Mathematical Tripos Part II - C Course Lent term 2014

Further Complex Methods, Examples sheet 2 Dr Maciej Dunajski

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1 Evaluate the following integrals, where $|f(z)/z| \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

$$\text{(i)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \qquad \text{(ii)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)} dx \qquad \text{(iii)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} \qquad \text{(iv)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x} dx.$$

2 The function $\sin^{-1} z$ is defined, for $0 \leq \arg z < 2\pi$, by

$$\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}} ,$$

where the integrand has a branch cut along the real axis from -1 to $+1$ and takes the value $+1$ at the origin on the upper side of the cut. The path of integration is a straight line for $0 \leq \arg(z) \leq \pi$ and is curved in a positive sense round the branch cut for $\pi < \arg z < 2\pi$. Express $\sin^{-1}(e^{i\pi} z)$ $(0 < \arg z < \pi)$ in terms of $\sin^{-1} z$ and deduce that $\sin(\phi - \pi) = -\sin\phi$. *Hint:* $\sin^{-1}(e^{i\pi} z) = -\pi + \sin^{-1} z$, as can be derived by calculating the integral half way round the cut and remembering that the integrand is an odd function.

3 Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u} du .$$

For what region of the z-plane does $F(z)$ define an analytic function?

Show by closing the contour (use a rectangle) in the upper half plane that

$$F(z) = \pi \operatorname{cosec} \pi z .$$

Explain how this result provides the analytic continuation of $F(z)$.

4 Find the analytic continuation of the function $f(z)$ defined by

$$f(z) = \int_0^{\infty} \frac{e^{-zt}}{1+t^2} dt, \qquad |\arg z| < \frac{\pi}{2},$$

to the domain $-\frac{\pi}{2} < \arg z < \pi$.

5 Find the two functions $\Phi^+(z)$ and $\Phi^-(z)$ analytic in the upper half plane and lower half plane respectively, which satisfy the following:

$$\text{(i)} \quad \Phi^+(x) - \Phi^-(x) = \frac{1-\cos x}{x}, \quad x \in \mathbb{R},$$

with

$$\Phi^{\pm}(x) = \lim_{\varepsilon \to 0} \Phi^{\pm}(x \pm i\varepsilon).$$

$$\text{(ii)} \quad \Phi^+(z) = O\left(\frac{1}{z}\right), \quad z \to \infty, \quad z = x + iy.$$

6 Solve the following singular integral equation:

$$\phi(x) + \frac{\alpha}{i\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\phi(\xi)}{\xi - x}d\xi = \frac{1-\cos x}{x}, \quad x \in \mathbb{R},$$

where α is a constant different that ± 1.
Hint: Use the formulae

$$\phi(x) = \Phi^{+}(x) - \Phi^{-}(x), \quad \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\phi(\xi)}{\xi - x}d\xi = \Phi^{+}(x) + \Phi^{-}(x),$$

to map to the previous problem.

7 Find two independent solutions of the Airy equation $w'' - zw = 0$ in the form

$$w(z) = \int_{\gamma} e^{zt} f(t)\,dt,$$

where γ is to be specified in each case. Show that there is a solution for which γ can be chosen to consist of two straight line segments in the left half t-plane ($\mathrm{Re}\; t \leq 0$).

For this solution show that, if $w(z)$ is normalised so that $w(0) = iA\,3^{-\frac{1}{6}}\Gamma(1/3)$, where A is a constant, then $w'(0) = -iA\,3^{\frac{1}{6}}\Gamma(2/3)$.

[Note: $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ for $\Re z > 0$.]

8 By writing $w(z)$ in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation $zw'' + (c-z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_{\gamma} t^{a-1}(1-t)^{c-a-1}e^{tz}\,dt,$$

provided the pathe γ is chosen such that $[t^a(1-t)^{c-a}e^{tz}]_\gamma = 0$.

In the case $\mathrm{Re}\, z > 0$, find paths which provide two independent solutions in each of the following cases (where m is a positive integer):

(i) $a = -m$, $c = 0$;
(ii) $\mathrm{Re}\, a < 0$, $c = 0$, a is not an integer;
(iii) $a = 0$, $c = m$;
(iv) $\mathrm{Re}\, c >\mathrm{Re}\, a > 0$, a and $c - a$ are not integers.