Part III Solitons and Instantons, Sheet One

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1. Starting with the Lagrangian of the Sine–Gordon theory

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - (1 - \cos\beta\phi)$$

derive the Sine–Gordon equation. Find a kink solution of the Sine–Gordon theory, and use the Bogomolny bound to find its energy. How many types of kinks are there?

2. The Lagrangian density for a complex scalar field ϕ in $1+1$ is

$$\mathcal{L} = \frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\phi_x|^2 - \frac{1}{2}\lambda^2(a^2 - |\phi|^2)^2, \qquad a \in \mathbb{R}.$$

Find the field equations, and verify that the real kink $\phi_0(x) = a\tanh(\lambda a x)$ is a solution. Now consider a small pure imaginary perturbation $\phi(x,t) = \phi_0(x) + i\eta(x,t)$ with η real and find the linear equation satisfied by η.

By considering $\eta =$sech $(\alpha x)e^{\omega t}$ show that the kink is unstable.

3. Let $\phi : \mathbb{R}^{2,1} \to S^2$. Set

$$\phi^1 + i\phi^2 = \frac{2u}{1+|u|^2}, \qquad \phi^3 = \frac{1-|u|^2}{1+|u|^2},$$

and deduce that the Bogomolny equations

$$\partial_i\phi^a = \pm\varepsilon_{ij}\varepsilon^{abc}\phi^b\partial_j\phi^c, \qquad \phi_t = 0$$

imply that u is holomorphic or antiholomorphic in $z = x_1 + ix_2$. Find the expression for the total energy

$$E = \frac{1}{2}\int \partial_j\phi^a\partial_j\phi^a \mathrm{d}^2x$$

in terms of u.

4. Show that in $SU(2)$ Yang–Mills–Higgs theory the general solution to the equation $D_i\hat{\Phi} = 0$ with $|\hat{\Phi}| = 1$ is

$$A_i^a = -\varepsilon^{abc}\partial_i\hat{\Phi}^b\hat{\Phi}^c + N_i\hat{\Phi}^a,$$

and calculate the gauge field corresponding to this potential. What can you deduce about the solution of the equating $D_i\Phi = 0$? [*Hint: Write* $\Phi = |\Phi|\hat{\Phi}$ *and use the covariant Leibniz rule*].

5. The Higgs field $\hat{\Phi}$ at infinity defines a map form S^2 to S^2. In polar coordinates the asymptotic magnetic field has non–zero components

$$F_{\theta\phi} = \varepsilon_{abc}\partial_\theta\hat{\Phi}^a\partial_\phi\hat{\Phi}^b\hat{\Phi}^c.$$

By writing

$$\hat{\Phi} = (\sin\lambda\cos\mu, \sin\lambda\sin\mu, \cos\lambda)$$

where $\lambda = \lambda(\theta,\phi), \mu = \mu(\theta,\phi)$ show that the magnetic charge satisfies

$$g = \int_{S^2} F_{\theta\phi}\mathrm{d}\theta\mathrm{d}\phi = 4\pi\mathrm{deg}\ (\hat{\Phi}).$$

6. Make the ansatz

$$\Phi^a = h(r)\frac{x^a}{r}, \qquad A_i{}^a = -\varepsilon^{aij}\frac{x^j}{r^2}(1-k(r))$$

and show that the Bogomolny equations for the non–Abelian magnetic monopole reduce to

$$h' = r^{-2}(1-k^2), \qquad k' = -kh.$$

Use the change of variables $H = h + r^{-1}, K = k/r$ to find the one–monopole solution.

7. Derive the pure $SU(2)$ Yang–Mills theory on $\mathbb{R}^4$ form the action. Let $A_\mu(x)$ be a solution to these equations. Show that $\widetilde{A}_\mu(x) = cA_\mu(cx)$ is also a solution and that it has the same action.

8. Let A be a 1–form gauge potential with values in $\mathbf{su(2)}$, and let F be its curvature. Verify that $Tr(A), Tr(A\wedge A), Tr(A\wedge A\wedge A\wedge A)$ and $Tr(F)$ all vanish.