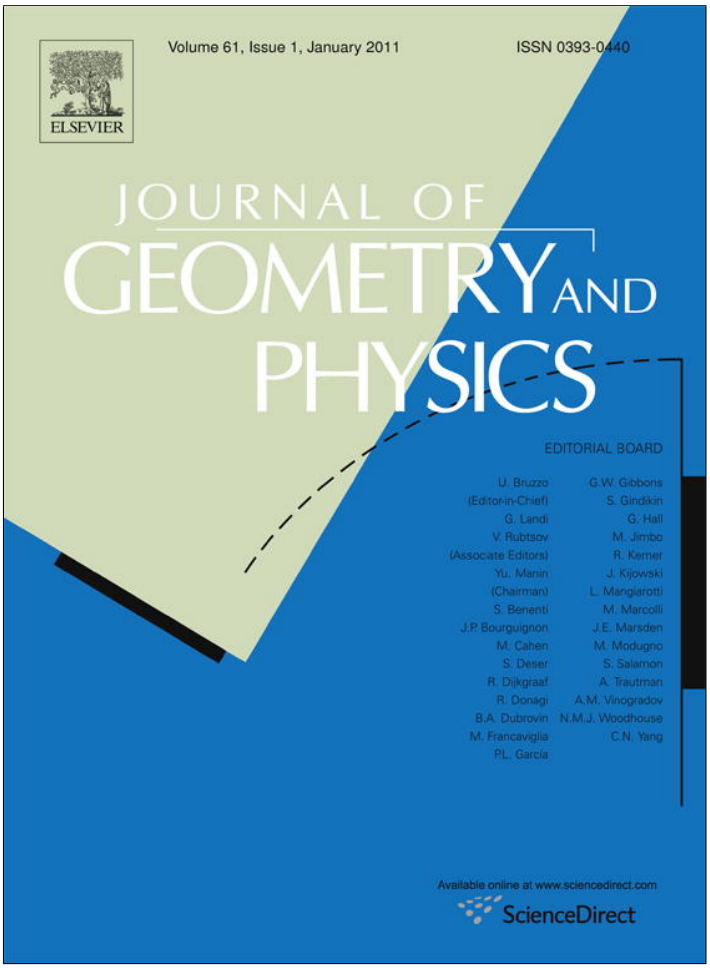

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On the 7th order ODE with submaximal symmetry

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ABSTRACT

We find a general solution to the unique 7th order ODE admitting a ten-dimensional group of contact symmetries. The integral curves of this ODE are rational contact curves in $\mathbb{P}^3$ which give rise to rational plane curves of degree six. The moduli space of these curves is a real form of the homogeneous space Sp(4)/SL(2).

The 7th order ODE

$$10(y^{(3)})^3y^{(7)} - 70(y^{(3)})^2y^{(4)}y^{(6)} - 49(y^{(3)})^2(y^{(5)})^2 + 280(y^{(3)})(y^{(4)})^2y^{(5)} - 175(y^{(4)})^4 = 0, \tag{1}$$

where $y = y(x)$ and $y^{(k)} = \mathrm{d}^k y/\mathrm{d}x^k$ has recently appeared explicitly [1–4] or implicitly [5] in several different contexts. It is the unique (up to contact transformations) equation admitting ten-dimensional algebra of contact symmetries [1], and the aim of this note is to show that its general solution is given by a degree six rational curve of the form

$$y^3 + \alpha(x)y^2 + \beta(x)y + \gamma(x) = 0, \tag{2}$$

where (α, β, γ) are a quadratic, a quartic, and a sextic respectively with the coefficients depending on seven parameters as in formula (13).

In fact the symmetry algebra of (1) was known to Lie [6] who also proved that this is the maximal algebra of contact vector fields on the plane. It is quite possible that Eq. (1) and its general solution were also known to Lie. We have been unable to find it in any of Lie's works, but we would be grateful to hear from anyone who has (the earliest reference to the equation – but not its solution – that we are aware of is [7]).

Contact Lie algebras. Let $U \subset \mathbb{R}^2$ be an open set and let $\mathbb{P}(T^*U)$ be a projectivized cotangent bundle with a contact one-form ω. A curve $\gamma \subset \mathbb{P}(TU)$ is called *contact* if $\omega|_\gamma = 0$. A contact transformation is a map $f : \mathbb{P}(TU) \to \mathbb{P}(TU)$ which takes contact curves into contact curves. Equivalently $f^*(\omega) = \lambda\omega$ for some function λ. Let (x, y) be the local coordinates on U and let z parametrise the fibres of $\mathbb{P}(TU)$, so that we can set $\omega = \mathrm{d}y - z\mathrm{d}x$. Consider a one parameter group of contact transformations. Close to the identity, this is characterized by a contact vector field X such that the contact condition $\mathcal{L}_X\omega = c\omega$ holds, where $\mathcal{L}_X = d(X\lrcorner) + X\lrcorner d$ is the Lie derivative and c is some function. The contact condition implies that locally there exist a function $H = H(x, y, z)$ such that

$$X_H = -(\partial_z H)\partial_x + (H - z\partial_z H)\partial_y + (\partial_x H + z\partial_y H)\partial_z. \tag{3}$$

If $H = a(x, y) + zb(x, y)$ then X_H generates a prolongation of a family of *point transformations* $f : U \to U$. Otherwise it generates a proper contact flow.

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The symmetry. A remarkable result of Lie is that a maximum dimension of a Lie algebra of proper contact vector fields on the plane is ten. This maximal, ten-dimensional Lie algebra is generated by vector fields (3) corresponding to functions

$$1, x, x^2, y, z, xz, x^2z - 2xy, z^2, 2yz - xz^2, 4xyz - 4y^2 - x^2z^2. \tag{4}$$

This algebra is isomorphic to $\mathfrak{so}(5)$, or equivalently to $\mathfrak{sp}(4)$ (this is, up to a choice of the real form, the maximal subalgebra of the 11 dimensional symmetry algebra of the trivial 7th order ODE). The first seven generators correspond to prolonged point vector fields and the last three are the proper contact vector fields.

The algebra (4) generates all contact symmetries of Eq. (1): if $y(x)$ is a solution to (1) then so is $\tilde{y}(\tilde{x})$, where

$$\tilde{x} = \tilde{x}(x, y, z, c_i), \qquad \tilde{y} = \tilde{y}(x, y, z, c_i), \qquad \tilde{z} = \tilde{z}(x, y, z, c_i) = \frac{\partial_x\tilde{y} + z\partial_y\tilde{y}}{\partial_x\tilde{x} + z\partial_y\tilde{x}}, \qquad \partial_z\tilde{y} = \tilde{z}\partial_z\tilde{x}$$

is the contact transformation generated by the vector fields (3) and $c_1, \ldots, c_{10}$ are parameters of this transformation. Equivalently, any of the generators (4) satisfies the linearisation of (1) when $z = y'$.

Another result of Lie [6] is that a maximal dimension of the contact symmetry algebra of an ODE of order $n > 3$ is $(n+4)$, with maximal symmetry occurring if only if the ODE is contact equivalent to a trivial equation $y^{(n)} = 0$. Therefore Eq. (1) is of submaximal type [2]—it is not equivalent to the trivial equation and its symmetry algebra has the largest possible dimension. Up to the contact equivalence (1) is the unique 7th order ODE with this property.

The solution. We verify that the algebraic curve $y^2 + x(x-1)^3 = 0$ solves (1). This curve is parametrised by

$$x(t) = \frac{1}{t^2+1}, \qquad y(t) = -\frac{t^3}{(t^2+1)^2}. \tag{5}$$

The 7-dimensional subalgebra of (4) consisting of prolonged point symmetries integrates to

$$y \to c_4y + c_1 + c_2x + c_3x^2, \qquad x \to c_5x + c_6, \quad \text{and} \quad y \to \frac{y}{(1+c_7x)^2}, \qquad x \to \frac{x}{1+c_7x}. \tag{6}$$

The contact transformations generated by (3) with $H_8 = z^2$, $H_9 = 2yz - xz^2$ and $H_{10} = 4xyz - 4y^2 - x^2z^2$ respectively are given by

$$\tilde{x} = x - 2c_8z, \qquad \tilde{y} = y - c_8z^2, \qquad \tilde{z} = z \tag{7}$$

$$\tilde{x} = \frac{x(1+c_9z) - 2c_9y}{1-c_9z}, \qquad \tilde{y} = \frac{y(1-2c_9z) + c_9xz^2}{(1-c_9z)^2}, \qquad \tilde{z} = \frac{z}{1-c_9z}, \quad \text{and}$$

$$\tilde{x} = \frac{x}{1+4c_{10}y-2c_{10}xz}, \qquad \tilde{y} = \frac{y + 4c_{10}y^2 - 4c_{10}xyz + c_{10}x^2z^2}{(1+4c_{10}y-2c_{10}xz)^2}, \qquad \tilde{z} = \frac{z}{1+4c_{10}y-2c_{10}xz}$$

where $c_8, c_9, c_{10} \in \mathbb{R}$.

Applying the group of point transformations to the given solution yields the six parameter family of solutions to (1) given by a family of algebraic curves of degree four

$$(y+Q)^2 + P = 0 \tag{8}$$

where $Q = Q(x)$ is an arbitrary quadratic, and $P = P(x)$ is a quartic with one simple and one triple root.

Recall [8] that a point p on an algebraic curve $f(x, y) = 0$ in $\mathbb{CP}^2$ is singular if the partial derivatives f_x and f_y vanish at p. Moreover p has multiplicity m if all $(m-1)$st derivatives of f vanish at p but at least one mth derivative does not. A singular point is called ordinary if the tangents to all branches at the point are distinct. Any singular point characterised by a triple of integers (m, δ, r), where m is the multiplicity, r is the number of branches and δ is the number of multiplicity two ordinary singular points concentrating at p. The arithmetic genus of a curve is given by

$$\mathrm{g} = \frac{(d-1)(d-2)}{2} - \sum \delta,$$

where d is the degree of the curve, and the summation is taken over all singular points. The curves in the family (8) have two singularities: a cusp of type (2, 1, 1) at $(x, y) = (x_0, -Q(x_0))$, where x_0 is the triple root of P and a point at ∞ of type (2, 2, 2). Calculating the genus of curves in this family yields

$$\mathrm{g} = \frac{3\cdot 2}{2} - 1 - 2 = 0.$$

Therefore the family is rational. Applying the point transformations in (6) to the parametrisation (5) and redefining the constants $c_1, \ldots, c_6$ we find that the rational parametrisation is given by

$$x(t) = \frac{b_5 + b_6t^2}{b_0 + t^2}, \qquad y(t) = \frac{b_4t^4 + b_3t^2 + b_2t + b_1}{(b_0+t^2)^2}.$$

The six parameters in (8) are algebraic expressions in the seven parameters $(b_0, \dots, b_6)$ one of which is irrelevant and arises only in the parametrisation.

The family of curves (8) can also be obtained applying H_9 and the point transformations to the trivial solution $y = x^2$. It is not the general solution to (1) as it depends on six parameters rather than seven. To introduce the additional parameter, and construct the general solution we use the contact transformation (7) generated by (3) with $H_8 = z^2$

$$\tilde{x}(t) = x(t) - 2bz(t), \qquad \tilde{y}(t) = y(t) - bz(t)^2, \qquad \tilde{z}(t) = z(t), \tag{9}$$

where

$$z(t) = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{(4b_4b_0 - 2b_3)t^3 - 3b_2t^2 + (2b_3b_0 - 4b_1)t + b_2b_0}{2(b_5 - b_0b_6)(b_0t + t^3)},$$

and $(b, b_0, \dots, b_6)$ are constant parameters. The relation (9) gives a seven-dimensional family of rational contact curves in $\mathbb{P}^3$. The connected component of the symplectic group $Sp(4)$ acts on $\mathbb{P}^3$ and preserves the family (9). The symmetry group of any fixed rational curve in this family is $SL(2)$, and so the seven-dimensional space of solutions to (1) is the symmetric space $Sp(4)/SL(2)$. In the holomorphic category, a rational curve in $\mathbb{P}^3$ can be characterised by a normal bundle, which in our case is $N = \mathcal{O}(5) \oplus \mathcal{O}(5)$. The contact modification of the Kodaira theorem described in [9] can be applied to deduce that we have constructed a complete analytic family of contact curves: an infinitesimal contact deformation of any fixed curve in the family (9) also belongs to this family.

Alternatively, the general solution to (1) can be given by an implicit relation

$$u(x, y, z) = 0, \qquad v(x, y, z) = 0 \tag{10}$$

where (x, y, z) are coordinates on an open set in $\mathbb{P}^3$. Using (8) and (9) we find

$$u = (y + bz^2 + Q(X))^2 + P(X), \qquad v = 4P(z + Q')^2 + (P')^2$$

where $Q = Q(X)$ is a quadratic, $P = P(X)$ is a quartic with one simple and one triple root, and $X = x + 2bz$. To find an explicit formula for $y(x)$ pick a real root z of the cubic $v = 0$, substitute this in $u = 0$ and solve the resulting quadratic for y. Alternatively we use the resultant to produce a planar curve birationally equivalent to (10) by eliminating z between u and v. Recall [10] that a resultant of two polynomials

$$u(z) = u_0 + u_1z + \cdots + u_mz^m, \qquad v(z) = v_0 + v_1z + \cdots + v_nz^n$$

is the determinant of the matrix

$$\begin{pmatrix}
u_m & u_{m-1} & \cdots & \cdots & u_0 & & \\
 & u_m & u_{m-1} & \cdots & \cdots & u_0 & \\
 & & \ddots & & & & \ddots \\
 & & & u_m & u_{m-1} & \cdots & \cdots & u_0 \\
v_n & v_{n-1} & \cdots & \cdots & v_0 & & & \\
 & v_n & v_{n-1} & \cdots & \cdots & v_0 & & \\
 & & \ddots & & & & \ddots & \\
 & & & v_n & v_{n-1} & \cdots & \cdots & v_0
\end{pmatrix}.$$

The resultant vanishes if u, v admit a common root. To obtain a manageable formula consider (10) with $Q = 0$ and $P = X(X-1)^3$, and find the resultant of u and v. This resultant factorises into two terms, each giving a rational curve of degree six. We choose one of these two curves (The second curve is not a solution to the ODE (1). It arises because the expressions in (10) are squares of actual solutions and contain the term z^2. Only one of the roots satisfies $z = y'$).

$$\begin{aligned}
&\left(64b + 1024b^3\right)y^3 + \left(\left(768b^2 + 16\right)x^2 - 768xb^2 + 288b^2\right)y^2 \\
&\quad + \left(264x^2b - 108b^3 + 192x^4b - 72xb - 384x^3b\right)y \\
&\quad + \left(48x^4 - 27b^2 + 54xb^2 - 16x^3 - 27x^2b^2 - 48x^5 + 16x^6\right) = 0.
\end{aligned} \tag{11}$$

This curve does not belong to the class (8): it has two (2, 1, 1) cusps and one non-ordinary (2, 2, 2) singularity at ∞ whereas (8) has one cusp (apart from the singularity at ∞). The curve (11) possesses the following property: its discriminant is a cube of a quartic $Q(x)$ in x with two real roots. These roots correspond to the positions of the finite cusps, as in Fig. 1. For any such curve (2) the cross-ratio ρ of the roots of Q is an invariant with respect to transformations (6). It is possible to verify that for (11) we have $\rho = \varepsilon$, where $\varepsilon = \exp\left(\frac{i\pi}{3}\right)$. To bring the curve (11) to a canonical form we first set the coefficient of y^2 to zero by a transformation of the form $y \to y + c_1 + c_2x + c_3x^2$. After that the remaining Möbius transformations can be used to set $Q(x) = x(x-1)(x-\rho)$. This, up to a scaling of y, yields

$$\begin{aligned}
&y^3 - 3y\Big(x - (1-\varepsilon)\Big)\Big(x - (1+\varepsilon)\Big)\Big(x - \frac{1}{3}(1+\varepsilon)\Big)\Big(x - (-1+\varepsilon)\Big) \\
&\quad + 2\Big(x - (1 - \mathrm{i}\varepsilon)\Big)\Big(x - (1 + \mathrm{i}\varepsilon)\Big)\Big(x - (\mathrm{i} - \mathrm{i}\varepsilon)\Big)\Big(x - (-\mathrm{i} + \mathrm{i}\varepsilon)\Big)\Big(x - (\mathrm{i} + \varepsilon)\Big)\Big(x - (-\mathrm{i} + \varepsilon)\Big) = 0.
\end{aligned}$$

The discriminant $Q(x)$ of this curve is proportional to $x^3(x-1)^3(x-\varepsilon)^3$.

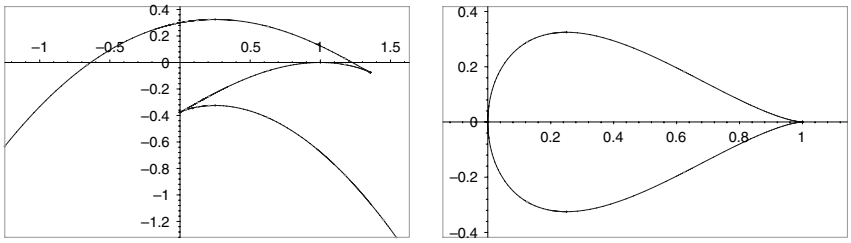

Fig. 1. Curve (11) with $b = 1/2$ and curve (8) with $Q = 0$, $P = x(x-1)^3$.

Using this complex canonical form, we easily can find a real one. To do that we find four roots of the sextic in x in the complex canonical form such that their cross-ratio is -1. Using Möbius transformations, we move these roots to $\pm 1 \pm \mathrm{i}$. The result (up to scaling of x and y) is given by

$$y^3 + 3(3x^4 - 6x^2 - 1)y + 12x(3x^4 + 1) = 0 \tag{12}$$

which can be parametrised by

$$x(t) = \frac{t(t^2-3)}{3(t^2+1)}, \qquad y(t) = -\frac{4t(t^4+3)}{3(t^2+1)^2}.$$

To get the general solution for the 7th order ODE (1) we apply the point transformations (6)–(12). The resulting degree six rational curve of the form (2) is given by

$$\begin{aligned}&\left(c_4y + c_1 + c_2x + c_3x^2\right)^3 + 3\left(3\left(c_5x + c_6\right)^4 - 6\left(c_5x + c_6\right)^2\left(1 - c_7x\right)^2 - \left(1 - c_7x\right)^4\right)\left(c_4y + c_1 + c_2x + c_3x^2\right)\\ &\quad + 12\left(c_5x + c_6\right)\left(3\left(c_5x + c_6\right)^4\left(1 - c_7x\right) + \left(1 - c_7x\right)^5\right) = 0.\end{aligned} \tag{13}$$

The formula (2) is the general solution as the generic initial data

$$\{y(0), y'(0), \ldots, y^{(6)}(0)\}$$

can be chosen arbitrarily by choosing the coefficients $c_1, \ldots, c_7$ in the solutions. There exist additional singular solutions corresponding to submanifolds of the initial data manifold which cannot be obtained by any choice of coefficients in (13). An example of such a singular solution with a co-dimension one initial data set is the curve (8). The leading term in this curve is y^2 and this cannot arise in (13). This type of behaviour should not be confused with the singular orbits of the symmetry group. The connected component of the group $Sp(4)$ is transitive on the solution space, and thus (13) and (8) are related by a contact transformation. This is indeed how we found (13).

Conclusions. We have found the general solution of the 7th order ODE (1). This equation is submaximal as its symmetry algebra is ten-dimensional, wheres the algebra of the trivial 7th order ODE is eleven-dimensional. The general solution to (1) is given by the seven-dimensional orbit of the point transformations (6) acting on the canonical solution (12). There also exists a six-dimensional submanifold in the space of solutions given by degree four curves (8). For each contact transformations generated by H_8, H_9 and H_{10} in (4) there exists a point transformation such that the composition of the two fixes (12). This gives a stabiliser $SL(2)$ of (12) and finally the solution space $Sp(4)/SL(2)$.

The analogous, submaximal 5th order ODE characterises conics in $\mathbb{RP}^2$. In the inhomogeneous coordinates (x, y) the five parameter family of conics is

$$y^2 = c_1x^2 + c_2xy + c_3y + c_4x + c_5.$$

Eliminating the parameters $(c_1, \ldots, c_5)$ between this equation and its fourth derivatives and substituting in the fifth derivative yields the ODE

$$9(y^{(2)})^2y^{(5)} - 45y^{(2)}y^{(3)}y^{(4)} + 40(y^{(3)})^3 = 0.$$

This construction goes back to Halphen [11] who wrote the equation as $((y^{(2)})^{-2/3})^{(3)} = 0$.

The seven-dimensional space of solutions M to (1) carries a $GL(2, \mathbb{R})$ structure in the sense of [12,3]: TM has a point-wise identification with a vector space of homogeneous degree six polynomials in two variables. The five Wünschmann–Doubrov–Wilczynski invariants vanish on the ODE (1) which implies [13] that the linearisation of (1) is equivalent to a trivial ODE $\delta y^{(7)} = 0$. Moreover [4], M also admits a conformal structure of signature (3, 4). The null vectors of this structure correspond to the six order polynomials

$$a_1x^6 + 6a_2x^5 + 15a_3x^4 + 20a_4x^3 + 15a_5x^2 + 6a_6x + a_7$$

with vanishing quadratic invariant [10]

$$a_1a_7 - 6a_2a_6 + 15a_3a_5 - 10a_4^2.$$

In [4] it was shown that the conformal structure associated to (1) contains a metric with weak holonomy $\tilde{G}_2$: there exists a three-form ϕ on M such that

$$\mathrm{d}\phi = \Lambda * \phi, \quad d * \phi = 0$$

where Λ is a constant, and $*$ is the Hodge operator. Taking an analytic continuation of this structure to the Riemannian signature yields the homology seven–sphere $M = SO(5)/SO(3)$ with its canonical weak G_2 structure originally constructed by Bryant [5].

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