Twistor Theory and Differential Equations

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Motivation: Integral Geometry

- 1917 Radon. \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) with decay condition of \( \infty \), \( L \subset \mathbb{R}^2 \) oriented line.

\[
\phi(L) := \int_L f.
\]

There exist an inversion formula \( \phi \rightarrow f \).
1938 Fritz John. $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, oriented line $L \subset \mathbb{R}^3$.

Define $\phi(L) = \int_L f$, or

$$
\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s)\,ds.
$$

The space of oriented lines is 4 dimensional, and $4 > 3$ so expect one condition on $\phi$.

Differentiate under the integral: ultrahyperbolic wave equation

$$
\frac{\partial^2 \phi}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \phi}{\partial \alpha_2 \partial \beta_1} = 0.
$$

Change coordinates $\alpha_1 = x + y, \alpha_2 = t + z, \beta_1 = t - z, \beta_2 = x - y$.

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 0.
$$

Relevant to physics with two times!

\[ \phi(L) = \int_L \frac{dI}{I} = \log I - \log I_0 = -\int_L f \]

1979 Nobel Prize (in medicine) for image reconstruction.
1967 Penrose (Twistor theory). Wave equation in Minkowski space.

\[ \phi(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} f((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda) d\lambda \]

verify

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \]

Mathematically sophisticated: Could modify a contour and add a holomorphic function inside the contour to \( f \). Needs sheaf cohomology.
Complex Numbers in Physics

- Quantum Physics. Complex wave function, Hilbert spaces, ... .
- Classical Physics. Complex numbers in the sky! Celestial sphere

\[(u_1)^2 + (u_2)^2 + (u_3)^2 = 1\]

Stereographic projection onto a plane

- From north pole \((0, 0, 1)\), \(\lambda = \frac{u_1 + iu_2}{1 - u_3}\).
- From south pole \((0, 0, -1)\), \(\tilde{\lambda} = \frac{u_1 - iu_2}{1 + u_3}\).

On the overlap \(\tilde{\lambda} = 1/\lambda\). This makes \(S^2\) into a complex manifold \(\mathbb{CP}^1\) (Riemann sphere).
- Möbius transformations \(\xrightarrow{2:1}\) Lorentz transformations.
Twistor Programme

- Twistor correspondence

  Space time $\longleftrightarrow$ Twistor space
  
  Point $\longleftrightarrow$ Complex line $\mathbb{CP}^1$
  
  Light ray $\longleftrightarrow$ Point.

- Space-time points are derived objects in twistor theory. They become ‘fuzzy’ after quantisation. Attractive framework for quantum gravity.

- 40 years of research: No major impact on physics (so far).
  Surprisingly major impact on pure mathematics: representation theory, differential geometry, solitons, instantons, integrable systems.
Differential equations and complex numbers

- Harmonic functions on $\mathbb{R}^2$. Complex numbers $\mathbb{R}^2 = \mathbb{C}$.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi = \text{Re}(f(\zeta)), \quad \zeta = x + iy.$$  

- Harmonic functions on $\mathbb{R}^3$? Problem: 3 is an odd number, $\mathbb{R}^3 \neq \mathbb{C}^n$.

- Twistor space $\mathbb{T} =$ space of oriented lines in $\mathbb{R}^3$. Line $v + tu$, $t \in \mathbb{R}$.

  $|u|=1$, $u.v=0$

Dimension of $\mathbb{T}$ is four (even!).
Oriented Lines in $\mathbb{R}^3$

- $T = \{ (u, v) \in S^2 \times \mathbb{R}^3, \ u \cdot v = 0 \}$. For each fixed $u$ this space restricts to a tangent plane to $S^2$. The twistor space is the union of all tangent planes – the tangent bundle $TS^2$.
- Topologically nontrivial: Locally $S^2 \times \mathbb{R}^2$ but globally twisted

- Reversing the orientation of lines $\tau : T \to T$, $\tau(u, v) = (-u, v)$.
- Points $p = (x, y, z)$ in $\mathbb{R}^3$ two–spheres in $T$; $\tau$–invariant maps

$$u \mapsto (u, v(u) = p - (p.u)u) \in T.$$
Holomorphic coordinates

\[ \lambda = \frac{u_1 + iu_2}{1 - u_3} \in \mathbb{C}P^1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2}v_3. \]

Need another coordinate patch \((\tilde{\lambda}, \tilde{\eta})\) containing \(u = (0, 0, 1)\). On the overlap \(\tilde{\lambda} = 1/\lambda, \tilde{\eta} = -\eta/\lambda^2\).

Points in \(\mathbb{R}^3\) are \(\tau\)-invariant holomorphic maps \(\mathbb{C}P^1 \to T\mathbb{C}P^1\)

\[ \lambda \rightarrow (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \]
Harmonic functions on $\mathbb{R}^3$

To find a harmonic function at $P = (x, y, z)$:

- Restrict a twistor function $f(\lambda, \eta)$ to $\hat{P} = \mathbb{CP}^1 = S^2$.
- Integrate along a closed contour

$$
\phi(x, y, z) = \oint_{\Gamma \subset \hat{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy)) d\lambda,
$$

- Differentiate under the integral to verify

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.
$$

(Whittaker, 1903).
**Magnetic monopole**

- **Dirac.** Are there point magnetic charges? If yes, quantisation of electric charge $e$ can be explained

  $$eg = 2\pi N, \quad N = 1, 2, 3, \ldots$$

- Maxwell equations = $U(1)$ gauge theory. Nonabelian gauge theory: Replace $U(1)$ by a non-abelian group $SU(2)$. (Relevant in electro–weak model).

- $(A_j(x), \Phi(x))$ anti-hermitian 2x2 matrices on $\mathbb{R}^3$.

- Nonabelian magnetic field

  $$F_{jk} = \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} + [A_j, A_k], \quad j, k = 1, 2, 3.$$ 

- Monopole equation

  $$\frac{\partial \Phi}{\partial x^j} + [A_j, \Phi] = \frac{1}{2} \varepsilon_{jkl} F_{kl}.$$ 

  System of non-linear PDEs.
Twistor solution to the monopole equation

- Given \((A_j(x), \Phi(x))\) solve a matrix ODE along each oriented line \(x(t) = v + tu\)
  \[
  \frac{dV}{dt} + (w^j A_j + i\Phi)V = 0.
  \]
  Space of solutions at \(p \in \mathbb{R}^3\) is a complex vector space \(\mathbb{C}^2\).
- Complex vector bundle over \(\mathbb{T}\) with patching matrix \(F(\lambda, \bar{\lambda}, \eta, \bar{\eta})\).

Open covering
\[T = U \cup \bar{U}\]

Patching matrix
\[F: U \cup \bar{U} \to GL(2, \mathbb{C})\]

- Monopole equation \(\leftrightarrow\) Cauchy–Riemann eq.
  \[
  \frac{\partial F}{\partial \lambda} = 0, \frac{\partial F}{\partial \bar{\eta}} = 0.
  \]
- Holomorphic vector bundles over \(T \mathbb{C} P^1\) - well understood. Take one and work backwards to construct a monopole. (Hitchin, 1982.)
**Geometry of ODEs**

- $\Psi(x, y; t_1, \ldots t_n) = 0$

  ![Diagram](image)

  point $\longleftrightarrow$ hypersurface
  curve $\longleftrightarrow$ point

- Implicit function theorem $y = Z(x; t_1, \ldots, t_n)$. Differentiate $N$ times, eliminate $t_1, \ldots, t_n$

  $$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \quad \text{ODE}$$

- Classical problem: Take $n = 3$, assume $z \subset M$ is null w.r.t. some conformal structure on $M$. Conditions on $F$? (Wünschmann (1905)).

  $$\frac{1}{3} F_2 \frac{d}{dx} F_2 - \frac{1}{6} \frac{d^2}{dx^2} F_2 + \frac{1}{2} \frac{d}{dx} F_1 - \frac{2}{27} (F_2)^3 - \frac{1}{3} F_2 F_1 - F_0 = 0.$$
**GL(2, \mathbb{R}) STRUCTURES**

- Bundle isomorphism $TM \cong S \circ S \circ \cdots \circ S = S^{(n-1)}(S)$, where $S \to M$ is a rank 2 symplectic vector bundle.
- Identifies vectors with $(n-1)$st order homogeneous polynomials. $V^{AB\cdots C} \pi_A \pi_B \cdots \pi_C$ in $\pi_A = (\pi_0, \pi_1)$. (conformal structure if $n = 3$).
  - Embedding $SL(2, \mathbb{R}) \subset O(n)$ given by $\pi_A \to t^B A \pi_B$.

(MD, Paul Tod.) Assume that the space of solutions to the $n$th order ODE has a $GL(2, \mathbb{R})$ structure such that surfaces $y = Z(x; t)$ are maximally null (i.e. normal vector has repeated root with multiplicity $(n-1)$). Then $F$ satisfies $(n-2)$ Doubrov conditions. Equivalently, $L_p \subset \mathbb{T}$ is a rational curve with a normal bundle $O(n-1)$.

- The same structure came up recently in many different contexts
  1. **Bryant:** exotic holonomy in dimension 4.
  2. **Doubrov:** Linearise the ODE around a given solution and demand that the resulting linear homogeneous ODE be trivialisable, i.e. equivalent to $y^{(n)} = 0$ by a coordinate transformation $(x, y) \to (b(x), a(x)y)$.
  3. **Godlinski–Nurowski:** A general framework: restricting torsion of a Cartan connection.
Is there a $G_2$ structure on a space of solutions to 7th order ODE? (work in progress with Michal Godlinski and Pawel Nurowski).

1. Take a family of rational curves $x \rightarrow (x, y(x; t))$ with self–intersection number 6 in a complex surface $\mathbb{T}$. Normal vector (section of $O(6)$)

$$
\delta y = \frac{\delta y}{\delta t_i} \delta t_i = a_0 x^6 + 6a_1 x^5 + 15a_2 x^4 + 20a_3 x^3 + 15a_4 x^2 + 6a_5 x + a_6
$$

2. Classical invariant theory: a conformal structure and a three–form

$$
g = a_0 a_6 - 6a_1 a_5 + 15a_2 a_4 - 10a_3^2,
$$

$$
\phi = 3 (a_1 \wedge a_2 \wedge a_6 + a_0 \wedge a_4 \wedge a_5) + a_3 \wedge (a_0 \wedge a_6 + 6 a_1 \wedge a_5 - 15 a_2 \wedge a_4).
$$

3. Example: rational curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bidegree $(1, 3)$

$$
y = \frac{r_0 + r_1 x + r_2 x^2 + r_3 x^3}{s_0 + s_1 x + s_2 x^2 + s_3 x^3}
$$

gives

$$
a_i = \binom{6}{i}^{-1} \sum_{\alpha + \beta = i} (r_\alpha ds_\beta - s_\beta dr_\alpha).
$$

Want to learn more - go to Godlinski’s talk tomorrow.
When are the curves $z \subset M$ are unparametrised geodesic of some torsion–free connection (projective structure)

One-to-one correspondence between holomorphic projective structures on $M$ and complex surfaces $\mathbb{T}$ with a family of rational curves.

points $\longleftrightarrow$ geodesics

rational curves with self intersection number 1 $\longleftrightarrow$ points

Double fibration $\mathbb{T} = \mathbb{P}(TM)/D \leftarrow \mathbb{P}(TM) \rightarrow M$, where $D$ is the geodesic spray.

Relevant to my next talk (in 15 minutes).
Twistor Theory

- Non-local construction with roots in the 19th century Klein correspondence (projective geometry).

  \[
  \begin{align*}
  \text{point} & \iff \text{line (complex)} \\
  \text{line} & \iff \text{point}.
  \end{align*}
  \]

- Complex numbers are essential

  \[
  \text{nonlinear PDEs} \iff \text{linear Cauchy–Riemann equations}.
  \]

- Its status as a physical theory is not clear
  - Weakness: effective only in low dimensions.
  - Strength: effective only in low dimensions.