Twistor Theory and Differential Equations

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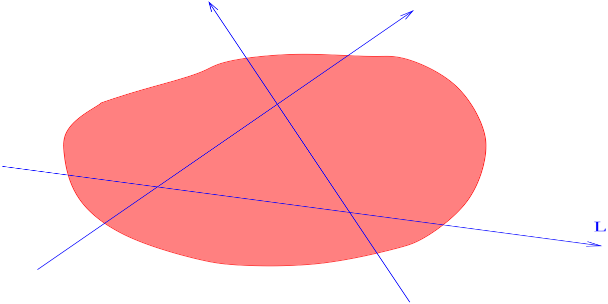
- 1917 Radon. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with decay condition ot ∞, $\quad L \subset \mathbb{R}^2$ oriented line.

$$\phi(L) := \int_L f.$$

There exist an inversion formula $\phi \longrightarrow f$.



- 1938 Fritz John. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$.
  Define $\phi(L) = \int_L f$, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds.$$

- The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on ϕ.
- Differentiate under the integral: ultrahyperbolic wave equation

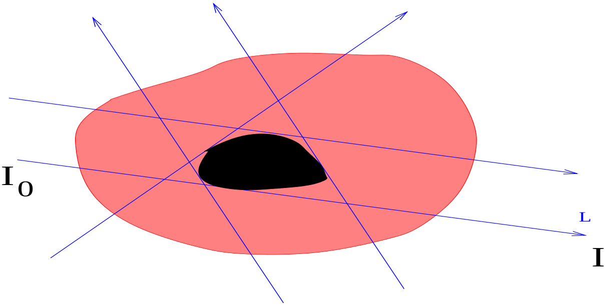
$$\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0.$$

- Change coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.$$

  Relevant to physics with two times!

- 1963 Cormack. Hole theorem.



$$\phi(L) = \int_L \frac{dI}{I} = \log I - \log I_0 = -\int_L f$$

1979 Nobel Prize (in medicine) for image reconstruction.

- 1967 Penrose (Twistor theory). Wave equation in Minkowski space.

$$\phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z+t) + (x+iy)\lambda, (x-iy) - (z-t)\lambda, \lambda) d\lambda$$

verify $$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0.$$

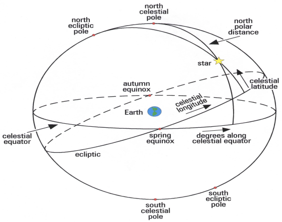
- Mathematically sophisticated: Could modify a contour and add a holomorphic function inside the contour to f. Needs sheaf cohomology.

Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ...
- Classical Physics. Complex numbers in the sky! Celestial sphere

$$(u_1)^2 + (u_2)^2 + (u_3)^2 = 1$$

Stereographic projection onto a plane



- From north pole $(0, 0, 1)$, $\lambda = \frac{u_1 + iu_2}{1 - u_3}$.
  From south pole $(0, 0, -1)$, $\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}$.
- On the overlap $\tilde{\lambda} = 1/\lambda$. This makes S^2 into a complex manifold $\mathbb{CP}^1$ (Riemann sphere).
- Möbius transformations $\xrightarrow{2:1}$ Lorentz transformations.

Twistor Programme

- Twistor correspondence

Space time	$\longleftrightarrow$	Twistor space
Point	$\longleftrightarrow$	Complex line $\mathbb{CP}^1$
Light ray	$\longleftrightarrow$	Point.

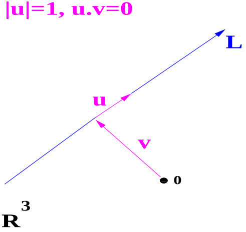
- Space-time points are derived objects in twistor theory. They become 'fuzzy' after quantisation. Attractive framework for quantum gravity.
- 40 years of research: No major impact on physics (so far). Surprisingly major impact on pure mathematics: representation theory, differential geometry, solitons, instantons, integrable systems .

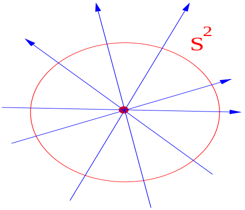
Differential equations and complex numbers

- Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad \phi = \mathrm{Re}(f(\zeta)), \quad \zeta = x + iy.$$

- Harmonic functions on $\mathbb{R}^3$? Problem: 3 is an odd number, $\mathbb{R}^3 \neq \mathbb{C}^n$.
- Twistor space $\mathbb{T}$ = space of oriented lines in $\mathbb{R}^3$. Line $\mathbf{v} + t\mathbf{u}, \quad t \in \mathbb{R}$.

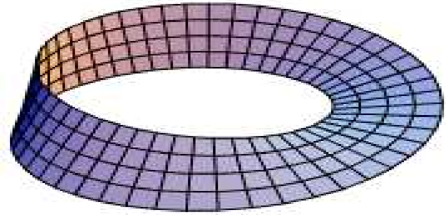




Dimension of $\mathbb{T}$ is four (even!).

Oriented Lines in $\mathbb{R}^3$

- $\mathbb{T} = \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3,\ \mathbf{u}.\mathbf{v} = 0\}$. For each fixed $\mathbf{u}$ this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes – the tangent bundle TS^2.
- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted



- Reversing the orientation of lines $\tau : \mathbb{T} \longrightarrow \mathbb{T}$, $\tau(\mathbf{u}, \mathbf{v}) = (-\mathbf{u}, \mathbf{v})$.
- Points $\mathbf{p} = (x, y, z)$ in $\mathbb{R}^3$ two–spheres in $\mathbb{T}$; τ–invariant maps

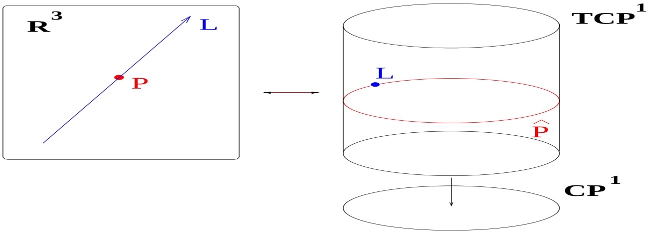
$$\mathbf{u} \longrightarrow (\mathbf{u}, \mathbf{v}(\mathbf{u}) = \mathbf{p} - (\mathbf{p}.\mathbf{u})\mathbf{u}) \in \mathbb{T}.$$

- Holomorphic coordinates

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{CP}^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3.$$

  Need another coordinate patch $(\tilde{\lambda}, \tilde{\eta})$ containing $\mathbf{u} = (0, 0, 1)$. On the overlap $\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2$.
- Points in $\mathbb{R}^3$ are τ-invariant holomorphic maps $\mathbb{CP}^1 \to T\mathbb{CP}^1$

$$\lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2 (x - iy)).$$



HARMONIC FUNCTIONS ON $\mathbb{R}^3$

To find a harmonic function at $P = (x, y, z)$:

- Restrict a twistor function $f(\lambda, \eta)$ to $\hat{P} = \mathbb{CP}^1 = S^2$.
- Integrate along a closed contour

$$\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2 (x - iy)) d\lambda,$$

- Differentiate under the integral to verify

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

(Whittaker, 1903).

Magnetic monopole

- Dirac. Are there point magnetic charges? If yes, quantisation of electric charge e can be explained

$$eg = 2\pi N, \qquad N = 1, 2, 3, \dots \quad .$$

- Maxwell equations = $U(1)$ gauge theory. Nonabelian gauge theory: Replace $U(1)$ by a non-abelian group $SU(2)$. (Relevant in elecro–weak model).
- $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ anti-hermitian 2x2 matrices on $\mathbb{R}^3$.
- Nonabelian magnetic field

$$F_{jk} = \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} + [A_j, A_k], \quad j, k = 1, 2, 3.$$

- Monopole equation

$$\frac{\partial \Phi}{\partial x^j} + [A_j, \Phi] = \frac{1}{2}\varepsilon_{jkl} F_{kl}.$$

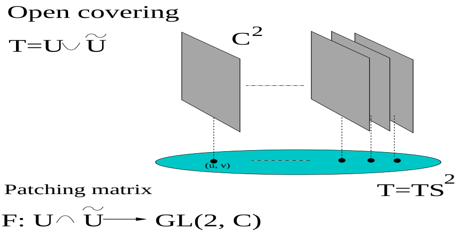
System of non-linear PDEs.

Twistor solution to the monopole equation

- Given $(A_j(\mathbf{x}), \Phi(\mathbf{x}))$ solve a matrix ODE along each oriented line $\mathbf{x}(t) = \mathbf{v} + t\mathbf{u}$

$$\frac{dV}{dt} + (u^j A_j + i\Phi)V = 0.$$

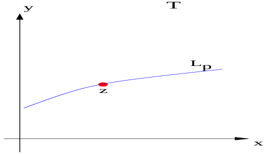
  Space of solutions at $p \in \mathbb{R}^3$ is a complex vector space $\mathbb{C}^2$.
- Complex vector bundle over $\mathbb{T}$ with patching matrix $F(\lambda, \overline{\lambda}, \eta, \overline{\eta})$.

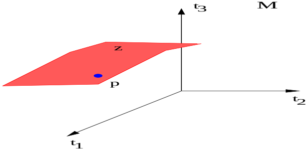


- Monopole equation $\longleftrightarrow$ Cauchy–Riemann eq. $\frac{\partial F}{\partial \overline{\lambda}} = 0, \frac{\partial F}{\partial \overline{\eta}} = 0.$
- Holomorphic vector bundles over $T\mathbb{CP}^1$ - well understood. Take one and work backwards to construct a monopole. (Hitchin, 1982.)

Geometry of ODEs

- $\Psi(x, y; t_1, \ldots t_n) = 0$





point $\longleftrightarrow$ hypersurface

curve $\longleftrightarrow$ point

- Implicit function theorem $y = Z(x; t_1, \ldots, t_n)$. Differentiate N times, elliminate $t_1, \ldots, t_n$

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \qquad \text{ODE}$$

- Classical problem: Take $n = 3$, assume $z \subset M$ is null w.r.t. some conformal structure on M. Conditions on F? (Wünschmann (1905)).

$$\frac{1}{3}F_2\frac{d}{dx}F_2 - \frac{1}{6}\frac{d^2}{dx^2}F_2 + \frac{1}{2}\frac{d}{dx}F_1 - \frac{2}{27}(F_2)^3 - \frac{1}{3}F_2F_1 - F_0 = 0.$$

- Bundle isomorphism $TM \cong \mathbb{S} \odot \mathbb{S} \odot \cdots \odot \mathbb{S} = \mathsf{S}^{(n-1)}(\mathbb{S})$, where $\mathbb{S} \to M$ is a rank 2 symplectic vector bundle.
- Identifies vectors with $(n-1)$st order homogeneous polynomials. $V^{AB\cdots C}\pi_A\pi_B \ldots \pi_C$ in $\pi_A = (\pi_0, \pi_1)$. (conformal structure if $n = 3$). Embedding $SL(2, \mathbb{R}) \subset O(n)$ given by $\pi_A \to t^B{}_A\pi_B$.
- (MD, Paul Tod.) Assume that the space of solutions to the nth order ODE has a $GL(2, \mathbb{R})$ structure such that surfaces $y = Z(x; t)$ are maximally null (i.e. normal vector has repeated root with multiplicity $(n-1)$). Then F satisfies $(n-2)$ Doubrov conditions. Equivalently, $L_p \subset \mathbb{T}$ is a rational curve with a normal bundle $\mathcal{O}(n-1)$.
- The same structure came up recently in many different contexts
  1. Bryant: exotic holonomy in dimension 4.
  2. Doubrov: Linearise the ODE around a given solution and demand that the resulting linear homogeneous ODE be trivialisable, i.e. equivalent to $y^{(n)} = 0$ by a coordinate transformation $(x, y) \to (b(x), a(x)y)$.
  3. Godlinski–Nurowski: A general framework: restricting torsion of a Cartan connection.

G_2 structures from $GL(2,\mathbb{R})$ structures

Is there a G_2 structure on a space of solutions to $7th$ order ODE? (work in progress with Michal Godlinski and Pawel Nurowski).

1. Take a family of rational curves $x \to (x, y(x;t))$ with self–intersection number 6 in a complex surface $\mathbb{T}$. Normal vector (section of $\mathcal{O}(6)$)

$$\delta y = \frac{\delta y}{\delta t_i}\delta t_i = a_0 x^6 + 6a_1 x^5 + 15a_2 x^4 + 20a_3 x^3 + 15a_4 x^2 + 6a_5 x + a_6$$

2. Classical invariant theory: a conformal structure and a three–form

$$g = a_0 a_6 - 6a_1 a_5 + 15a_2 a_4 - 10a_3^2,$$

$$\phi = 3\,(a_1\wedge a_2\wedge a_6 + a_0\wedge a_4\wedge a_5) + a_3\wedge(a_0\wedge a_6 + 6\,a_1\wedge a_5 - 15\,a_2\wedge a_4).$$

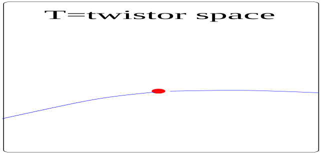
3. Example: rational curve in $\mathbb{CP}^1\times\mathbb{CP}^1$ of bidegree $(1,3)$

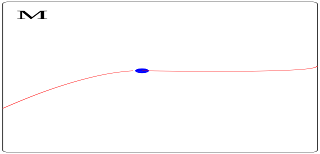
$$y = \frac{r_0 + r_1 x + r_2 x^2 + r_3 x^3}{s_0 + s_1 x + s_2 x^2 + s_3 x^3} \quad \text{gives} \quad a_i = \binom{6}{i}^{-1} \sum_{\alpha+\beta=i} (r_\alpha ds_\beta - s_\beta dr_\alpha).$$

Want to learn more - go to Godlinski's talk tomorrow.

2nd order ODEs: Nonlinear Radon transform

- When are the curves $z \subset M$ are unparametrised geodesic of some torsion–free connection (projective sturcture)
- One-to-one correspondence between holomorphic projective structures on M and complex surfaces $\mathbb{T}$ with a family of rational curves.





points $\longleftrightarrow$ geodesics

rational curves with self intersection number 1 $\longleftrightarrow$ points

- Double fibration $\mathbb{T} = \mathbb{P}(TM)/D \longleftarrow \mathbb{P}(TM) \longrightarrow M$, where D is the geodesic spray.
- Relevant to my next talk (in 15 minutes).

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

$$\text{point} \longleftrightarrow \text{line (complex)}$$
$$\text{line} \longleftrightarrow \text{point.}$$

- Complex numbers are essential

$$\text{nonlinear PDEs} \longleftrightarrow \text{linear Cauchy–Riemann equations.}$$

- Its status as a physical theory is not clear
  - Weakness: effective only in low dimensions.
  - Strength: effective only in low dimensions.