tilde{z}$ on $\mathbb{C}^2$. The Hitchin equations are invariant under the gauge transformations

$$A \to g^{-1} A g + g^{-1}\, dg, \qquad P \to g^{-1} P g, \quad Q \to g^{-1} Q g, \tag{4.3}$$

and later we shall also make use of the following coordinate freedom:

$$z \longrightarrow \hat{z}(z), \quad \tilde{z} \longrightarrow \hat{\tilde{z}}(\tilde{z}). \tag{4.4}$$

The Lax pair (4.1) for the ASDYM reduces to the following Lax pair for the holomorphic Hitchin equations:

$$[D_z + \lambda P, Q + \lambda D_{\tilde{z}}] = 0. \tag{4.5}$$

There are several gauge inequivalent ways to embed the Tzitzéica equation (3.16) as a special case of the Hitchin equations. The gauge used in [7] is

$$A_{\tilde{w}} = P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_w = Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^u & 0 & 0 \end{pmatrix}, \tag{4.6}$$

$$A_z = \begin{pmatrix} u_z & 0 & 0 \\ 1 & -u_z & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.7}$$

where $u(z, \tilde{z})$ is a complex valued function holomorphic in $(z, \tilde{z})$. With this ansatz the Hitchin equations yield the holomorphic Tzitzéica equation

$$u_{z\tilde{z}} = e^u - e^{-2u}. \tag{4.8}$$

Choosing the real form $SL(3, \mathbb{R})$ of $SL(3, \mathbb{C})$ and regarding $u = u(x, y)$ as a real function of real coordinates $z = x, \tilde{z} = y$ reduces (4.8) to (3.16).

On the other hand, performing the coordinate transformation

$$d\hat{z} = \left(\frac{U(z)}{2}\right)^{-\frac{1}{3}} dz, \quad d\hat{\tilde{z}} = \left(\frac{\tilde{U}(\tilde{z})}{2}\right)^{-\frac{1}{3}} d\tilde{z}$$

and setting

$$u = \psi(z, \tilde{z}) - \frac{1}{3}\log\left(\frac{U}{2}\right) - \frac{1}{3}\log\left(\frac{\tilde{U}}{2}\right) + \log\left(-\frac{1}{2}\right)$$

for any branch of $\log\left(-\frac{1}{2}\right)$ puts (4.8) in the form

$$\psi_{z\tilde{z}} + \frac{1}{2}e^{\psi} + U(z)\tilde{U}(\tilde{z})e^{-2\psi} = 0, \tag{4.9}$$

where we have dropped hats of the new variables. Equation (4.9) then reduces to the affine sphere equation (1.4) under the Euclidean reality conditions $\tilde{z} = \bar{z}$ and reducing the gauge group to $SU(2, 1)$, which implies the constraint $\tilde{U} = \bar{U}$.

Now we shall establish a gauge invariant characterisation of the ansatz (4.6), (4.7) in terms of the gauge and Higgs fields of the Hitchin equations. We will make use of the following lemma.

Lemma 4.1. *Consider* 3 *by* 3 *complex matrices P, Q such that*

$$P^2 = Q^2 = 0, \qquad Tr(PQ) = \omega \neq 0. \tag{4.10}$$

There exists a gauge transformation such that P, Q are in the form (4.6) *for some u.*

Proof. The conditions (4.10) are invariant under the gauge transformations

$$P \longrightarrow g^{-1}Pg, \qquad Q \longrightarrow g^{-1}Qg.$$

These conditions imply that the nullities (dimensions of the kernels of the associated linear maps) satisfy $n(QP) < 3$ and $n(P) = 2$. Thus

$$\mathrm{Ker}(QP) = \mathrm{Ker}(P).$$

Also $\mathrm{rank}(QP) = 1$ and $\mathrm{Im}(QP)$ is contained in the one-dimensional image of Q, therefore

$$\mathrm{Im}(QP) = \mathrm{Im}(Q). \tag{4.11}$$

Choose a Jordan basis $(\mathbf{v}, \mathbf{u}, \mathbf{w})$ of $\mathbb{C}^3$ such that

$$P(\mathbf{w}) = \mathbf{v}, \quad P(\mathbf{v}) = 0, \quad P(\mathbf{u}) = 0. \tag{4.12}$$

From (4.11) $\mathrm{Im}(Q) = \mathrm{span}\,(Q(\mathbf{v}))$, thus $Q(\mathbf{u}) = aQ(\mathbf{v})$, $Q(\mathbf{w}) = bQ(\mathbf{v})$ for some a, b so that $\mathrm{Ker}(Q) = \mathrm{span}\,(\mathbf{u} - a\mathbf{v}, \mathbf{w} - b\mathbf{v})$. Use the freedom in the basis (4.12) to set

$$\mathbf{w}' = \mathbf{w} - b\mathbf{v}, \quad \mathbf{u}' = \mathbf{u} - a\mathbf{v}, \quad \mathbf{v}' = \mathbf{v}.$$

Now

$$\begin{aligned} P(\mathbf{w}') &= \mathbf{v}', \quad P(\mathbf{v}') = 0, \quad P(\mathbf{u}') = 0, \\ Q(\mathbf{w}') &= 0, \quad Q(\mathbf{u}') = 0, \quad Q(\mathbf{v}') = c\mathbf{u}' + \omega\mathbf{w}', \end{aligned}$$

where $\omega \neq 0$ as $Tr(PQ) = \omega \neq 0$. There is still freedom in (4.12):

$$\mathbf{v}'' = \mathbf{v}', \quad \mathbf{u}'' = \mathbf{u}', \quad \mathbf{w}'' = \mathbf{w}' + (c/\omega)\mathbf{u}'$$

so that, dropping primes,

$$P(\mathbf{w}) = \mathbf{v}, \quad P(\mathbf{v}) = 0, \quad P(\mathbf{u}) = 0,$$

$$Q(\mathbf{w}) = 0, \quad Q(\mathbf{u}) = 0, \quad Q(\mathbf{v}) = \omega\mathbf{w}.$$

Ordering the basis $(\mathbf{v}, \mathbf{u}, \mathbf{w})$ yields the matrices in the desired form, i.e. $P_{13} = 1$, $Q_{31} = \omega$, and all other components vanish. The residual gauge freedom is

$$\mathbf{w} \to \alpha\mathbf{w}, \quad \mathbf{v} \to \alpha\mathbf{v}, \quad \mathbf{u} \to \beta\mathbf{u},$$

and the change of basis matrix gives the residual $GL(3, \mathbb{C})$ gauge transformation. In the $SL(3, \mathbb{C})$ case we set $\beta = \alpha^{-2}$. The statement of the lemma now follows by setting $\omega = e^u$. $\square$

We shall now give a set of necessary and sufficient conditions allowing solutions of the Hitchin equations (4.2a, b, c) to be transformed into (4.6), (4.7) by gauge and coordinate symmetries.

Proposition 4.2. *Let* $(Q, P, A = A_z dz + A_{\tilde{z}} d\tilde{z})$ *be a solution of the holomorphic Hitchin equations* (4.2a, b, c)*, with gauge group* $SL(3, \mathbb{C})$*. Then,* $(Q, P, A_z, A_{\tilde{z}})$ *can be transformed into* (4.6),(4.7) *by gauge symmetry and coordinate symmetry* (4.4) *if and only if the following conditions hold:*

(i) *P and Q have minimal polynomial t^2, with $Tr(PQ) \neq 0$.*
(ii) *$Tr\left((D_z P)^2\right) = 0 = Tr\left((D_{\tilde{z}} Q)^2\right)$ and $Tr\left((D_z P)^2 (D_{\tilde{z}} Q)^2\right) \neq 0$.*
(iii) *$Tr M = 0$, where*

$$M = (PQ)^4 + (PQ)^2 (D_z P)(D_{\tilde{z}} Q) - PQ(D_z P)QP(D_{\tilde{z}} Q).$$

Proof. The proof of the necessary conditions is straightforward. It can be shown by direct calculation that (4.6),(4.7) satisfy conditions (**i**), (**ii**), (**iii**). The three conditions are gauge invariant by the cyclic property of the trace. Under the coordinate transformation (4.4), the connection $(A_z, A_{\tilde{z}})$ and the Higgs fields (P, Q) transform as

$$\hat{A}_{\hat{z}} = \left(\frac{d\hat{z}}{dz}\right)^{-1} A_z, \quad \hat{A}_{\hat{\tilde{z}}} = \left(\frac{d\hat{\tilde{z}}}{d\tilde{z}}\right)^{-1} A_{\tilde{z}},$$

$$\hat{Q} = \left(\frac{d\hat{\tilde{z}}}{d\tilde{z}}\right)^{-1} Q, \quad \hat{P} = \left(\frac{d\hat{z}}{dz}\right)^{-1} P.$$

Thus, using condition (**i**), the square of the covariant derivative is given by

$$(\hat{D}_{\hat{z}} \hat{P})^2 = \left(\frac{d\hat{z}}{dz}\right)^{-4} (D_z P)^2$$

and similarly for $(D_{\tilde{z}} Q)^2$. Therefore, conditions (**i**) and (**ii**) are invariant under the coordinate transformation. A similar calculation shows that (**iii**) is also invariant under (4.4).

Conversely, we shall now show that any solution to (4.2a, b, c) such that all the conditions in Proposition 4.2 hold, can be gauge and coordinate transformed into the form (4.6),(4.7).

Firstly, by Lemma 4.1, condition (**i**) implies that we can use gauge symmetry to put the Higgs fields (Q, P) in the form (4.6). Equations (4.2a) and (4.2b) imply that $A_z, A_{\tilde{z}}$ are of the form

$$A_z = \begin{pmatrix} n & 0 & 0 \\ r & u_z - 2n & 0 \\ m & t & n - u_z \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} p & s & h \\ 0 & -2p & k \\ 0 & 0 & p \end{pmatrix}, \tag{4.13}$$

where n, r, m, t, p, s, h, k are some functions of $(z, \tilde{z})$. Note that we have also used the assumption that the fields are $\mathfrak{sl}(3, \mathbb{C})$ valued, hence traceless. Next, to set the diagonal elements of $(A_z, A_{\tilde{z}})$ to be as in (4.7), we consider the residual gauge freedom. Lemma 4.1 implies that the gauges preserving (Q, P) are given by

$$g(z, \tilde{z}) = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \tag{4.14}$$

for an arbitrary function $a(z, \tilde{z}) \neq 0$. Thus, using (4.3), we have

$$A_z \longrightarrow \begin{pmatrix} n + \frac{a_z}{a} & 0 & 0 \\ ra^3 & u_z - 2n - 2\frac{a_z}{a} & 0 \\ m & \frac{t}{a^3} & n - u_z + \frac{a_z}{a} \end{pmatrix},$$

$$A_{\tilde{z}} \longrightarrow \begin{pmatrix} p + \frac{a_{\tilde{z}}}{a} & \frac{s}{a^3} & h \\ 0 & -2p - 2\frac{a_{\tilde{z}}}{a} & ka^3 \\ 0 & 0 & p + \frac{a_{\tilde{z}}}{a} \end{pmatrix}.$$

We choose $a(z, \tilde{z})$ such that

$$(\ln a)_z = u_z - n, \quad \text{and} \quad (\ln a)_{\tilde{z}} = -p.$$

This is allowed because the compatibility condition

$$\partial_z p + \partial_z \partial_{\tilde{z}} u - \partial_{\tilde{z}} n = 0 \tag{4.15}$$

holds automatically as a consequence of condition (**iii**). To see it, note that Eq. (4.2c) implies

$$\partial_z p + \partial_z \partial_{\tilde{z}} u - \partial_{\tilde{z}} n + mh + tk = e^u.$$

Hence, condition (4.15) is equivalent to

$$mh + tk = e^u,$$

which holds by (**iii**).

Note that at this point elements of $(A_z, A_{\tilde{z}})$ will be transformed, however, for convenience we will label them with the same letters as in (4.13). Thus we have set $n = u_z$ and $p = 0$. We now proceed to deal with r, m, t, s, h, k. $\mathrm{Tr}\left((D_z P)^2 (D_{\tilde{z}} Q)^2\right) \neq 0$ in condition (**ii**) implies that $r, t, s, k \neq 0$, and

$$\mathrm{Tr}\left((D_z P)^2\right) = 0 = \mathrm{Tr}\left((D_{\tilde{z}} Q)^2\right)$$

gives

$$m = 0 = h.$$

Hence (4.2c) becomes

$$\begin{aligned} u_{z\tilde{z}} + rs &= e^u, \\ s_z + 2su_z &= 0, \\ r_{\tilde{z}} &= 0, \\ k_z - ku_z &= 0, \\ t_{\tilde{z}} &= 0, \\ tk &= e^u. \end{aligned}$$

Since $r, t, s, k \neq 0$, we can solve the above equations. The last three equations imply that t is a constant, and thus can be set to 1 by a constant gauge transformation of the form (4.14) with $a = t^{-1/3}$, and s is determined to be of the form $b(\tilde{z})e^{-2u}$. This results in

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^u & 0 & 0 \end{pmatrix},$$

$$A_z = \begin{pmatrix} u_z & 0 & 0 \\ r(z) & -u_z & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} 0 & b(\tilde{z})e^{-2u} & 0 \\ 0 & 0 & e^u \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Note that the gauge is now fixed. To get to ansatz (4.6),(4.7), we will now use the coordinate symmetry. Define $\hat{z}$, $\hat{\tilde{z}}$ such that

$$d\hat{z} = e^{j(z)}dz, \quad d\hat{\tilde{z}} = e^{l(\tilde{z})}d\tilde{z},$$

and set

$$\hat{u} := u - j(z) - l(\tilde{z}).$$

By choosing $j(z)$, $l(\tilde{z})$ such that $e^{3j(z)} = r(z)$ and $e^{3l(\tilde{z})} = b(\tilde{z})$, (4.16) becomes gauge equivalent to (4.6),(4.7) in the new variables $(\hat{z}, \hat{\tilde{z}}, \hat{u})$. The gauge transformation we need in the final step is given by (4.3) with

$$g(\hat{z}, \hat{\tilde{z}}) = \begin{pmatrix} e^{-j(z(\hat{z}))} & 0 & 0 \\ 0 & e^{j(z(\hat{z}))} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

We note that substituting (4.16) to the Hitchin equations yields

$$u_{z\tilde{z}} = e^u - r(z)b(\tilde{z})e^{-2u}. \tag{4.17}$$

Therefore, the change of coordinates can, roughly speaking, be regarded as setting $r(z)$ and $b(\tilde{z})$ to constants such that $r(z)b(\tilde{z}) = 1$.

We shall now choose the Euclidean reality condition and select the real form $SU(2, 1)$ of $SL(3, \mathbb{C})$ to deduce Theorem 1.1 from the last proposition.

Proof of Theorem 1.1. Consider the ansatz (4.16) and Eq. (4.17). By changing the dependent variable from u to

$$\psi = u - \log\left(-\frac{1}{2}\right)$$

for any branch of $\log\left(-\frac{1}{2}\right)$, Eq. (4.17) becomes

$$\psi_{z\tilde{z}} + \frac{1}{2}e^{\psi} + U(z)\tilde{U}(\tilde{z})e^{-2\psi} = 0, \tag{4.18}$$

where $U(z) = 2r(z)$, $\tilde{U}(\tilde{z}) = 2b(\tilde{z})$. Then, after an $SL(3, \mathbb{C})$ gauge transformation with

$$g(z, \tilde{z}) = \begin{pmatrix} 0 & 0 & -\sqrt{2}e^{-\frac{\psi}{2}} \\ 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the ansatz (4.16) becomes

$$
\begin{aligned}
A_w = Q &= \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_{\tilde{w}} = P &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 & 0 \end{pmatrix}, \\
A_z &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2}\psi_z & -U(z)e^{-\psi} \\ 0 & 0 & \frac{1}{2}\psi_z \end{pmatrix}, \\
A_{\tilde{z}} &= \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & \frac{1}{2}\psi_{\tilde{z}} & 0 \\ 0 & -\tilde{U}(\tilde{z})e^{-\psi} & -\frac{1}{2}\psi_{\tilde{z}} \end{pmatrix}.
\end{aligned}
\tag{4.19}
$$

Impose the Euclidean reality conditions $\tilde{z} = \bar{z}$, $\tilde{w} = -\bar{w}$, resulting in a positive-definite metric on $\mathbb{R}^4$. The ASDYM equations with these reality conditions are

$$F_{zw} = 0, \tag{4.20}$$

$$F_{z\bar{z}} + F_{w\bar{w}} = 0. \tag{4.21}$$

Take the gauge group to be $SU(2,1)$. A matrix $\mathcal{M}$ is in the Lie algebra $\mathfrak{su}(2,1)$ if it is trace-free and satisfies

$$\bar{\mathcal{M}}^t = -\eta\, \mathcal{M}\, \eta^{-1}, \tag{4.22}$$

where

$$\eta = \eta^{-1} = \mathrm{diag}(1, 1, -1).$$

Let $z = p + iq$, $w = r + is$, so (p, q, r, s) are standard flat coordinates on $\mathbb{R}^4$. The gauge fields A_p, A_q, A_r, A_s are $\mathfrak{su}(2,1)$ valued. The relations $A_z = (A_p - iA_q)/2$, $A_{\bar{z}} = (A_p + iA_q)/2$ together with (4.22) imply that

$$\bar{A_z}^t = -\eta A_{\bar{z}} \eta^{-1},$$

with a similar relation between A_w and $A_{\bar{w}}$. Concretely, this means that

$$A_{\bar{z}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}, \quad A_z = \begin{pmatrix} -\bar{a} & -\bar{d} & \bar{g} \\ -\bar{b} & -\bar{e} & \bar{h} \\ \bar{c} & \bar{f} & -\bar{k} \end{pmatrix},$$

where $a + e + k = 0$ (and of course A_w and $A_{\bar{w}}$ are related in the same way).

Choosing a real form $SU(2,1)$ of $SL(3,\mathbb{C})$ on restriction to the Euclidean slice imposes a constraint $\tilde{U} = \bar{U}$ and yields the affine sphere equation (1.4).

To sum up, one could achieve the characterisation of the ansatz (4.19), with $\tilde{z} = \bar{z}$, $\tilde{U} = \bar{U}$, analogous to Proposition 4.2. Let us again choose the double null coordinates such that the generators of the symmetry group of the ASDYM are given by $\partial_{\tilde{w}},\ \partial_w$.

With the chosen reality condition the ASDYM equations reduce to the $SU(2,1)$ Hitchin equations

$$D_z A_w = 0, \tag{4.23}$$

$$F_{z\bar{z}} + [A_w, A_{\bar{w}}] = 0, \tag{4.24}$$

where

$$A_{\bar{z}} = -\eta^{-1}\bar{A_z}^t\eta \quad \text{and} \quad A_{\bar{w}} = -\eta^{-1}\bar{A_w}^t\eta. \tag{4.25}$$

We now consider the reduction of the system (4.23),(4.24). Theorem 1.1 arises as a corollary of Proposition 4.2. □

4.2. Tzizéica equation. The Tzitzéica equation (3.16) is a different real form of (4.8). It arises from the ASDYM with the gauge group $SL(3,\mathbb{R})$ on restriction to the ultrahyperbolic real slice $\mathbb{R}^{2,2}$ in $\mathbb{C}^4$ with

$$(w, \tilde{w}, x = z, y = \tilde{z})$$

real. The Higgs fields are given by $P = A_{\tilde{w}}$, $Q = A_w$ and the metric on the space of orbits of $X_1 = \partial_{\tilde{w}}$ and $X_2 = \partial_w$ has signature $(1, 1)$.

The real version of the ansatz (4.6),(4.7) can be characterised analogously to the holomorphic case treated in Proposition 4.2. However, one needs to take care of the fact that $e^{u(x,y)} > 0$ for real valued function $u(x, y)$. There are two places where this needs to be considered. First is where we use condition (**i**) in Proposition 4.2 to put (Q, P) in the form (4.6),(4.7). To write $\mathrm{Tr}(PQ) = e^{u(x,y)}$, we require that $\mathrm{Tr}(PQ) > 0$. Assume that this can be done at a point (x_0, y_0) (if not then change coordinates $y \to -y$) and restrict the domain of u to a neighbourhood of this point where the positivity still holds.

The second place where the problem of the sign arises is when we use the coordinate symmetry to transform

$$u_{xy} = e^u - r(x)b(y)e^{-2u}$$

to the Tzitzéica equation (3.16). This can only be done for $r(x)b(y) > 0$. The sign of $r(x)b(y)$ is governed by the quantity $\mathrm{Tr}\left((D_x P)^2(D_y Q)^2\right)$ in condition (**ii**). To see it, note that in the notation of (4.16),

$$\mathrm{Tr}\left((D_x P)^2(D_y Q)^2\right) = (sktr)e^{2u}.$$

After we set $t = 1$, the condition (**iii**) implies that $k = e^u > 0$. Hence, the sign of sr, and thus the sign $r(x)b(y)$ is the same as the sign of $\mathrm{Tr}\left((D_x P)^2(D_y Q)^2\right)$. However, this cannot be changed by real coordinate transformation $x \to \hat{x}(x),\ y \to \hat{y}(y)$, because

$$\mathrm{Tr}\left((D_x P)^2(D_y Q)^2\right) \longrightarrow \left(\frac{d\hat{x}}{dx}\right)^{-4}\left(\frac{d\hat{y}}{dy}\right)^{-4}\mathrm{Tr}\left((D_x P)^2(D_y Q)^2\right),$$

where we have used $Q^2 = 0 = P^2$. Therefore, condition (**ii**) in Proposition 4.2 needs to be replaced by $\mathrm{Tr}\left((D_z P)^2\right) = 0 = \mathrm{Tr}\left((D_{\tilde{z}} Q)^2\right)$ and

$$\mathrm{Tr}\left((D_z P)^2(D_{\tilde{z}} Q)^2\right) > 0$$

in the domain of u.

We remark that $\mathrm{Tr}\left((D_x P)^2 (D_y Q)^2\right) < 0$ corresponds to the equation

$$u_{xy} = e^u + e^{-2u},$$

whereas $\mathrm{Tr}\left((D_x P)^2 (D_y Q)^2\right) = 0$ yields Louiville equation

$$u_{xy} = e^u.$$

Therefore, the sign of $\mathrm{Tr}\left((D_x P)^2 (D_y Q)^2\right)$ corresponds to the sign of ϵ in (3.17).

5. $\mathbb{Z}_3$ Two Dimensional Toda Chain

As a byproduct of the proof of Proposition 4.2, we find that, dropping condition (**iii**) in this proposition, the Hitchin equations can be reduced to a coupled system which includes the $\mathbb{Z}_3$ two dimensional Toda chain [24] as a special case. Recall that a two dimensional Toda chain is given by

$$(u_\alpha)_{xy} - e^{(u_{\alpha+1} - u_\alpha)} + e^{(u_\alpha - u_{\alpha-1})} = 0, \tag{5.1}$$

where $\alpha \in \mathbb{Z}$. In this paper (5.1) is called the $\mathbb{Z}_3$ two dimensional Toda chain when

i) $\alpha \in \mathbb{Z}/\mathbb{Z}_3$ and
ii) $u_1 + u_2 + u_3 = 0$.

We summarise the result in the following proposition.

Proposition 5.1. *Let u_1, u_2 be functions of (x, y). The coupled system of equations*

$$\begin{aligned}(u_1)_{xy} - \epsilon_1 e^{(u_2 - u_1)} + e^{2u_1 + u_2} &= 0,\\ (u_2)_{xy} + \epsilon_1 e^{(u_2 - u_1)} - \epsilon_2 e^{-2u_2 - u_1} &= 0,\end{aligned} \tag{5.2}$$

where $\epsilon_1, \epsilon_2 = \pm 1$, is gauge equivalent to the $SL(3, \mathbb{R})$ Hitchin equations (4.2a, b, c) *with $z = x, \tilde{z} = y$ real, and*

(**i**) *the Higgs fields P and Q have minimal polynomial t^2, with $Tr(PQ) \neq 0$,*
(**ii**) *$Tr\left((D_x P)^2\right) = 0 = Tr\left((D_y Q)^2\right)$ and $Tr\left((D_x P)^2 (D_y Q)^2\right) \neq 0$.*

Proof. These conditions are the first two conditions in Proposition 4.2. Following the proof and assuming condition (**i**) gives (4.13). However, now it is not possible to use gauge symmetry to set the diagonal elements of both A_x and A_y to be the same as in (4.7) without the compatibility condition. Instead, let us use only the gauge transformation (4.14) to eliminate the diagonal elements of A_y, by choosing $(\ln a)_y = -p$.

As before, condition (**ii**) implies that $m = h = 0$ and $sktr \neq 0$. The Hitchin equations (4.2a, b, c) imply that t is a function of x only. Hence, we can use the residual gauge freedom (4.14) with $a = a(x)$ to set $t = 1$. Equation (4.2c) then gives

$$n_y + r(x)s = e^u, \tag{5.3}$$

$$2n_y - u_{xy} + r(x)s - k = 0, \tag{5.4}$$

$$s_x + 3ns - su_x = 0, \tag{5.5}$$

$$k_x + 2ku_x - 3kn = 0. \tag{5.6}$$

Equations (5.5) and (5.6) imply that $sk = c(y)e^{-u}$, where $c(y)$ is some arbitrary function which arises from the integration. Now, since $s \neq 0$, let us write

$$k = \frac{c(y)}{s} e^{-u} \quad \text{and} \quad n = \alpha_x, \ s = \pm e^{\beta},$$

for some functions $\alpha(x, y)$ and $\beta(x, y)$. Then, (5.5) becomes

$$e^{\beta}(\beta_x + 3\alpha_x - u_x) = 0,$$

which can be integrated to give

$$s = b(y)e^{u-3\alpha} \quad \text{and} \quad n = \alpha_x$$

for some $b = b(y) \neq 0$. Finally, (5.3) and (5.4) give a coupled system

$$\begin{aligned} &\alpha_{xy} + r(x)b(y)e^{u-3\alpha} - e^{u} = 0, \\ &2\alpha_{xy} - u_{xy} + r(x)b(y)e^{u-3\alpha} - c(y)b^{-1}(y)e^{-2u+3\alpha} = 0. \end{aligned} \tag{5.7}$$

Set $u_1 = \alpha, \ u_2 = -2\alpha + u$, and change the coordinate $y \to -y$. The system (5.7) becomes

$$\begin{aligned} (u_1)_{xy} - r(x)b(y)e^{u_2-u_1} + e^{2u_1+u_2} &= 0, \\ (u_2)_{xy} + r(x)b(y)e^{u_2-u_1} - c(y)b^{-1}(y)e^{-2u_2-u_1} &= 0, \end{aligned}$$

which can be transformed into (5.2) by the change of dependent variables and coordinates. There are four distinct cases depending on the signs of ϵ_1, ϵ_2. Since the coordinates are real, the signs of ϵ_1, ϵ_2 are the same as those of $r(x)b(y)$ and $c(y)b^{-1}(y)$, respectively. Similar to the real version of Proposition 4.2 for the Tzitzéica equation, $r(x)b(y)$ and $c(y)b^{-1}(y)$ can be related to some gauge invariant quantities. It can be shown that at a given point (x_0, y_0) the signs of $r(x)b(y)$ and $c(y)b^{-1}(y)$ are determined by the signs of

$$\begin{aligned} (\mathbf{a}) &:= \mathrm{Tr}\left((D_x P)^2 (D_y Q)^2\right), \\ (\mathbf{b}) &:= \mathrm{Tr}\left((PQ)^2 (D_x P)(D_y Q) - PQ(D_x P)QP(D_y Q)\right). \end{aligned}$$

We shall analyse these signs and then restrict the domains of (u_1, u_2) to a neighbourhood of (x_0, y_0) where the signs remain constant. If $(\mathbf{a}) > 0$, setting $t = 1$ gives $skr > 0$, which gives $r(x)c(y) > 0$. This implies that $r(x)b(y)$ and $c(y)b^{-1}(y)$ have the same signs. Now if $(\mathbf{b}) > 0$, then $k > 0$ meaning $c(y)b^{-1}(y) > 0$, hence $r(x)b(y) > 0$. Similarly if $(\mathbf{b}) < 0$ then $c(y)b^{-1}(y)$ and $r(x)b(y) < 0$. On the other hand, $(\mathbf{a}) < 0$ implies that $r(x)b(y)$ and $c(y)b^{-1}(y)$ have opposite signs. Then, the sign of $(\mathbf{b})$ determines the sign of $c(y)b^{-1}(y)$. The important point is that the signs of $(\mathbf{a})$ and $(\mathbf{b})$ cannot be changed by real coordinate transformations. This completes the proof. □

6. Other Gauges

There are several gauge inequivalent ways to reduce the ASDYM equations to the Tzitzéica equation or to the definite affine sphere equation. The reductions are relatively easy to obtain, but their gauge invariant characterisation requires much more work. Here we shall mention one other possibility which is not gauge equivalent to (4.6, 4.7).

It can be shown that the holomorphic Tzitzéica equation (4.8) also arises from the Hitchin equations with

$$
\begin{aligned}
P &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^{u} \\ e^{u} & 0 & 0 \end{pmatrix}, \\
A_z &= \begin{pmatrix} u_z & 0 & 0 \\ 0 & -u_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = 0.
\end{aligned} \tag{6.1}
$$

The real version of this ansatz was implicitly used by E. Wang [31].

Let us comment on how this formulation is related to (4.6), (4.7). First note that the Lax pairs (4.5) with (4.6),(4.7) and (6.1) are equal for $\lambda = 1$. Now consider the ansatz (4.6),(4.7) and set $\lambda = 1$ in the Lax pair (4.5). Introduce the new spectral parameter by exploiting the Lorentz symmetry and rescaling the coordinates

$$
(z, \tilde{z}) \longrightarrow (\hat{\lambda} z, \hat{\lambda}^{-1} \tilde{z})
$$

and read off new $A_z, A_{\tilde{z}}, P, Q$ from (4.5) with λ replaced by $\hat{\lambda}$. This yields the ansatz (6.1).

Choosing the Euclidean reality conditions and reducing the gauge group to $SU(2,1)$ we find another reduction of ASDYM to the affine sphere equation. Take the following ansatz, in which the gauge fields are independent of w and $\bar{w}$, $\psi = \psi(z, \bar{z})$ is a real function, and $U(z, \bar{z})$ is a complex function:

$$
\begin{aligned}
A_w &= \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} e^{\psi/2} \\ \bar{U} e^{-\psi} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} e^{\psi/2} & 0 \end{pmatrix}, \\
A_{\bar{w}} &= \begin{pmatrix} 0 & -U e^{-\psi} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} e^{\psi/2} \\ \frac{1}{\sqrt{2}} e^{\psi/2} & 0 & 0 \end{pmatrix}, \\
A_z &= \begin{pmatrix} -\frac{1}{2}\psi_z & 0 & 0 \\ 0 & \frac{1}{2}\psi_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_{\bar{z}} &= \begin{pmatrix} \frac{1}{2}\psi_{\bar{z}} & 0 & 0 \\ 0 & -\frac{1}{2}\psi_{\bar{z}} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{6.2}
$$

Recall that $A_w = Q$ and $A_{\bar{w}} = -P$. The equation $F_{zw} = 0$ is satisfied provided that

$$
U_{\bar{z}} = 0,
$$

i.e. U must be holomorphic. The second ASDYM equation $F_{z\bar{z}} + F_{w\bar{w}} = 0$ is satisfied if and only if (1.4) holds.

7. Semi–Flat Calabi–Yau Metric

In this section we consider the semi–flat Calabi–Yau metric constructed by Loftin, Yau and Zaslow, and obtain the local expression of the metric explicitly in terms of solution of the definite affine sphere equation.

Let us first recall the Simon–Wang approach to affine spheres [27]. Consider the parametrisation of an elliptic affine sphere

$$(z, \bar{z}) \mapsto f = (f^1(z, \bar{z}), f^2(z, \bar{z}), f^3(z, \bar{z})) \in \mathbb{R}^3.$$

The structure equations[5] defining the affine sphere can be written as a linear first order system of PDEs in f, f_z and $f_{\bar{z}}$

$$\begin{aligned} \frac{\partial}{\partial z}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ -\frac{1}{2}e^{\psi} & 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix}, \\ \frac{\partial}{\partial \bar{z}}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2}e^{\psi} & 0 & 0 \\ 0 & \bar{U}e^{-\psi} & \psi_{\bar{z}} \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix}, \end{aligned} \tag{7.1}$$

where we have set the affine mean curvature to 1. The compatibility condition for this over-determined system is the affine sphere equation (1.4).

Therefore, given a solution ψ, one can find f and hence the cone over the sphere

$$(z, \bar{z}, r) \longmapsto (x^1 = rf^1(z, \bar{z}),\ x^2 = rf^2(z, \bar{z}),\ x^3 = rf^3(z, \bar{z})). \tag{7.2}$$

This expression can be inverted locally to give $r = r(x)$.

Proof of Proposition 1.2. The metric cone over an elliptic affine sphere is given by (3.15) with $\phi(x) = r^2/2$ and the corresponding semi-flat metric (1.3).

The matrix ϕ_{jk} in (1.3) can be obtained by contracting the metric (3.15) with $\partial/\partial x^j, \partial/\partial x^k$. Given a solution of the affine sphere equation ψ, we know g_B in the basis $(dr, dz, d\bar{z})$, thus we want to express $\partial/\partial x^j$ in terms of $\partial/\partial r, \partial/\partial z, \partial/\partial \bar{z}$. Now, from (7.2), we have that

$$\begin{pmatrix} \partial/\partial x^1 \\ \partial/\partial x^2 \\ \partial/\partial x^3 \end{pmatrix} = N^{-1}\begin{pmatrix} \partial/\partial r \\ r^{-1}\partial/\partial z \\ r^{-1}\partial/\partial \bar{z} \end{pmatrix}, \quad \text{where} \quad N = \begin{pmatrix} f^1 & f^2 & f^3 \\ f^1_z & f^2_z & f^3_z \\ f^1_{\bar{z}} & f^2_{\bar{z}} & f^3_{\bar{z}} \end{pmatrix}.$$

Moreover, N is the matrix solution of the linear system (7.1), whose existence and the existence of its inverse N^{-1} are guaranteed by the affine sphere equation. Writing

$$N^{-1} = \begin{pmatrix} p_1 & q_1 & \bar{q}_1 \\ p_2 & q_2 & \bar{q}_2 \\ p_3 & q_3 & \bar{q}_3 \end{pmatrix},$$

5 For the elliptic affine sphere with affine mean curvature set to 1, the shape operator is $S = I$. Now, with the affine metric (3.11), the affine normal chosen to point inward from the surface is given by minus the position vector $-f$, and the structure equations (3.12) and (3.13) become

$$\begin{aligned} D_X f_*(Y) &= f_*(\nabla_X Y) + h(X, Y)(-f), \\ D_X(-f) &= -f_*(X). \end{aligned}$$

Note that we have abused the notation so that f also denotes the immersion.

one calculates ϕ_{jk} and thus the metric on the fibre to be

$$\phi_{jk}dy^j dy^k = (p_j p_k + e^{\psi} q_j \bar{q}_k) dy^j dy^k.$$

Now, let us introduce new coordinates

$$\tau := p_i y^i, \quad \xi := q_i y^i, \quad \bar{\xi} := \bar{q}_i y^i$$

and write $p_i dy^i = d\tau - y^i dp_i$ etc. Denote the two matrices of coefficients in the linear system (7.1) by $-A^{(z)}$ and $-A^{(\bar{z})}$ respectively, so that (7.1) is

$$\partial_z N + A^{(z)} N = 0, \qquad \partial_{\bar{z}} N + A^{(\bar{z})} N = 0.$$

Then, by considering the corresponding equation for N^{-1}, the one–forms $y^i dp_i, y^i dq_i, y^i d\bar{q}_i$ can be written in terms of coordinates $\tau, \xi, \bar{\xi}$ and components of $A^{(z)}$ and $A^{(\bar{z})}$, which are known in terms of ψ.

Finally, we can write the metric (1.3) as

$$g = dr^2 + r^2 e^{\psi} |dz|^2 + |d\tau + \alpha|^2 + e^{\psi} |d\xi + \beta|^2,$$

where

$$\alpha = -\frac{1}{2} e^{\psi} (\bar{\xi} dz + \xi d\bar{z}), \quad \beta = (\tau + \xi \psi_z) dz + e^{-\psi} \bar{U} \bar{\xi} d\bar{z}.$$

By similar calculation, the Kähler form can be written as

$$\omega = dr \wedge (d\tau + \alpha) + \frac{r}{2} e^{\psi} (d\bar{z} \wedge (d\xi + \beta) + dz \wedge (d\bar{\xi} + \bar{\beta})).$$

Using the relation between the metric, the Kähler form and the complex structure, we find holomorphic basis $\{e_1, e_2, e_3\}$ (1.8) and write g and ω as in Proposition 1.2, where we have introduced a complex coordinate $w = r + i\tau$. $\square$

Remark 1. The Ricci flat condition for the metric (1.7) reduces to the affine sphere equation (1.4) for $\psi(z, \bar{z})$ and $U(z)$. Equation (1.4) is invariant under the transformations $\partial/\partial z \to \partial/\partial \hat{z}$, $\psi \to \hat{\psi}$, $U \to \hat{U}$, where

$$\partial/\partial_{\hat{z}} = e^{-j(z)} \partial/\partial_z, \quad \hat{\psi} = \psi - j(z) - \overline{j(z)}, \quad \text{and} \quad \hat{U} = e^{-3j(z)} U.$$

This can be understood geometrically, as $e^{\psi} dz d\bar{z}$ and $U dz^3$ are the affine metric and the cubic differential respectively of the affine sphere. The metric (1.7) is invariant under the above transformations, together with $\xi \to \hat{\xi} = e^{j(z)} \xi$.

Remark 2. One expects the linear system associated with the structure equations of affine spheres (7.1) to be equivalent to the Hitchin Lax pair (4.5) giving rise to the affine sphere equation. The matrices $A^{(z)}$ and $A^{(\bar{z})}$ in (7.1) are unique up to gauge transformations

$$A^{(z)} \longrightarrow g^{-1} A^{(z)} g + g^{-1} \partial_z g, \quad A^{(\bar{z})} \longrightarrow g^{-1} A^{(\bar{z})} g + g^{-1} \partial_{\bar{z}} g.$$

If we write

$$A^{(z)} = (A_z + \lambda P), \quad A^{(\bar{z})} = (A_{\bar{z}} + \lambda^{-1} Q) \tag{7.3}$$

for some value of λ, then it follows that $(A_z, A_{\tilde{z}}, Q, P)$ will satisfy the Hitchin equations (4.2a, b, c), with reality condition $\tilde{z} = \bar{z}$. Conversely, given a solution $(A_z, A_{\tilde{z}}, Q, P)$ to the Hitchin equations, we should be able to find a value of spectral parameter λ such that $(A_z + \lambda P)$ and $(A_{\tilde{z}} + \lambda^{-1} Q)$ can be gauge transformed to $A^{(z)}$ and $A^{(\tilde{z})}$ respectively.

For example, we can obtain $A^{(z)}$ and $A^{(\tilde{z})}$ in (7.1) from the ansatz (4.19), with $\tilde{z} = \bar{z}$ and $\tilde{U} = \bar{U}$, by gauge transformation with

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2}e^{-\psi/2} & 0 \\ 0 & 0 & -\sqrt{2}e^{-\psi/2} \end{pmatrix},$$

and choosing the value of spectral parameter in (7.3) to be $\lambda = 1$. Note that we need $\det g \neq 1$, since $A^{(z)}$ and $A^{(\tilde{z})}$ are not traceless.

8. Painlevé III

One of the main results of Loftin, Yau and Zaslow [20] is the existence of radially symmetric solutions of the affine sphere equation (1.4) for $U(z) = z^{-2}$, with prescribed behaviour near the singularity $z = 0$. In this section we shall show that the radially symmetric solutions of (1.4) are Painlevé III transcendents.

Proof of Proposition 1.3. Set $U = z^{-2}$, and look for solutions of (1.4) of the form $\psi = \psi(\rho)$, where $\rho = |z|$. Making a substitution $\psi(\rho) = \log(\rho^{-3/2} H(\rho))$ and introducing a new independent variable by $\rho = s^2$ yields the following ODE for $H = H(s)$:

$$H_{ss} = \frac{(H_s)^2}{H} - \frac{H_s}{s} - \frac{8H^2}{s} - \frac{16}{H}. \tag{8.1}$$

This is the celebrated Painlevé III equation [15]

$$H_{ss} = \frac{(H_s)^2}{H} - \frac{H_s}{s} + \frac{\alpha H^2 + \beta}{s} + \gamma H^3 + \frac{\delta}{H}$$

with special values of parameters

$$(\alpha, \beta, \gamma, \delta) = (-8, 0, 0, -16).$$

In the classification of Okamoto [26] it falls in the type $D7$. □

Remarks. • One can consider the radial symmetry reduction of the affine sphere equation (1.4) with $U = z^{-n}$ for general $n \in \mathbb{Z}$.

n ≠ 3. Changing the independent variable to

$$s = (z\bar{z})^{\frac{3-n}{4}}$$

and usin the ansatz

$$\psi = \log\left(s^{-\left(\frac{1+n}{3-n}\right)} H(s)^k\right)$$

with $k = \pm 1$ reduces (1.4) to the Painlevé III equation with parameters $(\alpha, \beta, \gamma, \delta) = \left(\frac{-8}{(3-n)^2}, 0, 0, \frac{-16}{(3-n)^2}\right)$ and $(\alpha, \beta, \gamma, \delta) = \left(0, \frac{8}{(3-n)^2}, \frac{16}{(3-n)^2}, 0\right)$ for $k = 1$ and $k = -1$, respectively. In both cases, the Painlevé III equations are of type $D7$ in Okamoto's classification.

n=3. Setting $\psi = \psi(s)$ where $s = (z\bar{z})^{\frac{1}{2}}$ in Eq. (1.4) yields

$$\psi_{ss} + \frac{\psi_s}{s} + \frac{4e^{-2\psi}}{s^6} + 2e^{\psi} = 0, \tag{8.2}$$

which, under multiplication by $\left(\frac{\psi_s}{2} + \frac{1}{s}\right)$, gives a first-order ODE

$$\frac{{\psi_s}^2}{4} + \frac{\psi_s}{s} + e^{\psi} - \frac{e^{-2\psi}}{s^6} + \frac{c}{s^2} = 0, \tag{8.3}$$

where c is a constant of integration. Hence any solution to (8.3) such that $s\psi_s \neq -1/2$ gives rise to a solution to (8.2), and conversely all solutions to (8.2) arise from (8.3). Equation (8.3) is integrable by quadratures in terms of the elliptic functions.

- In general, a Painlevé III equation may have two types of special (i.e. non–transcendental) solutions: the finite number of rational solutions and a one parameter family of Riccati type solutions expressible by special functions [15]. For the values of parameters in (8.1) the Riccati solutions do not exist, and there exists a unique algebraic solution

$$H = -(2s)^{1/3}.$$

This corresponds to

$$\psi = \frac{1}{3}\log(2) - \frac{4}{3}\log(|z|) + \log(-1)$$

which is not real. There are Bäcklund transformations leading to new solutions, but they change the value of the parameters. This shows that the desired radial solution to the affine sphere equation (1.4) is transcendental. In [3,17] it has been shown that the radial solutions of the Tzitzéica equation (3.16) also satisfies Painlevé III of type $D7$.

8.1. Lax pair for Painlevé III. The standard isomonodromic approach to Painlevé III identifies this equation with $SL(2, \mathbb{C})$ isomonodromic problem with two double poles. The connection with affine differential geometry and its underlying isospectral Lax pair suggests that there is an alternative isomonodromic Lax pair for PIII given in terms of 3 by 3 matrices, as opposed to the standard Lax pair with 2 by 2 matrices [16]. (See also [22] where $SL(2, \mathbb{C})$ ASDYM has been reduced to PIII.)

Let us now return to the holomorphic setting, and consider the Lax pair for ASDYM in $\mathbb{C}^4$ with gauge group $SL(3, \mathbb{C})$,

$$(D_w + \lambda D_{\tilde{z}})\Psi = 0, \quad (D_z + \lambda D_{\tilde{w}})\Psi = 0,$$

where Ψ is a vector-valued function of $w, \tilde{w}, z, \tilde{z}$ and λ. We require that the connection is invariant under the 3 dimensional subgroup of the conformal group $PGL(4, \mathbb{C})$ generated by

$$\{\partial_w,\ \partial_{\tilde{w}},\ z\partial_z - \tilde{z}\partial_{\tilde{z}}\}, \tag{8.4}$$

and introduce coordinates $(\rho, \theta) \in \mathbb{C}^2$ such that $z = \rho e^{i\theta}$, $\tilde{z} = \rho e^{-i\theta}$, and $z\partial_z - \tilde{z}\partial_{\tilde{z}} = -i\frac{\partial}{\partial\theta}$. Then the ASDYM Lax pair becomes

$$\left(-\zeta\,\partial_\rho + \rho^{-1}\zeta^2\partial_\zeta + 2(A_w - \zeta\,e^{-i\theta}A_{\tilde{z}})\right)\Psi = 0,$$
$$\left(\partial_\rho + \rho^{-1}\zeta\,\partial_\zeta + 2(e^{i\theta}A_z - \zeta\,A_{\tilde{w}})\right)\Psi = 0,$$

where the gauge fields are in an invariant gauge; $(A_w, A_{\tilde{w}}, e^{i\theta}A_z, e^{-i\theta}A_{\tilde{z}})$ are functions of ρ only, and $\zeta = -\lambda e^{i\theta}$ is an invariant spectral parameter[6]. Taking linear combinations of these two linear PDEs gives a Lax pair of the form

$$\frac{\partial\Psi}{\partial\zeta} = \hat{L}\;\Psi, \quad \frac{\partial\Psi}{\partial\rho} = \hat{M}\;\Psi, \tag{8.5}$$

where

$$\hat{L} = \rho\zeta^{-2}\left(\zeta^2 A_{\tilde{w}} - A_w + \zeta(e^{-i\theta}A_{\tilde{z}} - e^{i\theta}A_z)\right),$$
$$\hat{M} = \zeta^{-1}\left(A_w + \zeta^2 A_{\tilde{w}} - \zeta(e^{i\theta}A_z + e^{-i\theta}A_{\tilde{z}})\right).$$

The calculation leading to Painlevé III (8.1) implies that if we gauge transform ansatz (4.19) with $U(z) = z^{-2}$, $\tilde{U}(\tilde{z}) = \tilde{z}^{-2}$ into an invariant gauge and substitute it into (8.5), then in the new coordinate $s = \rho^{1/2}$ the system (8.5) becomes Lax pair of the Painlevé III with special values of parameters (8.1). We shall now present this calculation:

An invariant gauge of (4.19) can be obtained using the gauge transformation with

$$g = \begin{pmatrix} e^{i\theta/3} & 0 & 0 \\ 0 & e^{-i2\theta/3} & 0 \\ 0 & 0 & e^{i\theta/3} \end{pmatrix},$$

which does not change A_w and $A_{\tilde{w}}$, but gives

$$e^{i\theta}A_z = \begin{pmatrix} \frac{1}{6\rho} & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 0 & -\left(\frac{1}{4}\psi_\rho + \frac{1}{3\rho}\right) & -\frac{1}{\rho^2}e^{-\psi} \\ 0 & 0 & \frac{1}{4}\psi_\rho + \frac{1}{6\rho} \end{pmatrix},$$
$$e^{-i\theta}A_{\tilde{z}} = \begin{pmatrix} -\frac{1}{6\rho} & 0 & 0 \\ -\frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & \frac{1}{4}\psi_\rho + \frac{1}{3\rho} & 0 \\ 0 & -\frac{1}{\rho^2}e^{-\psi} & -\left(\frac{1}{4}\psi_\rho + \frac{1}{6\rho}\right) \end{pmatrix}.$$

Then, in terms of $s = \rho^{1/2}$ and $H(s) = s^3 e^{\psi}$, the system (8.5) gives a Lax pair for the Painlevé III equation (8.1) as

$$\frac{\partial\Psi}{\partial\zeta} = L\;\Psi, \quad \frac{\partial\Psi}{\partial s} = M\;\Psi, \tag{8.6}$$

[6] The spectral parameter λ is not constant along the lift of the generators (8.4) to $\mathbb{C}^4 \times \mathbb{CP}^1 \in (w, \tilde{w}, z, \tilde{z}, \lambda)$ where Ψ is defined. However, the invariant spectral parameter ζ is constant along the lift, and hence we are allowed to express Ψ as a function of ρ and ζ only.

where

$$L = -\frac{1}{\zeta^2}\begin{pmatrix} \frac{\zeta}{3} & \frac{1}{\sqrt{2}}\zeta(sH)^{1/2} & \frac{1}{\sqrt{2}}(sH)^{1/2} \\ \frac{1}{\sqrt{2}}\zeta(sH)^{1/2} & \zeta\left(\frac{1}{12}-\frac{sH_s}{4H}\right) & -\zeta\frac{s}{H} \\ \frac{1}{\sqrt{2}}\zeta^2(sH)^{1/2} & \zeta\frac{s}{H} & \zeta\left(\frac{sH_s}{4H}-\frac{5}{12}\right) \end{pmatrix},$$

$$M = \sqrt{2}\left(\frac{H}{s}\right)^{1/2}\begin{pmatrix} 0 & -1 & \frac{1}{\zeta} \\ 1 & 0 & \sqrt{2}\left(\frac{s}{H^3}\right)^{1/2} \\ -\zeta & \sqrt{2}\left(\frac{s}{H^3}\right)^{1/2} & 0 \end{pmatrix}.$$

The matrix L has two double poles as expected for Painlevé III [16], at $\zeta = 0$ and $\zeta = \infty$.

We note here that a different (i.e. gauge inequivalent) 3×3 isomonodromic Lax pair for Painlevé III of type $D7$ was used by Kitaev in [17]. The Lax pair can also be derived from the ASDYM Lax pair, from a solution to Hitchin equations which is gauge equivalent to (6.1).

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