Einstein–Maxwell Gravitational Instantons

Maciej Dunajski
(joint work with Sean Hartnoll, CQG 2007,hep-th/0610261)

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

GRG18, Sydney, July 2007

Gravitational Instantons

- Euclidean solutions to classical field equations with finite action.
- Semiclassical insight into quantum gravity. Only requires the validity of GR as a low energy theory.
- Asymptotically Euclidean with $R_{ab} = 0$ implies flat $\mathbb{R}^4$. Gravitational instantons are Asymptotically Locally Euclidean ALE or Asymptotically Locally Flat ALF.
- Look for instantons in Einstein–Maxwell theory

$$G_{ab} = 2F_a{}^c F_{bc} - \frac{1}{2} g_{ab} F^{cd} F_{cd}, \qquad \nabla_a F^{ab} = 0.$$

There exist Asymptotically Euclidean solutions.

$$
\begin{aligned}
g^{(4)} &= \frac{1}{U\widetilde{U}}(d\tau+\omega)^2 + U\widetilde{U}d\mathbf{x}^2 \\
F &= U\widetilde{U}\star_3 d(U^{-1}+\widetilde{U}^{-1}) - \frac{1}{2}d(U^{-1}-\widetilde{U}^{-1})\wedge(d\tau+\omega),
\end{aligned}
$$

where $\nabla^2 U = \nabla^2\widetilde{U} = 0, \quad \nabla\times\omega = \widetilde{U}\nabla U - U\nabla\widetilde{U}.$

- Take

$$
U = \frac{4\pi}{\beta} + \sum_{m=1}^{N}\frac{a_m}{\mid \mathbf{x}-\mathbf{x}_m\mid}, \qquad \widetilde{U} = \frac{4\pi}{\tilde{\beta}} + \sum_{n=1}^{\widetilde{N}}\frac{\tilde{a}_n}{\mid \mathbf{x}-\tilde{\mathbf{x}}_n\mid},
$$

  where $\beta, \tilde{\beta}, a_m, \mathbf{x}_m, \tilde{a}_n, \tilde{\mathbf{x}}_n, N, \widetilde{N}$ are constants.
- In the Lorentzian case these are nakedly singular unless $U = \widetilde{U}$.
- Study the Euclidean solutions:
  1. Global behaviour.
  2. Uniqueness.
  3. Lift to five dimensions.

If $\mathbf{x}_m \neq \tilde{\mathbf{x}}_n$ then the regularity at the centers requires $\tau \sim \tau + 4\pi$ and

$$U(\tilde{\mathbf{x}}_n)\tilde{a}_n = 1\,, \qquad \widetilde{U}(\mathbf{x}_m)a_m = 1\,, \qquad \forall m, n\,.$$

- $\frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} \neq 0$ gives ALF metric, tending to an S^1 bundle over S^2 at infinity, with $c_1 = N - \widetilde{N}$. Regularity: $\sum a_m - N = \sum \tilde{a}_n - \widetilde{N}$. If $N = \widetilde{N}$ the bundle is trivial and we obtain AF ($\sim \mathbb{R}^3 \times S^1$) solutions.
- $\frac{4\pi}{\beta} = 0$, $\frac{4\pi}{\tilde{\beta}} = 1$ gives an ALE metric, tending to $\mathbb{R}^4/\mathbb{Z}_{|N-\widetilde{N}|}$. Regularity: $\sum a_m = N - \widetilde{N}$. If $N = \widetilde{N} + 1$ the solution is AE ($\sim \mathbb{R}^4$).
- $\frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} = 0$ leads to an AL Robinson-Bertotti metric, tending to $AdS_2 \times S^2$ or $AdS_2/\mathbb{Z} \times S^2$. The former case only arises if all of the centres are coincident, so that $U = \widetilde{U}$, and τ need not be made periodic. Need $N = \widetilde{N}$ for regularity.

Killing Spinor and Uniqueness

Euclidean IWP=Most general Einstein–Maxwell instanton with a Killing spinor $\varepsilon = (\alpha^A, \beta_{A'})$ s. t.

$$\nabla_{AA'}\alpha_B - i\sqrt{2}\phi_{AB}\beta_{A'} = 0\,, \qquad \nabla_{AA'}\beta_{B'} + i\sqrt{2}\tilde{\phi}_{A'B'}\alpha_A = 0\,,$$

where $F_{ab} = \phi_{AB}\epsilon_{A'B'} + \tilde{\phi}_{A'B'}\epsilon_{AB}$.

- Define $U = (\alpha_A\hat{\alpha}^A)^{-1}$, $\widetilde{U} = (\beta_{A'}\hat{\beta}^{A'})^{-1}$. Find that $K_a = \alpha_A\hat{\beta}_{A'} - \hat{\alpha}_A\beta_{A'}$ is a Killing vector, and deduce the form of the metric.
- Analogous to a Lorentzian result of Tod, but stronger:

$$F_{ab}F^{ab} = 2(\phi_{AB}\phi^{AB} + \tilde{\phi}_{A'B'}\tilde{\phi}^{A'B'}) = |\nabla U^{-1}|^2 + |\nabla \widetilde{U}^{-1}|^2.$$

- $|\nabla U^{-1}|$ and $|\nabla \widetilde{U}^{-1}|$ are bounded. Maximum principle implies that U and $\widetilde{U}$ are superpositions of fundamental solutions.

- Instantons in D dimensions are solitons in $D+1$ dimensions.
- Lift to $(4+1)$ Einstein–Maxwell Chern–Simons theory

$$S_5 = \int d^5x\sqrt{-g^{(5)}}\left[R^{(5)} - H_{\alpha\beta}H^{\alpha\beta}\right] - \frac{8}{3\sqrt{3}}\int H \wedge H \wedge W\,,$$

where $H = dW$ is the five dimensional Maxwell field.

- Equations of motion

$$G_{\alpha\beta} = 2H_{\alpha}{}^{\gamma}H_{\beta\gamma} - \frac{1}{2}g^{(5)}_{\alpha\beta}H^{\gamma\delta}H_{\gamma\delta}, \qquad d\star_5 H = -\frac{2}{\sqrt{3}}H \wedge H.$$

- Euclidean IWP $(g^{(4)}, F = dA)$ lift to

$$g^{(5)} = g^{(4)} - (dt + \Phi)^2, \qquad W = \frac{\sqrt{3}}{2}A, \qquad \text{where}$$

$$\Phi = -\left(U^{-1} + \widetilde{U}^{-1}\right)(d\tau + \omega)/2 + \chi, \quad \nabla \times \chi = \frac{1}{2}\nabla\left(U - \widetilde{U}\right).$$

Regularity and Causality in $5D$

- $U \neq \widetilde{U}$ leads to naked singularities OR closed time–like curves.
- $U = \widetilde{U}$ gives a regular and causal solution (solitonic string)

$$g^{(5)} = -(dt - d\tau/U)^2 + \frac{d\tau^2}{U^2} + U^2 d\mathbf{x}^2, \qquad H = -\sqrt{3} \star_3 dU.$$

- $\mathbb{R}^{4+1}$ or $AdS_3 \times S^2$ away from the centres.
- Near the centres, where $U \to \infty$ this is $AdS_3 \times S^2$

$$g^{(5)} = a_m^2 \left[\frac{dr^2}{r^2} + 2r dt d\tau - dt^2 + d\Omega^2_{S^2} \right].$$

$$Y = \frac{1}{r^{1/2} \cos\frac{t}{2}}, \; X = \frac{\tau}{2} - \frac{1}{2}\left[\frac{1}{r} - 1\right] \tan\frac{t}{2}, \; T = \frac{\tau}{2} - \frac{1}{2}\left[\frac{1}{r} + 1\right] \tan\frac{t}{2},$$

so that the metric becomes

$$g^{(5)} = \frac{4a_m^2}{Y^2}\left(-dT^2 + dX^2 + dY^2\right) + a_m^2 d\Omega^2_{S^2}.$$

Outlook

- Einstein–Maxwell vacuum as a function of temperature.
- M–theory interpretation of

$$g^{(11)} = g^{(5)} + g_{T^6}, \qquad \mathcal{F} = H \wedge \Omega_{T^6}.$$

- Metric on moduli space
  1. Slowly moving solitonic strings in $5D$.
  2. Measure of $4D$ instantons.
- Index theory for zero modes.