Mathematical Tripos Part II - C Course — Lent term 2014

Further Complex Methods, Examples sheet 4 — Dr Maciej Dunajski

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1 If the functions $p(z)$ and $q(z)$ are analytic for $R \leq |z| < \infty$, for some fixed R, outline a proof using analytic continuation of the solutions to show that the equation $w'' + p(z)w' + q(z)w = 0$ has one solution of the form $z^\alpha \sum_{-\infty}^{\infty} c_n z^n$ for some constant α. Explain (with proof) what can be said about the other solution.

2 Show that the equation $w'' + \sin z \; w = 0$ has at least one solution of the form $e^{az}v(z)$, where $v(z)$ is periodic.

Hint: express $w_1(z + 2\pi)$ and $w_2(z + 2\pi)$ as linear combinations of $w_1(z)$ and $w_2(z)$.

3 Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at $z = a$ and $z = b$ can be written in the form

$$w'' + \left[\frac{1-A}{z-a} + \frac{1+A}{z-b}\right] w' + \frac{B(a-b)^2}{(z-a)^2(z-b)^2} w \; = \; 0\,, \qquad (\dagger)$$

where A and B are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and ∞, in the two cases $A^2 \neq 4B$ and $A^2 = 4B$. Use a Möbius transformation to deduce the general solution of ($\dagger$). What is the significance of the two constants α and α' which satisfy $\alpha + \alpha' = A$ and $\alpha\alpha' = B$?

4 By expanding $(1 - tz)^{-a}$, show that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(c-b)}{\Gamma(a)} \sum_0^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!},$$

where $(1 - tz)^{-a}$ takes its principal value, provided $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, and $|z| < 1$. You should explain the reason for these conditions.

State the regions of the complex z-plane in which (i) the integral defines an analytic function and (ii) the sum define an analytic function. Explain how the integral provides an analytic continuation in z of the function defined by the sum.

Given that the above integral is a multiple of $F(a, b; c; z)$, show that

$$F(a, b; c; 1) = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)}.$$

when a, b and c satisfy a condition that you should give.

5 What can be said about the nature of the solutions of a second order linear ordinary differential equation in the neighbourhood of a regular singular point?

Explain carefully why the hypergeometric equation with the usual parameters a, b, and c (so that the exponents at $z = 0$ are 0 and $1 - c$), has, for any given value of the parameter c, a solution $w(z)$ that satisfies $w(0) = 1$. Is $w(z)$ uniquely determined?

Is it the case that, for any given value of the parameter c, there is a solution $w(z)$ that is analytic at $z = 0$, and satisfies $w(0) = 1$? If such a solution exists, is it unique?

6 Is $(1-z)^{c-a-b}F(c-a, c-b; c; z)$ analytic at $z = 0$? The branch of $(1-z)^{c-a-b}$ is defined by $|\arg(1-z)| < \pi$ (which implies a branch cut from 1 to ∞ along the positive real axis).

Show, by considering transformations of P-functions, that

$$(1-z)^{c-a-b}F(c-a, c-b; c; z) = F(a, b; c; z).$$

Show also that

$$(1-z)^{-a}F(a, c-b; c; \frac{z}{z-1}) = F(a, b; c; z).$$

7 The hypergeometric function, $F(a, b; c; z)$ may be defined for $|z| < 1$ (and as usual $c \neq 0, -1, -2, \ldots$) by

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Let

$$g(z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t dt$$

where the contour runs to the left of all poles of $\Gamma(-t)$ and to the right of all poles of $\Gamma(a+t)$ and $\Gamma(b+t)$, $|\arg(-z)| < \pi$ and $a \neq 0, -1, -2, \ldots$, $b \neq 0, -1, -2 \ldots$.

(i) Give a sketch of the complex t-plane showing the positions of the singularities of the integrand and the curve.

By closing the contour with a large semi-circle in the right half complex plane (which may be assumed to make a negligible contribution to the integral), show that $g(z) = F(a, b; c; z)$.

(ii) By closing the contour instead with a large semi-circle in the left half complex plane (which may be assumed to make a negligible contribution to the integral), show that the analytic continuation to $|z| > 1$ of the series for $F(a, b; c; z)$ in the case when a and b do not differ by an integer is provided by

$$\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F\big(a, 1-c+a; 1-b+a; z^{-1}\big) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}F\big(b, 1-c+b; 1-a+b; z^{-1}\big)$$