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Topology and energy of time-dependent unitons

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We consider a class of time-dependent finite energy multi-soliton solutions of the $U(N)$ integrable chiral model in $(2+1)$ dimensions. The corresponding extended solutions of the associated linear problem have a pole with arbitrary multiplicity in the complex plane of the spectral parameter. Restrictions of these extended solutions to any space-like plane in $\mathbb{R}^{2,1}$ have trivial monodromy and give rise to maps from a three-sphere to $U(N)$. We demonstrate that the total energy of each multi-soliton is quantized at the classical level and given by the third homotopy class of the extended solution. This is the first example of a topological mechanism explaining the classical energy quantization of moving solitons.

Keywords: solitons; integrable chiral model; homotopy

1. Introduction

The fact that the allowed energy levels of some physical systems can take only discrete values has been well known since the early days of quantum theory. The hydrogen atom and the harmonic oscillator are the two well-known examples. In these two cases, the boundary conditions imposed on the wave function imply discrete spectra of the Hamiltonians. The reasons are therefore global.

The quantization of energy can also occur at the classical level in nonlinear field theories if the energy of a smooth field configuration is finite. The reasons are again global, but one needs more subtle ideas from topology to understand what is going on. The potential energy of static soliton solutions in the Bogomolny limit of certain field theories must be proportional to integer homotopy classes of smooth maps. The details depend on the model. In pure gauge theories, the energy of solitons satisfying the Bogomolny equations is given by one of the Chern numbers of the curvature. In scalar $(2+1)$-dimensional sigma models, allowed energies of Bogomolny solitons are given by the elements of $\pi_2(\Sigma)$, where the manifold Σ is the target space. In both the cases, the boundary condition is used to show that the finite energy configurations extend to the compactification of space. See Manton & Sutcliffe (2004) for a detailed exposition of these constructions.

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The situation is different for moving solitons. The total energy is the sum of kinetic and potential terms, and the Bogomolny bound is not saturated. One expects that the moving (non-periodic) solitons will have continuous energy. Attempts to construct theories with quantized total energy based on compactifying the time direction are physically unacceptable, as they lead to paradoxes related to the existence of closed time-like curves. A soliton moving along such curve could eventually reach its own past, thus opening possibilities to sinister scenarios, usually involving a death of somebody's great grandparents.

In a recent publication, Ioannidou & Manton (2005) made the surprising observation that the total energy of the time-dependent $SU(2)$ 2-uniton solution of Ward's (2+1)-dimensional chiral model (Ward 1988*a*, 1995) is quantized in the units of 8π when the pole of the corresponding extended solution is at $\pm\mathrm{i}$. They have shown that the 2-uniton energy density calculated at any instant of time t is the same as the energy density of a static $\mathbb{CP}^3$ multi-lump with a parameter t. The total (potential) energy of the latter model is quantized (Zakrzewski 1989), which leads to the total (kinetic+potential) energy quantization of the time-dependent unitons. The quantization was also obtained by Lechtenfeld & Popov (2001*a*,*b*), whose method was based on a large-time asymptotic analysis.

One expects that there are deeper topological reasons for this quantization, and the purpose of this paper is to show that this is indeed the case.

The Ward chiral model is

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0, \tag{1.1}$$

where $J : \mathbb{R}^3 \to U(N)$ and $x^\mu = (t, x, y)$ are the coordinates on $\mathbb{R}^3$, such that the line element is $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$. Here, we use the notation $J_\mu := \partial_\mu J$. The equations are not fully Lorentz invariant, as the commutator term fixes a space-like direction. A positive-definite, conserved energy functional for (1.1) is

$$E = \int_{\mathbb{R}^2} \mathcal{E}\, \mathrm{d}x\, \mathrm{d}y, \tag{1.2}$$

where the energy density is given by

$$\mathcal{E} = -\frac{1}{2}\mathrm{Tr}((J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2). \tag{1.3}$$

The integrability of (1.1) allows a construction of explicit static and also time-dependent solutions by twistor or inverse scattering methods (Ward 1988*a*, 1990). There are time-dependent solutions with non-scattering solitons (Ward 1988*a*) and also solitons that scatter (Ward 1995). A class of scattering solutions to (1.1) is given by the so-called time-dependent unitons

$$J(x, y, t) = M_1 M_2 \dots M_n, \tag{1.4}$$

where the unitary matrices M_k, $k=1, \dots, n$ are given by

$$M_k = \mathrm{i}\left(\mathbf{1} - \left(1 - \frac{\mu}{\bar{\mu}}\right)R_k\right), \qquad R_k \equiv \frac{q_k^* \otimes q_k}{\|q_k\|^2}. \tag{1.5}$$

Here, $\mu \in \mathbb{C}\backslash\mathbb{R}$ is a non-real constant and $q_k = (1, f_{k1}, \ldots, f_{k(N-1)}) \in \mathbb{C}^N$, with $k=1, \ldots, n$, are the vectors whose components $f_{kj} = f_{kj}(x^\mu) \in \mathbb{C}$ are smooth functions that tend to a constant at spatial infinity.[1]

If $n=1$, then q_1 is holomorphic and rational in $\omega = x + (1/2)\mu(t+y) + (1/2)\mu^{-1}(t-y)$ (Ward 1988*a*). Note that if $\mu = \pm\mathrm{i}$, then q_1 does not depend on t, and the corresponding 1-uniton is static. If $n>1$, q_1 is still holomorphic and rational in ω, but $q_2, q_3, \ldots$ are not holomorphic. The exact form of these functions is known explicitly for $n=2$, 3 (Ward 1995; Ioannidou 1996), for the case $N=2$. For $n>3$, the Bäcklund transformations (Ioannidou & Zakrzewski 1998; Dai & Terng 2004) can be used to determine the fs recursively. The total energy (1.2) of n-uniton solutions is finite.

In general, the finiteness of E is ensured by imposing the boundary condition (valid for all t)

$$J = J_0 + J_1(\varphi)r^{-1} + O(r^{-2}) \quad \text{as} \quad r \to \infty, \quad x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\varphi}, \tag{1.6}$$

and hence for a fixed value of t, the matrix J extends to a map from S^2 (conformal compactification of $\mathbb{R}^2$) to $U(N)$. The homotopy group $\pi_2(U(N))=0$; hence, there is no topological information in J defined on $\mathbb{R}\times S^2$, which could be related to the total energy. We shall nevertheless show that the energy of (1.4) is quantized and given by the third homotopy class of the extended solution to (1.1). The existence of this extended solution is linked to the complete integrability of (1.1) and the associated Lax equations with the spectral parameter. The extended solution also depends on this parameter, and hence is defined on $\mathbb{R}^3 \times \mathbb{CP}^1$. Restricting it to a space-like plane in $\mathbb{R}^3$ and an equator in a Riemann sphere of the spectral parameter gives a map ψ, whose domain is $\mathbb{R}^2 \times S^1$. If J is an n-uniton solution (1.4), the corresponding extended solution satisfies stronger boundary conditions, which promote ψ to a map $S^3 \to U(N)$. In §2, we shall introduce the extended solution, impose boundary conditions on J, which are stronger than (1.6), and, in fact, provide a coordinate-free characterization of the uniton solutions (1.4). In §3, we shall establish the following result.

Theorem 1.1. *The total energy of the n-uniton solution* (1.4) *with complex number* $\mu = m\mathrm{e}^{\mathrm{i}\phi}$ *is quantized and equal to*

$$E_{(n)} = 4\pi\left(\frac{1+m^2}{m}\right)|\sin(\phi)|[\psi], \tag{1.7}$$

where for any fixed value of t, the map $\psi: S^3 \to U(N)$ *is given by*

$$\psi = \prod_{k=n}^{1}\left(\mathbf{1} + \frac{\bar{\mu}-\mu}{\mu + \cot\left(\frac{\theta}{2}\right)}R_k\right), \quad \theta \in [0, 2\pi], \tag{1.8}$$

[1] The matrix R_k is a Hermitian projection, satisfying $(R_k)^2 = R_k$, and the corresponding M_k is a Grassmanian embedding of $\mathbb{CP}^{N-1}$ into $U(N)$. The results in this paper apply to the more general class of unitons obtained from the complex Grassmanian embeddings of $Gr(K,N)$ into the unitary group. For μ pure imaginary, a complex K-plane $V \subset \mathbb{C}^N$ corresponds to a unitary transformation $\mathrm{i}(\pi_V - \pi_{V\perp})$, where π_V denotes the Hermitian orthogonal projection onto V. Formula (1.5) with $\mu = \mathrm{i}$ corresponds to $K=1$, where $Gr(1,N) = \mathbb{CP}^{N-1}$.

and

$$[\psi] = \frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}((\psi^{-1}\mathrm{d}\psi)^3), \tag{1.9}$$

is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$.

The model (1.1) is $SO(1, 1)$ invariant, and in §3 it will be shown that the Lorentz boosts correspond to rescaling μ by a real number. The rest frame corresponds to $|\mu|=1$, when the y-component of the momentum vanishes. The $SO(1, 1)$ invariant generalization of equation (1.7) will be given by theorem 3.1. Energies of soliton solutions more general than (1.4) are briefly discussed in §4.

2. Extended solution and its homotopy

(*a*) *Lax pair and trivial scattering*

The proof of theorem (1.1) relies on the integrability of equation (1.1) and its Lax formulation, which we set-up in the following. Let $A = A_\mu \mathrm{d}x^\mu$ and Φ be a one-form and a function on $\mathbb{R}^{2,1}$, with values in a Lie algebra of $U(N)$ determined up to gauge transformations

$$A \to bAb^{-1} - \mathrm{d}b\, b^{-1}, \qquad \Phi \to b\Phi b^{-1}, \qquad b = b(x^\mu) \in U(N).$$

The system of first-order equations

$$D\Phi = *F,$$

where $D\Phi = \mathrm{d}\Phi + [A, \Phi]$ and $F = \mathrm{d}A + A \wedge A$ give the integrability conditions $[L_0, L_1] = 0$ for an overdetermined system of linear equations

$$\begin{aligned} L_0\Psi &:= (D_y + D_t - \lambda(D_x + \Phi))\Psi = 0, \\ L_1\Psi &:= (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0, \end{aligned} \tag{2.1}$$

where Ψ is a $GL(N, \mathbb{C})$-valued function of x^μ and a complex parameter $\lambda \in \mathbb{CP}^1$, which satisfies the unitary reality condition

$$\Psi(x^\mu, \bar{\lambda})^*\Psi(x^\mu, \lambda) = \mathbf{1}.$$

The matrix Ψ is also subject to gauge transformation $\Psi \to b\Psi$. The integrability conditions for equation (2.1) imply the existence of a gauge $A_t = A_y$, and $A_x = -\Phi$, and a matrix $J : \mathbb{R}^3 \to U(N)$, such that

$$A_t = A_y = \frac{1}{2}J^{-1}(J_t + J_y), \qquad A_x = -\Phi = \frac{1}{2}J^{-1}J_x,$$

and equation (1.1) hold. Given a solution Ψ to the linear system (2.1), one can construct a solution to (1.1) by

$$J(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0), \tag{2.2}$$

and all solutions to (1.1) arise from some Ψs. The detailed exposition of this is presented, for example, by Hitchin *et al.* (1999).

Let us restrict Ψ from $\mathbb{R}^{2,1} \times \mathbb{CP}^1$ to the space-like plane $t=0$. We shall also restrict the spectral parameter to lie in the real equator $S^1 \subset \mathbb{CP}^1$ parameterized by θ,

$$\Psi(x, y, t, \lambda) \to \psi(x, y, \theta) := \Psi\left(x, y, 0, -\cot\frac{\theta}{2}\right), \tag{2.3}$$

where now $\psi:\mathbb{R}^2\times S^1\rightarrow U(N)$ and we made change of variable for real $\lambda=-\cot(\theta/2)$. Note that ψ automatically satisfies

$$(u^\mu D_\mu-\Phi)\psi=0, \tag{2.4}$$

where the operator anihilating ψ is the spatial part of the Lax pair (2.1), which is given by

$$\frac{\lambda L_0+L_1}{1+\lambda^2}=u^\mu D_\mu-\Phi,\quad \text{where}\quad \boldsymbol{u}=\left(0,\frac{1-\lambda^2}{1+\lambda^2},\frac{2\lambda}{1+\lambda^2}\right)=(0,-\cos\theta,-\sin\theta).$$

We impose the 'trivial scattering' boundary condition (Anand 1997; Ward 1998)

$$\psi(x,y,\theta)\rightarrow\psi_0(\theta)\quad\text{as}\quad r\rightarrow\infty, \tag{2.5}$$

where $\psi_0(\theta)$ is a $U(N)$-valued function on S^1. We shall now demonstrate that this enables us to extend ψ to a map from S^3 to $U(N)$.

First, note that (2.5) implies the existence of the limit of ψ at spatial infinity for all values of θ, while the finite energy boundary condition (1.6) implies only the limit at $\theta=\pi$. Thus, the condition (2.5) extends the domain of ψ to $S^2\times S^1$. However, it turns out that (2.5) is also a sufficient condition for ψ to extend to the suspension $SS^2=S^3$ of S^2. This can be seen as follows. The domain $S^2\times S^1$ can be considered as $S^2\times[0,1]$, with $\{0\}$ and $\{1\}$ identified. Recall that a suspension SX of a manifold X is the quotient space (Bredon 1993)

$$SX=([0,1]\times X)/((\{0\}\times X)\cup(\{1\}\times X)).$$

This definition is compatible with spheres, in the sense that $SS^d=SS^{d+1}$.

Now, the only condition that ψ needs to fulfil for the suspension is an equivalence relation between all the points in $S^2\times\{0\}$, since such relation for $S^2\times\{1\}$ will follow from the identification of $\{0\}$ and $\{1\}$. This equivalence can be achieved by choosing a gauge

$$\psi(x,y,0)=\mathbf{1}. \tag{2.6}$$

Therefore, ψ extends to a map from $SS^2=S^3$ to $U(N)$ if it satisfies the trivial scattering boundary condition.

In addition, after fixing the gauge (2.6), there is still some residual freedom in ψ given by

$$\psi\rightarrow\psi K, \tag{2.7}$$

where $K=K(x,y,\theta)\in U(N)$ is anihilated by $u^\mu\partial_\mu$. Setting $K=(\psi_0(\theta))^{-1}$ results in

$$\psi(\{\infty\},\theta)=\mathbf{1}. \tag{2.8}$$

The gauge (2.8) picks a base point $\{x_0=\infty\}\in S^2$, and this implies that the trivial scattering condition is also sufficient for ψ to extend to the reduced suspension of S^2, which is given by

$$S_{\text{red}}S^2=([0,1]\times S^2)/((\{0\}\times S^2)\cup(\{1\}\times S^2)\cup([0,1]\times\{x_0\})).$$

This is also homeomorphic to S^3. The idea of (reduced) suspension is illustrated in figure 1.

Now, let us justify the term trivial scattering in (2.5). Consider equation (2.4) and restrict it to a line $(x,y)=(x_0-\sigma\cos\theta,y_0-\sigma\sin\theta)$, $\sigma\in\mathbb{R}$. Now (2.4) becomes an ODE describing the propagation of

$$\psi=\psi(x_0-\sigma\cos\theta,y_0-\sigma\sin\theta,\theta),$$

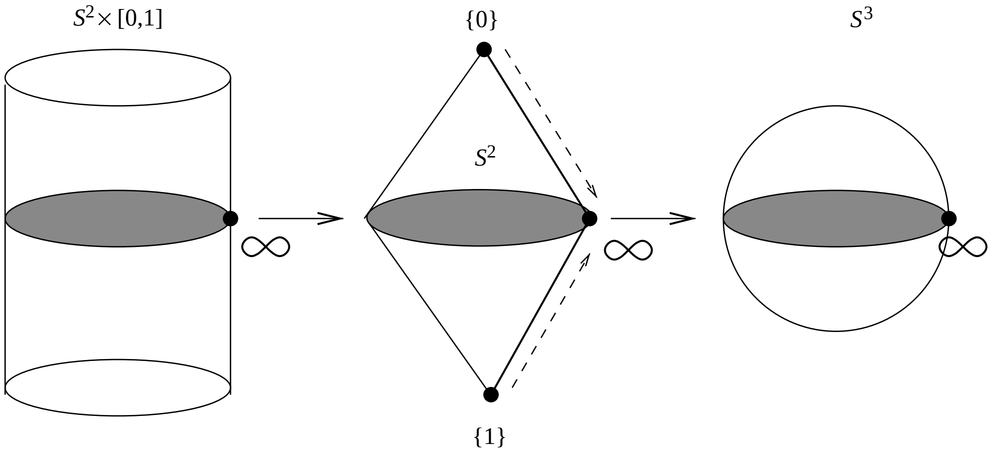

Figure 1. Suspension and reduced suspension.

along the oriented line through (x_0, y_0) in $\mathbb{R}^2$. We can choose a gauge, such that

$$\lim_{\sigma \to -\infty} \psi = \mathbf{1},$$

and define the scattering matrix $S: TS^1 \to U(N)$ on the space of oriented lines in $\mathbb{R}^2$ as

$$S = \lim_{\sigma \to \infty} \psi. \tag{2.9}$$

The trivial scattering condition (2.5) then implies that this matrix is trivial,

$$S = \mathbf{1}. \tag{2.10}$$

As we have explained, the boundary conditions (1.6) and (2.5) imply that for each value of θ, the function ψ extends to a one-point compactification S^2 of $\mathbb{R}^2$. The straight lines on the plane are then replaced by the great circles and, in this context, the trivial scattering condition implies that the differential operator $u^\mu D_\mu - \Phi$ has trivial monodromy along the compactification $S^1 = \mathbb{R} \cup \{\infty\}$ of a straight line parameterized by σ.

(*b*) *Topology of extended solution*

In §2*a*, we have explained that we can regard ψ as a map from S^3 to $U(N)$. All such maps are partially characterized by their homotopy type (Bredon 1993)

$$[\psi] = \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}((\psi^{-1} \mathrm{d}\psi)^3). \tag{2.11}$$

The element $[\psi]$ is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$ and is invariant under continuous deformations of ψ.

In §3, we will need the following result. Let g_1 and g_2 be maps from S^3 to $U(N)$ and $g_1 g_2: S^3 \to U(N)$ be given by

$$g_1 g_2(x) := g_1(x) g_2(x), \quad x \in S^3,$$

where the product on the r.h.s. is the pointwise group multiplication. Then,

$$[g_1 g_2] = [g_1] + [g_2]. \tag{2.12}$$

This is because

$$\mathrm{Tr}[((g_1g_2)^{-1}\mathrm{d}(g_1g_2))^3] = \mathrm{Tr}[(g_1^{-1}\mathrm{d}g_1)^3 + (g_2^{-1}\mathrm{d}g_2)^3] + \mathrm{d}\beta,$$

where β is a two-form and hence $\mathrm{d}\beta$ integrates to 0 by Stokes' theorem. This was explicitly demonstrated by Skyrme (1962) in the case of $SU(2)$.

Rather than exhibiting the exact form of β, we shall use the following general argument. The higher homotopy groups $\pi_d(G)$ of a Lie group G are abelian, and the group multiplication in G induces the addition in the homotopy groups, i.e. if g_1 and g_2 are maps from S^d to G, then the homotopy class of the map $g_1g_2:S^d\to G$ defined by the group multiplication is the sum of homotopy classes of g_1 and g_2. The proof of this is presented, for example, by Bredon (1993) and essentially follows the proof that the fundamental group of a topological group is abelian. Now $\pi_3(G)=\mathbb{Z}$ for any compact simple Lie group. If $G=SU(2)$, then this result just reproduces the calculation done by Skyrme, as two continuous maps from S^3 to itself are homotopic if and only if they have the same topological degree. Theorem 1.1 holds for unitons with values in $G=U(N)$, where $[\psi]$ in (1.7) is the sum of homotopy classes that arise from integrals of elements of $H^3(G)$. To find out a homotopy class of a map ψ, we can use formula (2.11), where the integrand is a left-invariant three-form on the group manifold pulled back to S^3. This is because $\pi_3(G)$ is isomorphic to the integral homology group $H_3(G,\mathbb{Z})$, and the r.h.s. of (2.11) coincides with the homology class of the cycle $\psi(S^3)\subset G$.

We remark that some part of this topological data is encoded in the Ward equation (1.1), which can be regarded as an ordinary chiral model with torsion (Ward 1988*b*). Any compact semi-simple group G admits a connection that parallel propagates left-invariant vector fields. This connection is flat, but necessarily has a torsion T. The torsion is totally antisymmetric, thus giving a preferred three-form in the third cohomology group that can then be pulled back to S^3. The first-order commutator term in equation (1.1) can be rewritten as

$$[\epsilon^{\mu\nu}(J^{-1}\partial_\mu J)(J^{-1}\partial_\nu J)],$$

where $\epsilon^{\mu\nu}$ is a totally antisymmetric constant matrix. In our case, $\epsilon^{\mu\nu}=\epsilon^{\mu\nu\alpha}V_\alpha$, where $V=(0, 1, 0)$ is the space-like unit vector and (1.1) takes the form

$$(\eta^{\mu\nu}+\epsilon^{\mu\nu})(\partial_\mu(J^{-1}\partial_\nu J)) = 0.$$

The choice of V reduces the symmetry group of (1.1) down to $SO(1, 1)$. The momenta $P_t=E$, and P_y are well defined, and conserved for (1.1).

The commutator term can be obtained from a Lagrangian density $\epsilon^{\mu\nu}(\partial_\mu\xi^i)(\partial_\nu\xi^j)e_{ij}(\xi)$, where the two-form e is a local potential for the torsion $\mathrm{d}e=T$, and ξ^j are local coordinates on G, where $i,j=1, \dots, \dim G$. The two-form e is defined only locally in G.

3. Time-dependent unitons and energy quantization

A class of extended solutions that satisfy the trivial scattering condition (2.5) gives rise to the n-uniton solutions defined in (1.4). These extended solutions factorize into the so-called n-uniton factors (Ward 1995)

$$\Psi = G_nG_{n-1}\dots G_1, \quad \text{where} \quad G_k = \left(\mathbf{1}-\frac{\bar\mu-\mu}{\lambda-\mu}R_k\right)\in GL(N,\mathbb{C}), \qquad R_k=\frac{q_k^*\otimes q_k}{\|q_k\|^2}. \tag{3.1}$$

Here, $q_k = q_k(x, y, t) \in \mathbb{C}^N$, $k=1, \ldots, n$, and μ is a non-real constant. The terminology here is rather confusing, as the maxima of the energy density of the corresponding soliton solutions of (1.1) do physically scatter. The exact form of q_ks is determined from (2.1) by demanding that the expressions

$$(\partial_x \Psi - \lambda(\partial_t - \partial_y)\Psi)\Psi^{-1} \quad \text{and} \quad ((\partial_t + \partial_y)\Psi - \lambda \partial_x \Psi)\Psi^{-1} \tag{3.2}$$

are independent of λ. In practice, one determines the q_ks by a limiting procedure from solutions of a Riemann problem with simple poles (Ward 1988*a*).

The restricted map ψ (2.3) corresponding to (3.1) is given by

$$\psi = g_n g_{n-1} \ldots g_1, \quad \text{where} \quad g_k = \mathbf{1} + \frac{\bar{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_k \in U(N), \tag{3.3}$$

where $\lambda = -\cot(\theta/2) \in S^1 \subset \mathbb{CP}^1$ as before and all the maps are restricted to the $t=0$ plane. Each element g_k has the limit at spatial infinity for all values of θ,

$$g_k(x, y, \theta) \to g_{0k}(\theta) = \mathbf{1} + \frac{\bar{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_{0k} \quad \text{as} \quad x^2 + y^2 \to \infty.$$

The existence of the limit at spatial infinity $R_{0k} = \lim_{r\to\infty} R_k(x, y) = \text{const}$ is guaranteed by the finite energy condition (1.6). Hence, ψ in (3.3) satisfies the trivial scattering condition (2.5) and extends to a map from S^3 to $U(N)$. The scattering matrix[2] (2.9) is $S=\mathbf{1}$.

Note, however, that the g_ks and ψ in (3.3) only extend to the ordinary suspension of S^2. One needs to perform the transformation (2.7) with $K = \prod_{k=1}^{n} g_{0k}^{-1}$ for ψ to extend to the reduced suspension of S^2. We shall use ψ as in (3.3), because (2.12) implies that the transformation (2.7) does not contribute to the degree and $[K(\theta)\psi] = [\psi]$.

Proposition 3.1. *The third homotopy class of* ψ *is given by*

$$[\psi] = \pm \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}^2} \sum_{k=1}^{n} \mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k]) \, \mathrm{d}x \, \mathrm{d}y \quad \begin{cases} 0 < \phi < \pi, \\ \pi < \phi < 2\pi, \end{cases} \tag{3.4}$$

where $\mu = m e^{\mathrm{i}\phi}$.

Proof. The recursive application of (2.12) implies that

$$[\psi] = \sum_{k=1}^{n} [g_k].$$

Using (2.11), with $z = x + \mathrm{i}y$,

$$\begin{aligned}[g_k] &= \frac{1}{8\pi^2} \int_{S^1 \times \mathbb{R}^2} \mathrm{Tr}\left(g_k^{-1} \partial_\theta g_k \left[g_k^{-1} \partial_z g_k, g_k^{-1} \partial_{\bar{z}} g_k\right]\right) \mathrm{d}\theta \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= \frac{1}{16\pi^2} \Theta(\mu) \int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_z R_k, \partial_{\bar{z}} R_k]) \mathrm{d}z \wedge \mathrm{d}\bar{z},\end{aligned}$$

[2] Novikov (2002) has demonstrated that given a scattering matrix on the space of oriented lines in $\mathbb{R}^D$ with $D>2$, it is always possible to reconstruct the gauge potential and the Higgs field on $\mathbb{R}^D$ by means of a non-abelian inverse Radon transform. The non-trivial initial data for the time-dependent n-unitons (3.1) have a trivial scattering matrix, which shows that the inversion is not, in general, possible if $D=2$.

where

$$\Theta(\mu) = \int_0^{2\pi} \frac{(\bar{\mu}-\mu)^3 \sin^2\left(\frac{\theta}{2}\right)}{\left(|\mu|^2 + (1-|\mu|^2)\cos^2\left(\frac{\theta}{2}\right) + (\mu+\bar{\mu})\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2} \mathrm{d}\theta = \pm 8\pi \mathrm{i} \quad \begin{cases} 0<\phi<\pi, \\ \pi<\phi<2\pi. \end{cases}$$

Hence, changing to the $(x,\, y)$ coordinates, we obtain

$$[g_k] = \pm \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\,\mathrm{d}y \quad \begin{cases} 0<\phi<\pi, \\ \pi<\phi<2\pi. \end{cases} \tag{3.5}$$

Therefore, the third homotopy class of ψ is given by (3.4).

■

The proof of theorem (1.1) makes use of proposition 3.1 and a recursive procedure of adding unitons to a given solution, which we shall now explain. Let Ψ be an extended solution to the Lax pair (2.1) corresponding to a solution J, which satisfies (1.1). Set

$$\hat{\Psi} = G\Psi = \left(\mathbf{1} - \frac{\bar{\mu}-\mu}{\lambda-\mu} R\right)\Psi, \qquad \hat{J} = \hat{\Psi}^{-1}|_{\lambda=0} = JM, \tag{3.6}$$

where M is of the form (1.5), up to a constant phase factor that is irrelevant. The matrix $\hat{\Psi}$ will be an extended solution if expressions (3.2) with Ψ replaced by $\hat{\Psi}$ are independent of λ. This leads to the Bäcklund relations (Ioannidou & Zakrzewski 1998; Dai & Terng 2004). These are first-order PDEs for M, which can be regarded as a generalization of Uhlenbeck's method of adding unitons for harmonic maps (Uhlenbeck 1989). In terms of the Hermitian projection R, these PDEs are

$$\begin{aligned} R(R_t - J^{-1}J_t(\mathbf{1}-R)) &= B, \\ RR_t &= C, \end{aligned} \tag{3.7}$$

where

$$B = (\mu R_x - R_y + RJ^{-1}J_y)(\mathbf{1}-R),$$

$$C = \frac{1}{\mu}((\mu R_y + R_x - RJ^{-1}J_x)(\mathbf{1}-R)).$$

Proof of theorem 1.1. We first consider a solution of the form $\hat{J} = JM$, where J is an arbitrary solution of (1.1). Noting that M is unitary and writing it in terms of R, the difference between the energy densities (1.3) of $\hat{J}$ and J is given by

$$\Delta\mathcal{E} \equiv \hat{\mathcal{E}} - \mathcal{E} = \sum_a \mathrm{Tr}(\kappa\bar{\kappa}R_a R_a R + \kappa(\mathbf{1}-\bar{\kappa}R)J^{-1}J_a R_a), \tag{3.8}$$

where a stands for $(t,\, x,\, y)$; $\hat{\mathcal{E}}$ and $\mathcal{E}$ are the energy densities of $\hat{J}$ and J, respectively; and $\kappa = (1-(\mu/\bar{\mu}))$.

Multiplying the relations (3.7) and their Hermitian conjugates yields the following identities:

$$\begin{aligned} \mathrm{Tr}(R_t R_t R) &= \mathrm{Tr}(CC^*), \\ \mathrm{Tr}(J^{-1}J_t R_t) &= \mathrm{Tr}(CB^* - BC^*), \\ \mathrm{Tr}(RJ^{-1}J_t R_t) &= \mathrm{Tr}((C-B)C^*). \end{aligned} \tag{3.9}$$

The terms involving the time derivatives in (3.8) are of the form R_tR_tR, $J^{-1}J_tR_t$ and $RJ^{-1}J_tR_t$, which, by (3.9), can be written in terms of the spatial derivatives only. Thus, by direct substitution and some rearrangements, equation (3.8) becomes

$$\Delta\mathcal{E} = -\frac{\kappa}{\mu}\mathrm{Tr}((1+|\mu|^2)R[R_x, R_y] + \mathcal{T}),$$

where $\mathcal{T} = \partial_x(RJ^{-1}J_y) - \partial_y(RJ^{-1}J_x)$ gives no contribution to the difference in the energy functionals of $\hat{J}$ and J. This is because

$$\begin{aligned}\mathrm{Tr}\int_{\mathbb{R}^2}\mathcal{T}\,\mathrm{d}x\wedge\mathrm{d}y &= \lim_{r\to\infty}\int_{D_r}\mathrm{d}(\mathrm{Tr}(RJ^{-1}\mathrm{d}J))\\ &= \lim_{r\to\infty}\oint_{C_r}\mathrm{Tr}(RJ^{-1}\mathrm{d}J) = \mathrm{Tr}\left(\lim_{r\to\infty}\oint_{C_r}(JR)^*\mathrm{d}J\right)\\ &\le \lim_{r\to\infty}\left(\mathrm{Tr}\left(\frac{((JR)_0)^*}{r}(J_1(\varphi=2\pi)-J_1(\varphi=0))\right)\right.\\ &\quad\left.+2\pi r\left\{\frac{|c_2|}{r^2}+\frac{|c_3|}{r^3}+\cdots\right\}\right) = 0,\end{aligned}$$

by Stokes' theorem, where C_r denotes the circle enclosing the disc D_r of radius r, φ is a coordinate on C_r and $|c_i|$ is the bound of $\mathrm{Tr}((JR)_i^*\partial_\varphi J)$, $i=1, 2, \ldots$. We have used the boundary condition

$$\lim_{r\to\infty} JR = (JR)_0 + (JR)_1(\varphi)r^{-1} + O(r^{-2}), \tag{3.10}$$

which follows from (1.6) for $\hat{J} = JM$, and the fact that integrands are continuous on the circle and hence bounded. Since $(JR)_0$ is a constant matrix, the first term in the series is a total derivative.

So far, we have only used the assumption that J is a solution of equation (1.1), but not that it has to be a uniton solution defined by equation (1.4). Therefore, we have a more general result for the total energy of a Ward solution of the form $\hat{J} = JM$, where J is an arbitrary solution to Ward equation. Let $\hat{E}$ and E be the total energies of $\hat{J}$ and J, respectively, then

$$\hat{E} = E + \frac{(\mu-\bar{\mu})(1+|\mu|^2)}{|\mu|^2}\int_{\mathbb{R}^2}\mathrm{Tr}(R[R_x, R_y])\mathrm{d}x\,\mathrm{d}y. \tag{3.11}$$

From (3.11), the explicit expression for the total energy of an n-uniton solution (1.4) follows. First, consider a 1-uniton solution $J_{(1)} = M_1$. It can be written as $J_{(1)} = J_{(0)}M_1$, where the constant matrix $J_{(0)}$, which satisfies (1.1) trivially, is chosen to be the identity matrix. Then, from (3.11), the total energy of a 1-uniton solution is given by

$$E_{(1)} = \frac{(\mu-\bar{\mu})(1+|\mu|^2)}{|\mu|^2}\int_{\mathbb{R}^2}\mathrm{Tr}(R_1[\partial_x R_1, \partial_y R_1])\mathrm{d}x\,\mathrm{d}y. \tag{3.12}$$

Therefore, using (3.11), we show by induction that the total energy of an n-uniton solution (1.4) is given by

$$E_{(n)} = \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \sum_{k=1}^{n} \int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\,\mathrm{d}y$$

$$= \pm 4\pi\left(\frac{1+m^2}{m}\right)\sin(\phi)[\psi] \quad \begin{cases} 0 < \phi < \pi, \\ \pi < \phi < 2\pi, \end{cases} \tag{3.13}$$

where $\mu = m\mathrm{e}^{\mathrm{i}\phi}$, and we have used (3.4).

■

We remark that formula (3.5) reveals another topological interpretation of the energy quantization, which is useful in practical calculations. Consider the group element (3.3) with the index k dropped. The Grassmanian projector R in (3.1) corresponds to a smooth map from the compactified space to the projective space $q: S^2 \to \mathbb{CP}^{N-1}$. The homotopy group $\pi_2(\mathbb{CP}^{N-1}) = \mathbb{Z}$ is non-trivial and the degree of q is obtained by evaluating the homology class on a standard generator for $H^2(\mathbb{CP}^{N-1})$ represented in a map $q = (1, f_1, \ldots, f_{N-1})$ by the Kähler form

$$\Omega = -4\mathrm{i}\partial\bar{\partial}\ln\left(1 + \sum_{j=1}^{N-1} |f_j|^2\right).$$

This evaluation is just the integration, thus

$$[q] = \frac{\mathrm{i}}{8\pi}\int_{\mathbb{R}^2} q^*(\Omega).$$

Evaluating the integrand, we verify that

$$\mathrm{i}\mathrm{Tr}(R[R_x, R_y]) = \frac{1}{4}q^*(\Omega).$$

We conclude that the energy is proportional to the sum of the topological degrees of Grassmanian projectors involved in the definition of unitons.

In the remaining part of this section, we shall prove a Lorentz invariant generalization of theorem (1.1). We start off by looking at the quantization of the momentum. Following Ward (1988*a*), we have chosen the conserved energy functional for a solution of (1.1) to be that obtained from the energy–momentum tensor of the associated standard chiral model. However, for (1.1), only the energy and the y-component of momentum are conserved, while the x-component of momentum is not. The conserved y-momentum is given by

$$P = \int_{\mathbb{R}^2} \mathcal{P}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.14}$$

where the momentum density is

$$\mathcal{P} = -\mathrm{Tr}(J^{-1}J_t J^{-1}J_y). \tag{3.15}$$

It turns out that this is also quantized and proportional to the third homotopy class of the restricted extended solution.

Proposition 3.2. *The y-momentum of the n-uniton solution* (1.4) *is given by*

$$P_{(n)} = -4\pi\left(\frac{1-m^2}{m}\right)|\sin(\phi)|[\psi]. \tag{3.16}$$

Thus, unless $[\psi]=0$, $P=0$ if and only if $m=1$.

Proof. We first consider $\hat{J} = JM$ as in proof of theorem 1.1. The difference between the y-momentum densities (3.15) of $\hat{J}$ and J is given by

$$\Delta\mathcal{P} \equiv \hat{\mathcal{P}} - \mathcal{P} = \kappa \mathrm{Tr}((\mathbf{1} - \bar{\kappa}R)(J^{-1}J_tR_y + J^{-1}J_yR_t) + \bar{\kappa}(R_yR_t)). \tag{3.17}$$

Then, the substitution

$$\mathrm{Tr}(J^{-1}J_tR_yR) = \mathrm{Tr}((C-B)R_y),$$
$$\mathrm{Tr}(J^{-1}J_tRR_y) = \mathrm{Tr}((B^* - C^*)R_y),$$

from the Bäcklund relations (3.7) gives

$$\Delta\mathcal{P} = \frac{\kappa}{\mu}\mathrm{Tr}((1-|\mu|^2)R[R_x, R_y] + \mathcal{T}), \quad \text{where} \quad \kappa = \left(1 - \frac{\mu}{\bar{\mu}}\right).$$

The term $\mathcal{T} = \partial_x(RJ^{-1}J_y) - \partial_y(RJ^{-1}J_x)$ gives no contribution to the difference in the y-momenta of $\hat{J}$ and J as to the difference in the energies. Thus, we have a result for Ward solution of the form in $\hat{J} = JM$, where J is an arbitrary solution to Ward equation; its y-momentum is given by

$$\hat{P} = P - \frac{(\mu - \bar{\mu})(1-|\mu|^2)}{|\mu|^2}\int_{\mathbb{R}^2}\mathrm{Tr}(R[R_x, R_y])\mathrm{d}x\,\mathrm{d}y, \tag{3.18}$$

where $\hat{P}$ and P are the y-momenta of $\hat{J}$ and J, respectively.

We then proceed by induction to obtain the expression for the y-momentum of an n-uniton solution (1.4), in the same way as for the total energy. This gives

$$\begin{aligned} P_{(n)} &= -\frac{(\mu - \bar{\mu})(1-|\mu|^2)}{|\mu|^2}\sum_{k=1}^{n}\int_{\mathbb{R}^2}\mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\,\mathrm{d}y \\ &= \mp 4\pi\left(\frac{1-m^2}{m}\right)\sin(\phi)[\psi] \quad \begin{cases} 0 < \phi < \pi, \\ \pi < \phi < 2\pi. \end{cases} \end{aligned} \tag{3.19}$$

■

We shall now exploit the $SO(1,\,1)$ invariance of (1.1) to combine theorem 1.1 and proposition 3.2 in the Lorentz invariant form.

Theorem 3.1. *For an n-uniton solution, the* $SO(1,\,1)$ *invariant relation*

$$E_{(n)}^2 - P_{(n)}^2 = 64\pi^2\sin^2(\phi)[\psi]^2, \tag{3.20}$$

holds.

Proof. Since equation (1.1) is invariant under $SO(1,\,1)$, we can generate new solutions from a given one by boosts in the y–t plane. In the coordinates $\{x, u = (1/2)(t+y), v = (1/2)(t-y)\}$, the boosts are given by $x \to x$, $u \to su$, $v \to s^{-1}v$, $s \in \mathbb{R}^*$. We shall show that a boost of an n-uniton solution with a pole μ in the extended solution gives rise to another n-uniton solution, with the pole $\mu' = s\mu$.

Consider the Bäcklund relations (3.7) expressed in the $\{x,u,v\}$ coordinates,

$$\begin{aligned} (\mu R_x - R_u + RJ^{-1}J_u)(\mathbf{1} - R) &= 0, \\ (\mu R_v - R_x + RJ^{-1}J_x)(\mathbf{1} - R) &= 0. \end{aligned} \tag{3.21}$$

Let J be an arbitrary solution of (1.1) and $R(x,u,v)$ be the Hermitian projector, satisfying (3.21). Under the boost to another solution $J \to J'$, we have

$R \to R' = R(x, su, s^{-1}v)$. By changing the coordinates, we see that R' will satisfy (3.21) with μ and J replaced by μ' and J', if $\mu' = s\mu$. That is, each restricted uniton factor transforms as

$$g_k = \mathbf{1} + \frac{\bar{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_k(x, u, v, \mu) \to g_k' = \mathbf{1} + \frac{s\bar{\mu} - s\mu}{s\mu + \cot\left(\frac{\theta}{2}\right)} R_k(x, su, s^{-1}v, \mu).$$

Since boost is a continuous transformation, it does not change the homotopy types and

$$[\psi(x, u, v)] = [\psi(x, su, s^{-1}v)].$$

Hence, under the transformation, $E_{(n)}$ and $P_{(n)}$ only change due to the explicit factors of μ in equations (1.7) and (3.16), respectively. The boosts rescale μ by $m \to sm$, keeping the phase ϕ fixed. This leads to the $SO(1, 1)$ invariance of $E_{(n)}^2 - P_{(n)}^2$. Formula (3.20) follows directly from (1.7) and (3.16).

■

Examples. Consider the $SU(2)$ case, where the third homotopy class is equal to the topological degree and set $\mu = \mathrm{i}$. The uniton factors are of the form

$$M_k = \frac{\mathrm{i}}{1 + |f_k|^2} \begin{pmatrix} |f_k|^2 - 1 & -2f_k \\ -2\overline{f_k} & 1 - |f_k|^2 \end{pmatrix}.$$

$n=1$. In the 1-uniton case, $\partial_t M_1 = 0$, and M_1 is given by equation (1.5), with $f_1 = f_1(z)$, a rational function of some fixed degree Q. The energy density is

$$\mathcal{E}_1 = \frac{8|f'_1|^2}{(1 + |f_1|^2)^2} = -\mathrm{i}\mathrm{Tr}(M_1[\partial_z M_1, \partial_{\bar{z}} M_1]),$$

and $E = 8\pi \deg(g_1)$ is in agreement with (1.7). In this case, g_1 is a suspension of a rational map $f_1 : \mathbb{CP}^1 \to \mathbb{CP}^1$ and $\deg(g_1) = \deg(f_1)$ is a simple illustration of the Freundenthal theorem, which says that a suspension of maps of d-spheres induces an isomorphism of the homotopy groups.

$n=2$. In the 2-uniton case, M_1 and M_2 are given by equation (1.5), with $\mu = \mathrm{i}$, and

$$q_1 = (1, f), \qquad q_2 = (1 + |f|^2)(1, f) - 2\mathrm{i}(tf' + h)(\bar{f}, -1),$$

where f and h are the rational functions of z (Ward 1995). Define $k = 2(tf' + h)$. The total energy density is

$$\mathcal{E} = \frac{8|(1 + |f|^2)k' - 2k\bar{f}f'|^2 + 16|kf'|^2 + 16(1 + |f|^2)^2|f'|^2}{(|k|^2 + (1 + |f|^2)^2)^2} \tag{3.22}$$

and

$$E = \int_{\mathbb{R}_2} \mathcal{E}\, \mathrm{d}x\, \mathrm{d}y = 8\pi(\deg(g_1) + \deg(g_2))$$

for all t. The quantization of energy in this case has first been observed by Ioannidou & Manton (2005), where it was shown that $E = 8\pi Q$, where generically $Q = 2\deg f + \deg h$. However, $Q = \max(2\deg f, \deg h)$ if both f and h are polynomials. Our formula (1.7) is valid for all pairs (f, h).

4. Conclusions

We have established the relation between the total energy of time-dependent solitons (1.4) and the homotopy classes of associated extended solutions. To the best of our knowledge, this is the first example of a topological mechanism ensuring the classical energy quantization of moving solitons.

The n-uniton solutions (1.4) form a subclass of all finite energy solitons, which satisfy the trivial scattering boundary condition (2.5). Dai & Terng (2004) have demonstrated that the extended solution corresponding to the general trivial scattering soliton has poles at non-real points $\mu_1, \ldots, \mu_r$, with multiplicities $n_1, \ldots, n_r$, and is a product of simple elements $G_{k,\alpha}$ $\alpha=1, \ldots, r$ of the form in (3.1). Our case (3.1) corresponds to $r=1$, but the method used in proof of theorem 1.1 applies to the general case, as one can choose a different μ at each iteration of the Bäcklund transformations (3.7). Formulae (3.5) and (3.11) lead to the general form of the total energy of trivial scattering solitons

$$E = 4\pi \sum_{\alpha=1}^{r} \sum_{k=1}^{n_r} \frac{1+m_\alpha^2}{m_\alpha} |\sin\phi_\alpha| [g_{k,\alpha}], \qquad \mu_\alpha = m_\alpha e^{i\phi_\alpha}, \tag{4.1}$$

where

$$g_{k,\alpha} = \mathbf{1} + \frac{\bar{\mu}_\alpha - \mu_\alpha}{\mu_\alpha + \cot\left(\frac{\theta}{2}\right)} R_{k,\alpha} \in U(N),$$

and $R_{k,\alpha}$ are Hermitian projections, whose form is determined by the Bäcklund relations. This agrees with the result of Lechtenfeld & Popov (2001). Formula (4.1) is not directly linked to the homotopy type of the extended solution, and the $SO(1, 1)$ invariance cannot be easily incorporated. This is why we have focused on the special case (1.4).

Dunajski & Manton (2005) have analysed the $SU(2)$ integrable chiral model (1.1) in the moduli space approximation, when the time-dependent slowly moving solitons correspond to curves in the moduli space of static solitons, which are geodesic with respect to the natural metric

$$h(\dot{\gamma}, \dot{\gamma}) = \frac{1}{2} \dot{\gamma}^p \dot{\gamma}^q \int_{\mathbb{R}^2} \frac{\left|\partial_p f \overline{\partial_q f}\right|}{(1+|f|^2)^2} \mathrm{d}x\, \mathrm{d}y,$$

on the space of rational maps. Here, $f=f(z, \gamma)$ is a rational meromorphic function of $z=x+iy$, which depends on real parameters (positions of zeros and poles) γ^p, and $\partial_p f = \partial f/\partial\gamma^p$.

The kinetic energy of these approximate solitons is small, and their total energy is close (in the units of 8π) to the degree of the associated rational map. Theorem (1.1) gives a class of exact solutions with quantized total energy, and one may expect that the approximate solitons of Dunajski & Manton (2005) arise from the time-dependent unitons by some limiting procedure.

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