INTEGRABLE SYSTEMS

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February 5, 2012

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Introduction

Integrable systems are nonlinear differential equations which 'in principle' can be solved analytically. This means that the solution can be reduced to a finite number of algebraic operations and integrations. Such systems are very rare - most nonlinear differential equations admit chaotic behaviour and no explicit solutions can be written down. Integrable systems nevertheless lead to a very interesting mathematics ranging from differential geometry and complex analysis to quantum field theory and fluid dynamics. The main reference for the course is [6]. There are other books which cover particular topics treated in the course:

- **Integrability of ODEs** [4] (Hamiltonian formalism, Arnold–Liouville theorem, action–angle variables). The integrability of ordinary differential equations is a fairly clear concept (i.e. it can be defined) based on existence of sufficiently many well behaved first integrals, or (as a physicist would put it) constant of motions.

- **Integrability of PDEs** [15], [5](Solitons, Inverse Scattering Transform). The universally accepted definition of integrability does not exist in this case. The phase space is infinite dimensional but having 'infinitely many' first integrals may not be enough - we could have missed every second one. Here one focuses on properties of solutions and solutions generation techniques. We shall study solitons - solitary non-linear waves which preserve their shape (and other characteristics) in the evolution. These soliton solutions will be constructed by means of an inverse problem: recovering a potential from the scattering data.

- **Lie symmetries** [9], [16] (Group invariant solutions, vector fields, symmetry reduction, Painlevé equations). The powerful symmetry methods can be applied to ODEs and PDEs alike. In case of ODEs a knowledge of sufficiently large symmetry group allows a construction of the most general solution. For PDEs the knowledge of symmetries is not sufficient to construct the most general solution, but it can be used to find new solutions from given ones and to reduce PDEs to more tractable ODEs. The PDEs integrable by inverse problems reduce to equations with Painlevé property.

Chapter 1

Integrability in classical mechanics

In this Chapter we shall introduce the integrability of ordinary differential equations. It is a fairly clear concept based on existence of sufficiently many well behaved first integrals.

1.1 Hamiltonian formalism

Motion of a system with n degrees of freedom is described by a trajectory in a $2n$ dimensional phase space M (locally think of an open set in $\mathbb{R}^{2n}$ but globally it can be topologically non-trivial manifold - e.g. a sphere or a torus. See Appendix A) with local coordinates

$$(p_j, q_j), \quad j = 1, 2, \ldots, n.$$

The dynamical variables are functions $f : M \times \mathbb{R} \longrightarrow \mathbb{R}$, so that $f = f(p, q, t)$ where t is called 'time'. Let $f, g : M \times \mathbb{R} \longrightarrow \mathbb{R}$. Define a Poisson bracket of f, g to be a function

$$\{f, g\} := \sum_{k=1}^{n} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}. \tag{1.1}$$

It satisfies

$$\{f, g\} = -\{g, f\}, \qquad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

The second property is called the Jacobi identity. The coordinate functions (p_j, q_j) satisfy the canonical commutation relations

$$\{p_j, p_k\} = 0, \quad \{q_j, q_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}.$$

Given a Hamiltonian $H = H(p, q, t)$ (usually $H(p, q)$) the dynamics is determined by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \qquad \text{for any} \qquad f = f(p, q, t).$$

Setting $f = p_j$ or $f = q_j$ yields Hamilton's equations of motion

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \qquad \dot{q}_j = \frac{\partial H}{\partial p_j}. \tag{1.2}$$

The system (1.2) of $2n$ ODEs is *deterministic* in a sense that $(p_j(t), q_j(t))$ are uniquely determined by $2n$ initial conditions $(p_j(0), q_j(0))$. Equations (1.2) also imply that volume elements in phase space are conserved. This system is essentially equivalent to Newton's equations of motion. The Hamiltonian formulation allows a more geometric insight to classical mechanics. It is also the starting point to quantisation.

Definition 1.1.1 *A function $f = f(p_j, q_j, t)$ which satisfies $\dot{f} = 0$ when equations* (1.2) *hold is called a first integral or a constant of motion. Equivalently,*

$$f(p(t), q(t), t) = const$$

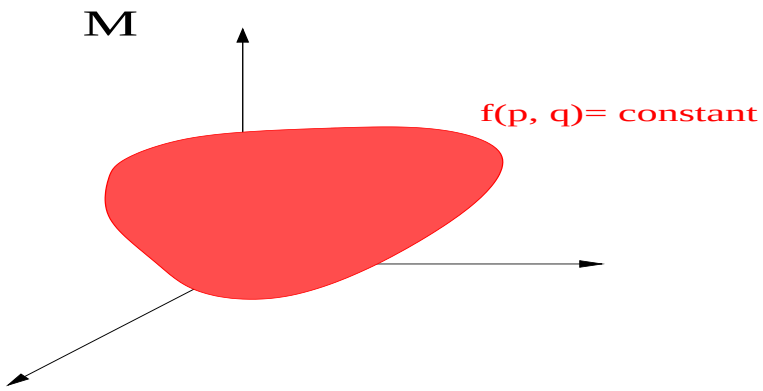
if $p(t), q(t)$ are solutions of (1.2).

In general the system (1.2) will be solvable if it admits 'sufficiently many' first integrals and the reduction of order can be applied. This is because any first integral eliminates one equation.

- **Example.** Consider a system with one degree of freedom with $M = \mathbb{R}^2$ and the Hamiltonian
$$H(p, q) = \frac{1}{2}p^2 + V(q).$$
Hamilton's equations (1.2) give
$$\dot{q} = p, \quad \dot{p} = -\frac{dV}{dq}.$$
The Hamiltonian itself is a first integral as $\{H, H\} = 0$. Thus
$$\frac{1}{2}p^2 + V(q) = E$$
where E is a constant called energy. Now
$$\dot{q} = p, \quad p = \pm\sqrt{2(E - V(q)}$$
and one integration gives a solution in the implicit form
$$t = \pm \int \frac{dq}{\sqrt{2(E - V(q))}}.$$
The explicit solution could be found if we can perform the integral on the RHS and invert the relation $t = t(q)$ to find $q(t)$. These two steps are not always possible to take but nevertheless we would certainly regard this system as integrable.

It is useful to adopt a more geometric approach. Assume that a first integral f does not explicitly depend on time, and that it defines a hypersurface $f(p, q) = \text{const}$ in M (Figure.1.1). Two hypersurfaces corresponding to two independent first integrals generically intersect in a surface of co–dimension 2 in M. In general the trajectory lies on a surface of dimension $2n - L$ where L is the number of independent first integrals. If $L = 2n - 1$ this surface is a curve - a solution to (1.2).

Figure 1.1: Level surface



How to find first integrals? Given two first integrals which do not explicitly depend on time their Poisson bracket will also be a first integral if it is not zero. This follows from the Jacobi identity and the fact all first integrals Poisson commute with the Hamiltonian. More generally, the Noether theorem gives some first integrals (they correspond to symmetries Hamilton's equation (1.2) may possess e.g. time translation, rotations) but not enough. The difficulty with finding the first integrals has deep significance. For assume we use some existence theorem for ODEs and apply it to (1.2). Now solve the algebraic equations

$$q_k = q_k(p^0, q^0, t), \quad p_k = p_k(p^0, q^0, t),$$

for the initial conditions (p^0, q^0) thus giving

$$q^0{}_k = q^0{}_k(p, q, t), \quad p^0{}_k = p^0{}_k(p, q, t).$$

This gives $2n$ first integrals as obviously (p^0, q^0) are constants which we can freely specify. One of these integrals determines the time parametrisations and others could perhaps be used to construct the trajectory in the phase space. However for some of the integrals the equations

$$f(p, q) = \text{const}$$

may not define a 'nice' surface in the phase space. Instead it defines a pathological (at least from the applied mathematics point of view) set which densely covers the phase space. Such integrals do not separate points in M.

One first integral - energy - always exist for Hamiltonian systems giving the energy surface $H(p, q) = E$, but often it is the only first integral. Sufficiently complicated, deterministic, systems may behave according to the laws of thermodynamics: probability that the system is contained in some element of the energy surface is proportional to the normalised volume of this element. This means that the time evolution covers uniformly the entire region of the constant energy surface in the phase space. It is not known wether this ergodic postulate can be derived from Hamilton's equations.

Early computer simulations in the 1960s revealed that some nonlinear systems (with infinitely many degrees of freedom!) are not ergodic. Soliton equations

$$u_t = 6uu_x - u_{xxx}, \quad u = u(x,t), \qquad KdV$$

or

$$\phi_{xx} - \phi_{tt} = \sin\phi, \quad \phi = \phi(x,t), \qquad Sine-Gordon$$

are examples of such systems. Both posses infinitely many first integrals. We shall study them in Chapter 2.

1.2 Integrability and action–angle variables

Given a system of Hamilton's equations (1.2) it is often sufficient to know n (rather than $2n-1$) first integrals as each of them reduces the order of the system by two. This underlies the following definition of an *integrable system.*

Definition 1.2.1 *An integrable system consists of a $2n$-dimensional phase-space M together with n independent functions (in a sense that the gradients ∇f_j are linearly independent vectors on a tangent space to any point in M) $f_1, \ldots, f_n : M \to \mathbb{R}$ such that*

$$\{f_j, f_k\} = 0, \qquad j,k = 1, \ldots, n. \tag{1.3}$$

The vanishing of Poisson brackets (1.3) means that the first integrals are in involution. We shall show that integrable systems lead to completely solvable Hamilton's equations of motion. Let us first explore the freedom in (1.2) given by a coordinate transformation of a phase-space

$$Q_k = Q_k(p,q), \qquad P_k = P_k(p,q).$$

This transformation is called *canonical* if it preserves the Poisson bracket

$$\sum_{k=1}^{n} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} = \sum_{k=1}^{n} \frac{\partial f}{\partial Q_k}\frac{\partial g}{\partial P_k} - \frac{\partial f}{\partial P_k}\frac{\partial g}{\partial Q_k}$$

for all $f, g : M \longrightarrow \mathbb{R}$. Canonical transformations preserve Hamilton's equation (1.2).

Given a function $S(q,P,t)$ such that

$$\det\left(\frac{\partial^2 S}{\partial q_j \partial P_k}\right) \neq 0$$

we can construct a canonical transformation by setting

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}, \quad \widetilde{H} = H + \frac{\partial S}{\partial t}.$$

The function S is an example of a generating function [4, 11, 19]. The idea behind the following Theorem is to seek a canonical transformation such that in the new variables $H = H(P_1, \ldots, P_n)$ so that

$$P_k(t) = P_k(0) = \text{const}, \qquad Q_k(t) = Q_k(0) + t\frac{\partial H}{\partial P_k}.$$

Finding a generating function for such canonical transformation is in practise very difficult, and deciding whether a given Hamiltonian system is integrable (without a priori knowledge of n Poisson commuting integrals) is still an open problem.

Theorem 1.2.2 (Arnold, Liouville) *Let*

$$(M, f_1, \ldots, f_n)$$

be an integrable system with a Hamiltonian $H = f_1$, and let

$$M_f := \{(p,q) \in M; f_k(p,q) = c_k\}, \qquad c_k = const, \qquad k = 1, \ldots, n$$

be an n-dimensional level surface of first integrals f_k. Then

- *If M_f is compact and connected then it is diffeomorphic to a torus*

$$T^n := S^1 \times S^1 \times \ldots \times S^1,$$

and (in a neighbourhood of this torus in M) one can introduce the 'action-angle' coordinates

$$I_1, \ldots, I_n, \phi_1, \ldots, \phi_n, \qquad 0 \leq \phi_k \leq 2\pi,$$

such that angles ϕ_k *are coordinates on M_f and* actions $I_k = I_k(f_1, \ldots, f_n)$ *are first integrals.*

- *The canonical equations of motion* (1.2) *become*

$$\dot{I}_k = 0, \qquad \dot{\phi}_k = \omega_k(I_1, \ldots, I_n), \qquad k = 1, \ldots, n \tag{1.4}$$

and so the integrable systems are solvable by quadratures (a finite number of algebraic operations, and integrations of known functions).

Proof. We shall follow the proof given in [4], but try to make it more accessible by avoiding the language of differential forms

- The motion takes place on the surface

$$f_1(p,q) = c_1, f_2(p,q) = c_2, \ldots, f_n(p,q) = c_n$$

of dimension $2n - n = n$. The first part of the Theorem says that this surface is a torus[1]. For each point in M there exists precisely one torus T^n passing through that point. This means that M admits a foliation by n–dimensional leaves. Each leaf is a torus and different tori correspond to different choices of the constants $c_1, \ldots, c_n$.

Assume

$$\det\Big(\frac{\partial f_j}{\partial p_k}\Big) \neq 0$$

so that the system $f_k(p,q) = c_k$ can be solved for the momenta p_i

$$p_i = p_i(q, c)$$

[1]This part of the proof requires some knowledge of Lie groups and Lie algebras. It is given in Appendix A.

and the relations $f_i(q, p(q, c)) = c_i$ hold identically. Differentiate these identities with respect to q_j

$$\frac{\partial f_i}{\partial q_j} + \sum_k \frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0$$

and multiply the resulting equations by $\partial f_m/\partial p_j$

$$\sum_j \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial q_j} + \sum_{j,k} \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0.$$

Now swap the indices and subtract $(mi) - (im)$. This yields

$$\{f_i, f_m\} + \sum_{j,k} \Big(\frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} - \frac{\partial f_i}{\partial p_j}\frac{\partial f_m}{\partial p_k}\frac{\partial p_k}{\partial q_j}\Big) = 0.$$

The first term vanishes as the first integrals are in involution. Rearranging the indices in the second term gives

$$\sum_{j,k} \frac{\partial f_i}{\partial p_k}\frac{\partial f_m}{\partial p_j}\Big(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k}\Big) = 0$$

and, as the matrices $\partial f_i/\partial p_k$ are invertible,

$$\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} = 0. \tag{1.5}$$

This condition implies that

$$\oint \sum_j p_j dq_j = 0$$

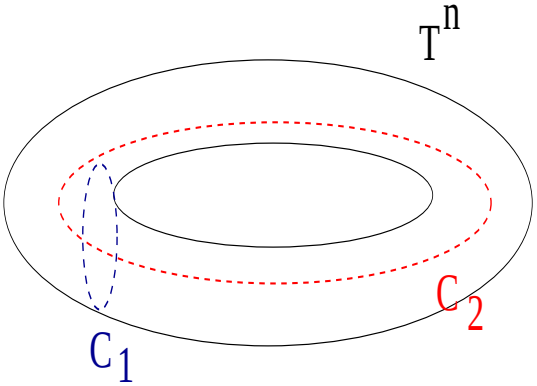
for any closed contractible curve on the torus T^n. This is a consequence of the Stokes theorem. To see it recall that in $n = 3$

$$\oint_{\delta D} \mathbf{p} \cdot d\mathbf{q} = \int_D (\nabla \times \mathbf{p}) \cdot d\mathbf{q}$$

where δD is a boundary of a surface D and

$$(\nabla \times \mathbf{p})_m = \frac{1}{2}\epsilon_{jkm}\Big(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k}\Big).$$

- There are n closed curves which can not be contracted down to a point, so that the corresponding integrals do not vanish.



Cycles on a Torus

Therefore we can define the action coordinates

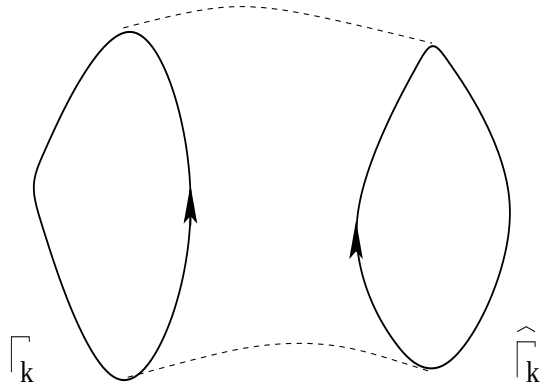
$$I_k := \frac{1}{2\pi} \oint_{\Gamma_k} \sum_j p_j dq_j, \tag{1.6}$$

where the closed curve Γ_k is the k-th basic cycle (the term 'cycle' in general means 'submanifold without boundary') of the torus T^n

$$\Gamma_k = \{(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) \in T^n; 0 \leq \tilde{\phi}_k \leq 2\pi, \tilde{\phi}_j = \text{const} \ \text{ for } j \neq k\},$$

where $\tilde{\phi}$ are some coordinates[2] on T^n.

The Stokes theorem implies that the actions (1.6) are independent on the choice of Γ_k.



Stokes Theorem

This is because

$$\oint_{\Gamma_k} \sum_j p_j dq_j + \oint_{\hat{\Gamma}_k} \sum_j p_j dq_j = \int \Big(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\Big) dq_j \wedge dq_i = 0$$

where we have chosen Γ and $\hat{\Gamma}$ to have opposite orientations.

- The actions (1.6) are also first integrals as $\oint p(q,c)dq$ only depends on $c_k = f_k$ and f_ks are first integrals. The actions are Poisson commuting

$$\{I_i, I_j\} = \sum_{r,s,k} \frac{\partial I_i}{\partial f_r}\frac{\partial f_r}{\partial q_k}\frac{\partial I_j}{\partial f_s}\frac{\partial f_s}{\partial p_k} - \frac{\partial I_i}{\partial f_r}\frac{\partial f_r}{\partial p_k}\frac{\partial I_j}{\partial f_s}\frac{\partial f_s}{\partial q_k} = \sum_{r,s} \frac{\partial I_i}{\partial f_r}\frac{\partial I_j}{\partial f_s}\{f_r, f_s\} = 0$$

and in particular $\{I_k, H\} = 0$.

The torus M_f can be equivalently represented by

$$I_1 = \tilde{c}_1, \qquad \ldots, \qquad I_1 = \tilde{c}_n.$$

for some constants $\tilde{c}_1, \ldots, \tilde{c}_n$ (We might have been tempted just to define $I_k = f_k$ but then the transformation $(p,q) \to (I,\phi)$ would not be canonical in general.)

[2]This is a non-trivial step. In practice it is unclear how to explicitly describe the n–dimensional torus and the curves Γ_k in $2n$ dimensional phase space. Thus, to some extend the Arnold–Liouville theorem has a character of the existence theorem.

- We shall construct the angle coordinates ϕ_k canonically conjugate to the actions using a generating function

$$S(q, I) = \int_{q_0}^{q} \sum_j p_j dq_j,$$

where q_0 is some chosen point on the torus. This definition does not depend on a path joining q_0 and q as a consequence of (1.5) and Stokes's theorem. Choosing a different q_0 just adds a constant to S thus leaving the *angles*

$$\phi_i = \frac{\partial S}{\partial I_i}$$

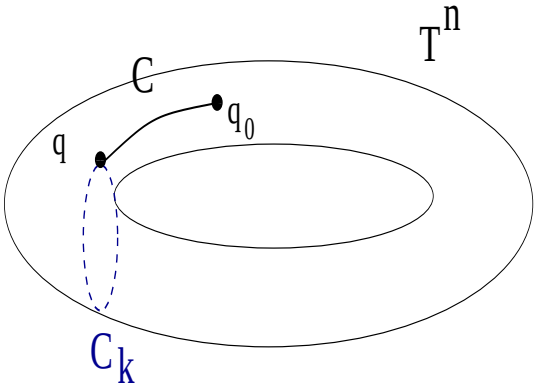
invariant.

- The angles are periodic coordinates with a period 2π. To see it consider two paths C and $C \cup C_k$ (where C_k represents the kth cycle) between q_0 and q and calculate

$$S(q, I) = \int_{C \cup C_k} \sum_j p_j dq_j = \int_C \sum_j p_j dq_j + \int_{C_k} \sum_j p_j dq_j = S(q, I) + 2\pi I_k$$

so

$$\phi_k = \frac{\partial S}{\partial I_k} = \phi_k + 2\pi.$$



Generating Function

- The transformations

$$q = q(\phi, I), \quad p = p(\phi, I), \qquad \text{and} \qquad \phi = \phi(q, p), \quad I = I(q, p)$$

are canonical (as they are defined by a generating function) and invertible. Thus

$$\{I_j, I_k\} = 0, \quad \{\phi_j, \phi_k\} = 0, \quad \{\phi_j, I_k\} = \delta_{jk}$$

and the dynamics is given by

$$\dot{\phi}_k = \{\phi_k, \widetilde{H}\}, \quad \dot{I}_k = \{I_k, \widetilde{H}\},$$

where

$$\widetilde{H}(\phi, I) = H(q(\phi, I), p(\phi, I)).$$

The I_ks are first integrals, therefore

$$\dot{I}_k = -\frac{\partial \widetilde{H}}{\partial \phi_k} = 0$$

so $\widetilde{H} = \widetilde{H}(I)$ and

$$\dot{\phi}_k = \frac{\partial \widetilde{H}}{\partial I_k} = \omega_k(I)$$

where the ω_ks are also first integrals. This proves (1.4). Integrating these canonical equations of motion yields

$$\phi_k(t) = \omega_k(I)t + \phi_k(0), \quad I_k(t) = I_k(0). \tag{1.7}$$

These are n circular motions with constant angular velocities.

□

The trajectory (1.7) may be closed on the torus or it may cover it densely. That depends on the values of the angular velocities. If $n = 2$ the trajectory will be closed if ω_1/ω_2 is rational and dense otherwise.

Interesting things happen to the tori under a small perturbation of the integrable Hamiltonian

$$H(I) \longrightarrow H(I) + \epsilon K(I, \phi).$$

In some circumstances the motion is still periodic and most tori do not vanish but become deformed. This is governed by the Kolmogorov–Arnold–Moser (KAM) theorem - not covered in this course. Consult the popular book by Schuster [18], or read the complete account given by Arnold [4].

- **Example.** All time-independent Hamiltonian system with two-dimensional phase spaces are integrable. Consider the harmonic oscillator with the Hamiltonian

  $$H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2).$$

  Different choices of the energy E give a foliation of M_f by ellipses

  $$\frac{1}{2}(p^2 + \omega^2 q^2) = E.$$

  For a fixed value of E we can take $\Gamma = M_f$. Therefore

  $$I = \frac{1}{2\pi} \oint_{M_f} pdq = \frac{1}{2\pi} \int \int_S dpdq = \frac{E}{\omega}$$

  where we used the Stokes's theorem to express the line integral in terms of the area enclosed by M_f.

The Hamiltonian expressed in the new variables is $\widetilde{H} = \omega I$ and

$$\dot{\phi} = \frac{\partial \widetilde{H}}{\partial I} = \omega, \qquad \phi = \omega t + \phi_0.$$

To complete the picture we need to express (I, ϕ) in terms of (p, q). We already know

$$I = \frac{1}{2}\Big(\frac{1}{\omega}p^2 + \omega q^2\Big).$$

Thus the generating function is

$$S(q, I) = \int p dq = \pm \int \sqrt{2I\omega - \omega^2 q^2} dq$$

and (choosing a sign)

$$\phi = \frac{\partial S}{\partial I} = \int \frac{\omega dq}{\sqrt{2I\omega - \omega^2 q^2}} = \arcsin\Big(q\sqrt{\frac{\omega}{2I}}\Big) - \phi_0.$$

This gives

$$q = \sqrt{\frac{2I}{\omega}} \sin(\phi + \phi_0)$$

and finally we recover the familiar solution

$$p = \sqrt{2E}\cos(\omega t + \phi_0), \qquad q = \sqrt{2E/\omega^2}\sin(\omega t + \phi_0).$$

1.3 Poisson structures

There is a natural way to extend the Hamiltonian formalism by generalising the notion of Poisson bracket (1.1). A geometric approach is given by symplectic geometry [4]. We shall take a lower level (but a slightly more general) point of view and introduce the Poisson structures. The phase space M is m dimensional with local coordinates $(\xi^1, \ldots, \xi^m)$. In particular we do not distinguish between the positions and momenta.

Definition 1.3.1 *A skew–symmetric matrix $\omega^{ab} = \omega^{ab}(\xi)$ is called a Poisson structure if the Poisson bracket defined by*

$$\{f, g\} = \sum_{a,b=1}^{m} \omega^{ab}(\xi) \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b} \tag{1.8}$$

satisfies

$$\{f, g\} = -\{g, f\},$$
$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

The second property is called the Jacobi identity. It puts restrictions on $\omega^{ab}(\xi)$ which can be seen noting that

$$\omega^{ab}(\xi) = \{\xi^a, \xi^b\}$$

and evaluating the Jacobi identity on coordinate functions.

Given a Hamiltonian $H : M \times \mathbb{R} \longrightarrow \mathbb{R}$ the dynamics is governed by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

and the Hamilton's equations generalising (1.2) become

$$\dot{\xi}^a = \sum_{b=1}^{m} \omega^{ab}(\xi) \frac{\partial H}{\partial \xi^b}. \tag{1.9}$$

- **Example**. Let $M = \mathbb{R}^3$ and $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc}\xi^c$, where ε^{abc} is the standard totally anti-symmetric tensor. Thus

  $$\{\xi^1, \xi^2\} = \xi^3, \quad \{\xi^3, \xi^1\} = \xi^2, \quad \{\xi^2, \xi^3\} = \xi^1.$$

  This Poisson structure admits a Casimir - any function $f(r)$ where

  $$r = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}$$

  Poisson commutes with the coordinate functions

  $$\{f(r), \xi^a\} = 0.$$

  This is independent on the choice of the Hamiltonian. With a choice

  $$H = \frac{1}{2}\Big(\frac{(\xi^1)^2}{a_1} + \frac{(\xi^2)^2}{a_2} + \frac{(\xi^3)^2}{a_3}\Big)$$

  where a_1, a_2, a_3 are constants, the Hamilton's equations (1.9) become the equations of motion of a rigid body fixed at its centre of gravity

  $$\dot{\xi}^1 = \frac{a_3 - a_2}{a_2 a_3}\xi^2\xi^3, \qquad \dot{\xi}^2 = \frac{a_1 - a_3}{a_1 a_3}\xi^1\xi^3, \qquad \dot{\xi}^3 = \frac{a_2 - a_1}{a_1 a_2}\xi^1\xi^2.$$

Assume that $m = 2n$ is even and the matrix ω is invertible with $W_{ab} := (\omega^{-1})_{ab}$. The Jacobi identity implies that the antisymmetric matrix $W_{ab}(\xi)$ is closed, i.e.

$$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0, \qquad \forall a, b, c = 1, \ldots, m.$$

In this case W_{ab} is called a symplectic structure. The Darboux theorem states that in this case there locally exists a coordinate system

$$\xi^1 = q_1, \cdots, \xi^n = q_n, \xi^{n+1} = p_1, \cdots, \xi^{2n} = p_n$$

such that

$$\omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

and the Poisson bracket reduces to the standard form (1.1). A simple proof can be found in [4]. One constructs a local coordinate system (p, q) by induction w.r.t half of the dimension of M. Choose a function p_1, and find q_1 by solving the equation $\{q_1, p_1\} = 1$. Then consider a level set of (p_1, q_1) in M which is locally a symplectic manifold. Now look for (p_2, q_2) etc.

- **Example.** The Poisson structure in the last example is degenerate as the matrix ω^{ab} is not invertible. This degeneracy always occurs if the phase space is odd dimensional or/and there exists a non-tivial Casimir. Consider the restriction of $\omega^{ab} = \sum_{c=1}^{3} \varepsilon^{abc}\xi^c$ to a two-dimensional sphere $r = C$. This gives a symplectic structure on the sphere given by

$$\{\xi^1, \xi^2\} = \sqrt{C^2 - (\xi^1)^2 - (\xi^2)^2}$$

or

$$W = \frac{1}{\sqrt{C^2 - (\xi^1)^2 - (\xi^2)^2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This of course has no Casimir functions apart from constants. It is convenient to choose a different parametrisation of the sphere: if

$$\xi^1 = C\sin\theta\cos\phi, \quad \xi^2 = C\sin\theta\sin\phi, \quad \xi^3 = C\cos\theta$$

then in the local coordinates (θ, ϕ) the symplectic structure is given by $\{\theta, \phi\} = \sin^{-1}\theta$ or

$$W = \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is equal to the volume form on the two–sphere. The radius C is arbitrary. Therefore the Poisson phase space $\mathbb{R}^3$ is foliated by symplectic phase spaces S^2 as there is exactly one sphere centred at the origin through any point of $\mathbb{R}^3$. This is a general phenomenon: fixing the values of the Casimir functions on Poisson spaces gives the foliations by symplectic spaces. The local Darboux coordinates on S^2 are given by $q = -\cos\theta, p = \phi$ as then

$$\{q, p\} = 1.$$

The Poisson generalisation is useful to set up the Hamiltonian formalism in the infinite–dimensional case. Formally one can think of replacing the coordinates on the trajectory $\xi^a(t)$ by a dynamical variable $u(x,t)$. Thus the discrete index a becomes the continuous independent variable x (think of m points on a string versus the whole string). The phase space $M = \mathbb{R}^m$ is replaced by a space of smooth functions on a line with appropriate boundary conditions (decay or periodic). The whole formalism may be set up making the following replacements

$$\begin{aligned} \text{ODEs} &\longrightarrow \text{PDEs} \\ \xi^a(t), a = 1, \ldots, m &\longrightarrow u(x,t), x \in \mathbb{R} \\ \sum_a &\longrightarrow \int_{\mathbb{R}} dx \\ \text{function } f(\xi) &\longrightarrow \text{functional } F[u] \\ \frac{\partial}{\partial \xi^a} &\longrightarrow \frac{\delta}{\delta u}. \end{aligned}$$

The functionals are given by integrals

$$F[u] = \int_{\mathbb{R}} f(u, u_x, u_{xx}, \ldots) dx$$

(we could in principle allow the t derivatives but we will not for the reasons to become clear shortly). Recall that the functional derivative is

$$\frac{\delta F}{\delta u(x)} = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial (u_x)} + \Big(\frac{\partial}{\partial x}\Big)^2 \frac{\partial f}{\partial (u_{xx})} + \dots$$

and

$$\frac{\delta u(y)}{\delta u(x)} = \delta(y-x)$$

where the δ on the RHS is the Dirac delta which satisfies

$$\int_{\mathbb{R}} \delta(x)dx = 1, \qquad \delta(x) = 0 \text{ for } x \neq 0.$$

The presence of the Dirac delta will constantly remind us that we have entered a territory which is rather slippery from a pure mathematics perspective. We should rest reassured that the formal replacements made above can nevertheless be given a solid functional-analytic foundation. This will not be done in this course.

The analogy with finite dimensional situation (1.8) suggests a following definition of a Poisson bracket

$$\{F, G\} = \int_{\mathbb{R}^2} \omega(x, y, u) \frac{\delta F}{\delta u(x)} \frac{\delta G}{\delta u(y)} dxdy$$

where the Poisson structure $\omega(x, y, u)$ should be such that the bracket is anti–symmetric and the Jacobi identity holds. A canonical (but not the only) choice is

$$\omega(x, y, u) = \frac{1}{2}\frac{\partial}{\partial x}\delta(x-y) - \frac{1}{2}\frac{\partial}{\partial y}\delta(x-y).$$

This is analogous to the Darboux form in which ω^{ab} is a constant and antisymmetric matrix and the Poisson bracket reduces to (1.1). This is because the differentiation operator $\partial/\partial x$ is anti–self–adjoint with respect to an inner product

$$< u, v >= \int_{\mathbb{R}} u(x)v(x)dx$$

which is analogous to a matrix being anti–symmetric. With this choice

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} dx \tag{1.10}$$

and the Hamilton's equations become

$$\begin{aligned} \frac{\partial u}{\partial t} &= \{u, H[u]\} = \int_{\mathbb{R}} \frac{\delta u(x)}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta H}{\delta u(y)} dy \\ &= \frac{\partial}{\partial x} \frac{\delta H[u]}{\delta u(x)}. \end{aligned} \tag{1.11}$$

- **Example.** The KdV equation mentioned earlier is a Hamiltonian system with the Hamiltonian given by the functional

  $$H[u] = \int_{\mathbb{R}} \Big(\frac{1}{2}u_x^2 + u^3\Big)dx.$$

  It is assumed that u belongs to the space of functions decaying sufficiently fast at when $x \to \pm\infty$.

Chapter 2

Soliton equations and Inverse Scattering Transform

The universally accepted definition of integrability does not exist for partial differential equations. The phase space is infinite dimensional but having 'infinitely many' first integrals may not be enough - we could have missed every second one. One instead focuses on properties of solutions and solutions generation techniques. We shall study solitons - solitary non-linear waves which preserve their shape (and other characteristics) in the evolution. These soliton solutions will be constructed by means of an inverse problem: recovering a potential from the scattering data.

2.1 History of two examples

Soliton equations originate in the 19th century. Some of them appeared in the study of non-linear wave phenomena and other arose in differential geometry of surfaces in $\mathbb{R}^3$

- The KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \qquad \text{where} \quad u = u(x,t) \tag{2.1}$$

has been written down, and solved in the simplest case, by Korteweg and de-Vires in 1895 to explain the following account of J. Scott Russell. Russell observed a soliton while ridding on horseback beside a narrow barge channel. The following passage has been taken from J . Scott Russell. Report on waves, Fourteenth meeting of the British Association for the Advancement of Science, 1844. *'I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it*

in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation'.

- The Sine–Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin\phi \qquad \text{where} \quad \phi = \phi(x,t) \tag{2.2}$$

locally describes the isometric embeddings of surfaces with constant negative Gaussian curvature in the Euclidean space $\mathbb{R}^3$. The function $\phi = \phi(x,t)$ is the angle between two asymptotic directions $\tau = (x+t)/2$ and $\rho = (x-t)/2$ on the surface along which the second fundamental form is zero. If the first fundamental form of a surface parametrised by (ρ, τ) is

$$ds^2 = d\tau^2 + 2\cos\phi\, d\rho d\tau + d\rho^2, \qquad \text{where} \quad \phi = \phi(\tau, \rho)$$

then the Gaussian curvature is constant and equal to -1 provided that

$$\phi_{\tau\rho} = \sin\phi.$$

which is (2.2).

The integrability of the Sine–Gordon equation have been used by Bianchi, Bäcklund, Lie and other classical differential geometers to construct new embeddings.

2.1.1 Physical derivation of KdV

Consider the linear wave equation

$$\Psi_{xx} - \frac{1}{v^2}\Psi_{tt} = 0$$

where $\Psi_{xx} = \partial_x^2\Psi$ etc. which describes a propagation of waves travelling with a constant velocity v. Its derivation is based on three simplifying assumptions:

- There is no dissipation i.e. the equation is invariant with respect to time inversion $t \to -t$.
- The amplitude of oscillation is small and so the nonlinear terms (like Ψ^2) can be omitted.
- There is no dispersion, i.e. the group velocity is constant.

In the derivation of the KdV we follow [15] and relax these assumptions.

The general solution of the wave equation is a superposition of two waves travelling in opposite directions

$$\Psi = f(x - vt) + g(x + vt)$$

where f and g are arbitrary functions of one variable. Each of these two waves is characterised by a linear 1st order PDE, e.g.

$$\Psi_x + \frac{1}{v}\Psi_t = 0 \qquad \longrightarrow \qquad \Psi = f(x - vt).$$

To introduce the dispersion consider a complex wave

$$\Psi = e^{i(kx - \omega(k)t)}$$

where $\omega(k) = vk$ and so the group velocity $d\omega/dk$ equals to the phase velocity v. We change this relation by introducing the dispersion

$$\omega(k) = v(k - \beta k^3 + \ldots)$$

where the absence of even terms in this expansion guarantees real dispersion relations. Let us assume that the dispersion is small and truncate this series keeping only the first two terms. The equation satisfied by

$$\Psi = e^{i(kx - v(kt - \beta k^3 t))}$$

is readily found to be

$$\Psi_x + \beta \Psi_{xxx} + \frac{1}{v}\Psi_t = 0.$$

This can be rewritten in a form of a conservation law

$$\rho_t + j_x = 0,$$

where the density ρ and the current j are given by

$$\rho = \frac{1}{v}\Psi, \qquad j = \Psi + \beta \Psi_{xx}.$$

To introduce nonlinearity modify the current

$$j = \Psi + \beta \Psi_{xx} + \frac{\alpha}{2}\Psi^2.$$

The resulting equation is

$$\frac{1}{v}\Psi_t + \Psi_x + \beta \Psi_{xxx} + \alpha \Psi \Psi_x = 0.$$

The non–zero constants (v, β, α) can be eliminated by a simple change of variables $x \to x - vt$ and rescaling Ψ. This leads to the standard form of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

The simplest 1–soliton solution found by Korteweg and de-Vires is

$$u(x,t) = -\frac{2\chi^2}{\cosh^2 \chi(x - 4\chi^2 t - \phi_0)}. \tag{2.3}$$

The KdV is not a linear equation therefore multiplying this solution by a constant will not give another solution. The constant ϕ_0 determines the location of the extremum at $t = 0$. We should therefore think of a one–parameter family of solutions labelled by $\chi \in \mathbb{R}$.

The one–soliton (2.3) was the only regular solution of KdV such that $u, u_x \to 0$ as $|x| \to \infty$ known until 1965 when Gardner, Green, Kruskal and Miura analysed KdV numerically. They took two waves with different amplitudes as their initial profile. The computer simulations revealed that the initially separated waves approached each-other distorting their shapes, but eventually the larger wave overtook the smaller wave and both waves re-emerged with their sizes and shapes intact. The relative phase shift was the only result of the non–linear interaction. This behaviour resembles what we usually associate with particles and not waves.

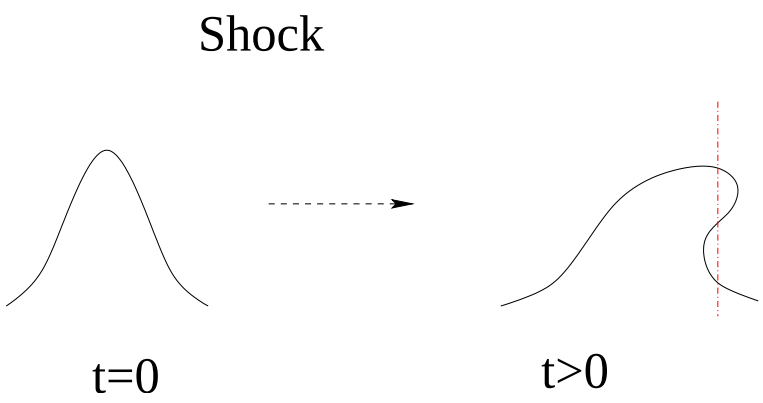
Thus Zabruski and Kruskal named these waves 'solitons' (like electrons, protons, barions and other particles ending with 'ons'). In this Chapter we shall construct more general N–soliton solutions describing the interactions of 1–solitons.

To this end we note that the existence of a stable solitary wave is a consequence of cancellations of effects caused by non–linearity and dispersion.

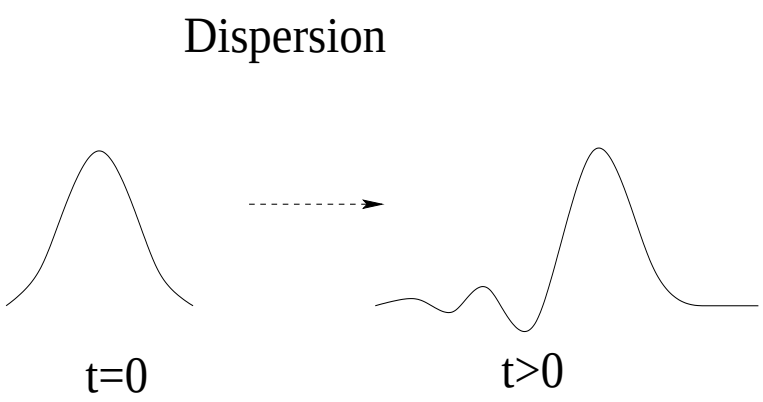
- If the dispersive term were not present the equation would be

$$u_t - 6uu_x = 0$$

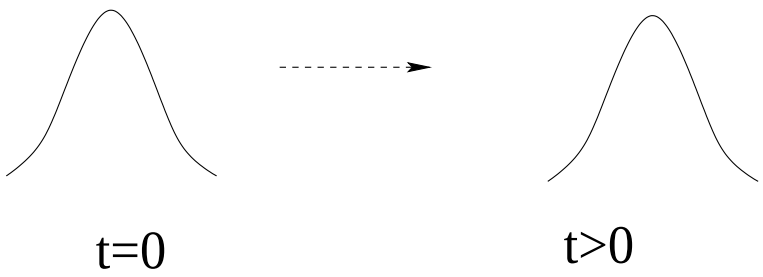
and the resulting solution would exhibit a discontinuity of first derivatives at some $t_0 > 0$ (shock, or 'breaking the wave'). This solution can be easily found using the method of characteristics.



- If the nonlinear term were not present the initial wave profile would disperse in the evolution $u_t + u_{xxx} = 0$.



- The presence of both terms allows smooth localised soliton solutions



of which (2.3) is an example (the plot gives $-u(x,t)$).

2.1.2 Bäcklund transformations for the Sine–Gordon equation

Let us consider the Sine–Gordon equation - the other soliton equation mentioned in the introduction to this Chapter. The simplest solution generating technique is the Bäcklund transformation. Set $\tau = (x+t)/2, \rho = (x-t)/2$ so that the equation (2.2) becomes

$$\phi_{\tau\rho} = \sin\phi.$$

Now define the Bäcklund relations

$$\partial_\rho(\phi_1 - \phi_0) = 2b\sin\Big(\frac{\phi_1+\phi_0}{2}\Big), \qquad \partial_\tau(\phi_1+\phi_0) = 2b^{-1}\sin\Big(\frac{\phi_1-\phi_0}{2}\Big), \qquad b = \text{const.}$$

Differentiating the first equation w.r.t τ, and using the second equation yields

$$\begin{aligned}\partial_\tau\partial_\rho(\phi_1-\phi_0) &= 2b\,\partial_\tau\sin\Big(\frac{\phi_1+\phi_0}{2}\Big) = 2\sin\Big(\frac{\phi_1-\phi_0}{2}\Big)\cos\Big(\frac{\phi_1+\phi_0}{2}\Big)\\ &= \sin\phi_1 - \sin\phi_0.\end{aligned}$$

Therefore ϕ_1 is a solution to the Sine–Gordon equation if ϕ_0 is. Given ϕ_0 we can solve the first order Bäcklund relations for ϕ_1 and generate new solutions form the ones we know. The trivial solution $\phi_0 = 0$ yields 1–soliton solution of Sine–Gordon

$$\phi_1(x,t) = 4\arctan\Big(\exp\Big(\frac{x-vt}{\sqrt{1-v^2}} - x_0\Big)\Big)$$

where v is a constant with $|v|<1$. This solution is called a kink (Figure 2.1). A static kink corresponds to a special case $v=0$.

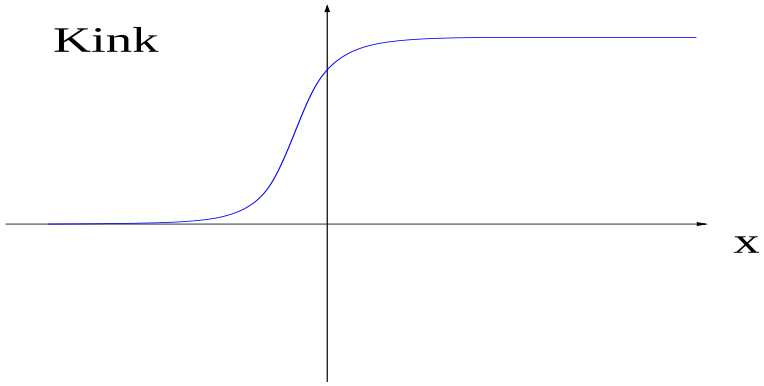
One can associate a topological charge

$$N = \frac{1}{2\pi}\int_{\mathbb{R}} d\phi = \frac{1}{2\pi}\Big(\phi(x=\infty,t) - \phi(x=-\infty,t)\Big)$$

with any solution of the Sine Gordon equation. It is an integral of a total derivative which depends only on boundary conditions. It is conserved if one insists on finiteness of the energy

$$E = \int_{\mathbb{R}}\Big(\frac{1}{2}\Big(\phi_t^2+\phi_x^2\Big) + (1-\cos(\phi))\Big)dx.$$

Figure 2.1: Sine–Gordon Kink



Note that the Sine–Gordon equations didn't enter the discussion at this stage. Topological charges, like N, are in this sense different form first integrals like E which satisfy $\dot{E} = 0$ as a consequence of (2.2). For the given kink solution $N(\phi) = 1$ and the kink is stable as it would take infinite energy to change this solution into a constant solution $\phi = 0$ with $E = 0$.

There exist interesting solutions with $N = 0$: a soliton–antisoliton pair has $N = 0$ but is non–trivial

$$\phi(x,t) = 4\arctan\Big(\frac{v\cosh\frac{x}{\sqrt{1-v^2}}}{\sinh\frac{vt}{\sqrt{1-v^2}}}\Big).$$

At $t \to -\infty$ this solution represents widely separated pair of kink and anti–kink approaching each-other with velocity v. A non–linear interaction takes place at $t = 0$ and as $t \to \infty$ kink and anti-kink reemerge unchanged.

2.2 Inverse scattering transform for KdV

One of the most spectacular methods of solving soliton equations comes from quantum mechanics. It is quite remarkable, as the soliton equations we have discussed so far have little to do with the quantum world.

Recall that the mathematical arena of quantum mechanics is the infinite–dimensional complex vector space $\mathcal{H}$ of functions [17]. Elements Ψ of this space are referred to as wave functions, or state vectors. In case of one–dimensional quantum mechanics we have $\Psi : \mathbb{R} \to \mathbb{C}$, $\Psi = \Psi(x) \in \mathbb{C}$. The space $\mathcal{H}$ is equipped with a unitary inner product

$$(\Psi, \Phi) = \int_{\mathbb{R}} \overline{\Psi(x)}\Phi(x)dx. \tag{2.4}$$

The functions which are square integrable, i.e. $(\Psi, \Psi) < \infty$ like $\Psi = e^{-x^2}$, are called bound states. Other functions, like e^{-ix}, are called the scattering states.

Given a real valued function $u = u(x)$ called the potential, the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + u\Psi = E\Psi$$

determines the x–dependence of a wave function. Here $\hbar$ and m are constants which we shall not worry about and E is the energy of the quantum system. The energy levels can be discrete for bound states or continuous for scattering states. This depends on the potential $u(x)$. We shall regard the Schrödinger equation as an eigen–value problem and refer to Ψ and E as eigenvector and eigenvalue respectively.

According to the Copenhagen interpretation of quantum mechanics the probability density for the position of a quantum particle is given by $|\Psi|^2$, where Ψ is a solution to the the Schrödinger equation. The time evolution of the wave function is governed by a time dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + u\Psi.$$

This equation implies that for bound states the quantum–mechanical probability is conserved in a sense that

$$\frac{d}{dt}\int_{\mathbb{R}} |\Psi|^2 dx = 0.$$

The way physicists discover new elementary particles is by scattering experiments. Huge accelerators collide particles through targets and, by analysing the changes to momenta of scattered particles, a picture of a target is built[1]. Given a potential $u(x)$ one can use the Schrödinger equation to find Ψ, the associated energy levels and the scattering data in the form of so called reflection and transmission coefficients. Experimental needs are however different: the scattering data is measured in the accelerator but the potential (which gives the internal structure of the target) needs to be recovered. This comes down to the following mathematical problem

- Recover the potential from the scattering data.

This problem was solved in the 1950s by the Gelfand, Levitan and Marchenko [8, 14] who gave a linear algorithm for reconstructing $u(x)$. Gardner, Green, Kruskal and Miura [7] used this algorithm to solve the Cauchy problem for the KdV equation. Their remarkable idea was to regard the initial data in the solution of KdV as a potential in the Schrödinger equation.

Set $\hbar^2/(2m) = 1$ and write the 1-dimensional Schrödinger equation as an eigenvalue problem

$$\Big(-\frac{d^2}{dx^2} + u(x)\Big)\Psi = E\Psi.$$

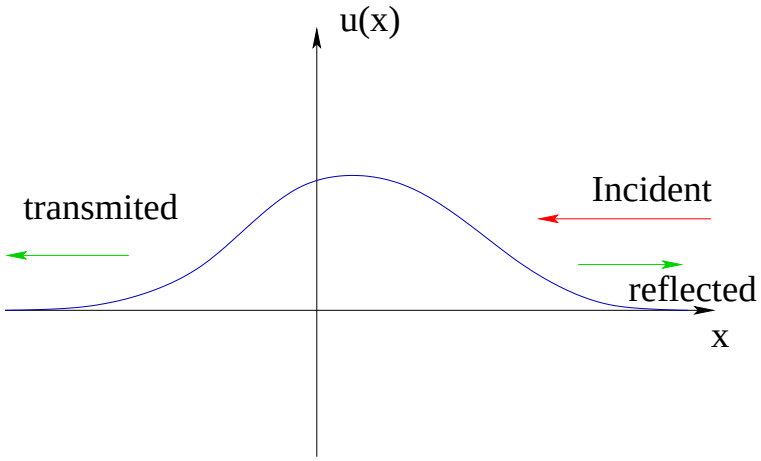
We allow u to depend on x as well as t which at this stage should be regarded as a parameter.

In the scattering theory one considers the beam of free particles incident from $+\infty$. Some of the particles will be reflected by the potential (which is assumed to decay sufficiently fast as $|x| \to \infty$) and some will be transmitted. There may also be a number of bound states with discrete energy levels. The Gelfand–Levitan–Marchenko theory shows that given

- energy levels E,
- transmission probability T,

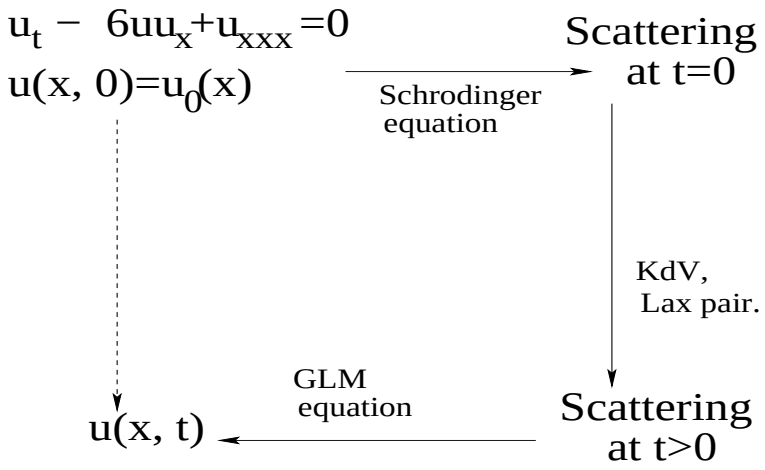
[1]These kind of experiments will take place in the Large Hadron Collider LHC opened in September 2008 at CERN. The LHC is located in a 27km long tunnel under the Swiss/French border outside Geneva. It is hoped that the elusive Higgs particle and a whole bunch of other exotic form of matter will be discovered.

Figure 2.2: Reflection and Transmission



- reflection probability R,

one can find the potential u: Given $u_0(x)$ one finds the scattering data at $t = 0$. If $u(x,t)$ is a solution to the KdV equation (2.1) with $u(x,0) = u_0(x)$ then the scattering data $(E(t), T(t), R(t))$ satisfies simple linear ODEs determining their time evolution. In particular E does not depend on t. Once this has been determined, $u(x,t)$ is recovered by solving a linear integral equation. The Gardner, Green, Kruskal and Miura scheme for solving KdV is summarised in the following table



We should stress that in this method the time evolution of the scattering data is governed by the KdV and not by the time dependent Schrödinger equation. In fact the time dependent Schrödinger equation will not play any role in the following discussion.

Appendix A

Manifolds

This course is intended to give an elementary introduction to the subject and the student is expected only to be familiar with basic real and complex analysis, algebra and dynamics as covered in the undergraduate syllabus. In particular no knowledge of differential geometry is assumed. One obvious advantage of this approach is that the course is suitable for advanced undergraduate students.

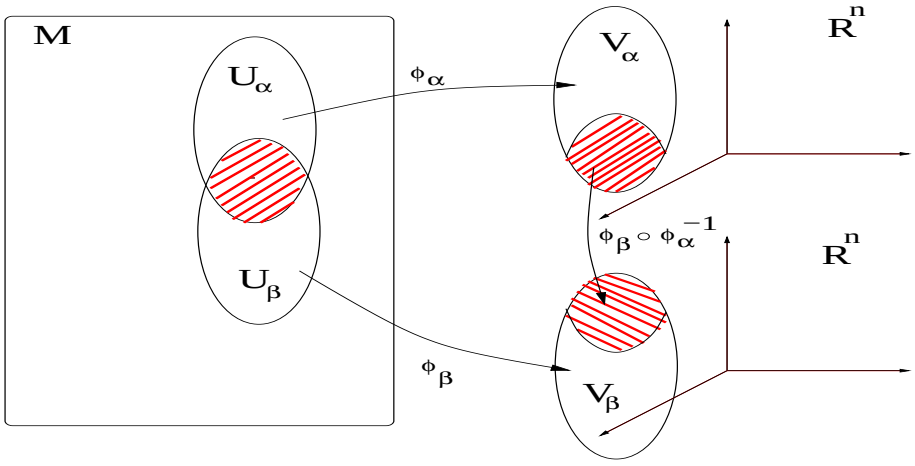
The disadvantage is that the discussion of Hamiltonian formalism and continuous groups of transformations in earlier chapters used phrases like 'spaces coordinatised by (p, q)', 'open sets in $\mathbb{R}^n$' or 'groups whose elements smoothly depend on parameters' instead calling these object by their real name - manifolds. This Appendix is intended to fill this gap.

Definition A.0.1 *An n–dimensional smooth manifold is a set M together with a collection of open sets U_α called the coordinate charts such that*

- *The open sets U_α labeled by a countable index α cover M.*
- *There exist one-to-one maps $\phi_\alpha : U_\alpha \to V_\alpha$ onto open sets in $\mathbb{R}^n$ such that for any pair of overlapping coordinate charts the maps*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth (i.e. infinitely differentiable) functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.



Thus a manifold is a topological space together with additional structure which makes the local differential calculus possible. The space $\mathbb{R}^n$ itself is of course a manifold which can be covered by one coordinate chart.

- **Example.** A less trivial example is the unit sphere

$$S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}.$$

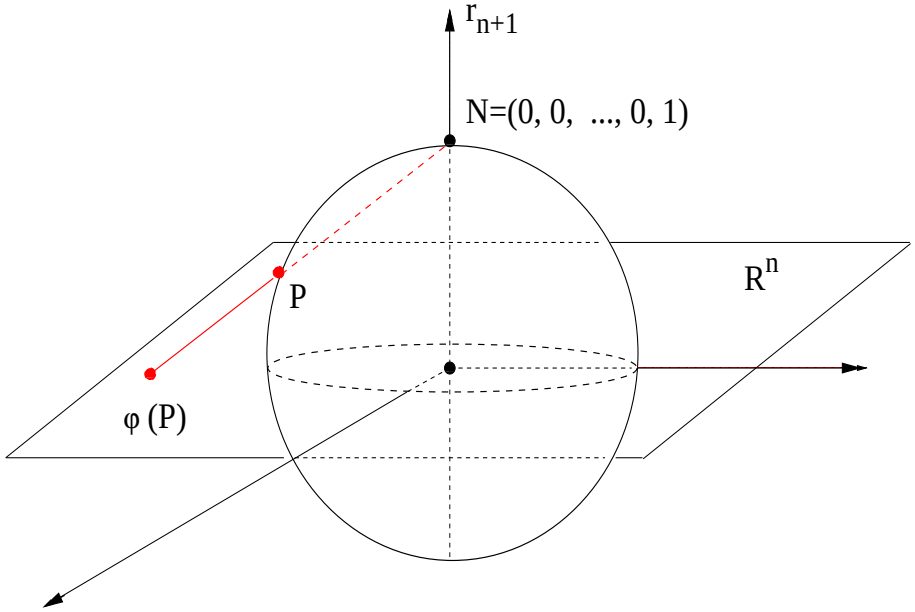
To verify that it is indeed a manifold, cover S^n by two open sets $U_1 = U$ and $U_2 = \widetilde{U}$

$$U = S^n/\{0, \ldots, 0, 1\}, \quad \widetilde{U} = S^n/\{0, \ldots, 0, -1\}$$

and define the local coordinates by stereographic projections

$$\phi(r_1, r_2, \ldots, r_{n+1}) = \Big(\frac{r_1}{1 - r_{n+1}}, \ldots, \frac{r_n}{1 - r_{n+1}}\Big) = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

$$\tilde{\phi}(r_1, r_2, \ldots, r_{n+1}) = \Big(\frac{r_1}{1 + r_{n+1}}, \ldots, \frac{r_n}{1 + r_{n+1}}\Big) = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$



Using

$$\frac{r_k}{1 + r_{n+1}} = \Big(\frac{1 - r_{n+1}}{1 + r_{n+1}}\Big)\frac{r_k}{1 - r_{n+1}}, \qquad k = 1, \ldots, n$$

where $r_{n+1} \neq \pm 1$ shows that on the overlap $U \cap \widetilde{U}$ the transition functions

$$\phi \circ \tilde{\phi}^{-1}(x_1, \ldots, x_n) = \Big(\frac{x_1}{x_1^2 + \ldots + x_n^2}, \ldots, \frac{x_n}{x_1^2 + \ldots + x_n^2}\Big),$$

are smooth.

A cartesian product of manifolds is also a manifold. For example the n–torus arising in the Arnold–Liouville theorem 1.2.2 is a cartesian product on n one–dimensional spheres.

Another way to obtain interesting manifolds is to define them as surfaces in $\mathbb{R}^n$ by a vanishing condition for a set of functions. If $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ then the set

$$M_f := (x \in \mathbb{R}^n, \quad f_i(x) = 0, \quad i = 1, \ldots, k) \tag{A1}$$

is a manifold if the rank of the k by n matrix of gradients ∇f_i is constant in a neighborhood of M_f in $\mathbb{R}^n$. If this rank is maximal and equal to k then $\dim M_f = n - k$. The manifold axioms can be verified using the implicit function theorem. For example the sphere S^{n-1} arises this way with $k = 1$ and $f_1 = 1 - |\mathbf{x}|^{\mathbf{2}}$. There is a theorem which says that every manifold arises as some surface in $\mathbb{R}^n$ for sufficiently large n. If the manifold is m dimensional then n is at most $2m + 1$. This useful theorem is now nearly forgotten - differential geometers like to think of manifolds as abstract objects defined by a collections of charts as in Definition A.0.1.

Lie groups

We can now give a proper definition of a Lie group

Definition A.0.2 *A Lie group G is a group and, at the same time, a smooth manifold such that the group operations*

$$G \times G \to G, \quad (g_1, g_2) \to g_1 g_2, \qquad \textit{and} \qquad G \to G, \quad g \to g^{-1}$$

are smooth maps between manifolds.

- **Example.** The general linear group $G = GL(n, \mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$ defined by the condition $\det g \neq 0, g \in G$. It is therefore a Lie group of dimension n^2. The special orthogonal group $SO(n)$ is defined by (A1), where the $n(n+1)/2$ conditions in $\mathbb{R}^{n^2}$ are

  $$gg^T - \mathbf{1} = 0, \quad \det g = 1.$$

  The determinant condition just selects a connected component in the set of orthogonal matrices, so it does not count as a separate condition. It can be shown that the corresponding matrix of gradients has constant rank and thus $SO(n)$ is an $n(n-1)/2$ dimensional Lie group.

In Chapter **??** a Lie algebra $\mathfrak{g}$ was defined as a vector space with an anti–symmetric bilinear operation which satisfies the Jacobi identity (**??**).

A Lie algebra of a Lie group G is the tangent space to G at the identity element, $\mathfrak{g} = T_e G$ with the Lie bracket defined by a commutator of vector fields at e.

Proof of the first part of Arnold–Liouville's theorem 1.2.2.

The gradients ∇f_k are independent, thus the set

$$M_f := \{(p, q) \in M; f_k(p, q) = c_k\}$$

where $c_1, c_2, \ldots, c_n$ are constant defines a manifold of dimension n. Let $\xi^a = (p, q)$ be local coordinates on M such that the Poisson bracket is

$$\{f, g\} = \omega^{ab} \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b}, \qquad a, b = 1, 2, \ldots, 2n$$

where ω is a constant anti–symmetric matrix

$$\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

The vanishing of Poisson brackets $\{f_j, f_k\} = 0$ implies that each Hamiltonian vector field

$$X_{f_k} = \omega^{ab} \frac{\partial f_k}{\partial \xi^b} \frac{\partial}{\partial \xi^a}$$

is orthogonal (in the Euclidean sense) to any of the gradients $\partial_a f_j, a = 1, \ldots, 2n, j, k = 1, \ldots, n$. The gradients are perpendicular to M_f, thus the Hamiltonian vector fields are tangent to M_f. They are also commuting as

$$[X_{f_j}, X_{f_k}] = -X_{\{f_j, f_k\}} = 0,$$

so the vectors generate an action of the abelian group $\mathbb{R}^n$ on M. This action restricts to an $\mathbb{R}^n$ action on M_f. Let $p_0 \in M_f$, and let Γ be a lattice consisting of all vectors in $\mathbb{R}^n$ which fix p_0 under the group action. Then Γ is a finite subgroup of $\mathbb{R}^n$ and (by an intuitively clear modification of the orbit–stabiliser theorem) we have

$$M_f = \mathbb{R}^n / \Gamma.$$

Assuming that M_f is compact, this quotient space is diffeomorphic to a torus T^n.

□

In fact this argument shows that we get a torus for any choice of the constants c_k. Thus, varying the constants, we find that the phase–space M is foliated by n–dimensional tori.

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