**Part III Solitons and Instantons, Sheet Two**

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1. The harmonic function $\rho = 1 + r^{-2}$ determines a one–instanton solution. Check, by explicit integration that the second Chern number of the corresponding bundle is 1.

2. Consider the map $g : S^3 \to SU(2)$ defined by

$$g(x_1, x_2, x_3, x_4) = x_4 + i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3),$$

where $\sigma_i$ are Pauli matrices and $x_1^2 + x_2^2 + x_3^3 + x_4^2 = 1$ and find its degree. By calculating $Tr((\mathrm{d}g\, g^{-1})^3)$ at the point on $S^3$ where $x_4 = 1$, or otherwise deduce that the formula

$$deg(g) = \frac{1}{24\pi^2}\int_{S^3} Tr((\mathrm{d}g\, g^{-1})^3)$$

is correctly normalised.

3. Show that $**F = \pm F$ depending on the signature.

Show that in the $U(1)$ theory $F \to *F$ interchanges the electric and magnetic fields with factors of $\pm 1$ or $\pm i$ and determine the different cases in the corresponding signatures.

4. Show that the two–forms

$$\omega_1 = \mathrm{d}w \wedge \mathrm{d}z, \qquad \omega_2 = \mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z}, \qquad \omega_3 = \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z}$$

span the space of SD two–forms in $\mathbb{C}^4$, where

$$\mathrm{d}s^2 = 2(\mathrm{d}z\mathrm{d}\tilde{z} - \mathrm{d}w\mathrm{d}\tilde{w}), \qquad \mathrm{vol} = \mathrm{d}w \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z}.$$

Show that a two form $F$ is ASD iff $F \wedge \omega_i = 0$.

5. Use the Lax pair formulation of ASDYM to

   (a) Deduce the existence of a gauge such that $A = A_{\tilde{w}}\mathrm{d}\tilde{w} + A_{\tilde{z}}\mathrm{d}\tilde{z}$

   (b) Deduce the existence of a $\mathfrak{g}$ valued function $K = K(w, z, \tilde{w}, \tilde{z})$ such that $A_{\tilde{w}} = \partial_z K, A_{\tilde{z}} = \partial_w K$

   (c) Reduce the ASDYM to a single second order PDE

$$\partial_z\partial_{\tilde{z}}K - \partial_w\partial_{\tilde{w}}K + [\partial_w K, \partial_z K] = 0.$$

What is the residual gauge freedom in $K$?

6. Show that iff $f(\lambda)$ is a meromorphic function on $\mathbb{CP}^1$ (so its only singularities are a finite collection of poles), then $f = s_1/s_2$ for some holomorphic sections $s_1, s_2$ of $\mathcal{O}(n)$ for some $n$.

Show that the holomorphic tangent and cotangent bundles of $\mathbb{CP}^1$ are, respectively, $\mathcal{O}(2)$ and $\mathcal{O}(-2)$.

7. **Exercises on two component spinor notation**. Show that

   - $\xi_A \rho^A = 0$ iff $\xi_A, \rho_A$ are proportional
   - A vector in $\mathbb{C}^4$ is null iff $V^{AA'} = \lambda^A \xi^{A'}$ for some spinors $\lambda^A, \xi^{A'}$.
   - $\tau_{AB} = \tau_{(AB)} + \frac{1}{2}\tau\varepsilon_{AB}$ where (...) denotes symmetrisation, and $\tau$ should be determined.
   - Any two–form $F_{AA'BB'}\mathrm{d}x^{AA'} \wedge \mathrm{d}x^{BB'}$ in $\mathbb{C}^4$ is of the form

     $$F_{AA'BB'} = \psi_{AB}\varepsilon_{A'B'} + \phi_{A'B'}\varepsilon_{AB},$$

     where $\psi_{AB}, \phi_{A'B'}$ determine the ASD and SD parts of $F$ respectively.
   - ASDYM equations are of the form

     $$\partial_{B(B'}A_{C')}{}^{B} + A_{B(B'}A^{B}_{C')} = 0$$

     and are equivalent to $[L_0, L_1] = 0$, where $L_B = \pi^{B'}(\partial_{BB'} + A_{BB'})$ for some constant spinor $\pi^{B'}$.

8. Show that the factorisation of the patching matrix $F = \widetilde{H}H^{-1}$ in the Ward correspondence is unique up to multiplication of $H$ and $\tilde{H}$ on the left by a nonsingular matrix $g$ depending on the space-time coordinates, but not $\lambda$. Show that different choices of factorisation give gauge-equivalent connections.

9. Let the patching matrix for a rank-two Ward bundle over the twistor space $\mathcal{PT}$ be given by

   $$F = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix},$$

   where $f = f(\omega^A, \pi_{A'})$ is en element of $H^1(\mathcal{PT}, \mathcal{O})$. Find a YM potential $A$ in terms of the electromagnetic field generated by $f$.