

## Four-dimensional metrics conformal to Kähler

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### *Abstract*

We derive some necessary conditions on a Riemannian metric  $(M, g)$  in four dimensions for it to be locally conformal to Kähler. If the conformal curvature is non anti-self-dual, the self-dual Weyl spinor must be of algebraic type  $D$  and satisfy a simple first order conformally invariant condition which is necessary and sufficient for the existence of a Kähler metric in the conformal class. In the anti-self-dual case we establish a one to one correspondence between Kähler metrics in the conformal class and non-zero parallel sections of a certain connection on a natural rank ten vector bundle over  $M$ . We use this characterisation to provide examples of ASD metrics which are not conformal to Kähler.

We establish a link between the ‘conformal to Kähler condition’ in dimension four and the metrisability of projective structures in dimension two. A projective structure on a surface  $U$  is metrisable if and only if the induced  $(2, 2)$  conformal structure on  $M = TU$  admits a Kähler metric or a para-Kähler metric.

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### 1. Introduction

A Kähler structure on a  $2n$ -dimensional real manifold  $M$  consists of a pair  $(g, J)$  where  $g$  is a Riemannian metric and  $J: TM \rightarrow TM$  is a complex structure such that, for any vector fields  $X, Y$ ,

$$g(X, Y) = g(JX, JY)$$

and the two-form  $\Sigma$  defined by

$$\Sigma(X, Y) = g(X, JY)$$

is closed. Consider the following local problem

Given a Riemannian metric  $g$ , is there a Kähler metric in the conformal class

$$[g] = \{cg \mid c: M \rightarrow \mathbb{R}^+\}$$

of  $g$ ?

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We expect the answer to depend only on the conformal class  $[g]$ , thus the local obstructions to the existence of a Kähler metric in  $[g]$  should be given by conformally invariant tensors.

The general Riemannian metric in  $2n$  dimensions depends on  $n(2n+1)$  arbitrary functions of  $2n$  variables. The diffeomorphism freedom can be used to reduce this number to  $2n^2 - n$  and finally the freedom of conformally rescaling the metric leaves one with  $2n^2 - n - 1$  arbitrary functions. On the other hand the general Kähler metric can be locally described by the Kähler potential: there exists a function  $\mathcal{K}: M \rightarrow \mathbb{R}$  and a holomorphic coordinate system  $(z^1, \dots, z^n)$  such that

$$g = \frac{\partial^2 \mathcal{K}}{\partial z^j \partial \bar{z}^k} dz^j d\bar{z}^k.$$

There is some freedom in the Kähler potential  $\mathcal{K}$ , but this freedom depends on functions of  $n$  variables. Thus essentially the Kähler metric depends on one arbitrary function of  $2n$  variables. The difference between the number of arbitrary functions in the general conformal class and the general Kähler metric is

$$2n^2 - n - 2$$

which is positive if  $n > 1$ . This gives a lower bound for the number of conditions a Riemannian metric needs to satisfy in order to be conformal to a Kähler metric.

If  $n = 1$  then the general Riemannian metric is conformally flat. In this case one does not need conformal rescaling – any Riemannian metric on a surface is Kähler as in two dimension a conformal structure is equivalent to a complex structure. The first non-trivial case is  $n = 2$ , where  $(M, g)$  is a real four-manifold. There the conformal to Kähler condition is overdetermined: the numerology tells us to expect at least four conditions on  $g$ .

In Section 2 we derive some necessary conditions on a Riemannian metric in four dimensions for it to be locally conformal to Kähler. If the self-dual part of the conformal curvature of  $g$  is non-zero then necessary and sufficient conditions are relatively easy to find: The self-dual (SD) Weyl spinor must be of algebraic type  $D$  (i.e. it must have two repeated roots when viewed as a symmetric homogeneous polynomial in two variables of degree four), and a differential obstruction (given in Theorem 2.2) must vanish.

In the anti-self-dual (ASD) case, when the SD Weyl tensor vanishes, the analysis is more complicated. We shall construct a natural connection  $\mathcal{D}$  on a rank ten vector bundle

$$E = \Lambda_+^2(M) \oplus \Lambda^1(M) \oplus \Lambda_-^2(M)$$

over  $M$  and show (Theorem 2.3) that there is a one-to-one correspondence between Kähler metrics in the conformal class and non-zero parallel sections of  $\mathcal{D}$ . Readers familiar with the tractor approach to conformal geometry [3] may want to note that the vector bundle  $E$  can be thought of as the space of self-dual three-forms for the standard tractors  $T$  of  $(M, [g])$ , but the induced tractor connection on  $(\Lambda^3 T)_+$  is different from  $\mathcal{D}$ . These two connections coincide only in the conformally-flat case.

In Section 3 we provide examples of metrics which are not conformal to Kähler. In the non-ASD case any metric not of type  $D$  gives such an example. In the ASD case we find that an Einstein metric with non-zero cosmological constant can be conformal to Kähler if and only if it admits a Killing vector. We then argue, using the construction of LeBrun [11], that ASD Einstein metrics with no symmetries exist. Another class of examples is provided by a family of hyper-hermitian metrics (these are automatically ASD) which depends on a

(locally) harmonic function on the three-sphere [6, 16]. In Section 3.4 we shall show that the harmonic function can be chosen so that no Kähler metric exists in the conformal class.

In Section 4 we shall make a link between the ‘conformal to Kähler condition’ in dimension four and the metrisability of projective structures in dimension two. A projective structure on a manifold  $U$  is an equivalence class of torsion-free connections which share the same unparametrised geodesics. In [5] necessary and sufficient conditions have been determined in the case when  $\dim U = 2$  for the existence of a (pseudo) Riemannian metric on  $U$  whose geodesics coincide with the geodesics of the given projective structure. If such a metric exists, the projective structure is called metrisable. Any  $n$ -dimensional projective structure on  $U$  gives rise to a natural conformal structure of signature  $(n, n)$  on  $TU$  [24, 26]. If  $n = 2$  this conformal structure is necessarily ASD [9, 15] and we shall show (Theorem 4.1) that the projective structure on a surface  $U$  is metrisable if and only if the induced conformal structure on  $TU$  admits a Kähler metric or a para-Kähler metric. This establishes a conjecture made in [5]. In Section 4.1 we shall characterise the resulting  $(2, 2)$  ASD conformal structures as those which admit a real parallel section of the unprimed spin bundle.

All considerations in this paper are local. We shall end this introduction listing some global obstructions.<sup>1</sup> If  $(M, [g])$  is a compact ASD conformal Riemannian four-manifold then the existence of a Kähler metric in the conformal class  $[g]$  is equivalent to the following pair of conditions:

- (i) there is a metric in  $[g]$  with vanishing scalar curvature.
- (ii) the number  $b_+(M)$  of positive eigenvalues of the intersection form is non-zero.

The second condition is purely topological, while the first condition is equivalent to vanishing of the Yamabe constant of  $[g]$ . These conditions impose constraints on the diffeomorphism type of  $M$ . The only simply connected  $M$  that are allowed are  $K3$  and  $\mathbb{C}P_2 \# k\overline{\mathbb{C}P}_2$ ,  $k \geq 10$ . See [10, 12, 14, 21] for details.

### 1.1. Spinors in four dimensions

Given an oriented Riemannian four-manifold  $(M, g)$ , the Hodge operator  $*$  :  $\Lambda^2 \rightarrow \Lambda^2$  satisfies  $*^2 = \text{Id}$  and induces a decomposition

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2 \tag{1.1}$$

of 2-forms into self-dual and anti-self-dual components. The rank three vector bundles  $\Lambda_\pm^2 \rightarrow M$  are the eigenspaces of  $*$  with eigenvalues  $\pm 1$  respectively. The Riemann tensor has the index symmetry  $R_{abcd} = R_{[ab][cd]}$  so can be thought of as a map  $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$ . This map decomposes under (1.1) as follows:

$$\mathcal{R} = \left( \begin{array}{c|c} C_+ + \frac{R}{12} & \phi \\ \hline \phi & C_- + \frac{R}{12} \end{array} \right). \tag{1.2}$$

The  $C_\pm$  terms are the self-dual and anti-self-dual parts of the Weyl tensor, the  $\phi$  terms are the trace-free Ricci curvature, and  $R$  is the scalar curvature which acts by scalar multiplication. The Weyl tensor is conformally invariant, so can be thought of as being defined by the conformal structure  $[g]$ .

<sup>1</sup> We thank Claude LeBrun for bringing these obstructions to our attention.

Locally there exist complex rank two vector bundles  $\mathbb{S}, \mathbb{S}'$  (called spin-bundles) over  $M$  equipped with parallel symplectic structures  $\varepsilon, \varepsilon'$  such that

$$\mathbb{C} \otimes TM \cong \mathbb{S} \otimes \mathbb{S}' \quad (1.3)$$

is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2)$$

for  $v_1, v_2 \in \Gamma(\mathbb{S})$  and  $w_1, w_2 \in \Gamma(\mathbb{S}')$ . We use the standard convention [18] in which spinor indices are capital letters, unprimed for sections of  $\mathbb{S}$  and primed for sections of  $\mathbb{S}'$ . For example  $\mu_A$  denotes a section of  $\mathbb{S}^*$ , the dual of  $\mathbb{S}$ , and  $\nu^{A'}$  a section of  $\mathbb{S}'$ . The symplectic structures  $\varepsilon_{AB}$  and  $\varepsilon^{A'B'}$  (such that  $\varepsilon_{01} = \varepsilon_{0'1'} = 1$ ) are used to raise and lower indices by  $\mu_A := \mu^B \varepsilon_{BA}, \mu^A = \varepsilon^{AB} \mu_B$ .

The decomposition (1.1) of two-forms takes a simple form in the spinor notation. If  $F_{ab} = F_{[ab]}$  is a two-form then

$$F_{AA'BB'} = f_{AB}\varepsilon_{A'B'} + \tilde{f}_{A'B'}\varepsilon_{AB}, \quad (1.4)$$

where  $f_{AB}$  and  $\tilde{f}_{A'B'}$  are symmetric in their indices. This is precisely the decomposition of  $F$  into self-dual and anti-self dual parts. Thus

$$\Lambda_+^2 \cong \mathbb{S}'^* \odot \mathbb{S}'^*, \quad \Lambda_-^2 \cong \mathbb{S}^* \odot \mathbb{S}^*.$$

In terms of the spinor notation the decomposition (1.1) of the Riemann tensor is

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad + \phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &\quad + \frac{R}{12}(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}), \end{aligned} \quad (1.5)$$

where  $\psi_{ABCD}$  and  $\psi_{A'B'C'D'}$  are ASD and SD Weyl spinors which are symmetric in their indices and  $\phi_{A'B'CD} = \phi_{(A'B')(CD)}$  is the traceless Ricci spinor. A conformal structure is called ASD iff  $\psi_{A'B'C'D'} = 0$ .

## 2. Conformal-to-Kähler conditions

Let  $(M, g)$  be a four manifold such that

$$\hat{g}_{ab} = \Omega^2 g_{ab} \quad (2.6)$$

is a Kähler metric with a Kähler form  $\hat{\Sigma}$ . The two-form  $\hat{\Sigma}$  on  $M$  induces a natural orientation given by the volume form  $\hat{\Sigma} \wedge \hat{\Sigma}$ . With respect to this orientation  $\hat{\Sigma}$  is self-dual. Thus, in terms of 2-component spinors, the Kähler form may be written as

$$\hat{\Sigma}_{ab} = \hat{\omega}_{A'B'}\hat{\varepsilon}_{AB},$$

so that

$$\begin{aligned} \hat{\nabla}_{AA'}\hat{\omega}_{B'C'} &= 0, \\ \hat{\omega}_{A'B'}\hat{\omega}^{A'B'} &= 2. \end{aligned}$$

Set

$$\Upsilon_a = \Omega^{-1}\nabla_a\Omega.$$

Under conformal rescaling we shall make the choice that  $\hat{\omega}_{A'B'}$  transforms as

$$\hat{\omega}_{A'B'} = \Omega^2 \omega_{A'B'}, \quad (2.7)$$

since then

$$\hat{\nabla}_{AA'} \hat{\omega}_{B'C'} = \Omega^2 (\nabla_{AA'} \omega_{B'C'} - \Upsilon_{B'A} \omega_{A'C'} - \Upsilon_{C'A} \omega_{B'A'} + 2\Upsilon_{AA'} \omega_{B'C'}) = 0, \quad (2.8)$$

so that

$$\hat{\nabla}_{A(A'} \hat{\omega}_{B'C')} = \Omega^2 \nabla_{A(A'} \omega_{B'C')}.$$

We have therefore arrived at the following result of Pontecorvo [19].

LEMMA 2.1. *The metric  $g_{ab}$  is conformal to a Kähler metric if and only if there exists a real, symmetric spinor field  $\omega_{A'B'}$  satisfying*

$$\nabla_{A(A'} \omega_{B'C')} = 0, \quad (2.9)$$

and such that  $\omega_{A'B'} \omega^{A'B'} \neq 0$ .

*Proof.* The condition (2.9) is equivalent to

$$\nabla_{AA'} \omega_{B'C'} = \varepsilon_{A'B'} K_{C'A} + \varepsilon_{A'C'} K_{B'A}. \quad (2.10)$$

for some one-form  $K_a$ . Note that (2.10) follows from (2.8) with

$$K_{AA'} = -\omega_{A'}^{B'} \Upsilon_{B'A}, \quad (2.11)$$

and  $\Omega$  can be found from

$$\omega_{A'B'} \omega^{A'B'} = 2\Omega^{-2}. \quad (2.12)$$

Suppose instead we have a real solution of (2.9), so that also (2.10) holds for some  $K_{AA'}$ . Define  $\Omega$  by (2.12), then one may calculate that

$$\Upsilon_{AA'} = \Omega^2 \omega_{A'}^{C'} K_{C'A}, \quad (2.13)$$

and then the right-hand side of (2.8) is zero, so that  $g_{ab}$  is indeed conformal to Kähler, provided  $\Omega$  has no zeroes. Thus the existence of a real solution of (2.9) is necessary and sufficient for the metric to be conformal to Kähler, in regions where  $\Omega \neq 0$ .

We want to find conditions on curvature for a suitable solution of (2.10) to exist. We shall use the procedure of prolongation to introduce new variables, and rewrite (2.9) as a closed overdetermined system of first order PDEs which then can be investigated for Frobenius integrability.

Differentiate and commute derivatives on (2.10) to obtain

$$\psi_{(A'B'C')\omega^{D')E'}}^{E'} = 0, \quad (2.14)$$

in terms of the SD Weyl spinor  $\psi_{E'A'B'C'}$ , and

$$\nabla_{AA'} K_{BB'} = -P_{ABA'C'} \omega_{B'}^{C'} + \varepsilon_{A'B'} \rho_{AB} \quad (2.15)$$

in terms of some as yet unknown but symmetric  $\rho_{AB}$ , where  $P_{ABA'B'}$  is given in terms of the Ricci spinor  $\phi_{ABA'B'}$  and curvature scalar  $\Lambda = R/24$  by

$$P_{ABA'B'} = \phi_{ABA'B'} - \Lambda \varepsilon_{AB} \varepsilon_{A'B'}.$$

2.1. *The non anti-self-dual case*

Suppose that  $\psi_{A'B'C'D'} \neq 0$ , then (2.14) has no solution for non-null (hence for real)  $\omega_{A'B'}$  unless

$$\psi_{A'B'C'D'} = c \Theta_{(A'B'} \Theta_{C'D')}, \quad (2.16)$$

for some function  $c$  and real symmetric  $\Theta_{A'B'}$  (one can always choose  $\Theta_{A'B'}$  to set  $c$  to  $\pm 1$ ). If (2.16) does hold, which is equivalent to type-D SD Weyl spinor in the Petrov–Pirani–Penrose classification ( see e.g. [18], or [2, 7] for discussion relevant in the compact case), then any solution  $\omega_{A'B'}$  of (2.14) is proportional to  $\Theta_{A'B'}$ . Thus in the case  $\psi_{A'B'C'D'} \neq 0$ , one may determine whether the metric is conformal to Kähler by simply checking if any candidate  $\omega_{A'B'} = f \Theta_{A'B'}$  solves (2.10) and is appropriately nonzero. Following [18] define two conformal invariants

$$\mathcal{I} = |\psi|^2 = \psi_{A'B'C'D'} \psi^{A'B'C'D'}, \quad \mathcal{J} = \psi_{A'B'}{}^{C'D'} \psi_{C'D'}{}^{E'F'} \psi_{E'F'}{}^{A'B'}$$

of conformal weight  $-4$  and  $-6$  respectively. Define a spinor

$$T_{AA'B'C'D'E'} := \nabla_{A(A'} \psi_{B'C'D'E')} - V_{A(A'} \psi_{B'C'D'E')}, \quad (2.17)$$

where

$$V_{AA'} = \frac{1}{|\psi|^2} \left( \frac{1}{6} \nabla_{AA'} |\psi|^2 + \frac{4}{3} \psi^{B'C'D'E'} \nabla_{AB'} \psi_{C'D'E'A'} \right). \quad (2.18)$$

**THEOREM 2.2.** *Let  $(M, [g])$  be a conformal manifold such that the self-dual part of the conformal curvature of  $[g]$  is non-zero. Then there exists a Kähler metric in the conformal class if and only if the following conformally invariant conditions hold*

$$\mathcal{I}^3 = 6\mathcal{J}^2 \quad (2.19)$$

$$T_{AA'B'C'D'E'} = 0, \quad (2.20)$$

$$\nabla_{[a} V_{b]} = 0. \quad (2.21)$$

*Proof.* Consider a metric  $g$  with a SD Weyl tensor satisfying (2.16). The Weyl spinor corresponds to a homogeneous polynomial in two variables of degree 4 with two repeated roots, and (2.19) holds identically. If the conformal class of  $g$  contains a Kähler metric then  $\Theta_{A'B'}$  satisfies (2.9). Differentiating (2.16) and symmetrising over the primed indices gives

$$\nabla_{A(A'} \psi_{B'C'D'E')} = V_{A(A'} \psi_{B'C'D'E')} \quad (2.22)$$

where  $V_{AA'}$  is a gradient. We shall analyse (2.22) for general  $V_{AA'}$ . Contracting both sides with  $\psi^{B'C'D'E'}$  and using the identity

$$\psi^{B'C'D'E'} \psi_{B'C'D'A'} = \frac{1}{2} \delta_{A'}^{E'} |\psi|^2$$

yields the expression (2.18) for  $V$ . Substituting this back to (2.22) gives the vanishing of  $T_{AA'B'C'D'}$  defined by (2.17). Now we need to establish the conformal invariance of conditions (2.20) and (2.21). Under the conformal rescaling (2.6) we have

$$\hat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}, \quad \hat{\varepsilon}^{A'B'} = \Omega^{-1} \varepsilon^{A'B'}$$

and

$$\hat{\psi}_{A'B'C'D'} = \psi_{A'B'C'D'}, \quad \hat{\psi}^{A'B'C'D'} = \Omega^{-4} \psi^{A'B'C'D'}.$$

Using

$$\begin{aligned} \hat{\nabla}_{AB'}\hat{\psi}_{C'D'E'A'} &= \nabla_{AB'}\psi_{C'D'E'A'} \\ &\quad - \Upsilon_{AC'}\psi_{B'D'E'A'} - \Upsilon_{AD'}\psi_{B'C'E'A'} - \Upsilon_{AE'}\psi_{B'C'D'A'} - \Upsilon_{AA'}\psi_{B'C'D'E'} \end{aligned}$$

gives

$$\hat{V}_{AA'} = V_{AA'} - 4\Upsilon_{AA'} \quad (2.23)$$

and

$$\hat{\nabla}_{A(B'}\hat{\psi}_{C'D'E'A')} = \nabla_{A(B'}\psi_{C'D'E'A')} - 4V_{A(B'}\psi_{C'D'E'A')}.$$

Therefore

$$\hat{T}_{AA'B'C'D'} = T_{AA'B'C'D'}$$

and condition (2.20) is conformally invariant. The closure of  $V$  is also conformally invariant because of (2.23).

Conversely, assume that conditions (2.19), (2.20) and (2.21) hold. The algebraic condition (2.19) implies that  $\psi_{A'B'C'D'}$  has a repeated root. There are various possibilities, but the only one compatible with the Riemannian signature of  $g$  is that  $\psi_{A'B'C'D'}$  is given by (2.16) for some  $\Theta_{A'B'}$ . Now construct  $V_{AA'}$  and  $T_{AA'B'C'D'}$ . The closure condition (2.21) implies that locally  $V$  is a gradient. The relation (2.23) can now be used to find a conformal factor such that  $V = 0$  and (2.20) reduces to

$$\nabla_{A(A'}\psi_{B'C'D'E')} = 0$$

which implies

$$\Theta_{(B'C'}\nabla^A_{A'}\Theta_{D'E')} = 0$$

where we have reabsorbed  $\phi$  into  $\Theta$ . This is equivalent to

$$\nabla_{A(A'}\Theta_{B'C')} = 0.$$

Thus, by Lemma 2.1, there exists a Kähler metric in the conformal class of  $g_{ab}$ .

## 2.2. The anti-self-dual case

The harder case is therefore zero SD Weyl spinor, so henceforth we shall assume that  $\psi_{A'B'C'D'} = 0$  but  $\psi_{ABCD} \neq 0$  i.e. the Weyl tensor is nonzero but ASD.

Commute derivatives on (2.15) to obtain

$$\nabla_{AA'}\rho_{BC} = \omega_{A'}^{E'}\nabla_{E'}^D\psi_{ABCD} - K_{A'}^D\psi_{ABCD} - 2P_{A'E'(A}K_{C)}^{E'} \quad (2.24)$$

using curvature assumptions already made.

Suppose for a moment that  $g_{ab}$  actually is Kähler. Then (2.9, 2.10, 2.15) must hold with constant  $\Omega$ , in which case  $K_a$  vanishes but from (2.15) we deduce

$$\Lambda = 0; \quad \phi_{ABA'B'} = \omega_{A'B'}\rho_{AB}$$

and we may identify  $\rho_{AB}\epsilon_{A'B'}$  as the Ricci form. In this case (2.24) becomes an identity and this gives an understanding of what  $\rho$  is in general.

In the general case, we can next commute derivatives on  $\rho$ . In terms of  $\Delta_{A'B'} := \nabla_{A(A'}\nabla_{B')}^A$ , we obtain an identity from  $\Delta_{A'B'}\rho_{AB}$  (using the vanishing of the Bach tensor, which holds

for any metric with ASD or SD Weyl tensor [18]) but something new arises from  $\Delta_{AB}\rho_{CD}$  namely the necessary condition:

$$4\psi_{E(ABC\rho_D)^E} + K^{EE'}\nabla_{EE'}\psi_{ABCD} + 4K_{E'(A}\nabla^{EE'}\psi_{BCD)E} + \omega_{E'A'}(\nabla_{(A'}\nabla^{EE'}\psi_{BCD)E} + \psi_{E(ABC}\phi_{D)}^{E'E'}) = 0. \quad (2.25)$$

This represents five linear restrictions on the ten-component column vector

$$X_{\mathcal{A}} = (\omega_{A'B'}, K_{AA'}, \rho_{AB})^T.$$

This is not enough to be overdetermined, but it and its derivatives have to hold, which eventually gives constraints on the conformal curvature, as we shall see with an example below.

### 2.3. The prolongation bundle

The formalism has led us to consider a natural rank-10 vector bundle<sup>2</sup>

$$E = \Lambda_+^2(M) \oplus \Lambda^1(M) \oplus \Lambda_-^2(M)$$

with sections  $X_{\mathcal{A}}$  which have the following behaviour under conformal rescaling of the metric:

$$\hat{X}_{\mathcal{A}} := \begin{pmatrix} \hat{\omega}_{A'B'} \\ \hat{K}_{AA'} \\ \hat{\rho}_{AB} \end{pmatrix} = \begin{pmatrix} \Omega^2\omega_{A'B'} \\ \Omega(K_{AA'} + \Upsilon_{AB'}\omega_{A'}^{B'}) \\ \rho_{AB} + 2\Upsilon_{C'(A}K_{B)}^{C'} + \omega^{C'D'}\Upsilon_{C'A}\Upsilon_{D'B} \end{pmatrix}. \quad (2.26)$$

This transformation recalls that of a tractor, [3], and as we noted in the introduction the bundle  $E$  can be understood as the bundle of self-dual 3-forms  $\Lambda_+^3(T)$  where  $T$  is the usual tractor bundle (although with a different connection). Define a derivative  $\mathcal{D}$  on this vector bundle by

$$\mathcal{D}_a X_B := \begin{pmatrix} \nabla_{AA'}\omega_{B'C'} - \varepsilon_{A'B'}K_{C'A} - \varepsilon_{A'C'}K_{B'A} \\ \nabla_{AA'}K_{BB'} + P_{ABA'C'}\omega_{B'}^{C'} - \varepsilon_{A'B'}\rho_{AB} \\ \nabla_{AA'}\rho_{BC} - \omega_{A'}^{E'}\nabla_{E'}^D\psi_{ABCD} + K_{A'}^D\psi_{ABCD} + 2P_{A'E'A(B}K_{C)}^{E'} \end{pmatrix}. \quad (2.27)$$

We can now state our result in the following way

**THEOREM 2.3.** *A four-dimensional Riemannian manifold  $(M, g)$  with ASD conformal curvature is locally conformal to Kähler if and only if there exists a parallel non-zero section  $X_B$  for  $\mathcal{D}$ :*

$$\mathcal{D}_a X_B = 0 \quad (2.28)$$

such that  $\omega_{A'B'}\omega^{A'B'} \neq 0$ .

We calculate the curvature of this connection from the general formula

$$\mathcal{D}_{[a}\mathcal{D}_{b]}X_C = \frac{1}{2}\mathcal{R}_{ab}{}^{\varepsilon}{}_{C}X_{\varepsilon},$$

to find that  $\mathcal{R}_{ab}{}^{\varepsilon}{}_{C}$  is ASD on the index pair  $ab$ , so that the only nonzero components are  $\mathcal{R}_{ABC}{}^{\varepsilon}$ .

<sup>2</sup> A similar construction, albeit not in a conformally invariant setting, was used in [22] in the study of conformal Killing forms.

From (2.28) we may deduce a necessary condition to be conformal to Kähler as

$$\mathcal{R}_{ab\dot{C}}^{\dot{\mathcal{E}}} X_{\dot{\mathcal{E}}} = 0,$$

and we claim that (2.25) is precisely

$$\mathcal{R}_{AB\dot{C}}^{\dot{\mathcal{E}}} X_{\dot{\mathcal{E}}} = 0.$$

To see how many conditions follow by differentiating (2.25), we calculate the Bianchi identity:

$$\mathcal{D}_{[a}\mathcal{R}_{bc]\dot{C}}^{\dot{\mathcal{E}}} = 0$$

which given ASD reduces to

$$\mathcal{D}_{A'}^A \mathcal{R}_{AB\dot{C}}^{\dot{\mathcal{E}}} = 0.$$

Thus from differentiating (2.25), we obtain just

$$\mathcal{D}_{A'(A}\mathcal{R}_{BC)\dot{C}}^{\dot{\mathcal{E}}} X_{\dot{\mathcal{E}}} = 0. \quad (2.29)$$

This turns out to be a further twelve conditions, so that we have seventeen linear conditions on the ten-component  $X_{\mathcal{A}}$ .

### 3. Examples of Riemannian four-metrics not conformal to Kähler

#### 3.1. Nonzero SD Weyl spinor

A Riemannian 4-metric whose Weyl tensor has a non-zero SD Weyl spinor which is not type D cannot be conformal to Kähler. Examples could be found among the self-dual vacuum metrics, though these would be Kähler with the opposite orientation.

Other examples may be given as follows: consider the metric

$$g = \left( \delta_{ab} + \frac{1}{2} R_{abcd}^0 x^c x^d \right) dx^a dx^b,$$

where  $R_{abcd}^0$  is a constant tensor with Riemann tensor symmetries. The Riemann tensor of this metric, calculated at the origin of the coordinates  $x^a$ , is  $R_{abcd}(0) = R_{abcd}^0$  so that, if this does not have type-D SD Weyl tensor then this metric is not conformal to Kähler in any neighbourhood of the origin. By including more terms in the Taylor series, one may construct examples which satisfy the first condition of Theorem 2.2 but not the later ones.

#### 3.2. ASD Einstein

Examples of ASD conformal classes which do not contain a Kähler metric would be given by ASD Einstein metrics with non-zero cosmological constant, which did not admit any Killing vectors. This is because of:

**PROPOSITION 3.1.** *An anti-self-dual Einstein metric  $g$  with  $\Lambda \neq 0$  is conformal to a Kähler metric iff  $g$  admits a Killing vector.*

*Proof.* Let  $g$  be ASD Einstein, and assume that there exists a Kähler metric in  $[g]$ . Then (2.10) and (2.15) imply that  $K = 0$  or  $K$  is Killing.

If  $K = 0$  then  $\omega$  is parallel so  $g$  is already Kähler-Einstein-ASD. But Kähler-ASD implies scalar-flat. Thus  $g$  is Ricci-flat Kähler and so hyper-Kähler.

If  $K$  is Killing then one can locally choose coordinates such that  $K = \partial/\partial t$  and  $g$  is given by [20], [23]

$$g = \frac{P}{z^2}(e^u(dx^2 + dy^2) + dz^2) + \frac{1}{Pz^2}(dt + \alpha)^2$$

where  $u = u(x, y, z)$  is a solution of the  $SU(\infty)$  Toda equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

the function  $P$  is given by  $2\Lambda P = zu_z - 2$ , and

$$d\alpha = -P_x dy \wedge dz - P_y dz \wedge dx - (Pe^u)_z dx \wedge dy.$$

Thus  $\hat{g} = z^2 g$  is of the form given by LeBrun ansatz [13] because  $P$  satisfies the linearised  $SU(\infty)$  Toda equation. Therefore  $\hat{g}$  is Kähler with vanishing Ricci scalar. All such metrics have ASD conformal curvature.

It is worth pointing out that such an ASD Einstein metric may be conformal to Kähler in more than one way. Thus for example  $\mathbb{C}P^2$  with the Fubini-Study metric, but the wrong orientation so that it is not Kähler, is ASD Einstein and is conformal to Kähler in different ways via the above construction using the Killing vectors  $\partial/\partial\psi$  and  $\partial/\partial\phi$  in the usual Euler angles.

To find ASD Einstein metrics with negative cosmological constant not conformal to Kähler, one could apply the LeBrun construction, [11], which fills in a collar neighbourhood of a compact metric at infinity with such a metric. Now if one takes a metric on  $S^3$  with no symmetries as the metric at infinity, then the resulting Einstein metric will also have no symmetries and so cannot be conformal to Kähler.

### 3.3. Compact hyperbolic four-manifolds

This construction, suggested to us by the referee, gives examples of Riemannian manifolds which are locally conformal to Kähler in many ways, but are not globally conformally Kähler. Consider a compact hyperbolic four manifold  $(M, g)$ . Its traceless Ricci and Weyl tensors vanish and the Ricci scalar is equal to  $-1$ . In particular the Weyl curvature of  $g$  is ASD, so  $g$  is not Kähler as its Ricci scalar is not zero. Let us assume that there exists globally defined non-degenerate  $\omega_{A'B'}$  such that (2.9) holds. The spinor  $\omega_{A'B'}$  is not parallel (otherwise  $g$  would be Kähler), thus (2.10) defines a non-zero one-form  $K_{AA'}$  and formula (2.15) implies that  $K^{AA'}$  is a Killing vector. However this is not possible as the Bochner argument implies that non-trivial Killing vector fields can not exist on compact manifold with negative Ricci curvature: A Killing vector field  $K^a$  satisfies  $\square K_a + R_{ab}K^b = 0$ , where in our case  $R_{ab} = -g_{ab}/4$ . Multiplying this identity by  $K^a$  and integrating over a compact manifold  $M$  yields

$$\int_M |\nabla K|^2 \sqrt{g} d^4x = -\frac{1}{4} \int_M |K|^2 \sqrt{g} d^4x.$$

Thus  $K^a$  must vanish everywhere which contradicts our assumption about  $\omega_{A'B'}$ .

### 3.4. Hypercomplex examples

Families of these were given by [16] and [6]. The metric takes the form:

$$g = V^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + V^{-1}(dt + \alpha)^2, \quad (3.30)$$

where the basis  $(\sigma_i)$  of invariant one-forms on  $S^3$  satisfies

$$d\sigma_1 = \sigma_2 \wedge \sigma_3 \text{ etc}$$

and  $V$  and  $\alpha$  satisfy

$$d\alpha = V_1\sigma_2 \wedge \sigma_3 + V_2\sigma_3 \wedge \sigma_1 + V_3\sigma_1 \wedge \sigma_2$$

where we define  $V_i$  by

$$dV = V_1\sigma_1 + V_2\sigma_2 + V_3\sigma_3.$$

Necessarily then

$$\Delta V := V_{11} + V_{22} + V_{33} = 0,$$

so that  $V$  is harmonic on  $S^3$ . For a metric which isn't conformally flat,  $V$  isn't constant and so necessarily has singularities, so that these metrics are only defined locally. It is straightforward to check that the Weyl tensor is ASD and the Killing vector  $T = \partial/\partial t$  has ASD derivative.

For the examples, the idea is to choose  $V$  so that, at a point  $p \in S^3$ ,  $V = 1$  while the first and second derivatives  $V_i$  and  $V_{ij}$  all vanish. By considering harmonic functions expanded in spherical harmonics on  $S^3$  it is clear that this can be done and then the third partials  $V_{ijk}(p)$  are freely disposable, subject to being symmetric and trace-free; that gives a 7-dimensional vector space of choices for  $V_{ijk}(p)$  (but an infinite-dimensional family of choices for  $V$ ).

For the derivative of the Killing vector, we calculate

$$\nabla_a T_b = \varepsilon_{A'B'}\phi_{AB}$$

with

$$\phi_{AB} = \frac{1}{V} T_A^{B'} \nabla_{BB'} V.$$

From the Killing vector identity

$$\nabla_a \nabla_b T_c = R_{bcad} T^d$$

we may calculate

$$\psi_{ABCD} = 2V T_D^{C'} \nabla_{C'A} \phi_{BC}.$$

Thus, at  $p$ ,

$$\psi_{ABCD} = 0 = \nabla_a T_b = \nabla_a \nabla_b T_c.$$

We need explicit formulae for the Ricci curvature, for which we use Cartan calculus to find

$$R_{ab} = \frac{1}{2} \left( T_a T_b - \frac{1}{V} g_{ab} \right) \quad (3.31)$$

so that, at  $p$ ,

$$\nabla_a R_{bc} = 0 = \nabla_a \nabla_b R_{cd}.$$

From the Bianchi identity this means that, at  $p$ ,

$$\nabla^{AA'} \psi_{ABCD} = 0 = \nabla^{FF'} \nabla^{AA'} \psi_{ABCD}.$$

The Lie derivative of the Weyl curvature along the Killing vector necessarily vanishes everywhere, so that, at  $p$ ,

$$T^{EE'} \nabla_{EE'} \psi_{ABCD} = 0 = T^{EE'} \nabla_{FF'} \nabla_{EE'} \psi_{ABCD}. \quad (3.32)$$

If we use all that we have so far established in (2.25) evaluated at  $p$ , then it collapses to

$$K^{EE'} \nabla_{EE'} \psi_{ABCD} = 0. \quad (3.33)$$

Clearly this is satisfied with  $K^a(p)$  any multiple of the Killing vector  $T^a(p)$ ; we claim that generically there are no other solutions. To prove this we need explicit formulae for the components of the Weyl tensor. These are readily found via Cartan calculus in the tetrad:

$$\theta^0 = V^{-1/2}(dt + \alpha), \quad \theta^1 = V^{1/2}\sigma_1, \quad \theta^2 = V^{1/2}\sigma_2, \quad \theta^3 = V^{1/2}\sigma_3.$$

We find

$$C_{0101} = -\frac{V_{11}}{2V^2} - \frac{1}{2V^3}(V_2^2 + V_3^2 - 2V_1^2) \quad (3.34)$$

$$C_{0102} = -\frac{V_{12}}{2V^2} - \frac{3}{2V^3}V_1V_2 - \frac{V_3}{4V^2} \quad (3.35)$$

from which the rest follow by permutations and anti-self-duality. Now it follows that, at  $p$ , the components of the derivative  $\nabla_a C_{bcde}$  are proportional to the third derivatives  $V_{ijk}$ . Condition (3.33) is equivalent to

$$K^a \nabla_a C_{bcde} = 0$$

which reduces to an equation on the components of  $K^a$  orthogonal to  $T^a$ :

$$K^i V_{ijk} = 0 \quad (3.36)$$

and this implies  $K^i = 0$  for generic  $V_{ijk}$  (and in particular for the choice we make below). Thus

$$K^a(p) = c_1 T^a(p),$$

for some real constant  $c_1$ , which could be zero. We move on to the derivative of (2.25), evaluated at  $p$ . This is messy but straightforward. We are assisted by an identity obtained from the Bianchi identity

$$\nabla^{AA'} \psi_{ABCD} = \nabla_{(B}^{B'} \phi_{CD)B'}^{A'} \quad (3.37)$$

$$= -\frac{3}{4} \phi_{(AB} T_{C)}^{A'} \quad (3.38)$$

by using (3.31) for Ricci. This simplifies the last term in (2.25) to

$$\omega_{E'A'} T_{(A}^{E'} T^{A'E} \psi_{BCD)E}$$

in which form its derivative is easier to see. We take the derivative  $\nabla_F^{F'}$  of (2.25), and evaluate at  $p$ , using  $K^a(p) = c_1 T^a(p)$ . The resulting expression is simplified by defining

$$\hat{\rho}_{AB} = \rho_{AB} - \frac{1}{4} T_A^{E'} T_B^{F'} \omega_{E'F'}$$

when it becomes

$$5\hat{\rho}_{(A}^E \nabla_B^{F'} \psi_{CDF)E} + \frac{1}{8} T_{A'}^E \omega^{F'A'} T_{G'F} \nabla_E^{G'} \psi_{ABCD} - \Lambda \omega^{E'F'} \nabla_{E'F'} \psi_{ABCD} = 0.$$

At  $p$ ,  $V = 1$  so that  $T_a T^a = 1$  and, by (3.31),  $\Lambda = -1/16$ . The second term in the above expression simplifies further using (3.32) and some algebra as follows:

$$\frac{1}{8} T_{A'}^E \omega^{F'A'} T_{G'F} \nabla_E^{G'} \psi_{ABCD} = \frac{1}{16} \omega^{F'A'} \nabla_{FA'} \psi_{ABCD},$$

which duplicates the third term. We finally obtain

$$5\hat{\rho}_{(A}^E \nabla_B^{F'} \psi_{CDF)E} + \frac{1}{8} \omega^{E'F'} \nabla_{E'F'} \psi_{ABCD} = 0. \quad (3.39)$$

Note that (3.39) is totally symmetric on the indices  $ABCD$  (as expected - see (2.29)) so it is a system of twelve (real) linear equations on the unknowns  $(\omega_{A'B'}, \rho_{AB})$ . If the relevant determinant is non-zero then the only solution will be

$$\omega_{A'B'} = 0 = \rho_{AB}$$

holding at  $p$ . By (2.12), a zero in  $\omega_{A'B'}$  corresponds to a pole in  $\Omega$ , which of course isn't allowed. This will show that this metric is not conformally Kähler in any neighbourhood of  $p$ . Thus we must analyse (3.39) further.

We introduce two spinor dyads  $(o^A, o^{\dagger A})$  and  $(\tilde{o}^{A'}, \tilde{o}^{\dagger A'})$  normalised by

$$o_A o^{\dagger A} = 1 = \tilde{o}_{A'} \tilde{o}^{\dagger A'}$$

and related to the tetrad by

$$\begin{aligned} \frac{1}{\sqrt{2}}(\theta^0 + i\theta^1) &= o_A \tilde{o}_{A'} dx^a \\ \frac{1}{\sqrt{2}}(\theta^0 - i\theta^1) &= o_A^{\dagger} \tilde{o}_{A'}^{\dagger} dx^a \\ \frac{1}{\sqrt{2}}(\theta^2 + i\theta^3) &= o_A^{\dagger} \tilde{o}_{A'} dx^a \\ \frac{1}{\sqrt{2}}(\theta^2 - i\theta^3) &= -o_A \tilde{o}_{A'}^{\dagger} dx^a, \end{aligned}$$

with the corresponding operators:

$$\begin{aligned} D &= \frac{1}{\sqrt{2}}(e_0 + ie_1) \\ \bar{D} &= \frac{1}{\sqrt{2}}(e_0 - ie_1) \\ \delta &= \frac{1}{\sqrt{2}}(e_2 + ie_3) \\ \bar{\delta} &= \frac{1}{\sqrt{2}}(e_2 - ie_3). \end{aligned}$$

We introduce a spinor field  $\chi_{ABCDE}$  by

$$\nabla_{E'E} \psi_{ABCD} = \tilde{o}_{E'} \chi_{ABCD} + \text{H.c.}$$

where, at  $p$ ,  $\chi_{ABCDE}$  is totally symmetric (though not Hermitian). For (3.39) we need the dyad components of  $\chi_{ABCDE}$ . In the Newman–Penrose formalism [18] at  $p$  these can be written

$$\begin{aligned} \chi_0 &= \bar{\delta}^3 V \\ \chi_1 &= -\bar{D} \bar{\delta}^2 V \\ \chi_2 &= \bar{D}^2 \bar{\delta} V \\ \chi_3 &= -\bar{D}^3 V \\ \chi_4 &= D^2 \delta V \\ \chi_5 &= D \delta^2 V \end{aligned}$$

The Laplace equation on  $V$  and stationarity forces relations among these, since

$$D\bar{D}V + \delta\bar{\delta}V = 0 = (D + \bar{D})V.$$

We may write the components of  $\chi$  in terms of three complex numbers  $\lambda, \mu, \nu$  and a real  $a$  as

$$\chi_0 = \nu; \quad \chi_1 = -\bar{\chi}_5 = \mu; \quad \chi_2 = \bar{\chi}_4 = \lambda; \quad \chi_3 = ia.$$

Also at  $p$

$$T^{EE'} \nabla_{EE'} \psi_{ABCD} = 0$$

which forces relations between the components of  $\chi_{ABCDE}$  and its Hermitian conjugate:

$$\begin{aligned} \chi_0^\dagger &= \chi_1 \\ \chi_1^\dagger &= \chi_2 \\ \chi_2^\dagger &= \chi_3 \\ \chi_3^\dagger &= \chi_4 \\ \chi_4^\dagger &= \chi_5 \end{aligned}$$

with also

$$\chi_5^\dagger = \bar{\chi}_0.$$

from Hermiticity of  $\nabla_{E'E} \psi_{ABCD}$ .

We expand  $\omega_{A'B'}$  and  $\rho_{AB}$  in the dyad as

$$\omega_{A'B'} = \omega_2 \tilde{o}_A \tilde{o}_{B'} - 2\omega_1 \tilde{o}_{(A} \tilde{o}_{B')}^\dagger + \omega_0 \tilde{o}_A^\dagger \tilde{o}_{B'}^\dagger \quad (3.40)$$

$$\rho_{AB} = \rho_2 o_A o_B - 2\rho_1 o_{(A} o_{B)}^\dagger + \rho_0 o_A^\dagger o_B^\dagger \quad (3.41)$$

and take dyad components of (3.39) to obtain the system

$$\begin{pmatrix} -5\mu & 5\nu & 0 & -\nu & -\mu \\ -4\lambda & 3\mu & \nu & -\mu & -\lambda \\ -3ia & \lambda & 2\mu & -\lambda & -ia \\ -2\bar{\lambda} & -ia & 3\lambda & -ia & -\bar{\lambda} \\ \bar{\mu} & -3\bar{\lambda} & 4ia & -\bar{\lambda} & \bar{\mu} \\ 0 & 5\bar{\mu} & 5\bar{\lambda} & \bar{\mu} & -\bar{\nu} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \omega_1/8 \\ \omega_2/8 \end{pmatrix} = 0. \quad (3.42)$$

This is a system of six (complex) equations in five unknowns, since  $\omega_0$  is absent (but recall that reality of  $\omega_{A'B'}$  entails  $\omega_0 = \bar{\omega}_2$ ). We may conclude that the only solution of (3.42) has zero  $\omega_{A'B'}(p)$ , if the rank of the coefficient matrix is five. This is generically true, as we see by taking the special case  $\nu = \lambda = 0$  and  $\mu = 1$ . We omit the fourth row of (3.42) and calculate the determinant of the resulting  $5 \times 5$  matrix to be  $64(1 - a^2)$ . This is nonzero if  $a^2 \neq 1$ , and then the only solution has  $\omega_{A'B'}(p) = 0$ . We need to check back to (3.36), and with these choices we do find that we may deduce  $K^i = 0$  provided  $a^2 \neq 1$ . Thus there is an open set of choices for  $V$  which force  $\omega_{A'B'}(p) = 0$  and provide ASD metrics which are not conformal to Kähler in any neighbourhood of the chosen point  $p$ .

#### 4. The relation with projective invariants

A two-dimensional projective structure on  $(U, [\gamma])$  is an equivalence class of torsion-free connections on  $TU$

$$\gamma_{A'B'}^{C'} \cong \gamma_{A'B'}^{C'} + \delta_{A'}^{C'} \beta_{B'} + \delta_{B'}^{C'} \beta_{A'}, \quad A', B', C' = 1, 2,$$

(the reasons for using primed indices will become clear shortly) where  $\beta_{A'}$  is a one-form on  $U$ .

Consider the symmetric projective connection  $\nabla^\Pi$  with connection symbols

$$\Pi_{A'B'}^{C'} = \gamma_{A'B'}^{C'} - \frac{1}{3}\gamma_{D'A'}^{D'}\delta_{B'}^{C'} - \frac{1}{3}\gamma_{D'B'}^{D'}\delta_{A'}^{C'}.$$

These symbols do not depend on a choice of  $\gamma$  in the projective class. Let  $x^{A'}$  be local coordinates on  $U$  and let  $(x^{A'}, z^{A'})$  be local coordinates on  $TU$ . Define a (2, 2) conformal structure on  $TU$  by

$$g = dz_{A'} \otimes dx^{A'} - \Pi_{A'B'}^{C'} z_{C'} dx^{A'} \otimes dx^{B'}. \quad (4.43)$$

This is a projectively invariant version of the Riemannian extension studied by Walker [24] (also introduced in [26] as a horizontal lift).

This conformal structure is anti-self-dual and admits a twisting, null conformal Killing vector  $K = z^{A'}\partial/\partial z^{A'}$ . Thus it fits into the classification [9] as explained in [5]. Choose a spin frame<sup>3</sup>

$$\nabla_{A'1} = \frac{\partial}{\partial z^{A'}}, \quad \nabla_{A'0} = \frac{\partial}{\partial x^{A'}} - \Pi_{A'B'}^{C'} z^{B'} \frac{\partial}{\partial z^{C'}}$$

and set  $\delta_{A'} = \iota^A \nabla_{AA'} = \partial/\partial z^{A'}$ . Then the spin connection, and curvature in terms of projective curvature of the projective covariant derivative  $\nabla^\Pi$  on  $U$  are

$$\Gamma_{AB} = \iota_A \iota_B \mathbf{P}_{A'B'} z^{A'} dx^{B'}, \quad \Gamma_{A'B'} = \Pi_{A'B'}^{C'} dx_{C'}, \quad \Lambda = 0, \quad (4.44)$$

$$\phi_{ABA'B'} = \iota_A \iota_B \mathbf{P}_{A'B'}, \quad \psi_{A'B'C'D'} = 0, \quad \psi_{ABCD} = \iota_A \iota_B \iota_C \iota_D (z^{A'} Y_{A'}) \quad (4.45)$$

where  $\mathbf{P}_{A'B'}$  is the symmetric Ricci tensor of the 2D projective structure and

$$Y_{C'} = \varepsilon^{A'B'} Y_{A'B'C'}, \quad Y_{A'B'C'} = \frac{1}{2} (\nabla_{A'}^\Pi \mathbf{P}_{B'C'} - \nabla_{B'}^\Pi \mathbf{P}_{A'C'})$$

is the Cotton tensor of  $[\gamma]$ . The components of  $Y_{C'}$  are given by (2.14) in [5] and referred to as Liouville's invariants. In particular  $(U, [\gamma])$  is flat if  $Y_{C'} = 0$ . Note that the spinor  $\iota^A = (0, 1)$  is covariantly constant on  $TU$ .

In [5] it was shown that a projective structure comes from a (possibly Lorentzian) metric on  $U$  if and only if there exists a covariantly constant section  $(\sigma^{A'B'}, \mu^{A'}, \rho)$  of a connection

$$\begin{pmatrix} \sigma^{B'C'} \\ \mu^{B'} \\ \rho \end{pmatrix} \xrightarrow{\mathcal{D}_{A'}} \begin{pmatrix} \nabla_{A'} \sigma^{B'C'} - \delta_{A'}^{B'} \mu^{C'} - \delta_{A'}^{C'} \mu^{B'} \\ \nabla_{A'} \mu^{B'} - \delta_{A'}^{B'} \rho + \mathbf{P}_{A'C'} \sigma^{B'C'} \\ \nabla_{A'} \rho + 2\mathbf{P}_{A'B'} \mu^{B'} - 2Y_{A'B'C'} \sigma^{B'C'} \end{pmatrix}, \quad (4.46)$$

on a rank 6 vector bundle over  $U$  for which  $\sigma^{A'B'} = \sigma^{(A'B')}$  is non-degenerate. This condition is projectively invariant (when appropriate projective weights are used) so is also true when the covariant derivative  $\nabla$  with respect to a representative  $\gamma \in [\gamma]$  is replaced by an invariant derivative  $\nabla^\Pi$ . Given such a section, the contravariant metric is constructed by

$$h^{A'B'} = \det(\sigma) \sigma^{A'B'}. \quad (4.47)$$

<sup>3</sup> In this frame  $K^a = o^{A'} \iota^A$  where  $\nabla_{AA'} \iota^B = 0$  and  $o^{A'} = z^{A'}$ . The conformal Killing vector has an ASD derivative  $dK = dz_{A'} \wedge dx^{A'}$  and a non-zero twist. It defines two integrable distributions

$$\left\{ K, \mathcal{S} = z^{A'} \frac{\partial}{\partial x^{A'}} - \Pi_{A'B'}^{C'} z^{A'} z^{B'} \frac{\partial}{\partial z^{C'}} \right\} \quad \text{and} \quad \left\{ \partial/\partial z^1, \partial/\partial z^2 \right\}$$

where  $\mathcal{S}$  is the geodesic spray of the projective structure.

The necessary condition for the existence of the parallel section is obtained by commuting derivatives. It gives

$$5Y_{A'}\mu^{A'} + \nabla_{A'}^{\Pi} Y_{B'}\sigma^{A'B'} = 0. \quad (4.48)$$

(This is [5, (7.46) or (3.20)]).

Now form a 10-tractor  $(\omega^{A'B'}, K^{AA'}, \rho^{AB})$  from the 6-tractor  $(\sigma^{A'B'}, \mu^{A'}, \rho)$  by

$$\omega^{A'B'} = \sigma^{A'B'}, \quad K^{AA'} = \iota^A \mu^{A'}, \quad \rho^{AB} = \iota^A \iota^B \rho. \quad (4.49)$$

Then the first and the last term in the condition (2.25) vanish. Using

$$\iota^A \nabla_{AA'}(\iota^{B'} Y_{B'}) = Y_{A'}, \quad \nabla_{A(A'} Y_{B')} = \iota_A \nabla_{(A'}^{\Pi} Y_{B')}$$

reduces the five conditions (2.25) to one condition

$$\iota_A \iota_B \iota_C \iota_D (5Y_{A'}\mu^{A'} + \nabla_{A'}^{\Pi} Y_{B'}\sigma^{A'B'}) = 0$$

which holds if (4.48) does. Thus, given the conformal structure (4.43), the 6-tractor bundle with connection over  $U$  embeds in a 10-tractor bundle with connection over  $M = TU$  and the rank 5 curvature of the latter is given by a rank 1 curvature of the former.

Differentiating (2.25) gives 25 conditions on 10 unknowns so some constraints must hold for the conformal structure. But we know that these will hold automatically for (4.43): in [5] it was shown that the first constraint arises after taking two derivatives of (4.48). There is perhaps no surprise here – (4.43) is type  $N$  and the lowest order obstructions tend to vanish in this case.

A point of caution is needed: if the projective structure is metrisable by a Riemannian metric, then (4.49) implies that  $\omega_{A'B'}\omega^{A'B'} > 0$  and thus  $\omega_{A'B'}$  gives rise to a Kähler metric, albeit in  $(2, 2)$  signature. If on the other hand, the metric underlying the projective structure is Lorentzian then  $|\omega|^2 < 0$  and one instead obtains a para-Kähler structure: there exists a  $(2, 2)$  metric  $g$  and an almost-product structure

$$J : TM \longrightarrow TM, \quad J^2 = \text{Id}$$

such that;

- (i) the structure  $J$  is integrable in the sense that the eigenspaces of  $TM$  corresponding to eigenvalues  $\pm 1$  of  $J$  are integrable distributions;
- (ii)  $g(JX, JY) = -g(X, Y)$  for all  $X, Y \in TM$ ;
- (iii) the two-from  $\Sigma := g(J \cdot, \cdot)$  is closed.

We have proved the ‘only if’ part of the following:

**THEOREM 4.1.** *The projective structure  $(U, [\gamma])$  is metrisable if and only if its Riemannian extension (4.43) contains a Kähler or a para-Kähler metric in its conformal class.*

*Proof.* It remains to prove the ‘if’ part, and show that if  $(U, [\gamma])$  is metrisable then the  $(2, 2)$  metric (4.43) on  $TU$  is Kähler. Let  $h$  be a metric on  $U$ . First assume that  $h$  is Riemannian. Its conformal class defines a complex structure  $j : TU \rightarrow TU$ ,  $j^2 = -\text{Id}$ . Let  $\omega$  be a canonical symplectic structure on  $T^*U$  and let

$$T(TU) = V \oplus H$$

be the splitting of the tangent space to  $TU$  into vertical and horizontal components with respect to the Levi-Civita connection of  $h$ . The complex structure  $J$  on  $TU$  defined by taking the complex structure  $j$  on each factor  $H$  and  $V$ .

We regard  $h$  as an isomorphism between  $TU$  and  $T^*U$ , and define a metric  $g$  on  $TU$  by

$$g(X, Y) = h^*(\omega)(JX, Y), \quad (4.50)$$

This agrees with (4.43) if local coordinates are adapted. To see it use the definition of  $J$  to find

$$J\left(\frac{\partial}{\partial z^{A'}}\right) = j_{A'}^{B'} \frac{\partial}{\partial z^{B'}}, \quad J\left(\frac{\partial}{\partial x^{A'}}\right) = j_{A'}^{B'} \frac{\partial}{\partial x^{B'}} + z^{B'} (\gamma_{A'B'}^{C'} j_{C'}^{D'} - \gamma_{C'B'}^{D'} j_{A'}^{C'}) \frac{\partial}{\partial z^{D'}},$$

where  $\gamma$  is the Levi-Civita connection of  $h$ . A triple  $(g, J, h^*(\omega))$  is a Kähler structure on  $TU$ .

If the metric  $h$  on  $U$  is Lorentzian, then its conformal structure defines a product structure  $j$  on  $TU$  with  $j^2 = \text{Id}$ . The argument given in the proof still applies, but it leads to a product structure  $J$  on  $TM = T(TU)$  and eventually to a para-Kähler structure on  $M$ .

#### 4.1. Anti-self-dual null Kähler structures

In this Section we shall invariantly characterise the Riemannian extensions as a subclass of all  $(2, 2)$  ASD conformal structures which admit a parallel real section of  $\mathbb{S}$ .

In [8]  $(2, 2)$  ASD metrics which admit a covariantly constant real spinor  $\iota^{A'}$  were studied. These were called null Kähler structures as the endomorphism  $N_b^a = \iota^{A'} \iota_B \varepsilon_B^{A'}$  satisfies

$$N^2 = 0, \quad g(NX, Y) + g(X, NY) = 0, \quad \nabla N = 0. \quad (4.51)$$

It resembles the Kähler condition albeit null.

The condition (4.51) is equivalent to the existence of parallel  $\iota^{A'}$  or  $\iota^A$  and in the following we shall choose the spinor to be  $\iota^A$ , so that (4.51) holds with

$$N_b^a = \iota^A \iota_B \varepsilon_B^{A'}.$$

Given a null Kähler structure there exist a local coordinate system  $(x^{A'}, z^{A'})$  and a function  $\Theta = \Theta(x^{A'}, z^{A'})$  such that

$$g = dz_{A'} \otimes dx^{A'} + \frac{\partial^2 \Theta}{\partial z^{A'} \partial z^{B'}} dx^{A'} \otimes dx^{B'}, \quad N = dx^{A'} \otimes \frac{\partial}{\partial z^{A'}}$$

where the indices are raised and lowered using  $\varepsilon_{A'B'}$  [4, 8].

The self-duality conditions  $\psi_{ABCD} = 0$  imposed on  $g$  lead to a fourth order integrable PDE for  $\Theta$ . This was shown in [8], where the opposite orientation was used. If on the other hand the anti-self-duality conditions  $\psi_{A'B'C'D'} = 0$  are imposed  $\Theta$  can be found explicitly and we obtain the following

**PROPOSITION 4.2.** *There is a one-to-one correspondence between ASD Null Kähler structures where the parallel spinor is a section of  $\mathbb{S}$ , and Riemannian extensions of projective structures of the form (4.43).*

*Proof.* Choose a spin frame

$$\nabla_{A'1} = \iota^A \nabla_{AA'} = \frac{\partial}{\partial z^{A'}}, \quad \nabla_{A'0} = o^A \nabla_{AA'} = \frac{\partial}{\partial x^{A'}} + \frac{\partial \Theta}{\partial z^{A'} \partial z^{B'}} \frac{\partial}{\partial z_{B'}},$$

and set

$$f = \frac{\partial^2 \Theta}{\partial x^{A'} \partial z_{A'}} + \frac{1}{2} \frac{\partial^2 \Theta}{\partial z^{A'} \partial z^{B'}} \frac{\partial^2 \Theta}{\partial z_{A'} \partial z_{B'}}.$$

Then

$$\begin{aligned} \psi_{A'B'C'D'} &= \delta_{A'}\delta_{B'}\delta_{C'}\delta_{D'}\Theta, & \psi_{ABCD} &= \iota_A\iota_B\iota_C\iota_D\Box f, & \phi_{ABA'B'} &= \iota_A\iota_B\delta_{A'}\delta_{B'}f, \\ \Gamma_{A'B'} &= \delta_{A'}\delta_{B'}\delta_{C'}\Theta dx^{C'}, & \Gamma_{AB} &= \iota_A\iota_B\delta_{A'}f dx^{A'}, & \Lambda &= 0, \end{aligned}$$

where

$$\delta_{A'} = \frac{\partial}{\partial z^{A'}}, \quad \text{and} \quad \Box = \frac{\partial^2}{\partial x^{A'}\partial z^{A'}} + \frac{\partial^2\Theta}{\partial z_{A'}\partial z_{B'}} \frac{\partial^2}{\partial z^{A'}\partial z^{B'}}.$$

Therefore the self-duality condition implies that

$$\Theta = -\frac{1}{6}\Pi_{A'B'}^{C'}z^{A'}z^{B'}z_{C'}$$

for some  $\Pi_{A'B'}^{C'} = \Pi_{A'B'}^C(x^{D'})$  such that

$$\Pi_{A'B'}^{C'} = \Pi_{B'A'}^{C'}, \quad \Pi_{A'B'}^{A'} = 0$$

(the terms of order lower than 3 in  $z^{A'}$  can be eliminated by redefining  $\Theta$  and translating the coordinates  $z^{A'} \rightarrow z^{A'} + t^{A'}(x^{B'})$ ). Now  $f = (1/2)P_{A'B'}z^{A'}z^{B'}$  and the curvature and connection coefficients agree with those of (4.43).

### 5. Twistor Theory.

We shall end the paper by briefly describing the twistor origins of the ‘conformal to Kähler’ obstructions.

Let  $B$  be a twistor-space (a complex three-fold with an embedded rational curve with normal bundle  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ ) corresponding to an ASD conformal structure  $(M, [g])$  [1, 17]. A Kähler structure in  $[g]$  corresponds to a preferred section of the anti-canonical divisor bundle  $\kappa_B^{-1/2}$ , where  $\kappa_B$  is the canonical bundle of  $B$  [19]. Restriction of  $\kappa_B$  to a rational curve gives a line bundle isomorphic to the fourth power of the tautological line bundle  $\mathcal{O}(-1)$ . Therefore the restriction of the preferred section to a curve, pulled back to the total space of  $S' \rightarrow M$  is of the form  $\pi^{A'}\pi^{B'}\omega_{A'B'}$ , where  $\pi^{A'}$  are coordinates on the fibres of  $S'$  and  $\omega_{A'B'}$  satisfies the conformally invariant linear equation (2.10).

In Section 2.3 we constructed a rank-10 vector bundle  $E \rightarrow M$  with connection, such that the parallel sections of this bundle correspond to solutions to (2.10), and therefore to sections of  $\kappa_B^{-1/2}$ . The forward Ward transform [1, 25] of  $E$  gives rise to a rank-10 holomorphic vector bundle  $\mathcal{E} \rightarrow B$  (with no connection) which is holomorphically trivial on the twistor curves. This holomorphic vector bundle can be also constructed directly from the twistor data and is given by the second-jet bundle  $\mathcal{E} = J^2(\kappa_B^{-1/2})$ .

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