## Part III Solitons, Instantons, and Geometry, Sheet Three.

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- 1. Action from Yang–Mills. Derive the SU(2) Yang–Mills theory on  $\mathbb{R}^4$  form the action. Let  $A_a(x)$  be a solution to these equations. Show that, for any nonzero constant c, the potential  $\widetilde{A}_a(x) = cA_a(cx)$  is also a solution and that it has the same action.
- 2.  $\mathfrak{so}(4)$ , and self-duality. Let  $T_1, T_2, T_3$  form a basis of  $\mathfrak{su}(2)$  such that

$$[T_{\alpha}, T_{\beta}] = -\varepsilon_{\alpha\beta\gamma}T_{\gamma}, \quad \alpha, \beta, \gamma = 1, 2, 3,$$

and let the symbols  $\sigma_{ab} = -\sigma_{ba}$  where  $a, b = 1, \ldots, 4$  be defined by

$$\sigma_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} T_{\gamma}, \quad \sigma_{\alpha4} = T_{\alpha}.$$

Show that

$$\sigma_{ab} = \frac{1}{2} \varepsilon_{ab}{}^{cd} \sigma_{cd}, \quad \text{and} \quad \sigma_{ab} \sigma_{ac} = -\frac{3}{4} \mathbf{1} \delta_{bc} - \sigma_{bc}$$

Identify  $\Lambda^2(\mathbb{R}^4)$  with the Lie algebra  $\mathfrak{so}(4)$  and deduce that  $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ .

3. An explicit instanton. Let  $V = 1 + r^{-2}$ , where  $r^2 := \delta_{ab} x^a x^b$ . Show that the one-form

$$A = \sigma_{ab} \ \frac{1}{V} \frac{\partial V}{\partial x^b} dx^a \tag{1}$$

is a solution of the anti-self-dual Yang-Mills equations on  $\mathbb{R}^4$ .

The one-form A is singular at r = 0. What can you say about the behaviour of the field strength F at r = 0?

- 4. ... and its Chern number. Find, by explicit integration, the Chern number of the solution (1).
- 5. Hodge operator in various signatures. Let F be a two-form on  $\mathbb{R}^4$ . Show, from the definition of the Hodge operator, that
  - (a)  $**F = \pm F$  depending on the signature.
  - (b)  $*F \wedge *F = F \wedge F$ .

Show that in the U(1) theory  $F \to *F$  interchanges the electric and magnetic fields with factors of  $\pm 1$  or  $\pm i$  and determine the different cases in the corresponding signatures.

Let F be a non-zero real self-dual two-form on  $\mathbb{R}^4$  such that  $F \wedge F = 0$ . What is the signature of the underlying metric?

6. Self-duality in null coordinates. Show that the two-forms

 $\omega_1 = dw \wedge dz, \qquad \omega_2 = dw \wedge d\tilde{w} - dz \wedge d\tilde{z}, \qquad \omega_3 = d\tilde{w} \wedge d\tilde{z}$ 

span the space of SD two–forms in  $\mathbb{C}^4$ , where

$$ds^{2} = 2(dzd\tilde{z} - dwd\tilde{w}), \qquad \text{vol} = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}.$$

Show that a two form F is ASD iff  $F \wedge \omega_i = 0$ .

- 7. *K*-equation from the Lax pair. Use the Lax pair formulation of ASDYM to
  - (a) Deduce the existence of a gauge such that  $A = A_{\tilde{w}} d\tilde{w} + A_{\tilde{z}} d\tilde{z}$
  - (b) Deduce the existence of a  $\mathfrak{g}$  valued function  $K = K(w, z, \tilde{w}, \tilde{z})$ such that  $A_{\tilde{w}} = \partial_z K, A_{\tilde{z}} = \partial_w K$
  - (c) Reduce the ASDYM to a single second order PDE

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K + [\partial_w K, \partial_z K] = 0.$$

What is the residual gauge freedom in K?

8. Chern–Simons three–form. Let A be a 1–form gauge potential on  $\mathbb{R}^n$  with values in  $\mathfrak{su}(2)$ , and let F be its curvature. Verify that  $Tr(A), Tr(A \wedge A), Tr(A \wedge A \wedge A \wedge A)$  and Tr(F) all vanish.

Verify that  $C_2 = dY_3$ , where  $C_2$  and  $Y_3$  are the second Chern form, and the Chern–Simons three–form respectively.

9. ASD Yang-Mills as gradient flow. Let  $A = A_i dx^i$ , i = 1, 2, 3 be a gauge potential on  $\mathbb{R}^3$  with values in the Lie algebra  $\mathfrak{g}$ . Find the Euler-Lagrange equations arising from varying the Chern-Simons functional

$$W[A] = \int_{\mathbb{R}^3} \operatorname{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

with respect to A.

Now consider a one parameter family of  $\mathfrak{g}$ -valued one-forms A = A(t)on  $\mathbb{R}^3$ , and define a one-form on  $\mathbb{R}^4$  by  $\mathcal{A} = A + \phi dt$ , where the function  $\phi = \phi(x^i, t)$  takes its values in  $\mathfrak{su}(2)$ . Show that, in a gauge where  $\phi = 0$ , the anti-self-dual Yang-Mills equations on  $\mathcal{A}$  take the gradient flow form

$$\frac{dA_i(t)}{dt} = \frac{\delta W[A]}{\delta A_i}.$$

10. Left and right invariant vector fields on a Lie group. Let  $G \subset GL(n+1,\mathbb{R})$  be a Lie group consisting of matrices of the form

$$g = \begin{pmatrix} x^0 & x^1 & x^2 & \dots & x^n \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where  $x^0 \in \mathbb{R}^+, x^i \in \mathbb{R}$  for  $i = 1, \dots n$ .

- By considering g near the identity matrix find the structure constants of the Lie algebra  $\mathfrak{g}$  of G.
- Construct the left invariant one-forms  $\sigma^a, a = 0, 1, ..., n$ , and the left invariant vector fields of G. Show that

$$d\sigma^a = \sum_{b,c=0}^n \kappa^a{}_{bc} \sigma^b \wedge \sigma^c$$

for some  $\kappa^a{}_{bc}$  which should be determined

- 11. Connection and curvature of a principal bundle. Consider a connection  $\omega = \gamma^{-1}A\gamma + \gamma^{-1}d\gamma$  on a principal *G*-bundle  $P \to B$ , where *A* is a one-form on *B* and  $\gamma^{-1}d\gamma$  is the Maurer-Cartan form on *G*.
  - (a) Show that the transformation of the fibres  $\gamma' = g\gamma$ , where  $g \in G$  depends on the coordinates on B, does not change  $\omega$  if A transforms like a gauge potential.

- (b) Let  $\Omega = d\omega + \omega \wedge \omega$ . Show that  $\Omega = \gamma^{-1} F \gamma$  for some F which should be found.
- (c) Let  $D_a, a = 1, ..., \dim(B)$  be linearly independent vector fields on P such that

$$D_a \,\lrcorner\, \omega = 0.$$

Show that  $D_a = \partial_a - A^{\alpha}_a R_{\alpha}$ , where  $\partial_a = \partial/\partial x^a$  are vector fields on *B* and  $R_{\alpha}$  are right-invariant vector fields on *G*. Demonstrate that

$$[D_a, D_b] = -F_{ab}^{\alpha} R_{\alpha}.$$