Part III Solitons, Instantons, and Geometry, Sheet Two.

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1. **Abelian Higgs model.** Find the Euler–Lagrange equations for the $U(1)$ gauge potential A and the complex Higgs field ϕ resulting from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{\lambda}{8}(1-|\phi|^2)^2.$$

2. **Variational equations for one vortex**. Consider the static vortex solution with $N = 1$ in a gauge where $A = f(r)d\theta$ and $\phi = h(r)e^{i\theta}$. Find a coupled system of 2nd order ODEs for $f(r), h(r)$ resulting from extremizing the potential energy functional in the polar coordinates.

3. **Boundary integral in the Bogomolny equations.** Verify the covariant Leibniz rule

$$\partial_j(\overline{\phi}D_k\phi) = \overline{(D_j\phi)}D_k\phi + \overline{\phi}D_jD_k\phi,$$

and the identity

$$[D_j, D_k]\phi = -iF_{jk}\phi.$$

Use these to complete the derivation of the Bogomolny energy bound and the Bogomolny equations in the Abelian Higgs model at critical coupling $\lambda = 1$.

4. **Taubes equations for one vortex.** Consider the one–vortex solution in a gauge as in Question 2. Show that, if $\lambda = 1$ the Bogomolny equations for f and h reduce to a pair of 1st order ODEs

$$h' = \frac{1}{r}(1-f)h, \quad f' = \frac{r}{2}(1-h^2) \tag{1}$$

   (a) Show that these equations imply the second order ODEs from Question 2 with $\lambda = 1$.

   (b) Eliminate f from (1) to find a radial form of the Taubes equation for $u = 2\log h$.

5. **Vortices on a Riemann surface.** Consider a surface Σ with a curved metric
$$g = \Omega(z, \bar{z}) dz d\bar{z}, \tag{2}$$
where $z = x^1 + ix^2$, and $\Omega = \Omega(z, \bar{z})$ is the conformal factor (locally, any curved metric on a surface takes this form for some Ω).

   (a) Show that the potential energy functional takes the form
$$V = \frac{1}{2} \int_\Sigma \Big(\Omega^{-1} B^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{\lambda}{4} \Omega (1 - |\phi|^2)^2 \Big) dx^1 dx^2, \tag{3}$$
   where $F = B dx^1 \wedge dx^2$ is the gauge field, $|\psi|^2 \equiv \psi\bar{\psi}$ for any complex number, and $N = \frac{1}{2\pi} \int_\Sigma F$ is the vortex number.

   (b) Complete the square with $\lambda = 1$, and show that $V \geq \pi N$, with the equality if
$$\bar{D}\phi = 0, \quad B = \frac{1}{2} \Omega (1 - |\phi|^2). \tag{4}$$

6. **Taubes equation.** Let $u : \Sigma \to \mathbb{R}$ be a function such that $|\phi|^2 = \exp(u)$. Show that the Bogomolny equations (4) reduce to the Taubes equation
$$\Delta h + \Omega(1 - \exp(u)) = 0 \quad \text{where} \quad \Delta = \Big(\frac{\partial}{\partial x^1} \Big)^2 + \Big(\frac{\partial}{\partial x^2} \Big)^2. \tag{5}$$
What are the boundary conditions for u if Σ is non–compact, and $\Omega \to 1$ as $|z|^2 \to \infty$? What if instead Σ is compact with no boundary?

7. **Vortices on the hyperbolic space.** Compute Gaussian curvature of the metric (2) in terms of Ω and its derivatives, and find a constant value K_0 of the Gaussian curvature for which the change of variables $u = \sigma - \log \Omega$ reduces the Taubes equation (5) to the Liouville equation
$$\Delta \sigma = e^\sigma.$$

   (a) Verify that the hyperbolic metric
$$g = \frac{8}{(1 - |z|^2)^2} dz d\bar{z}$$
   has the Gaussian curvature equal to K_0.

(b) Show that

$$\sigma = \log\left(\frac{32|z|^2}{(1-|z|^4)^2}\right)$$

satisfies the Liouville equation, and find the norm of the Higgs field ϕ of the corresponding vortex solution. What is its vortex number? Are the required boundary conditions satisfied?

8. **Vortex from Sinh–Gordon.** Find a conformal factor Ω in terms of u, such that the Taubes equation (5) becomes the Sinh–Gordon equation.

$$\Delta(u/2) = \sinh(u/2) \tag{6}$$

In this example the intrinsic geometry of the surface with the metric (2) is interpreted as a vortex. Verify this by showing that the $U(1)$ gauge potential is gauge equivalent to the $SO(2)$ Levi–Civita connection one–form, and the curvature two–form of g is a constant multiple of the magnetic–field two–form $F = Bdx^1 \wedge dx^2$.

Show that if $u = u(r)$ (where $r^2 = |z|^2$) is a circularly symmetric solution with vortex number $N = 1$, then the corresponding metric (2) has a conical singularity at the origin, and find its deficit angle.

The Sinh–Gordon equation imposes conditions on the metric g which are equivalent to the statement that the background surface (Σ, g) is a space–like immersion with constant mean curvature in the flat Lorentzian three–space $\mathbb{R}^{2,1}$. It can be shown, that for any vortex number N there exists a unique circularly symmetric solution to (6) corresponding to a vortex.