How to recognise a conformally Einstein metric?

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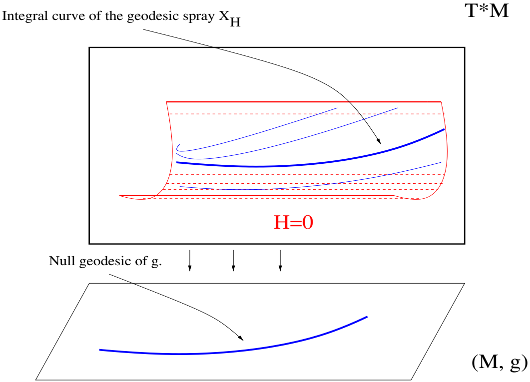
- MD, Paul Tod `arXiv:1304.7772.`, Comm. Math. Phys. (2014).

Conformal geometry and null geodesics

- Two (pseudo) Riemannian metrics g and $\hat{g} = \Omega^2 g$ on M (where $\Omega : M \to \mathbb{R}^+$) are conformally equivalent.
- Conformal rescalings preserve angles, and null geodesics:

$$H = g^{ab} p_a p_b, \quad \hat{H} = \Omega^{-2} H,$$

X_H Hamiltonian vector field on $T^*M, \omega = dp_a \wedge dx^a$



Conformal geometry and field equations

- Four dimensions

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \quad \hat{\epsilon}_{abcd} = \Omega^4 \epsilon_{abcd}, \quad \text{so} \quad \hat{\epsilon}_{ab}{}^{cd} = \epsilon_{ab}{}^{cd}$$

  and so $\star : \Lambda^2(M) \to \Lambda^2(M)$ is conformally invariant.

- Maxwell's equations

$$dF = 0, \quad d \star F = 0$$

  are conformally invariant.

- Einstein equations are NOT conformally invariant.

Conformal to Einstein

- (M, g) (pseudo) Riemannian n–dimensional manifold with Levi–Civita connection ∇.
- Curvature decomposition: Riemann=Weyl+Ricci+Scalar

$$[\nabla_a, \nabla_b]V^c = R_{ab}{}^c{}_d V^d$$

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b}.$$

- (M, g) is
  1. Einstein, if $R_{ab} = \frac{1}{n}Rg_{ab}$.
  2. Ricci–flat if $R_{ab} = 0$.
  3. Conformal to Einstein (Ricci–flat) if there exists $\Omega : M \to \mathbb{R}^+$ such that $\hat{g} = \Omega^2 g$ is Einstein (Ricci–flat).

- (M, g) Lorentzian four-manifold. Does there exist $\Omega : M \longrightarrow \mathbb{R}^+$, and a local coordinate system (x, y, z, t) such that

$$\hat{g} = \Omega^2 g = -dt^2 + dx^2 + dy^2 + dz^2.$$

  Answer. Need ${C_{ab}}^c{}_d = 0$ (Conformally invariant, as $\hat{C} = C$).

- Given an n–dimensional manifold (M, g), does there exist $\Omega : M \longrightarrow \mathbb{R}^+$ such that $\Omega^2 g$ is Einstein?
  1. $n = 2$. All metrics are Einstein.
  2. $n = 3$. Einstein=constant curvature. Need $Y_{abc} = 0$ (Cotton tensor).
  3. $n = 4$?? Brinkman 1920s, Szekeres 1963, Kozameh–Newman–Tod 1985, ...

Conformal gravity

$$B_{bc} := (\nabla^a\nabla^d - \frac{1}{2}R^{ad})C_{abcd}, \quad \text{Bach}$$

$$E_{abc} := C^{ef}{}_{cb}\nabla^d(*C)_{afed} - (*C)^{ef}{}_{cb}\nabla^d(C)_{afed}, \quad \text{Eastwood-Dighton.}$$

- Necessary conditions: $B_{ab} = 0, E_{abc} = 0$.
- Sufficient conditions (locally): $B_{ab} = 0, E_{abc} = 0$ + genericity on Weyl.
- Conformal gravity:

$$\int_M |C|^2 \text{vol}_g \longrightarrow \text{Euler-Lagrange} \longrightarrow B_{ab} = 0.$$

Anti-self-duality (genericity fails)

- An oriented 4-manifold (M, g) is anti-self-dual (ASD) if

$$C_{abcd} = -\frac{1}{2}\epsilon_{ab}{}^{ef}C_{cdef}.$$

- $B_{ab} = 0$, $E_{abc} = 0$. ...and yet (in the real analytic category)
  - ASD depends on 6 functions of 3 variables.
  - $R_{ab} = 0+$ASD depends o 2 functions of 3 variables.
- Signature $(4, 0)$ or $(2, 2)$. (Lorentzian+ASD=conformal flatness).
- Question: Given a 4-manifold (M, g) with ASD Weyl tensor, how to determine whether g is conformal to a Ricci–flat metric?

Twistor equation

- $\mathbb{C} \otimes TM \cong \mathbb{S} \otimes \mathbb{S}'$, where $(\mathbb{S}, \varepsilon), (\mathbb{S}', \varepsilon')$ are rank–two complex symplectic vector bundles (spin bundles) over M.
- Spinor indices (love it or hate it): $a = AA'$, where $A, B, \dots = 0, 1$, and $A', B', \dots = 0, 1$. Metric

$$g = \varepsilon \otimes \varepsilon'.$$

- Twistor operator. $\mathbb{D} : \Gamma(\mathbb{S}') \to \Gamma(T^*M \otimes \mathbb{S}')$.

$$\pi_{A'} \longrightarrow \nabla_{A(A'}\pi_{B')}$$

- Conformally invariant if

$$g \to \Omega^2 g, \quad \varepsilon \to \Omega\, \varepsilon, \quad \varepsilon' \to \Omega\, \varepsilon', \quad \pi \to \Omega\, \pi.$$

- Proposition. An ASD metric g is conformal to a Ricci flat metric if and only if $\dim(\text{Ker } \mathbb{D}) = 2$.
- Prolongation

$$\nabla\pi - \alpha \otimes \epsilon' = 0, \quad \nabla\alpha + \pi \lrcorner P = 0,$$

where $\alpha \in \Gamma(\mathbb{S})$, and $P_{ab} = (1/12)Rg_{ab} - (1/2)R_{ab}$.
- Connection $\mathcal{D}$ on a rank-four complex vector bundle $E = \mathbb{S} \oplus \mathbb{S}'$.
- Spinor conjugation in Riemannian signature: $\sigma : \mathbb{S}' \to \mathbb{S}'$.

$$\pi = (p, q), \quad \sigma(\pi) = (-\bar{q}, \bar{p}), \quad \text{so} \quad \pi \in \text{Ker } \mathbb{D} \leftrightarrow \sigma(\pi) \in \text{Ker } \mathbb{D}.$$

- Theorem 1. There is a one-to-one correspondence between parallel sections of $(E, \mathcal{D})$ and Ricci–flat metrics in an ASD conformal class.

Tensor Obstructions

- Integrability conditions for parallel sections: $\mathcal{R}\Psi = 0$ where $\mathcal{R} = [\mathcal{D}, \mathcal{D}]$ is the curvature (4 by 4 matrix), and $\Psi = (\pi, \alpha)^T$.
- If $\mathcal{R} = 0$ then g is conformally flat. Otherwise differentiate: $(\mathcal{D}\mathcal{R})\Psi = 0, \quad (\mathcal{D}\mathcal{D}\mathcal{R})\Psi = 0, \quad \dots$
- Necessary conditions: rank $(\mathcal{R}) = 2$. Sufficient conditions: rank $(\mathcal{R},\ \mathcal{D}\mathcal{R}) = 2$.
- Set $V_a = 4|C|^{-2}C^{bcd}{}_a\nabla^e C_{bcde}$.
  Theorem 2. An ASD Riemannian metric g is conformal to a Ricci–flat metric if and only if

$$\begin{aligned} \text{det}(\mathcal{R}) &:= 4\nabla^e C_{bcde}\nabla_f C^{bcdf} - |V|^2|C|^2 = 0, \quad \text{and} \\ T_{ab} &:= P_{ab} + \nabla_a V_b + V_a V_b - \frac{1}{2}|V|^2 g_{ab} = 0. \end{aligned}$$

Example

- Two parameter family of Riemannian metrics (LeBrun 1988)

$$g = f^{-1}dr^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2 + f\sigma_3^2), \quad \text{where} \quad f = 1 + \frac{A}{r^2} + \frac{B}{r^4},$$

  where σ_j are 1-forms on $SU(2)$ such that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.
- Kahler and vanishing Ricci scalar, so ASD. Conformal to Ricci flat?

$$\det(\mathcal{R}) = 0, \quad T_{ab} = \frac{A(4B - A^2)}{(Ar^2 + 2B)^2}\, g_{ab}.$$

- If $A = 0$ then g is Ricci flat (Eguchi–Hanson).
- If $B = A^2/4$ then $\hat{g} = (2r^2 + A)^{-1}g$ is Ricci flat. ASD Taub–NUT

$$\hat{g} = U\Big(dR^2 + R^2(\sigma_1^2 + \sigma_2^2)\Big) + U^{-1}\sigma_3^2, \quad \text{where} \quad U = \frac{1}{R} - 4A.$$

- g is ASD Kahler, $\Omega^2 g$ is hyper-Kahler! Are there more such examples?? (Clue: The Kahler form for g becomes a conformal Killing–Yano tensor for $\hat{g}$).

Another example

- Left invariant metrics on a four–dimensional nilpotent Lie group

$$g = {\sigma_0}^2 + {\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2, \quad \text{where}$$

$$d\sigma_0 = 2\sigma_0\wedge\sigma_3 - \sigma_1\wedge\sigma_2, \quad d\sigma_1 = \sigma_1\wedge\sigma_3, \quad d\sigma_2 = \sigma_2\wedge\sigma_3, \quad d\sigma_3 = 0.$$

- Find $C_+ = 0$, but $R_{ab} \neq 0$. Conf. to. Einstein obstructions vanish.
- Let $\sigma_3 = d\,\ln(z)$ where $z : M \to \mathbb{R}^+$.

$$\hat{g} = z^3\, g \quad \text{ASD and Ricci-flat.}$$

- Local coordinates (τ, x, y, z) s.t.

$$\sigma_0 = z^{-2}(d\tau + ydx), \quad \sigma_1 = z^{-1}dx, \quad \sigma_2 = z^{-1}dy, \quad \text{and}$$

$$\hat{g} = U(dx^2 + dy^2 + dz^2) + U^{-1}(d\tau + \alpha)^2,$$

where ($U = z$, $\alpha = ydx$) satisfy the monopole eq $*_3 dU = d\alpha$ on $\mathbb{R}^3$.

- Domain Wall in $(M \times \mathbb{R}, g - dt^2)$ (Gibbons+Rychenkova 2000, MD+Hoegner 2012). Approximate form of a regular metric on a complement of a cubic in $\mathbb{CP}^2$.

Thank You!