Part II Integrable Systems, Revision.

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Revise the lecture notes, and try last year's exam questions (do not worry about questions from the previous years as the course has changed). The following three questions could form a basis of an exam. Attempt these questions without notes.

1. State the Arnold–Liouville theorem.

   Let $M = \mathbb{R}^2$ be a phase–space with canonical coordinates (p, q) such that $\{q, p\} = 1$. By considering $\{\phi, I\}$ or otherwise, prove that the transformation $(p, q) \to (I, \phi)$ given by

$$p = \sqrt{2I}\cos\phi, \quad q = \sqrt{2I}\sin\phi \tag{1}$$

   is canonical, and use it to solve Hamilton's equations with the Hamiltonian given by $H = (p^2 + q^2)/2$.

2. Put the modified KdV equation (mKdV)

$$v_t - 6v^2 v_x + v_{xxx} = 0, \qquad v = v(x, t) \tag{2}$$

   in the Hamiltonian form, and verify explicitly that the Hamiltonian functional you have found does not depend on t if v is a solution to (2) and v together with its x derivatives tend to zero as $x \to \pm\infty$.

   Assume that a solution of mKdV is of the form $v(x, t) = f(\xi)$ where $\xi = x - ct$ and find the ODE satisfied by f. Express the solution to this ODE in the form

$$\xi = \int F(f) df$$

   for some F which should be found.

3. Consider a one–parameter group of transformations of $\mathbb{R}^n \times \mathbb{R}$

$$(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{u}) = (c^{\alpha_1} x^1, \ldots, c^{\alpha_n} x^n, c^{\alpha} u), \tag{3}$$

   where $c \neq 0$ is the parameter of the transformation and $(\alpha, \alpha_1, \ldots, \alpha_n)$ are fixed constants, and find the vector field generating this group.

Find all Lie point symmetries of the PDE

$$u_t = uu_x$$

of the form (3), where $(x^1, x^2) = (x, t)$. Guess two more Lie point symmetries of this PDE not of the form (3) and calculate the Lie brackets of the corresponding vector fields.