Part II Integrable Systems, Sheet One

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1. **Jacobi identity.** Assume that (p_j, q_j) satisfy the Hamilton equations and show that any function $f = f(p, q, t)$ satisfies

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\},$$

where H is the Hamiltonian.

Show that the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0 \qquad (1)$$

holds for Poisson brackets.

Deduce that if functions f_1 and f_2 which do not explicitly depend on time are first integrals of a Hamiltonian system then so is $f_3 = \{f_1, f_2\}$.

2. **Canonical transformations.**

- Find the canonical transformation generated by

$$S = \sum_{k=1}^{n} q_k P_k.$$

- Show that the canonical transformations preserve volume in the two–dimensional phase space, i.e.

$$\frac{\partial(P, Q)}{\partial(p, q)} = 1.$$

[This result also holds in phase spaces of arbitrary dimension.]

- Show that the transformation

$$Q = \cos(\beta)q - \sin(\beta)p, \quad P = \sin(\beta)q + \cos(\beta)p$$

is canonical for any constant $\beta \in \mathbb{R}$. Find the corresponding generating function. Is it defined for all β?

3. **Action variables for the Kepler problem.** Consider the four–dimensional phase space coordinatised by

$$q_1 = \phi, \quad q_2 = r, \quad p_1 = p_\phi, \quad p_2 = p_r$$

equipped with a Hamiltonian

$$H = \frac{{p_\phi}^2}{2r^2} + \frac{{p_r}^2}{2} - \frac{\alpha}{r}$$

where $\alpha > 0$ is a constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that this system is integrable in a sense of the Arnold–Liouville theorem.

Construct the action variables. Express the Hamiltonian in terms of the action variables to show that the frequencies associated to the corresponding angles are equal.

[Hint: ϕ and one function of (r, p_r) parametrise M_f. Varying ϕ and fixing the other coordinate gives one cycle $\Gamma_\phi \subset M_f$. To find the second action coordinate fix ϕ (on top of H and p_ϕ).]

4. **Poisson Structures.** A Poisson on structure on $\mathbb{R}^{2n}$ is an anti–symmetric matrix ω^{ab} whose components depend on the coordinates $\xi^a \in \mathbb{R}^{2n}, a = 1, \cdots, 2n$ and such that the Poisson bracket

$$\{f, g\} = \sum_{a,b=1}^{2n} \omega^{ab}(\xi) \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b}$$

satisfies the Jacobi identity (1).

Show that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

Assume that the matrix ω is invertible with $W := (\omega^{-1})$ and show that the antisymmetric matrix $W_{ab}(\xi)$ satisfies

$$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0. \qquad (2)$$

[Hint: note that $\omega^{ab} = \{\xi^a, \xi^b\}$.] Deduce that if $n = 1$ then any anti-symmetric invertible matrix $\omega(\xi^1, \xi^2)$ gives rise to a Poisson structure (i.e. show that the Jacobi identity holds automatically in this case).

[In differential geometry the invertible antisymmetric matrix W which satisfies (2) is called a symplectic structure. We have therefore deduced that symplectic structures are special cases of Poisson structures.]

5. **KdV.** Verify that the equation

$$\frac{1}{v}\Psi_t + \Psi_x + \beta\Psi_{xxx} + \alpha\Psi\Psi_x = 0.$$

where $\Psi = \Psi(x,t)$ and (v, β, α) are non–zero constants is equivalent to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \qquad u = u(x,t) \tag{3}$$

after a suitable change of dependent and independent variables.

6. **1–soliton solution.** Assume that a solution of the KdV equation (3) is of the form

$$u(x,t) = f(\xi), \qquad \text{where} \quad \xi = x - ct$$

for some constant c. Show that the function $f(\xi)$ satisfies the ODE

$$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}cf^2 + \alpha f + \beta$$

where (α, β) are arbitrary constants. Assume that f and its first two derivatives tend to zero as $|\xi| \to \infty$ and solve the ODE to construct the one–soliton solution to the KdV equation.

7. **Sine–Gordon soliton from Backlund transformations.**

The Sine–Gordon equation is

$$\phi_{xx} - \phi_{tt} = \sin(\phi), \qquad \phi = \phi(x,t).$$

Set $\tau = (x+t)/2, \rho = (x-t)/2$ and consider the Bäcklund transformations

$$\partial_\rho(\phi_1 - \phi_0) = 2b\sin\left(\frac{\phi_1 + \phi_0}{2}\right), \qquad \partial_\tau(\phi_1 + \phi_0) = 2b^{-1}\sin\left(\frac{\phi_1 - \phi_0}{2}\right),$$

where $b = \text{const}$ and ϕ_0, ϕ_1 are functions of (τ, ρ). Take $\phi_0 = 0$ and construct the 1-soliton (kink) solution ϕ_1.

Draw the graph of $\phi_1(x,t)$ for a fixed value of t. What happens when t varies?

8. **Miura transformation**. Let $v = v(x,t)$ satisfy the modified KdV equation
$$v_t - 6v^2 v_x + v_{xxx} = 0.$$
Show that the function $u(x,t)$ given by
$$u = v^2 + v_x \tag{4}$$
satisfy the KdV equation. Is it true that any solution u to the KdV equation gives rise, via (4), to a solution of the modified KdV equation?

Books. The course follows the first four chapters of
Dunajski, M. (2009) *Solitons, Instantons and Twistors*, OUP.
Other interesting books are

- Hamiltonian Systems.

  Arnold, V. I. *Mathematical Methods of Classical Mechanics.* (This uses a language of differential forms but has the best possible exposition of the Arnold–Liouville theorem. Chapter 10 is most relevant).

  Schuster, H. G. *Deterministic Chaos: An Introduction.* (A popular introduction to KAM theorem and ergodicity with some mention of integrable systems).

- Solitons and Inverse Scattering.

  Novikov S., Manakov S. V., Pitaevskii L. P., Zaharov V. E., *Theory of Solitons.* (The lectures follow Chapter 1 of this book in the treatment of the KdV equation and solitons).

  Drazin, P. G., Johnson, R.S. *Solitons: an introduction.* (A very readable text. Chapters 3, 4, 5 are most relevant).

- Lie symmetries, Painleve equations.

  Hydon P. E. *Symmetry Methods for Differential Equations: A Beginner's Guide.* (Elementary and very easy to follow)

  Olver, P. J. *Applications of Lie groups to differential equations.*

  Fokas, A.S. et. al. *Painleve transcendents. The Riemann-Hilbert approach.*