Mathematical Tripos Part II - C Course | Lent term 2014

Further Complex Methods, Examples sheet 1 | Dr Maciej Dunajski

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1 *Show that:*

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$;

(ii) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\, dx}{x^2+a^2} = \frac{\pi}{a}e^{-a}$ where $a>0$;

(iii) $\displaystyle\int_{-\infty}^\infty \frac{x-\sin x}{x^3}\, dx = \frac{\pi}{2}$;

2 *Write down* the range of values of α (real) or β (complex) for which the following integrals converge.

(i) $\displaystyle\int_\gamma e^{z^2} dz$ where $\{\gamma : z = se^{i\alpha}\ ,\ -\infty < s < \infty\}$

(ii) $\displaystyle\int_\gamma e^{1/z} dz$ where $\{\gamma : z = se^{i\alpha},\ 0 \le s \le 1\}$

(iii) $\displaystyle\int_0^\infty \frac{x^\beta\, dx}{1+x}$

(iv) $\displaystyle\int_\gamma (1+\tanh z)\, dz,$ where $\{\gamma\ :\ z = se^{i\alpha},\ 0 \le s < \infty\}$

3 Let $f(t)$ be analytic at $t=0$ with $f(0)=0$ and $f'(0)\neq 0$. Let C be a circle centred on the origin, with interior D, such that f is analytic in D and the inverse of f exists on $f(D)$.

For a fixed point z within C, let $w=f(z)$. Assuming that w is small, show (using the residue theorem) that

$$z = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)-w}dt,$$

and hence that $z = \sum\limits_{n=1}^{\infty} b_n w^n$, where

$$b_n = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{(f(t))^{n+1}}dt = \frac{1}{2\pi i n}\int_C \frac{1}{(f(t))^n}dt = \frac{1}{n!}\lim_{t\to 0}\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{f(t)}\right)^n.$$

Show that the equation $w = ze^{-z}$ has a solution, for sufficiently small w (how small?),

$$z = \sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}w^n.$$

Find also one solution of the equation $w = 2z - z^2$.

4 Let $\phi(x,y)$ be a harmonic function. Show that ϕ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi\big((z+1)/2, (z-1)/2i\big) - \phi(1,0) + ic$$

where c is a real constant (provided ϕ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2+y^2)$ and (ii) $\tan^{-1} y/x$.

5 Let $f_1(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by branch cuts in the z-plane along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by a cut along the real axis from -1 to $+1$, with $f_2(x)$ real and positive for $(x-1)$ real and positive. Show that $f_1(z)=f_1(-z)$ but $f_2(z)=-f_2(-z)$.

6 Let $P(z)$ be a polynomial of degree n, with n simple roots, none of which lie on a simple close contour L. Show that

$$\frac{1}{2\pi i}\int_L \frac{P'(z)}{P(z)}dz = \text{number of roots lying within } L.$$

7 By integrating the function

$$\frac{(\ln z)^2}{z^2+1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{(\ln x)^m}{x^2+1}dx, \quad m=1,2.$$

8 Evaluate

$$\int_0^\infty \frac{x^{m-1}}{x^2+1}dx, \quad 0<m<2.$$

Why is it necessary for m to satisfy the above restrictions?