$\mathrm{DIFF}(\Sigma^2)$ DISPERSIONLESS INTEGRABLE SYSTEMS AND TWISTOR THEORY

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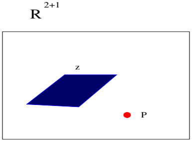
Outline

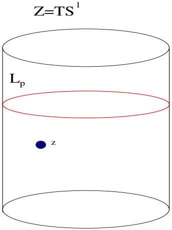
- Twistor theory in $(2+1)$ dimensions
- Einstein–Weyl geometry
- Dispersioness integrable system

- $M = 2+1$–dimensional Minkowski space, $ds^2 = dy^2 - 4dxdt$.

$$\eta = x + \lambda y + \lambda^2 t, \qquad (\eta, \lambda) \in TS^1$$

Minkowski Space	$\longleftrightarrow$	Twistor Space $Z = TS^1$
null plane	$\longleftrightarrow$	point
point	$\longleftrightarrow$	projective line





Twistor Theory

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- Extend this:

Wave equation	$\longrightarrow$	Dispersionless systems
Minkowski space	$\longrightarrow$	Curved metrics (Einstein–Weyl structures)
$T\mathbb{CP}^1$	$\longrightarrow$	Complex Surface Z containing a rational curve $L_p \subset Z$ with self–intersection number 2

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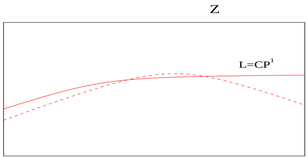

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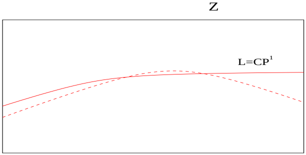

- $Z = M \times \mathbb{CP}^1/(L_0, L_1)$. Lax pair:

$$L_0 = W - \lambda V + f_0 \frac{\partial}{\partial \lambda}, \quad L_1 = V - \lambda \widetilde{W} + f_1 \frac{\partial}{\partial \lambda},$$

where $(W, \widetilde{W}, V)$ are vector fields on M, and (f_0, f_1) are cubic polynomials in $\lambda \in \mathbb{CP}^1$.

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- Lax formulation: Einstein–Weyl condition = existence of 2 parameter family of totally geodesic null surfaces in M.

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$$u_y + w_x = 0, \qquad u_t + w_y - c(uw_x - wu_x) + buu_x = 0.$$

where $u = u(x, y, t), w = w(x, y, t)$ and (b, c) are constants.

$$\begin{aligned} L_0 &= \frac{\partial}{\partial t} + (cw + bu - \lambda cu - \lambda^2)\frac{\partial}{\partial x} + b(w_x - \lambda u_x)\frac{\partial}{\partial \lambda}, \\ L_1 &= \frac{\partial}{\partial y} - (cu + \lambda)\frac{\partial}{\partial x} - bu_x\frac{\partial}{\partial \lambda}. \end{aligned}$$

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- Einstein–Weyl: $h = (dy - cu\, dt)^2 - 4(dx - (cw + bu)\, dt)\, dt,$

$$\omega = -cu_x\, dy + (4bu_x + c^2uu_x - 2cu_y)\, dt.$$

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  - Explicit solutions with regular metrics.