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Metrisability of Painlevé equations

Felipe Contatto[a)] and Maciej Dunajski[a)]
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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We solve the metrisability problem for the six Painlevé equations, and more generally for all 2nd order ordinary differential equations with the Painlevé property, and determine for which of these equations their integral curves are geodesics of a (pseudo) Riemannian metric on a surface. *Published by AIP Publishing.* https://doi.org/10.1063/1.4998147

I. INTRODUCTION

A geometric approach to nonlinear 2nd order ordinary differential equations (ODEs) was initiated in the work of Liouville[21] and developed by Cartan. A general 2nd order ODE defines a path geometry on a surface U coordinatised by the dependent and independent variables: there is a unique integral curve through each point of U in each direction. The paths are unparametrised geodesics of a torsion-free connection ∇ on TU with Christoffel symbols Γ_{ab}^{c} if and only if the ODE is of the form

$$\frac{d^2y}{dx^2} = A_3(x,y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x,y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x,y)\Big(\frac{dy}{dx}\Big) + A_0(x,y), \tag{1.1}$$

where

$$A_0 = -\Gamma_{11}^{2}, \quad A_1 = \Gamma_{11}^{1} - 2\Gamma_{12}^{2}, \quad A_2 = 2\Gamma_{12}^{1} - \Gamma_{22}^{2}, \quad A_3 = \Gamma_{22}^{1}. \tag{1.2}$$

Conversely, with any ODE of the form (1.1), one can associate a *projective structure*[2,25] that is an equivalence class of torsion-free connections which share the same unparametrised geodesics. Two connections ∇ and $\hat{\nabla}$ belong to the same projective equivalence class if their geodesic flows on TU project to the same foliation of $\mathbb{P}(TU)$. Equivalently, there exists a one-form Υ on U such that

$$\hat{\Gamma}_{bc}^{a} = \Gamma_{bc}^{a} + \Upsilon_b\delta_c^a + \Upsilon_c\delta_b^a. \tag{1.3}$$

Definition 1.1. A second order ODE is called metrisable if its integral curves are unparametrised geodesics of a Levi–Civita connection of some (pseudo) Riemannian metric.

A problem of characterising metrisable ODEs by differential invariants was posed by Liouville,[21] who has reduced it to an overdetermined system of linear PDEs (see Theorem 2.1 in Sec. II). The complete solution was provided relatively recently,[1] where it was shown that an ODE is metrisable if and only if three point invariants of differential orders five and six vanish and certain genericity assumptions hold.

A different approach was developed by Painlevé, Kowalevskaya, and Gambier who studied 2nd order ODEs in the complex domain.[16,24]

Definition 1.2. The ODE $y'' = R(x, y, y')$, where R is a rational function of y and y' has the Painlevé property (PP) if its movable singularities (i.e., singularities whose locations depend on the initial conditions) are poles.

The solutions of equations with the Painlevé property are single-valued thus giving rise to proper functions on $\mathbb{C}$. There exist fifty canonical types of second order ODEs with PP up to the change of

[a)]Electronic addresses: felipe.contatto@damtp.cam.ac.uk and m.dunajski@damtp.cam.ac.uk

variables

$$y \to Y(x, y) = \frac{a(x)y + b(x)}{c(x)y + d(x)}, \quad x \to X(x) = \phi(x), \tag{1.4}$$

where functions (a, b, c, d, ϕ) are analytic in x. Forty-four of these are solvable in terms of 'known' functions (sine, cosine, elliptic functions or in general solutions to linear ODEs). The remaining six types define new transcendental functions and are given by the Painlevé equations

$$y'' = 6y^2 + x, \tag{PI}$$

$$y'' = 2y^3 + xy + \alpha, \tag{PII}$$

$$y'' = \frac{1}{y}y'^2 - \frac{1}{x}y' + \alpha\frac{y^2}{x} + \frac{\beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{PIII}$$

$$y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{PIV}$$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1}, \tag{PV}$$

$$\begin{aligned} y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \\ &\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]. \end{aligned} \tag{PVI}$$

Here α, β, γ, and δ are constants. Thus PVI belongs to a four-parameter family of ODEs, etc. Some work towards characterising the Painlevé equations by point invariants of (1.1) has been done in Refs. 17, 13, and 18.

The aim of this paper is to determine which of the Painlevé equations are metrisable. In Sec. II, we shall prove the following.

Theorem 1.3. *The only metrisable Painlevé equations are as follows:*

1. *Painlevé III, where* $\alpha = \gamma = 0$ *or* $\beta = \delta = 0$.
2. *Painlevé V, where* $\gamma = \delta = 0$.
3. *Painlevé VI, where* $\alpha = \beta = \gamma = 0$ *and* $\delta = 1/2$.

If $\alpha = \beta = \gamma = \delta = 0$ *then the projective structures defined by PIII and PV are flat. The metrisable PVI projective structure is also flat.*

The flatness of a projective structure is equivalent to the existence of a point transformation $(x, y) \to (X(x, y), Y(x, y))$ such that the corresponding ODE (1.1) becomes

$$\frac{d^2Y}{dX^2} = 0. \tag{1.5}$$

Here we recall that a second order ODE $y'' = R(x, y, y')$ is equivalent to (1.5) under a point transformation if and only if it is of the form (1.1) and the following quantities, called *Liouville invariants*, vanish:

$$L_1 = \frac{2}{3}\frac{\partial^2 A_1}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_0}{\partial y^2} + A_0\frac{\partial A_2}{\partial y} + A_2\frac{\partial A_0}{\partial y} - A_3\frac{\partial A_0}{\partial x} - 2A_0\frac{\partial A_3}{\partial x} - \frac{2}{3}A_1\frac{\partial A_1}{\partial y} + \frac{1}{3}A_1\frac{\partial A_2}{\partial x},$$

$$L_2 = \frac{2}{3}\frac{\partial^2 A_2}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_3}{\partial x^2} - A_3\frac{\partial A_1}{\partial x} - A_1\frac{\partial A_3}{\partial x} + A_0\frac{\partial A_3}{\partial y} + 2A_3\frac{\partial A_0}{\partial y} + \frac{2}{3}A_2\frac{\partial A_2}{\partial x} - \frac{1}{3}A_2\frac{\partial A_1}{\partial y}.$$

In Sec. III, we shall clarify a connection between the metrisability of Painlevé equations and the existence of first integrals: all metrisable cases are reducible to quadratures. In Sec. IV, we shall extend the analysis to the remaining forty-four equations with PP.

We end this introduction with a comment about the formalism used in the paper: it is elementary and admittedly brute force (which should make the results and their proofs accessible to undergraduate students). There are other more sophisticated approaches using Cartan and tractor connections or twistor theory which could be adopted in line with Refs. 1, 9, 17, 14, and 15.

II. PROOF OF THE MAIN THEOREM

Our approach to proving Theorem 1.3 is based on the seminal result of Liouville.

Theorem 2.1 (Roger Liouville[21]). *A projective structure corresponding to the second order ODE (1.1) is metrisable on a neighbourhood of a point $p \in U$ if and only if there exist functions ψ_1, ψ_2, ψ_3 defined on a neighbourhood of p such that $\Delta \equiv \psi_1\psi_3 - {\psi_2}^2 \neq 0$ at p and the equations*

$$\frac{\partial \psi_1}{\partial x} = \frac{2}{3}A_1\psi_1 - 2A_0\psi_2, \tag{2.1a}$$

$$\frac{\partial \psi_3}{\partial y} = 2A_3\psi_2 - \frac{2}{3}A_2\psi_3, \tag{2.1b}$$

$$\frac{\partial \psi_1}{\partial y} + 2\frac{\partial \psi_2}{\partial x} = \frac{4}{3}A_2\psi_1 - \frac{2}{3}A_1\psi_2 - 2A_0\psi_3, \tag{2.1c}$$

$$\frac{\partial \psi_3}{\partial x} + 2\frac{\partial \psi_2}{\partial y} = 2A_3\psi_1 - \frac{4}{3}A_1\psi_3 + \frac{2}{3}A_2\psi_2, \tag{2.1d}$$

hold on the domain of definition. The corresponding metric is then given by

$$g = \Delta^{-2}(\psi_1 dx^2 + 2\psi_2 dxdy + \psi_3 dy^2). \tag{2.2}$$

The system (2.1a)–(2.1d) is overdetermined, as there are more equations than unknowns. In Ref. 1, the integrablity conditions were established in terms of point invariants (1.1). The invariants obstructing metrisability vanish identically for the projective structures arising from all six Painlevé equations, as these equations are non-generic in the sense explained in Ref. 1: we will see that a non-trivial solution to (2.1a)–(2.1d) always exists, but it is degenerate as in general $\psi_2 = \psi_3 = 0$. Thus the metrisability analysis of the Painlevé equations needs to be carried over by analysing the linear system (2.1a)–(2.1d) directly on a case by case basis.

Proof of Theorem 1.3. The metrisability of Painlevé equations depends on the values of the parameters $(\alpha, \beta, \gamma, \delta)$. When necessary, we will indicate them in parentheses in front of the equation label, for instance, PII(α), PIII(α, β, γ, δ), and so on. The Painlevé equations do not have a cubic term in y' [so that $A_3 = 0$ in Eq. (1.1) which makes step 2 possible]. A general approach to seek solutions to the metrisability problem of this kind of projective structure is the following:

Step 1. Calculate the invariants of Ref. 1. If they do not vanish identically, then there is no non-trivial solution to (2.1a)–(2.1d).
Step 2. Solve Eq. (2.1b) for ψ_3.
Step 3. Substitute ψ_3 in (2.1d) and solve it for ψ_2.
Step 4. Apply the integrability condition $\partial_x\partial_y\psi_1 = \partial_y\partial_x\psi_1$, $\forall x, y$, to the remaining equations (2.1a) and (2.1c).
Step 5. If step 4 is successful, solve Eqs. (2.1a) and (2.1c).

Step 1 is optional because it is equivalent to step 4. After steps 2 and 3, in general, we end up with a solution for ψ_2 and ψ_3 depending on arbitrary functions of one variable. Step 4 is then necessary to fix those functions up to constants of integration. The above steps may be troublesome to be performed by hand, but they are easily implemented on the computer.

We find that Painlevé I, II, and IV are never metrisable. On the other hand, PIII, PV, and PVI are metrisable for special values of parameters, as we discuss below. The values of the parameters are found in step 4. For other choices of parameters, step 4 forces us to choose $\psi_2 = \psi_3 = 0$ which leads to a degenerate solution. An obvious degenerate solution is the trivial one $\psi_i = 0$. However, for the Painlevé equations, there always exist non-trivial solutions to the Liouville system (2.1a)–(2.1d) spanning a 1-dimensional space, which is the maximal dimension allowed for degenerate solutions (cf. Lemma 4.3 of Ref. 1). To see this, set $\psi_2 = \psi_3 = 0$. Then, (2.1a)–(2.1d) reduce to a closed overdetermined system for ψ_1 which has a non-vanishing solution if and only if $\partial_y A_1 = 2\partial_x A_2$. It is straightforward that this condition is fulfilled by all equations PI–PVI, which explains why all invariants of Ref. 1 vanish for Painlevé equations. The degenerate solutions corresponding to each Painlevé equation are, up to a multiplicative constant, given by

$$\mathrm{PI, PII}: \psi_1 = 1, \quad \mathrm{PIII}: \psi_1 = \frac{y^{4/3}}{x^{2/3}}, \quad \mathrm{PIV}: \psi_1 = y^{2/3},$$
$$\mathrm{PV}: \psi_1 = \frac{(1-y)^{4/3} y^{2/3}}{x^{2/3}}, \quad \mathrm{PVI}: \psi_1 = (x-y)^{2/3}\left[\frac{(y-1)y}{(x-1)x}\right]^{2/3}.$$

- **Painlevé III.** Applying steps 2 to 5 implies that a metric exists if and only if

$$\alpha = \gamma = 0 \quad \text{or} \quad \beta = \delta = 0.$$

Both cases are essentially the same since the change of coordinates $y \mapsto y^{-1}$ induces PIII(α, β, γ, δ) $\to$ PIII($-\beta$, $-\alpha$, $-\delta$, $-\gamma$) and all results from one case can be recovered from the other through this map. Therefore, we only present the detailed results for $\beta = \delta = 0$. If all parameters are zero, then the projective structure is flat (which can be seen by evaluating the Liouville invariants L_1, L_2). If $\beta = \delta = 0$ and $(\alpha, \gamma) \neq (0, 0)$, there exists a two-dimensional family of solutions to (2.1a)–(2.1d) giving rise to the metric

$$g = \Omega\left(\frac{B - Axy(2\alpha + \gamma xy)}{Ax^2}dx^2 + \frac{2}{xy}dxdy + \frac{1}{y^2}dy^2\right), \quad \text{where} \tag{2.3}$$
$$\Omega = A^{-1}(A - B + 2A\alpha xy + A\gamma x^2 y^2)^{-2}.$$

The metric admits a one-parameter family of isometries $(x, y) \mapsto (e^{\epsilon}x, e^{-\epsilon}y)$. Setting $r = xy$ and $\theta = \ln|x|$ and rescaling the metric by A^3 yields

$$g = \frac{1}{(-C + 2\alpha r + \gamma r^2)^2 r^2}dr^2 - \frac{1}{(-C + 2\alpha r + \gamma r^2)}d\theta^2,$$

where $C = B/A - 1$ is a constant. By rescaling r, we can set either α to 1 if $\alpha \neq 0$ or γ to $\gamma/|\gamma|$ if $\gamma \neq 0$.

If $\alpha = \beta = \gamma = \delta = 0$, we have a six-dimensional family of solutions to (2.1a)–(2.1d); all rise to metrics of constant curvature. The projective structure is flat, and PIII(0, 0, 0, 0) can be put in the form (1.5) with $Y = e^y$ and $X = \ln x$.

- **Painlevé V**. The projective structure is metrisable if and only if $\gamma = \delta = 0$ and is projectively flat if and only if $\alpha = \beta = \gamma = \delta = 0$.

If $\gamma = \delta = 0$ and $(\alpha, \beta) \neq (0, 0)$, we have a two-dimensional family of solutions giving rise to the metric

$$g = \frac{y}{A^2x^2[By + 2A(\beta - \alpha y^2)]}dx^2 + \frac{y}{A(y-1)^2[By + 2A(\beta - \alpha y^2)]^2}dy^2, \tag{2.4}$$

which admits $(x, y) \mapsto (e^{\epsilon}x, y)$ as one-parameter family of isometries, Defining $r = y$, $\theta = \ln|x|$, the metric becomes

$$g = \frac{r}{A^3(r-1)^2[Cr + 2(\beta - \alpha r^2)]^2}dr^2 + \frac{r}{A^3[Cr + 2(\beta - \alpha r^2)]}d\theta^2,$$

where $C = B/A$. By redefining C, B, and θ we can set either β to $\frac{\beta}{|\beta|}$ if $\beta \neq 0$ or α to $\frac{\alpha}{|\alpha|}$ if $\alpha \neq 0$.

If $\alpha = \beta = \gamma = \delta = 0$, there exists a six-dimensional family of solutions to the Liouville system, each giving rise to a projectively flat metric. Equation PV(0, 0, 0, 0) can be put in the form (1.5) with $Y = \ln\left(\frac{1+\sqrt{y}}{\sqrt{1-y}}\right)$ and $X = \ln x$.

- **Painlevé VI.** PVI is metrisable if and only if $\alpha = \beta = \gamma = 0$, $\delta = \frac{1}{2}$. In this case, PVI has a solution given in terms of the elliptic integral[3,12,22]

$$\int_0^{y(x)} \frac{dw}{\sqrt{w(w-1)(w-x)}} = a\omega_1(x) + b\omega_2(x), \tag{2.5}$$

where the right-hand side is the general solution of the Picard-Fuchs equation

$$4x(x-1)\omega''(x) + 4(2x-1)\omega'(x) + \omega(x) = 0, \tag{2.6}$$

with a and b constants. Since the constants of integration appear linearly in (2.5), the projective structure is flat (this is actually the definition of projective flatness used by Liouville[21]). In fact, $\text{PVI}\left(0,0,0,\frac{1}{2}\right)$ is equivalent to (1.5) in the variables $Y = \frac{1}{\omega_2(x)}\int_0^y \frac{dw}{\sqrt{w(w-1)(w-x)}}$, $X = \frac{\omega_1(x)}{\omega_2(x)}$. □

A. Coalescence

The first five Painlevé equations PI–PV can be derived from PVI by the process of coalescence of the parameters.[16] In particular, PIII arises from PV in the limit $\epsilon \to 0$ where

$$x \mapsto x^2, \quad y \mapsto 1 + \epsilon xy, \quad \alpha \mapsto \frac{\gamma}{8\epsilon^2} + \frac{\alpha}{4\epsilon}, \quad \beta \mapsto -\frac{\gamma}{8\epsilon^2}, \quad \gamma \mapsto \frac{\epsilon\beta}{4}, \quad \delta \mapsto \frac{\epsilon^2\delta}{8}.$$

We can use this process to recover the metric (2.3) of PIII$(\alpha, 0, \gamma, 0)$ from a metric of PV$(\alpha, \beta, 0, 0)$. To do so, it is necessary to start with (2.4) with the constants of integration

$$A = \left(\frac{4\gamma}{2\alpha\epsilon + \gamma}\right)^{\frac{2}{3}}, \quad B = \frac{(-\alpha\epsilon + \gamma)(4\alpha\epsilon + 2\gamma)^{\frac{1}{3}}}{\epsilon^2\gamma^{\frac{1}{3}}}.$$

Then, in the limit $\epsilon \to 0$, we find the metric (2.3) with $A_{III} = 1$ and $B_{III} = 1 - \frac{4\alpha^2}{\gamma}$, where we have attached the index *III* to indicate that these constants A_{III} and B_{III} correspond to the metric of PIII$(\alpha, 0, \gamma, 0)$. This is valid only if $\gamma \neq 0$. In the case $\gamma = 0$, we need $A = 4^{2/3}A_{III}$ and $B = \frac{2\alpha A_{III} + (-A_{III}+B_{III})\epsilon}{2^{2/3}\epsilon}$, so we still have freedom to choose two constants of integration A_{III} and B_{III}.

III. REDUCIBILITY AND FIRST INTEGRALS

The metrisable cases of PIII and PV do not define new transcendental functions but admit a quadrature and are reducible to 1st order ODEs. We shall explain this in the context of Theorem 1.3 using the following lemma.

Lemma 3.1. Let

$$g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$$

be a metric on U which admits a linear first integral $K = K_1(x,y)\dot{x} + K_2(x,y)\dot{y}$. Then

$$I(x,y,y') = \frac{1}{(K_1 + K_2 y')^2}\left(E + 2Fy' + Gy'^2\right) \tag{3.1}$$

is a first integral of the unparametrised geodesic equation (1.1).

Proof. Set $x^a = (x, y)$ and consider the geodesic equations for g parametrised by t,

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0. \tag{3.2}$$

Let t be a value of the affine parameter such that $\dot{x} \neq 0$ (if no such t exists, then swap x and y). Using the chain rule $d/dx = \dot{x}^{-1}d/dt$ to eliminate t between the two equations, (3.2) yields (1.1) with (1.2). The geodesic Hamiltonian $H = g_{ab}\dot{x}^a\dot{x}^b$ is a first integral of (3.2), but it depends on $\dot{x}$, so it does not give rise to a first integral of (1.1). However dividing H by the square of the linear first integral K is independent on $\dot{x}^a$ and yields the first integral (3.1) for (1.1).

□

Let us apply this lemma to the metrisable Painlevé cases. In the case of PIII$(\alpha, 0, \gamma, 0)$ and PV$(\alpha, \beta, 0, 0)$, we shall recover the known first integrals.[12]

- **Painlevé III.** The metric (2.3) admits a Killing vector $x\partial_x - y\partial_y$ which gives rise to a first integral (3.1) for PIII$(\alpha, 0, \gamma, 0)$,

$$I = x^2\left(\frac{y'}{y}\right)^2 + 2x\frac{y'}{y} - 2\alpha xy - \gamma x^2y^2.$$

- **Painlevé V.** The metric (2.4) admits a Killing vector $K = x\partial_x$ which leads to a first integral for PV(α, β, 0, 0),

$$I = \frac{1}{y}\left(\frac{xy'}{y-1}\right)^2 + \frac{2\beta}{y} - 2\alpha y.$$

- **Painlevé VI.** The first integrals in this case are linear in y',[3] and we will construct them from the Killing vectors (rather than a quadratic integral) of the associated metric $g = dX^2 + dY^2$. The ratios of linear integrals $\dot{Y}$ and $Y\dot{X} - X\dot{Y}$ by a linear integral $\dot{X}$ give dY/dX and $Y - XdY/dX$. Evaluating these integrals by implicitly differentiating Y and using the ODE satisfied by the Wronskian of (2.6) gives

$$I = \frac{y'B(x)}{\sqrt{y(y-1)(y-x)}} + \int_0^y \left[A(x) + \frac{B(x)}{2(w-x)}\right] \frac{dw}{\sqrt{w(w-1)(w-x)}}, \tag{3.3}$$

where A and B are solutions to the Picard-Fuchs adjoint equations

$$\begin{cases} A'(x) & = B(x)\frac{1}{4x(x-1)}, \\ B'(x) & = -B(x)\frac{1-2x}{x(x-1)} - A(x). \end{cases}$$

A prolongation of the metrisability equations (2.1a)–(2.1d) leads to a closed system of six linear PDEs for six unknowns.[1] The dimension $m([\nabla])$ of the vector space of solutions to this system is called the degree of mobility of the projective structure. In the generic, non-metrisable case, $m([\nabla])$ = 0, and in the projectively flat case, $m([\nabla]) = 6$. The Koenigs theorem[19] states that $m([\nabla]) \neq 5$. The construction below applies to projective structures where $m([\nabla]) > 1$.

Proposition 3.2. If a projective structure $[\nabla]$ *in two dimensions admits two linearly independent solutions* $\psi^{(1)}$ *and* $\psi^{(2)}$ *to the metrisability equations (2.1a)–(2.1d), then*

$$I(x, y, y') := \frac{\psi_1^{(1)} + 2\psi_2^{(1)}y' + \psi_3^{(1)}y'^2}{\psi_1^{(2)} + 2\psi_2^{(2)}y' + \psi_3^{(2)}y'^2} \tag{3.4}$$

is a first integral of the unparametrised geodesic equation (1.1).

If there exists a linear combination of $\psi^{(1)}$ *and* $\psi^{(2)}$ *which is degenerate, then any metric* g *compatible with* $[\nabla]$ *admits a Killing vector.*

Proof. The constancy of (3.4) could be established by explicitly evaluating dI on solutions to (2.1a)–(2.1d), which gives 0. Below, we shall use a less direct method that will allow us to prove both parts of the proposition. Two connections ∇ and $\hat{\nabla}$ belong to the same projective equivalence class $[\nabla]$ if there exists a one-form Υ on U such that (1.3) holds. Consider a connection $D \in [\nabla]$ with Christoffel symbols given by

$$\Pi^1_{11} = \frac{1}{3}A_1, \quad \Pi^1_{12} = \frac{1}{3}A_2, \quad \Pi^1_{22} = A_3, \quad \Pi^2_{11} = -A_0, \quad \Pi^2_{21} = -\frac{1}{3}A_1, \quad \Pi^2_{22} = -\frac{1}{3}A_2. \tag{3.5}$$

Set $\psi_1 = \sigma_{11}$, $\psi_2 = \sigma_{12}$, and $\psi_3 = \sigma_{22}$. Then the metrisability equations (2.1a)–(2.1d) are equivalent to the Killing tensor equation[1]

$$D_{(a}\sigma_{bc)} = 0. \tag{3.6}$$

Therefore $\sigma^{(i)}$ $(i = 1, 2)$ are Killing tensors, and $I^{(i)}(t) := \sigma^{(i)}_{11}\dot{x}^2 + 2\sigma^{(i)}_{12}\dot{x}\dot{y} + \sigma^{(i)}_{22}\dot{y}^2$ are conserved along geodesics of D

$$\frac{d}{dx}I(x, y(x), y'(x)) = \frac{1}{\dot{x}}\frac{d}{dt}\frac{I^{(1)}(t)}{I^{(2)}(t)} = 0,$$

where we have used $\dot{y}/\dot{x} = y'$ to write $I = I^{(1)}/I^{(2)}$.

For the second part, the projective structure is metrisable (this is true even if both $\psi^{(i)}$ are degenerate, as there always exists a non-degenerate linear combination, i.e., two degenerate solutions can only differ by a constant multiple. See Lemma 4.3 in Ref. 1). Without loss of generality, say that $\psi^{(2)}$ is degenerate. Then there exists a non-vanishing one-form ω such that $\sigma^{(2)}_{ab} = \omega_a\omega_b$. Then the metrisability equations (3.6) yield

$$D_{(a}\omega_{b)} = 0.$$

The Levi-Civita connection ∇ of the metric $g = \sigma/\Delta^2$ is obtained from D by applying a transformation (1.3) with an equi-tensor $\Upsilon_a = \nabla_a \left(-\frac{1}{2}\ln|\Delta|\right)$, and we verify that

$$\nabla_{(a}K_{b)} = 0,$$

where $K = \Delta^{-1}\omega$. Thus K is a linear first integral of the geodesic flow of g.

□

Remarks.

- Not all projective structures with $m([\nabla]) > 1$ admit a linear first integral. The metrics[6]

$$g_1 = (X(x) - Y(y))\left(dx^2 + dy^2\right) \quad \text{and} \quad g_2 = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$$

are projectively equivalent with an unparametrised geodesic equation

$$y'' + \frac{1}{2\,(X(x) - Y(y))}\left(Y' + X'y' + Y'y'^2 + X'y'^3\right) = 0. \tag{3.7}$$

These metrics in general do not admit a Killing vector, but clearly $m([\nabla]) > 1$. The first integral (3.4) is

$$I = \frac{Y(y) + X(x)y'^2}{1 + y'^2}.$$

- Each Painlevé equation admits a degenerate solution to the metrisability equations. This implies that the corresponding projective class $[\nabla]$ contains a representative ∇ which has a symmetric Ricci tensor and admits a linear first integral. In Ref. 5, it was shown that for such affine connections $\nu_5 = 0$, where ν_5 is a point invariant for (1.1) defined by Liouville.[21] This is in agreement with Ref. 13, where it was stated that ν_5 vanishes for all Painlevé equations.
- A more invariant way to define the connection (3.5) is as follows. Pick a connection $\nabla \in [\nabla]$ and set

$$\Pi^c_{ab} = \Gamma^c_{ab} - \frac{1}{3}\Gamma^d_{da}\delta^c_b - \frac{1}{3}\Gamma^d_{db}\delta^c_a.$$

The object Γ^d_{da} is not a 1-form, and thus Π^c_{ab} does not transform as an affine connection in general, but only under coordinate transformations of constant Jacobian. So once we choose this representative, we can only apply this kind of coordinate transformations in (3.6). Thomas[25] introduced the terminology *equi-transformation* for coordinate changes preserving the volume (of Jacobian identically 1), *projective connection* for Π^c_{ab}, and *equi-tensor* for entities such as Γ^d_{ad} transforming like tensors under equi-transformations.
- In Ref. 23, it was shown that all two-dimensional projective structures are locally Weyl-metrisable. For a given ODE (1.1), finding an explicit expression for the Weyl connection reduces to constructing a point transformation such that $A_0 = A_2$ and $A_1 = A_3$. This should in principle be possible of all six Painleve equations, but the resulting ODEs may not have the Painlevé property if the point transformation is not of the form (1.4).
- In the recent work,[20] some connections between the Painlevé property and Lie point symmetries have been uncovered. While the problems studied in Ref. 20 are different than those addressed in our work, some of the results appear to be related. In particular, among the six Painlevé transcendents only PIII and PV have non-trivial symmetry algebras and that only for special values of the parameters.

IV. METRISABILITY OF EQUATIONS WITH THE PAINLEVÉ PROPERTY

All fifty equivalence classes of 2nd order ODEs with the Painlevé property are of the form (1.1) and so they define projective structures. Six of them are the Painlevé equations and their metrisability is determined by Theorem 1.3. In this section, we summarise the results of the analysis of the remaining forty-four cases listed in Ref. 16 in their most general form. We use the same numbering as this reference. We can divide these equations into five sets, according to their metrisability properties:

1. Metrisable with one degenerate solution and $4 > m([\nabla]) > 1$: II, III, VII, VIII, XII, XVIII, XIX, XXI, XXIII, XXIX, XXX, XXXIII, XXXVIII, XLIV, XLIX.
2. Metrisable with degenerate solution and $m([\nabla]) = 4$: XXII, XXXII.
3. Non-metrisable but admitting a degenerate solution: XIV, XX, XXXIV.
4. Not metrisable and no non-trivial solutions to the metrisability equations: V, X, XV, XVI, XXIV, XXV, XXVI, XXVII, XXVIII, XXXV, XXXVI, XL, XLII, XLVII, XLV, XLVI, XLVIII.
5. Metrisable and projectively flat: I, VI, XI, XVII, XXXVII, XLI, XLIII.

The Painlevé equations are IV, IX, XIII, XXXI, XXXIX, and L, which we did not include in the list but would fit in group 3 in general. The metrisable cases all admit a degenerate solution (thus their metrics admit a Killing vector, from Proposition 3.2), and their ODEs admit a quadratic first integral.

The submaximal (i.e., degree of mobility $m([\nabla]) = 4$) equations XXII and XXXII are related by a point transformation which is however not rational. The ODE XXXII

$$y'' = \frac{1}{2y}(1 + (y')^2)$$

is metrisable by $g = y(dx^2 + dy^2)$. The four quadratic first integrals for the parametrised geodesic motion give rise to three functionally dependent integrals quadratic in y'. Two independent integrals are as follows:

$$I_1 = \frac{1}{y}(1 + (y')^2), \quad I_2 = 2y' - \frac{x}{y}(1 + (y')^2).$$

V. SUMMARY

We have established which 2nd order ODEs with the Painlevé property are metrisable, i.e., all their integral curves are geodesics of some (pseudo) Riemannian metric. Out of the six Painlevé equations only PIII(α, 0, γ, 0), PIII(0, β, 0, δ), PV(α, β, 0, 0), and PVI(0, 0, 0, 1/2) are metrisable, the last case being projectively flat. In all cases, the metrisable equations with PP admit a first integral, and the degree of mobility is at least two. Thus metrisability picks out non-transcendental cases in the Painlevé analysis.

It would be interesting to extend Theorem 1.3 to systems of two second-order ODEs,

$$y'' = F(x, y, z, y', z'), \quad z'' = G(x, y, z, y', z'). \tag{5.1}$$

It is known how to characterise the systems resulting from a three-dimensional projective structure,[4,7,11] and some necessary and sufficient conditions for metrisability have recently been constructed in Refs. 8 and 10. The classification of systems (5.1) which admit that the Painlevé property is however missing.

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