G_2-STRUCTURES AND TWISTOR THEORY

Maciej Dunajski

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University of Cambridge

- Joint with Tod, Godliński, Sokolov, Doubrov.
- Bulids on Cayley, Sylvester, Penrose, Hitchin, Bryant.

MD, Tod `arXiv:math/0502524`, J. Geom. Phys. (2006)
MD, Godliński `arXiv:math/1002.3963`, Quart. J. Math. (2012)
MD, Sokolov `arXiv:math/1002.1620`. J. Geom. Phys. (2011)
Doubrov, MD `arXiv:math/1107.2813`. Ann. Glob. Anal. (2012)

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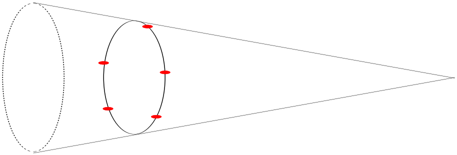

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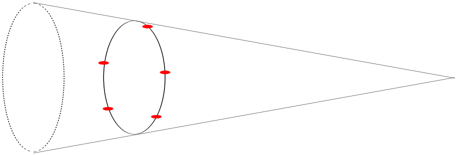
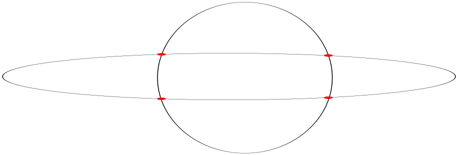


- Fourth jet at a point determines a conic (Halphen 1879)
$y^2 + \alpha x^2 + \beta\, xy + \gamma\, y + \delta\, x + \epsilon = 0$. Differentiate five times

$$9(y^{(2)})^2 y^{(5)} - 45 y^{(2)} y^{(3)} y^{(4)} + 40 (y^{(3)})^3 = 0.$$

Geometry of plane conics

- $GL(2)$ structure on $M = SL(3)/SL(2)$. $T_cM = \mathsf{Sym}^4(\mathbb{C}^2)$. Vectors=binary quartics $a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.



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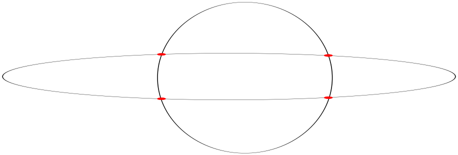

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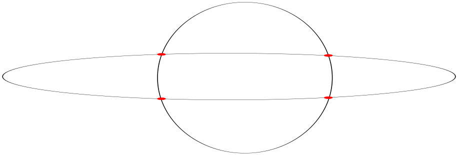

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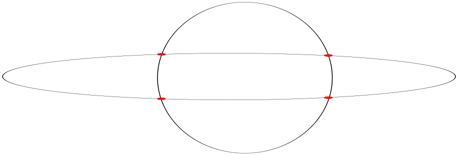

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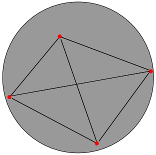
- Conformal structure on M: $V \in \Gamma(TM)$ is null iff $I(V) = 0$.



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- Examples from twistor theory/algebraic geometry.
- Mixture of *old* and *new*: Classical invariant theory (Young, Sylvester), algebraic geometry, twistor theory (Penrose, Hitchin).

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where $\tau_0 \in \Lambda^0(M), \tau_1 = \Lambda^1(M), \tau_2 = \Lambda^2(M), \tau_3 \in \Lambda^3(M)$ satisfy

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G_2 structures and Fernandez–Gray types

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2. Weak G_2 holonomy $\tau_1 = \tau_2 = \tau_3 = 0$. Implies g is Einstein.

$$\begin{aligned} d\phi &= \tau_0 * \phi + \frac{3}{4}\tau_1 \wedge \phi + *\tau_3 \\ d*\phi &= \tau_1 \wedge *\phi - \tau_2 \wedge \phi, \end{aligned}$$

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- Transvectants (Grace, Young 1903), or two component spinors (Penrose).

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SEVEN DIMENSIONS AND G_2 GEOMETRY

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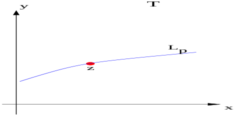
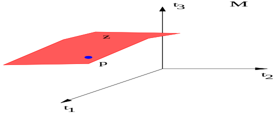
- $GL(2) \subset (G_2)^{\mathbb{C}} \times \mathbb{C}^*$. Really follows from Morozov's theorem.

- Assume that the space of solutions to the 7th order ODE

$$y^{(7)} = F(x, y, y', \dots, y^{(6)})$$

has a $GL(2,\mathbb{R})$ structure such that normals to surfaces $y = y(x;t)$ have root with multiplicity 6. Then F satisfies five contact–invariant conditions $W_1[F] = \cdots = W_5[F] = 0$.





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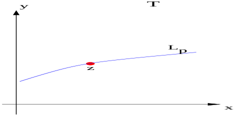

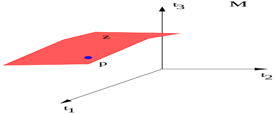

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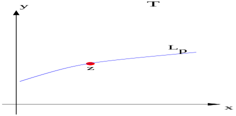

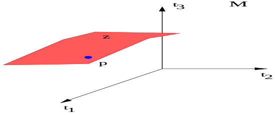

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- If $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$ then g is conformally flat and $y^{(7)} = 0$.

- Family of rational curves L_t parametrised by $t \in M$. $x \to (x, y(x;t))$ with self–intersection number six in a complex surface Z. Normal vector
$$\delta y = \sum_{\alpha=1}^{6} \frac{\delta y}{\delta t_\alpha} \delta t_\alpha$$
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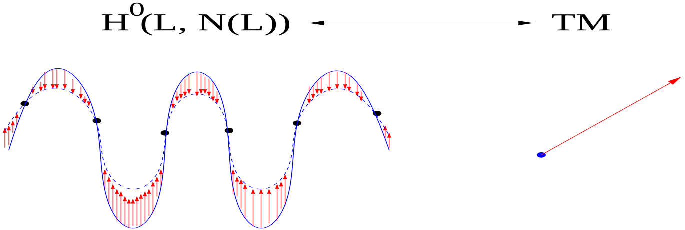
- $H^1(L, N(L)) = 0$. Kodaira Theory: $T_t M \cong H^0(L_t, N(L_t))$.



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- Sections of $\mathcal{O}(6) \to \mathbb{CP}^1$=homogeneous polynomials of degree 6. $GL(2)$ structure.
- In practice: $f(x, y, t_\alpha) = 0$ with rational parametrisation $x = p(\lambda, t_\alpha), y = q(\lambda, t_\alpha)$. Polynomial in λ giving rise to a null vector is given by

$$\sum_\alpha \frac{\partial f}{\partial t_\alpha}|_{\{x=p,y=q\}} \delta t_\alpha.$$

Three examples

1. Example 1.
   - Rational curve: cuspidial cubic. (Neil 1657).
   - 7th order ODE: (Halphen 1879, Sylvester 1888, Wilczynski 1905).
   - Co–calibrated G_2 structure on $SU(2,1)/U(1)$. (MD, Doubrov 2011).

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2. Example 2.
   - Rational curve: Bihorn sextic.
   - 7th order ODE: (Wilczynski).
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   - Rational curve: cuspidial cubic. (Neil 1657).
   - 7th order ODE: (Halphen 1879, Sylvester 1888, Wilczynski 1905).
   - Co–calibrated G_2 structure on $SU(2,1)/U(1)$. (MD, Doubrov 2011).
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   - Rational curve: Bihorn sextic.
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   - Closed G_2 structure (MD, Godliński 2010).
3. Example 3.
   - Rational curve: (MD, Sokolov 2010).
   - 7th order ODE: (Noth 1904).
   - Weak G_2 holonomy on $SO(5)/SO(3)$ (Bryant 1987).

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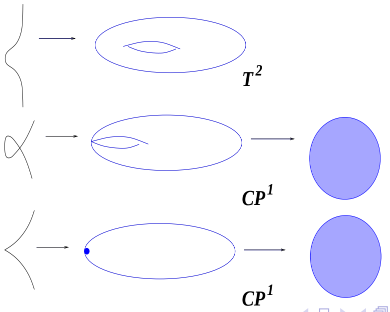

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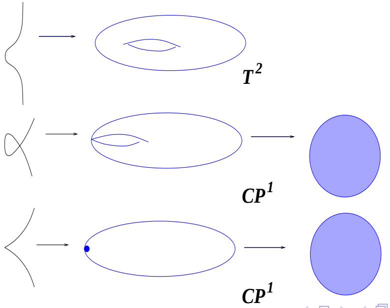

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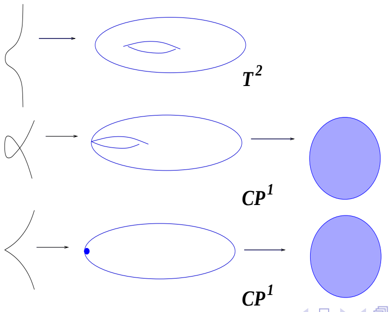

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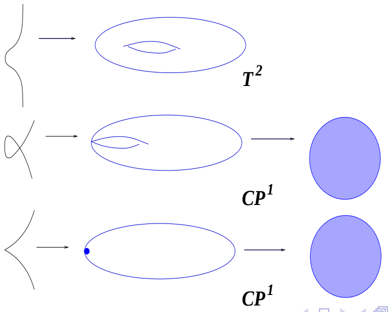

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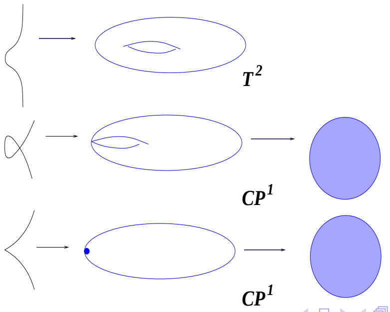

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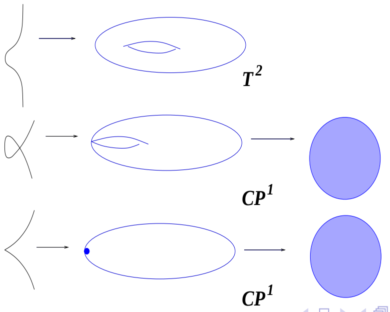

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- Set $\sigma = N^{-1}dN \in \Lambda^1(SL(3,\mathbb{C})) \otimes \mathfrak{sl}(3,\mathbb{C})$.

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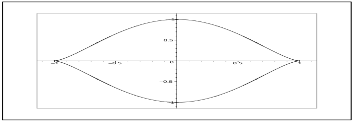
- Co–calibrated G_2 structure $d\phi = \lambda * \phi + \tau, \; d * \phi = 0$.

Example 2: Closed G_2 from bihorn sextics.

- $(y + Q(x))^2 + P(x)^3 = 0$, where

$$Q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3, \quad P(x) = p_3(x - p_2)(x - p_1).$$



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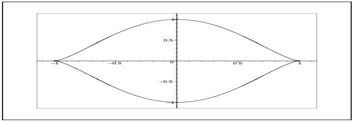

- Two double points and one irregular quadruple point at ∞. g $= 0$.

$$x(\lambda) = \frac{p_1 + p_2\lambda^2}{\lambda^2 + 1}, \quad y(\lambda) = {p_3}^{3/2}(p_1 - p_2)^3 \frac{\lambda^3}{(\lambda^2 + 1)^3} - Q(x(\lambda)).$$

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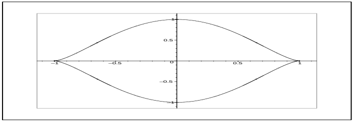

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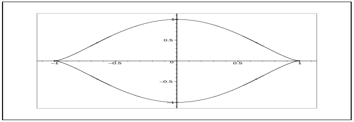

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- Closed Riemannian G_2 structure - explicit but messy.

Example 3: Weak G_2 from submaximal ODE

- Contact geometry: $(x, y) \in Z$, $(x, y, z) \in P(TZ)$, contact form $\omega = dy - zdx$. Generators of contact transformations, $H = H(x, y, z)$

$$X_H = -(\partial_z H)\partial_x + (H - z\partial_z H)\partial_y + (\partial_x H + z\partial_y H)\partial_z.$$

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- Two real forms of G_2 structures on $Sp(4)/SL(2)$, one of which is a Riemannian homogeneous space $SO(5)/SO(3)$ (Bryant 1987).

Example 3: Weak G_2 from submaximal ODE

$$\left(c_4y + c_1 + c_2x + c_3x^2\right)^3 + 3\,\left(c_4y + c_1 + c_2x + c_3x^2\right)$$
$$\left(3\,\left(c_5x + c_6\right)^4 - 6\,\left(c_5x + c_6\right)^2\left(1 - c_7x\right)^2 - \left(1 - c_7x\right)^4\right)$$
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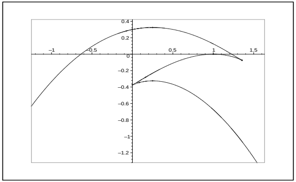

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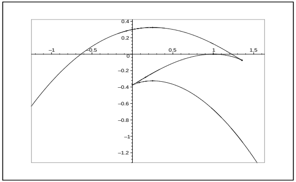

Discriminant of this cubic (in y) is a 3rd power of a quartic with equianharmonic cross–ratio.

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  - Sextic (relevant in this talk) -??