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1   Let $\omega_{m,n} = m\omega_1 + n\omega_2$, where (m,n) are integers not both zero, and let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n}^{\infty} \frac{1}{(z - \omega_{m,n})^2} - \frac{1}{\omega_{m,n}{}^2}$$

be the Weierstrass elliptic function with periods (ω_1, ω_2) such that ω_1/ω_2 is not real. Show that, in a neighbourhood of $z = 0$,

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6)$$

where

$$g_2 = 60 \sum_{m,n} (\omega_{m,n})^{-4}, \quad g_3 = 140 \sum_{m,n} (\omega_{m,n})^{-6}.$$

Deduce that $\mathcal{P}$ satisfies a 1st order nonlinear ODE

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3.$$

2   By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be stated)

$$\int_0^\infty t^{z-1} e^{-it} \, dt = e^{-\frac{1}{2}\pi i z} \Gamma(z).$$

Hence evaluate (again, for ranges of z to be stated)

$$\int_0^\infty t^{z-1} \cos t \, dt \quad \text{and} \quad \int_0^\infty t^{z-1} \sin t \, dt.$$

Use your results to evaluate $\displaystyle\int_0^\infty \frac{\cos t}{t^{1/2}} \, dt$, $\displaystyle\int_0^\infty \frac{\sin t}{t} \, dt$ and $\displaystyle\int_0^\infty \frac{\sin t}{t^{3/2}} \, dt$.

3   Derive the formula $B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta$ and prove that

$$B(z,z) = 2^{1-2z} B(z, \tfrac{1}{2}).$$

For which values of z does this result hold?

4   Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \tfrac{1}{\sqrt{32\pi}} \left(\Gamma(\tfrac{1}{4})\right)^2$$

Using the change of variable $x = t(2-t^2)^{-\frac{1}{2}}$, deduce that

$$K(\tfrac{1}{\sqrt{2}}) = \tfrac{4}{\sqrt{\pi}} \left(\Gamma(\tfrac{5}{4})\right)^2 ,$$

where $K(k)$ is the complete elliptic integral $\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$.

5   By using the infinite product representation of the Gamma function (Weierstrass canonical product), prove that

$$\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

is a constant independent of z. Then, by letting $z \to 0$ evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z).$$

Furthermore, show that for $m = 1, 2, 3, \ldots$, the following identity is valid:

$$\Gamma(z)\Gamma(z+\frac{1}{m})\Gamma(z+\frac{2}{m})\ldots\Gamma(z+\frac{m-1}{m}) = m^{\frac{1}{2}-mz}(2\pi)^{\frac{m-1}{2}}\Gamma(mz).$$

6   Using $t = s\tau$, $s > 0$, it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}d\tau.$$

Letting $z = 1$ and integrating the resulting formula with respect to s from 1 to t, show that

$$\ln t = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}.$$

Using this formula in the expression for $\Gamma'(z)$, prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right)\frac{d\tau}{\tau}.$$

7   Starting with the infinite product representation of the Gamma function and using the definition of γ, derive Euler's product representation, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(1+z)(2+z)\ldots(n+z)}.$$

8   (a) Prove that for $\mathrm{Re}\,z > 1$,

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt = \frac{\Gamma(1-z)}{2i\pi} \int_\gamma \frac{t^{z-1}}{e^{-t} - 1}\, dt,$$

where γ denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers B_n are defined by

$$\frac{1}{e^t - 1} = \sum_{m=0}^\infty B_m \frac{t^{m-1}}{m!},$$

and $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_{2m+1} = 0$ for $m = 1, 2, \ldots$.

Use (a) and the residue theorem to compute $\zeta(-n)$, $n = 0, 1, 2, \ldots$ in terms of B_n. Hence, deduce that the negative even integers are zeros of $\zeta(z)$.

9   Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\, dt = -\gamma - \ln k + k - \frac{k^2}{4} + O(k^3), \quad k \to 0^+.$$

Hint:

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1} \right) \frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1} \right) \frac{dt}{t}.$$