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Einstein–Maxwell gravitational instantons and five-dimensional solitonic strings

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Abstract

We study various aspects of four-dimensional Einstein–Maxwell multicentred gravitational instantons. These are half-BPS Riemannian backgrounds of minimal $\mathcal{N} = 2$ supergravity, asymptotic to $\mathbb{R}^4$, $\mathbb{R}^3 \times S^1$ or $\mathrm{AdS}_2 \times S^2$. Unlike for the Gibbons–Hawking solutions, the topology is not restricted by boundary conditions. We discuss the classical metric on the instanton moduli space. One class of these solutions may be lifted to causal and regular multi 'solitonic strings', without horizons, of (4 + 1)-dimensional $\mathcal{N} = 2$ supergravity, carrying null momentum.

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1. Introduction

1.1. Motivation

Instantons play a key role in the nonperturbative dynamics of Yang–Mills theories, and indeed in a wide range of quantum mechanical systems. One useful property of instantons is that they can allow a semiclassical description where a full treatment is either difficult or even ill defined, as in the case of gravity. At the other extreme, in supersymmetric theories instantons are crucial in obtaining exact results.

Within the programme of Euclidean quantum gravity, multicentred gravitational instantons followed hotly on the tails of their Yang–Mills counterparts [1, 2]. However, while the Gibbons–Hawking metrics have found a surprising range of physical applications, their dynamical role within quantum gravity remains unclear. One reason for this is that if the instanton contains more than one centre, it is no longer asymptotically Euclidean ($\sim\mathbb{R}^4$) or asymptotically flat ($\sim\mathbb{R}^3 \times S^1$). These are the most natural asymptotics for infinite volume quantum gravity at zero or finite temperature, respectively. In contrast, at a constant large radius the multicentred Gibbons–Hawking spaces tend to S^1 fibred over S^2 with an increasingly high Chern number.

Said differently, the boundary conditions determine the gravitational instanton topology. There is no sum over different spacetime topologies for fixed asymptotics. In this sense, the Gibbons–Hawking spaces do not provide a semiclassical realization of spacetime foam.

It is therefore of interest to study gravitational theories in which an arbitrarily high instanton number is allowed with fixed asymptotics. One example of such a theory is conformal gravity, in which the Einstein–Hilbert term is replaced by the Weyl curvature squared [3–5]. Despite some rather attractive features of the gravitational instantons in this theory, the physical status of the theory itself is uncertain due to problems with higher derivative Lagrangians and unitarity.

In this paper, we emphasize that the Einstein–Maxwell theory also admits regular multicentred instantons with arbitrarily complicated topology for fixed asymptotics. These solutions have essentially appeared before in the literature [6, 7]. Various unsatisfactory aspects of these previous treatments, for instance we have preferred to use a Riemannian Maxwell field that is real, have led us to carry out a systematic study *de novo*. We furthermore extend our understanding of Einstein–Maxwell gravitational instantons through discussions of uniqueness, supersymmetry, moduli space metrics and lifts to five dimensions. This final point may be of independent interest.

1.2. Summary

In section 2 we present the instanton solutions. We detail the possible asymptotics: $\mathbb{R}^4$, $\mathbb{R}^3 \times S^1$ and $\mathrm{AdS}_2 \times S^2$, and local versions thereof. We show that the solutions are half-BPS when embedded into minimal $\mathcal{N} = 2$ supergravity and that they are all the regular Riemannian half-BPS solutions. Finally, we evaluate the action of the solutions. The asymptotically Euclidean case is found to be only well defined when a certain linear combination of the charge and potential is fixed at infinity.

In section 3, we discuss the moduli space metric on the Einstein–Maxwell instantons. We consider in some detail the ambiguities involved in finding an inner product on the space of metric fields. We show that there is a preferred inner product which is inherited from the action and for which zero modes are orthogonal to pure gauge modes.

Section 4 shows how the four-dimensional instantons may be lifted to solitons of the five-dimensional Einstein–Maxwell–Chern–Simons (EMCS) theory, or minimal $\mathcal{N} = 2$ supergravity in five dimensions. Generically, the lifted solutions are either singular or contain closed time-like curves. However, we find that one class of solutions lift to regular, causal plane fronted wave spacetimes with the fields localized in lumps orthogonal to the wave propation. We call these ‘solitonic strings’ as they do not have an event horizon.

Section 5 briefly discusses the slow motion of the five-dimensional solitons. Unlike in the case of the Gibbons–Hawking instantons and their lift to Kaluza–Klein monopoles, it seems that there is not a direct connection between the four-dimensional instanton moduli space metric and the five-dimensional soliton slow motion moduli space metric in our case.

We end with a discussion of possible physical applications of these multicentred Einstein–Maxwell instantons and directions for future work.

2. The gravitational instantons

2.1. The solutions

The gravitational instantons on a four-dimensional manifold M_4 are solutions to the Einstein–Maxwell equations with Riemannian signature

$$G_{ab} = 2F_a{}^c F_{bc} - \tfrac{1}{2} g_{ab} F^{cd} F_{cd}, \qquad \nabla_a F^{ab} = 0. \tag{1}$$

The metric is given by

$$g^{(4)} = \frac{1}{U\widetilde{U}} (\mathrm{d}\tau + \boldsymbol{\omega})^2 + U\widetilde{U}\, \mathrm{d}\mathbf{x}^2, \tag{2}$$

where the functions $U, \widetilde{U}$ and the 1-form ω depend on $\mathbf{x} = (x, y, z)$ and satisfy

$$\nabla^2 U = \nabla^2 \widetilde{U} = 0, \qquad \nabla \times \omega = \widetilde{U}\nabla U - U\nabla\widetilde{U}. \tag{3}$$

We will work with four-dimensional tangent space indices, $a, b, \ldots$, and the vierbeins

$$e^4 = \frac{1}{(U\widetilde{U})^{1/2}} (\mathrm{d}\tau + \boldsymbol{\omega}), \qquad e^i = (U\widetilde{U})^{1/2}\, \mathrm{d}x^i. \tag{4}$$

The electromagnetic field strength may now be written as

$$F_{4i} = \tfrac{1}{2}\partial_i [U^{-1} - \widetilde{U}^{-1}], \qquad F_{ij} = \tfrac{1}{2}\varepsilon_{ijk}\partial_k [U^{-1} + \widetilde{U}^{-1}], \tag{5}$$

where the derivatives are partial derivatives with respect to the corresponding spacetime indices. One can check that this field strength satisfies the Bianchi identities, and thus locally at least we can write $F = \mathrm{d}A$. Our expressions for the field strength in the Riemannian signature differ slightly from the others in the literature [6, 7], which were not real. In particular, the Riemannian Majumdar–Papapetrou metrics with $U = \widetilde{U}$ have purely magnetic field strength $F = -2 \star_3 \mathrm{d}U$.

These backgrounds were first found in the Lorentzian regime by Israel and Wilson [8] and by Perjés [9] as a stationary generalization of the static Majumdar–Papapetrou multi black hole solutions. However, it was shown by Hartle and Hawking that all the non-static solutions suffered from naked singularities [10, 11].

With the Riemannian signature, however, regular solutions exist [6, 7]. We can take

$$U = \frac{4\pi}{\beta} + \sum_{m=1}^{N} \frac{a_m}{|\mathbf{x} - \mathbf{x}_m|}, \qquad \widetilde{U} = \frac{4\pi}{\tilde{\beta}} + \sum_{n=1}^{\widetilde{N}} \frac{\tilde{a}_n}{|\mathbf{x} - \tilde{\mathbf{x}}_n|}; \tag{6}$$

in these expressions $\beta, \tilde{\beta}, a_m, \mathbf{x}_m, \tilde{a}_n, \tilde{\mathbf{x}}_n, N, \widetilde{N}$ are constants. For the signature to remain $(+, +, +, +)$ throughout, we can require $U, \widetilde{U} > 0$ which in turn requires $a_m, \tilde{a}_n > 0$. From the explicit forms for U and $\widetilde{U}$ in (6) we can write down explicit expressions for the 1-forms $\boldsymbol{\omega}$ and A, which so far we have only defined implicitly. These are given in appendix A.

If there is at least one non-coincident centre, $\mathbf{x}_m \neq \tilde{\mathbf{x}}_n$, regularity requires that τ is identified with period 4π and that the constants satisfy the following constraints at all the non-coincident centres:

$$U(\tilde{\mathbf{x}}_n)\tilde{a}_n = 1, \qquad \widetilde{U}(\mathbf{x}_m)a_m = 1, \qquad \forall m, n. \tag{7}$$

Given the locations of the centres $\{\mathbf{x}_n, \tilde{\mathbf{x}}_m\}$, these constraints may be solved uniquely for $\{a_n, \tilde{a}_m\}$ [7]. When $\frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} = 0$, the solution is only unique up to the overall scaling

$$U \to e^s U, \qquad \widetilde{U} \to e^{-s}\widetilde{U}. \tag{8}$$

In general, this scaling leaves the metric invariant and induces a linear duality transformation on the Maxwell field mapping solutions to solutions

$$\mathbf{E} \to \cosh s\mathbf{E} + \sinh s\mathbf{B}, \qquad \mathbf{B} \to \sinh s\mathbf{E} + \cosh s\mathbf{B}. \tag{9}$$

The rescaling does not leave the action and other properties of the solutions invariant.

The constants β and $\tilde{\beta}$ determine the asymptotics of the solution. There are three possibilities which are as follows.

- The case $\frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} \neq 0$ gives an asymptotically locally flat (ALF) metric, tending to an S^1 bundle over S^2 at infinity, with the first Chern number $N - \widetilde{N}$. Without loss of generality, we have rescaled the harmonic functions using (8) so that $\beta = \tilde{\beta}$. Equation (7) now implies that $\sum a_m - N = \sum \tilde{a}_n - \widetilde{N}$. If $N = \widetilde{N}$ the asymptotic bundle is trivial and we obtain asymptotically flat ($\sim\mathbb{R}^3 \times S^1$) solutions.
- The case $\frac{4\pi}{\beta} = 0$, $\frac{4\pi}{\tilde{\beta}} = 1$ gives an asymptotically locally Euclidean (ALE) metric, tending to $\mathbb{R}^4/\mathbb{Z}_{|N-\widetilde{N}|}$. We have used the rescaling (8) to set $\frac{4\pi}{\tilde{\beta}} = 1$ without loss of generality. In this case, the constraints (7) require that $\sum a_m = N - \widetilde{N}$. Of course, we can reverse the roles of β and $\tilde{\beta}$. If $N = \widetilde{N} + 1$, the solution is asymptotically Euclidean ($\sim\mathbb{R}^4$).
- The case $\frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} = 0$ leads to an asymptotically locally Robinson–Bertotti metric, tending to $\mathrm{AdS}_2 \times S^2$ or $\mathrm{AdS}_2/\mathbb{Z} \times S^2$. The former case only arises if all of the centres are coincident, so that $U = \widetilde{U}$, and τ need not be made periodic. For both these asymptotics, the constraints (7) require that $N = \widetilde{N}$. We may further use the rescaling (8) to set $\sum a_m = \sum \tilde{a}_n$.

As Riemannian solutions, the backgrounds are naturally thought of as generalizations of the Gibbons–Hawking multicentre metrics which in fact they include as the special case $\widetilde{U} = 1$, albeit with an additional anti-self-dual Maxwell field. A crucial new aspect of the asymptotically locally Euclidean and asymptotically locally flat Israel–Wilson–Perjés solutions is that when

$$N = \widetilde{N} \pm 1 (\text{for ALE}) \quad \text{or} \quad N = \widetilde{N} (\text{for ALF}), \tag{10}$$

the fibration of the τ circle over S^2 at infinity is trivial and the metrics do not require the $\mathbb{Z}_N$ identifications at infinity that are needed in the Gibbons–Hawking case. The spacetimes are therefore strictly asymptotically Euclidean and asymptotically flat respectively in these cases. The Euler number is given by $\chi = N + \widetilde{N} - 1$ in the ALF and ALE cases [7]. Thus the spaces admit arbitrarily complicated topology, not restricted by the asymptotic topology, and provide a semiclassical realization of spacetime foam in the quantum Einstein–Maxwell theory.

The metric (2) has vanishing scalar curvature. If U or $\widetilde{U}$ is constant, then (2) is Ricci flat and hyper-Kähler. It is natural to ask whether any other special choices of harmonic functions U and $\widetilde{U}$ lead to scalar flat Kähler metrics with a symmetry $\partial/\partial\tau$ preserving the Kähler structure. Such metrics would be conformally anti-self-dual and thus interesting in twistor theory. The answer is negative. From [4] any such metric is of the form

$$g^{(4)} = \frac{1}{\mathcal{W}}(\mathrm{d}\tau + \omega)^2 + \mathcal{W}h^{(3)}, \tag{11}$$

where the metric $h^{(3)}$ on the three-dimensional orbit space of $\partial/\partial\tau$ and the function $\mathcal{W}$ on this space satisfy a coupled nonlinear system of PDEs. In the case that $h^{(3)}$ is flat, the equations reduce to

$$\nabla \times \boldsymbol{\omega} = \nabla\mathcal{W}. \tag{12}$$

Therefore $\mathcal{W} = U\widetilde{U}$ is harmonic, and then (3) implies that $\widetilde{U}$ is a constant.

2.2. *Killing spinors*

The solutions have the further important property of admitting two complex Killing spinors. These satisfy

$$e^{\mu}{}_{a}\partial_{\mu}\varepsilon + \tfrac{1}{4}\left[\omega^{bc}{}_{a}\Gamma_{bc} + \mathrm{i}F^{bc}\Gamma_{bc}\Gamma_{a}\right]\varepsilon = 0, \tag{13}$$

where ${\omega^{bc}}_a$ are the components of the spin connection 1-form ω^{bc} defined by $\mathrm{d}e^b = \omega^{bc} \wedge e^c$. We use Greek letters $\mu, \nu, \ldots$ to denote Euclidean spacetime indices. Our gamma matrix conventions are given in appendix B, as is the spin connection for the background. With these conventions, one may solve equation (13) to find

$$\varepsilon = \begin{pmatrix} U^{-1/2}\varepsilon_0 \\ \mathrm{i}\widetilde{U}^{-1/2}\varepsilon_0 \end{pmatrix}, \tag{14}$$

where ε_0 is a constant two-component complex spinor: $\partial_\mu \varepsilon_0 = 0$.

Within the Lorentzian Einstein–Maxwell theory, the Killing spinors imply that the solutions saturate a Bogomolny bound [12]. It is also natural to view the solutions as half-BPS states of four-dimensional $\mathcal{N} = 2$ supergravity [13]. This theory has a complex spin-$\frac{3}{2}$ Rarita–Schwinger field as well as the graviton and photon. In fact, in a paper that anticipated current interest in classifying supersymmetric solutions, Tod has shown that the Lorentzian versions of these solutions are all the supersymmetric solutions to $\mathcal{N} = 2$ supergravity with a time-like Killing spinor [14].

In the following subsection, we shall repeat Tod's analysis in the Riemannian case. As well as recovering the local form of the metric, we will find that $|\nabla U^{-1}|$ and $|\nabla \widetilde{U}^{-1}|$ are both bounded[3]. Combined with a result from analysis [15], it will follow that (2) together with (6) is the most general regular supersymmetric solution to minimal $\mathcal{N} = 2$ supergravity. To put it differently, only harmonic functions with a finite number of point sources lead to regular metrics.

As usual, given Killing spinors ε and η we can build differential forms. In particular, we have the 1-forms

$$V = \tfrac{1}{2}\bar{\eta}\Gamma_a \varepsilon e^a, \qquad K = \tfrac{1}{2}\bar{\eta}\Gamma_5\Gamma_a\varepsilon e^a, \tag{15}$$

and the 2-form

$$\Omega = -\frac{\mathrm{i}}{2}\bar{\eta}\Gamma_{ab}\varepsilon e^a \wedge e^b. \tag{16}$$

In our representation of the Clifford algebra, given in the appendix, all the gamma matrices are Hermitian and therefore the bar simply denotes complex conjugation. From the Killing spinor condition (13), we have that

$$\mathrm{d}\Omega = -2V \wedge F, \qquad \mathrm{d}V = 0, \qquad \nabla_{(a}K_{b)} = 0. \tag{17}$$

With a little more work, one can also show that

$$\nabla_a \Omega_{bc} = 2V_a F_{bc} - 4F_{a[b}V_{c]} + 4V^d F_{d[b}g_{c]a}, \qquad \nabla_a V_b = \tfrac{1}{4}F^{cd}\Omega_{cd}g_{ab} + {F^c}_{(a}\Omega_{b)c}. \tag{18}$$

In fact there is more structure. The 2-form Ω can be split into self-dual and anti-self-dual parts: $\Omega = \Omega^+ + \Omega^-$. One can then show that Ω^+ and Ω^- separately satisfy the first equation in (18) with F replaced by its self-dual, F^+, and anti-self-dual, F^-, parts respectively.

Three important cases giving real forms are when $\eta = \varepsilon = \varepsilon^I$, for $I = 1, 2, 3$, which are defined by ε_0 in (14) satisfying $\bar{\varepsilon}_0^I \tau^J \varepsilon_0^I = \delta^{IJ}$ and $\bar{\varepsilon}_0^I \varepsilon_0^J = \delta^{IJ}$. For these cases, we find

$$V^I = \mathrm{d}x^I, \qquad K = \frac{1}{U\widetilde{U}}(\mathrm{d}\tau + \omega), \tag{19}$$

and

$$\Omega^I = (U^{-1} - \tilde{U}^{-1})e^4 \wedge e^I + \tfrac{1}{2}(U^{-1} + \tilde{U}^{-1})\epsilon^{Ijk}e^j \wedge e^k. \tag{20}$$

Raising the index, the Killing vector is $K = \partial/\partial\tau$ as we should expect.

[3] This is stronger than the Lorentzian result of [11] where the separate bounds cannot be established.

2.3. Uniqueness of the solutions

Here we show that solutions (2), (5) with the harmonic functions described by (6) and satisfying the constraints (7) are the most general regular Einstein–Maxwell instanton with a complex Killing spinor.

In this section, it will be convenient to write the Dirac spinor $\varepsilon = (\alpha^A, \beta_{A'})$ as a pair of complex two-component spinors. When dealing with these spinors, we use the conventions given in appendix C. With positive signature, spinor conjugation preserves the type of spinors. Thus if $\alpha_A = (p, q)$, we can define $\hat{\alpha}_A = (\overline{q}, -\overline{p})$ so that $\hat{\hat{\alpha}}_A = -\alpha_A$. This Hermitian conjugation induces a positive inner product

$$\alpha_A\hat{\alpha}^A = \epsilon_{AB}\alpha^B\hat{\alpha}^A = |p|^2 + |q|^2. \tag{21}$$

We define the inner product on the primed spinors in the same way. Here, ϵ_{AB} and $\epsilon_{A'B'}$ are covariantly constant symplectic forms with $\epsilon_{01} = \epsilon_{0'1'} = 1$. These are used to raise and lower spinor indices according to $\alpha_B = \epsilon_{AB}\alpha^A$, $\alpha^B = \epsilon^{BA}\alpha_A$, and similarly for primed spinors. In terms of our gamma matrices, $\hat{\varepsilon} = \Gamma^{31}\bar{\varepsilon}$.

The Killing spinor equation (13) becomes

$$\nabla_{AA'}\alpha_B - \mathrm{i}\sqrt{2}\phi_{AB}\beta_{A'} = 0, \qquad \nabla_{AA'}\beta_{B'} + \mathrm{i}\sqrt{2}\tilde{\phi}_{A'B'}\alpha_A = 0, \tag{22}$$

where the spinors ϕ and $\tilde{\phi}$ are symmetric in their respective indices and give the anti-self-dual and self-dual parts of the electromagnetic field, respectively,

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \tilde{\phi}_{A'B'}\epsilon_{AB}. \tag{23}$$

Suppose that $\varepsilon = (\alpha^A, \beta_{A'})$ solves the Killing spinor equation (22). Now we can reconstruct the spacetime metric and Maxwell field.

Define

$$U = (\alpha_A\hat{\alpha}^A)^{-1}, \qquad \widetilde{U} = (\beta_{A'}\hat{\beta}^{A'})^{-1}. \tag{24}$$

In our positive definite case, these two inverted functions do not vanish unless α or β vanishes. In the Lorentzian case, their possible vanishing leads to plane wave spacetimes [14]. If α or β vanishes identically, we recover the Gibbons–Hawking solutions. Now define a (complex) null tetrad

$$X_a = \alpha_A\beta_{A'}, \qquad \overline{X}_a = \hat{\alpha}_A\hat{\beta}_{A'}, \qquad Y_a = \alpha_A\hat{\beta}_{A'}, \qquad \overline{Y}_a = -\hat{\alpha}_A\beta_{A'}. \tag{25}$$

We can check that $\hat{\varepsilon}$ is also a solution to the Killing spinor equation (22). It therefore follows from (22) that $X_a, \overline{X}_a, Y_a - \overline{Y}_a$ are gradients and that $K_a = Y_a + \overline{Y}_a$ is a Killing vector. Now define local coordinates (x, y, z, τ) by

$$X = \frac{1}{\sqrt{2}}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y), \qquad (Y - \overline{Y}) = \mathrm{i}\sqrt{2}\,\mathrm{d}z, \qquad K^a\nabla_a = \sqrt{2}\frac{\partial}{\partial\tau}, \tag{26}$$

where the form $X = X_a e^a = X_{AA'}e^{AA'}$ and similarly for $Y, \overline{Y}$. The vector K Lie derives the spinors $(\alpha_A, \beta_{A'})$, implying that U and $\widetilde{U}$ are independent of τ.

The metric is now given by $\mathrm{d}s^2 = \epsilon_{AB}\epsilon_{A'B'}e^{AA'}e^{BB'}$. This expression may be evaluated by noting that from (24) we have $\epsilon_{AB} = U(\alpha_A\hat{\alpha}_B - \alpha_B\hat{\alpha}_A)$ and similarly for $\epsilon_{A'B'}$. Using the fact that from the above definitions $K_aK^a = 2(U\widetilde{U})^{-1}$, we find that the metric takes the form (2) for some 1-form $\boldsymbol{\omega}$. The next step is to find $\boldsymbol{\omega}$.

The definitions of $U, \widetilde{U}$ and K together with (22) imply

$$\nabla_a K_b = \mathrm{i}\sqrt{2}[\widetilde{U}^{-1}\phi_{AB}\epsilon_{A'B'} + U^{-1}\tilde{\phi}_{A'B'}\epsilon_{AB}], \tag{27}$$

and

$$\nabla_a U^{-1} = \mathrm{i}\sqrt{2}\phi_{AB}K^B_{A'}, \qquad \nabla_a\widetilde{U}^{-1} = -\mathrm{i}\sqrt{2}\tilde{\phi}_{A'B'}K^{B'}_A. \tag{28}$$

The formulae in (28) may be inverted to find expressions for ϕ_{AB} and $\tilde{\phi}_{A'B'}$, using $K_B^{A'}K^{BC'} = \frac{1}{2}\epsilon^{A'C'}K_{DE'}K^{DE'}$. Substituting the result into (27) yields expression (3) for $\nabla \times \boldsymbol{\omega}$.

Finally, differentiating relations (22) shows that the energy momentum tensor is that of the Einstein–Maxwell theory: $T_{ab} = 2\phi_{AB}\tilde{\phi}_{A'B'}$. The Maxwell equations

$$\nabla^{AA'}\phi_{AB} = 0, \qquad \nabla^{AA'}\phi_{A'B'} = 0 \tag{29}$$

now imply that U and $\widetilde{U}$ are harmonic on $\mathbb{R}^3$. This completes the local reconstruction of the solution from the Killing spinors.

So far everything has proceeded as in [14] with minor differences in the reality conditions. The main difference arises in global regularity considerations which lead us to consider the invariant

$$\begin{aligned} F_{ab}F^{ab} &= 2(\phi_{AB}\phi^{AB} + \tilde{\phi}_{A'B'}\tilde{\phi}^{A'B'}) \\ &= |\nabla U^{-1}|^2 + |\nabla \widetilde{U}^{-1}|^2, \end{aligned} \tag{30}$$

where the norm of the gradients is taken with respect to the flat metric on $\mathbb{R}^3$, and we have used (28). Regularity requires that this invariant be bounded. Therefore, both $|\nabla U^{-1}|$ and $|\nabla \widetilde{U}^{-1}|$ must be bounded. The various boundary conditions we have described imply that U and $\widetilde{U}$ are regular as $|\mathbf{x}| \to \infty$. In particular, they are both regular outside a ball B_R of sufficiently large radius R in $\mathbb{R}^3$.

The coordinates $\{\mathbf{x}, \tau\}$ cover $\mathbb{R} \times (\mathbb{R}^3 \backslash \mathcal{S})$, where $\mathcal{S}$ is the compact subset of B_R on which U or $\widetilde{U}$ blows up. A theorem from [15] can now be applied separately to both harmonic functions to prove that $\mathcal{S}$ consists of a finite number of points. In fact,

$$\#\mathcal{S} < \max\{|\nabla U^{-1}|, |\nabla \widetilde{U}^{-1}|\}|U(p) + \widetilde{U}(p)|R + 1, \tag{31}$$

where p is any point in B_R which does not belong to $\mathcal{S}$. This combined with the maximum principle shows that (6) are the most general harmonic functions leading to regular metrics. It also follows from (24) and the positivity of the spinor inner product that a_m and $\tilde{a}_n$ in (6) are all non-negative.

The spinors $\alpha_A, \beta_{A'}$ and their conjugates give a preferred basis for the space $\Lambda^2(M)$ of 2-forms. The anti-self-dual 2-forms are given in terms of α_A by

$$\mathrm{Re}(\alpha_A\alpha_B\epsilon_{A'B'}), \qquad \mathrm{Im}(\alpha_A\alpha_B\epsilon_{A'B'}), \qquad \mathrm{i}\alpha_{(A}\hat{\alpha}_{B)}\epsilon_{A'B'}, \tag{32}$$

and the self-dual 2-forms are given in terms of $\beta_{A'}$ by analogous expressions. The three 2-forms (20) can be expressed in this basis as

$$\begin{aligned} \Omega^1 + \mathrm{i}\Omega^2 &= -(\alpha_A\alpha_B\epsilon_{A'B'} + \beta_{A'}\beta_{B'}\epsilon_{AB})e^{AA'} \wedge e^{BB'}, \\ \Omega^3 &= \mathrm{i}(\beta_{(A'}\hat{\beta}_{B')}\epsilon_{AB} - \alpha_{(A}\hat{\alpha}_{B)}\epsilon_{A'B'})e^{AA'} \wedge e^{BB'}. \end{aligned} \tag{33}$$

The spinor expressions for (18) can now be easily derived using (22).

2.4. Action of the instantons

The contribution of instantons to physical processes is of course weighted by their actions. Therefore, it is important to evaluate the actions of the spacetimes we are considering. Previous computations on this subject should be approached with caution: there are computational errors in [6] leading to unphysical results such as an action unbounded from below, while in [7] the Maxwell contribution to the action is not considered. Both of these papers also work with imaginary electric fields which lead to some undesirable properties of the actions.

The Riemannian Einstein–Maxwell action, including the Gibbons–Hawking boundary term, is

$$S = -\int_{M_4} \mathrm{d}^4x\sqrt{g^{(4)}}[R^{(4)} - F_{ab}F^{ab}] - 2\int_{\partial M_4} \mathrm{d}^3x\sqrt{\gamma}\mathcal{K}, \tag{34}$$

where γ is the induced metric on the boundary and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary.

Evaluated on the Einstein–Maxwell instantons we are considering, one finds

$$S = -2\pi \lim_{r\to\infty} \int_{S^2} \mathrm{d}\Omega^2 r^2 \left[\frac{(U+\widetilde{U})^2\partial_r(U\widetilde{U})}{(U\widetilde{U})^2} + \frac{8}{r}\right]. \tag{35}$$

Here, we have introduced spherical polar coordinates $\mathrm{d}\mathbf{x}^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$. Expression (35) is divergent and needs to be regularized by subtracting off the action of a reference geometry. This must be done separately for the asymptotically locally flat, Euclidean and Robinson–Bertotti cases. We have assumed in (35) that τ is identified with period 4π.

The easiest case is asymptotic local flatness, with $\beta = \tilde{\beta} \neq 0$. Here the background simply has $U = \widetilde{U} = \frac{4\pi}{\beta}$, giving flat $S^1 \times \mathbb{R}^3$ and a vanishing Maxwell field. One finds

$$\Delta S_{\text{ALF}} = 8\pi\beta\left(\sum a_m + \sum \tilde{a}_n\right). \tag{36}$$

Recall that furthermore $\sum a_m = \sum \tilde{a}_n + N - \widetilde{N}$ in this case.

The asymptotically locally Robinson–Bertotti case is also straightforward. Here the background is the Robinson–Bertotti spacetime with τ identified, $\mathrm{AdS}_2/\mathbb{Z} \times S^2$, supported by magnetic flux, that is, $U = \widetilde{U} = \frac{\sum a_m}{r} = \frac{\sum \tilde{a}_n}{r}$. The regularized action turns out to vanish

$$\Delta S_{\text{ALRB}} = 0. \tag{37}$$

Now consider the asymptotically locally Euclidean case, with $\frac{4\pi}{\beta} = 0$ and $\frac{4\pi}{\tilde{\beta}} = 1$. The required background is Euclidean space with an anti-self-dual Maxwell field, that is, $U = \frac{\sum a_m}{r}$ and $\widetilde{U} = 1$. Subtracting this background regularizes the gravitational action, but it does not remove all the divergences from the Maxwell action. The divergence of the regularized action tells us that we have not imposed the correct boundary conditions for the Maxwell field with these asymptotics.

The standard action (34) is appropriate for fixing the potential at infinity: $\delta A_a = 0$. Different boundary conditions may be implemented by adding a boundary term to the action. To obtain a finite action for ALE asymptotics, we need to add a boundary term [16] that entirely cancels the bulk Maxwell action when evaluated on solutions. The required term is

$$S_{\text{ALE}}|_{\text{bdy.}} = 2\int_{\partial M_4} \mathrm{d}^3x\sqrt{\gamma}A^aF_{ab}n^b, \tag{38}$$

where n^b is a unit normal vector to the boundary. The resulting boundary condition is

$$A_a\delta(F^{ab}n_b) = \delta A_a F^{ab}n_b \quad \text{on} \quad \partial M_4. \tag{39}$$

Physically, this equation corresponds to keeping a certain linear combination of the charge and potential fixed at infinity.

The boundary term (38) is gauge invariant when evaluated on configurations satisfying the classical equations of motion, which is what we require. Off shell it is not gauge invariant under gauge transformations that do not vanish at infinity. This reflects the standard fact that such transformations include global gauge transformations which are generated by the charge operator. In the quantum theory, these transformations do not leave charged states invariant.

With the boundary term (38) added, the action is found to be given by

$$\Delta S_{\mathrm{ALE}} = 16\pi^2 \sum \tilde{a}_n. \tag{40}$$

At this moment, we do not have a physical understanding of why the ALE instantons only contribute to processes in which the particular boundary condition (39) is imposed.

3. Instanton moduli space metric

The analysis done in section 2.3 has demonstrated that the Einstein–Maxwell gravitational instantons with a Killing spinor have $3(N + \widetilde{N})$ free parameters or moduli. The Euclidean group in three dimensions can be used to fix six of these, except in the case when $N + \widetilde{N} = 2$, in which case it only fixes five, due to the axisymmetry. To obtain the moduli space one should also quotient by the symmetric group $S_N \times S_{\widetilde{N}}$ acting on the centres. Note that fixing the action then adds a further constraint on the centres in the asymptotically locally flat and Euclidean cases.

While computation of the measure and metric on the moduli space of Yang–Mills instantons is by now a highly developed field, the case of gravitational instantons in four dimensions appears to have been less systematically treated in the literature. In two dimensions, of course, the measure plays a fundamental role in string theory. Reflecting this state of affairs, we now give a fairly general exposition of the formalism needed to compute moduli space metrics for gravitational instantons in pure gravity and Einstein–Maxwell theory.

3.1. Inner products

Let us recall the Yang–Mills procedure, but work with just the $U(1)$ Maxwell case both for simplicity and because this is what we need anyhow. One begins by writing down a natural ultralocal inner product on the space of field perturbations. Strictly speaking, it is an inner product on the tangent bundle to the space of fields

$$\langle \delta A, \delta A' \rangle = 2 \int_{M_4} \mathrm{d}^4x \sqrt{g} g^{\mu\nu} \delta A_\mu \delta A'_\nu. \tag{41}$$

In this section, it is appropriate to work with spacetime indices $\mu, \nu \ldots$. One now restricts to considering only perturbations that are orthogonal to pure gauge transformations. Thus, one requires

$$0 = \langle \delta A, \mathrm{d}\Omega \rangle = -2 \int_{M_4} \mathrm{d}^4x \sqrt{g} g^{\mu\nu} \Omega \nabla_\mu \delta A_\nu, \tag{42}$$

for all Ω. Therefore, perturbations must be considered in the Lorenz gauge

$$\nabla_\mu \delta A^\mu = 0. \tag{43}$$

Given this gauge, we can note that the inner product (41) should be thought of as coming from the quadratic terms in the action. In particular, this determines the normalization. The quadratic action is

$$\begin{aligned} S^{(2)}_{\delta A} &= 2 \int_{M_4} \mathrm{d}^4x \sqrt{g} (\nabla^\mu \delta A^\rho \nabla_\mu \delta A_\rho - \nabla^\mu \delta A^\rho \nabla_\rho \delta A_\mu) \\ &\to -2 \int_{M_4} \mathrm{d}^4x \sqrt{g} g^{\rho\sigma} \delta A_\rho \nabla^2 \delta A_\sigma + \text{non-derivative terms}, \end{aligned} \tag{44}$$

where the arrow denotes imposition of the Lorenz gauge. We can see that the index structure of the gauge field is now that of the inner product (41). That is to say, the term in the last line of (44) is just $-\langle\delta A, \nabla^2\delta A\rangle$, where ∇^2 should be regarded as an operator on M_4. In this way, the inner product is inherited from the action. The metric on the moduli space is obtained by restricting the inner product (41) to zero modes. To summarize the logic: the metric on the moduli space is inherited from the quadratic kinetic terms in the action written in a specific gauge. However, that gauge must simultaneously imply that field fluctuations are orthogonal to pure gauge transformations.

We should note at this point that imposing orthogonality to gauge transformations, with a consequent choice of gauge imposed, is not completely essential. However, it does greatly simplify instanton computations and gives a clear physical meaning to the moduli space metric itself.

For the case of metric fluctuations, there is not a unique ultralocal inner product with the correct symmetries. Rather we have the family of de Witt metrics parametrized by $\lambda \in \mathbb{R}$

$$\langle\delta g, \delta g'\rangle_\lambda = \int_{M_4} d^4x\, G^{\mu\nu\rho\sigma}_{(\lambda)}\delta g_{\mu\nu}\delta g'_{\rho\sigma}, \tag{45}$$

where

$$G^{\mu\nu\rho\sigma}_{(\lambda)} = \tfrac{1}{8}\sqrt{g}[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2\lambda g^{\mu\nu}g^{\rho\sigma}]. \tag{46}$$

Thus, λ parametrizes the possible inner products. The metric is positive definite for $\lambda < 1/4$ and non-degenerate for $\lambda \neq 1/4$. In appendix D, we demonstrate that different values of λ indeed give non-equivalent inner products on moduli space.

The de Witt metric with $\lambda = 1$ also appears in Hamiltonian treatments of gravity. This is not what we are doing here; the metric we want is on four-dimensional Riemannian geometries. In the case of pure gravity there is a connection, as the four-dimensional Euclidean theory can be lifted to the 4 + 1 Einstein theory. The gravitational instantons become Kaluza–Klein monopoles in five dimensions. In this context, the moduli space on the multicentred Gibbons–Hawking spaces has been computed as the slow motion moduli space metric of the Kaluza–Klein monopoles [17]. We will describe a lift of our solutions in a later section, but for the moment we are pursuing a four-dimensional treatment.

The ambiguity in the inner product translates into a choice of gauge. Imposing orthogonality to pure gauge transformations now requires

$$0 = \langle\delta g, \mathcal{L}_\xi g\rangle_\lambda = -\frac{1}{2}\int_{M_4} d^4x\sqrt{g}\xi^\mu\left[\nabla^\nu\delta g_{\mu\nu} - \lambda\nabla_\mu\delta g^\nu{}_\nu\right]. \tag{47}$$

Here, $\mathcal{L}$ is the Lie derivative. Therefore, metric fluctuations must be considered in the gauge

$$\nabla^\nu\delta g_{\mu\nu} = \lambda\nabla_\mu\delta g^\nu{}_\nu. \tag{48}$$

In appendix D, we discuss the extent to which the different choices of λ lead to isometric inner products. The result will certainly not depend on λ if all fluctuations are trace free. All the gauges are equivalent in that case. Indeed, for non-compact gravitational instantons, all normalizable zero modes are trace free. This is not true for the compact gravitational instanton, K3. However, we now need to check compatibility with the quadratic kinetic terms in the action. The quadratic action, only keeping track of the derivative terms, is

$$\begin{aligned} S^2_{\delta g} &= \frac{1}{4}\int_{M_4} d^4x\sqrt{g}\big(\nabla^\mu\delta g^{\rho\sigma}\nabla_\mu\delta g_{\rho\sigma} - \nabla^\mu\delta g^\rho{}_\rho\nabla_\mu\delta g^\sigma{}_\sigma \\ &\quad - 2\nabla^\mu\delta g^\rho{}_\mu\nabla^\sigma\delta g_{\rho\sigma} + 2\nabla^\mu\delta g^\rho{}_\rho\nabla^\sigma\delta g_{\mu\sigma}\big) \\ &\to -\frac{1}{8}\int_{M_4} d^4x\sqrt{g}[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2(1+2\lambda^2-2\lambda)g^{\mu\nu}g^{\rho\sigma}]\delta g_{\mu\nu}\nabla^2\delta g_{\rho\sigma}, \end{aligned} \tag{49}$$

where the arrow denotes imposition of the gauge (48). Generically, this does not correspond to the de Witt (46) inner product which we started with. For consistency, we now need to impose

$2\lambda^2 - 3\lambda + 1 = 0$. The two solutions to this equation are $\lambda = 1$ and $\lambda = \frac{1}{2}$. These are in fact rather interesting values. The first is that obtained from viewing the instanton moduli space as the slow motion moduli space of (4 + 1)-dimensional Kaluza–Klein monopoles [17]. The second corresponds to the de Donder gauge, perhaps the most natural gauge for the theory, and was considered recently because the gradient flow on the space of metrics with this inner product is the Ricci flow [18].

It follows from the previous few paragraphs that for gravitational instantons there are two preferred gauges, which correspond to taking $\lambda = 1$ or $\lambda = \frac{1}{2}$ in the de Witt metric. However, we are interested in the Einstein–Maxwell theory, so we furthermore need to take into account the fact that the Maxwell field also transforms under infinitesimal diffeomorphisms $A \to A + \mathcal{L}_\xi A$. Orthogonality to such diffeomorphisms therefore requires

$$\langle \delta g, \mathcal{L}_\xi g\rangle_\lambda + \langle \delta A, \mathcal{L}_\xi A\rangle = 0. \tag{50}$$

Using the Lorenz condition on the gauge field perturbation (43), one finds that the orthogonality condition (50) requires that the following gauge be implemented for the moduli:

$$\nabla^\nu \delta g_{\mu\nu} - \lambda \nabla_\mu \delta g^\nu{}_\nu = -4\delta A^\nu F_{\nu\mu}. \tag{51}$$

Once again, we need to substitute this gauge choice into the quadratic term of the action. This is similar to the case of pure gravity (49) except that there are two extra terms due to the right-hand side of the gauge condition (51). One of these does not involve any derivatives of δA_μ or $\delta g_{\mu\nu}$ and so does not contribute to the quadratic terms. However, the other term involves a single derivative. This latter term is always present unless $\lambda = \frac{1}{2}$, suggesting that this is the preferred gauge for Einstein–Maxwell instantons.

3.2. *Towards the moduli space metric*

To find the moduli space metric, we need to find the general solution to the linearized Einstein–Maxwell equations satisfying the gauge conditions (43) and (51). Once we have the solution, we should then evaluate the norm of the fluctuations using the results of the previous section. Given that we have the general solution at a nonlinear level, we can easily solve the linearized Einstein–Maxwell system by perturbing the full solutions. However, these solutions will not be in the required gauge. Finding a gauge transformation to map the solution into the correct gauge does not appear easy.

An alternative and more elegant approach is that employed in [17] to find the moduli space metric on the Gibbons–Hawking gravitational instantons. This uses the existence of N closed self-dual 2-forms on the background, F^J, as well as the three self-dual Kähler forms Ω^i to write the metric fluctuation

$$\delta g^{iJ}_{\mu\nu} = \Omega^{i\rho}{}_{(\mu} F^J_{\nu)\rho}. \tag{52}$$

This perturbation solves the linearized Einstein equations. Furthermore, it is transverse and trace free and therefore solves the gauge condition required for pure Einstein gravity.

Note that this approach combines supersymmetry, which provides the three Kähler forms, and the topology of solution, which has $b_2^+ = N$ and hence implies the existence of the closed self-dual forms F^J. Using these modes, Ruback [17] shows that the moduli space metric is given in terms of the intersection matrix of the Gibbons–Hawking background and is flat.

So far, we have not been able to adapt this argument to the Einstein–Maxwell case in a way consistent with the gauge condition (51). We hope that the framework presented in this section will be a useful starting point for future work on the moduli space metric.

4. Lift to five dimensions

4.1. Lifting the solutions

Recall the following feature of field theory instantons: instantons in D dimensions may be viewed as solitons in $(D+1)$ dimensions. Furthermore, the L^2 instanton metric coincides with a natural Riemannian metric on the moduli space of solitons that is induced from the kinetic term in the $(D+1)$-dimensional action. This is interesting given the differing interpretations of the metrics in each case. The metric is relevant at the classical level in $(D+1)$ dimensions, as its geodesic motion approximates the soliton dynamics in the non-relativistic limit [19]. However, in D dimensions the metric is only important in quantum field theory, where measures on solution spaces are needed.

This procedure can also be applied to the four-dimensional Einstein–Maxwell gravitational instantons (2) if it is possible to lift them to Lorentzian metrics which are solitons of some theory in higher dimensions. Of course, the resulting moduli space metric could depend on the choice of higher dimensional theory. In this section, we study one possible theory in (4 + 1) dimensions. The five-dimensional metrics resulting from the lift are interesting in their own right, and we clarify some of their properties in this section. In section 5, we shall discuss the metric on the slow motion moduli space of these solitons.

The Einstein–Maxwell theory without a dilaton cannot be consistently lifted to pure gravity in five dimensions[4]. However, Einstein–Maxwell configurations may be lifted to solutions of the five-dimensional Einstein–Maxwell theory with a Chern–Simons term. This lift is the bosonic sector of the lift from $\mathcal{N}=2$ supergravity in four dimensions to $\mathcal{N}=2$ supergravity in five dimensions [20, 21]. We are interested in lifting the four-dimensional Riemannian theory to a Lorentzian theory on a five-dimensional manifold M_5. The four-dimensional action is

$$S_4 = \int \mathrm{d}^4x\sqrt{g^{(4)}}[R^{(4)} - F_{ab}F^{ab}], \tag{53}$$

with equations of motion given by (1). The five-dimensional action is

$$S_5 = \int \mathrm{d}^5x\sqrt{-g^{(5)}}[R^{(5)} - H_{\alpha\beta}H^{\alpha\beta}] - \frac{8}{3\sqrt{3}}\int H\wedge H\wedge W, \tag{54}$$

where $H = \mathrm{d}W$ is the five-dimensional Maxwell field. We use Greek indices ranging from 0 to 4 in five dimensions. The equations of motion in five dimensions are

$$G_{\alpha\beta} = 2H_\alpha{}^\gamma H_{\beta\gamma} - \tfrac{1}{2}g^{(5)}_{\alpha\beta}H^{\gamma\delta}H_{\gamma\delta}, \qquad d\star_5 H = -\tfrac{2}{\sqrt{3}}H\wedge H. \tag{55}$$

Given a solution, $g^{(4)}$ and $F = \mathrm{d}A$, to the four-dimensional equation (1), we may lift the solution to five dimensions as follows:

$$g^{(5)} = g^{(4)} - (\mathrm{d}t + \Phi)^2, \qquad W = \tfrac{\sqrt{3}}{2}A, \tag{56}$$

where Φ is a 1-form determined by $g^{(4)}$ and F through

$$\mathrm{d}\Phi = \star_4 F. \tag{57}$$

One may then check that the five-dimensional configuration (56) solves the equations of motion (55). Note that solutions to (57) exist because $d\star_4 F = 0$ on shell. In our case, we may solve for Φ explicitly to find

$$\Phi = -\tfrac{1}{2}(U^{-1} + \widetilde{U}^{-1})(\mathrm{d}\tau + \omega) + \chi, \tag{58}$$

[4] The need for a dynamical scalar field was not originally appreciated in the 1920s by Kaluza and Klein who set it to a constant. This mistake was corrected more than 20 years later by Jordan and Thiry.

where χ satisfies

$$\nabla \times \boldsymbol{\chi} = \tfrac{1}{2}\nabla(U - \widetilde{U}). \tag{59}$$

Lifting the Riemannian instantons to five dimensions leads to supersymmetric Lorentzian solutions. The five-dimensional Killing spinor equation,

$$e^{\alpha}{}_{a}\partial_{\alpha}\varepsilon + \frac{1}{4}\left[\omega^{bc}{}_{a}\Gamma_{bc} + \frac{\mathrm{i}}{4\sqrt{3}}\left(\Gamma_{a}{}^{bc} - 4\delta_{a}{}^{b}\Gamma^{c}\right)W_{bc}\right]\varepsilon = 0, \tag{60}$$

is found to be implied by the four-dimensional Killing spinor equation (13) together with the formulae for the lift (56). Thus, the four- and five-dimensional Killing spinors coincide. We are using a mostly plus convention for the metric, which is why (60) has an extra i relative to the expression in [22]. In equation (60), and nowhere else in this paper, the gamma matrices are five dimensional and the roman indices $a, b, \ldots$ are five-dimensional tangent space indices.

The supersymmetric solutions of $\mathcal{N} = 2$ supergravity in five dimensions have been classified [22]. For the case of a time-like Killing spinor, the general solution is given as a $U(1)$ fibration over a four real dimensional hyper-Kähler manifold. It was shown in [22] how the lift of the Lorentzian Israel–Wilson–Perjés solutions to five dimensions could be expressed as a fibration over the multicentred Gibbons–Hawking metrics [2].

It turns out that the lift of the Riemannian Israel–Wilson–Perjés solutions we are considering may also be expressed as a fibration over the multicentred Gibbons–Hawking metrics. The five-dimensional metric (56) can be written as follows[5]:

$$g^{(5)} = -f^{2}(\mathrm{d}\tau + \omega')^{2} + f^{-1}g^{\mathrm{GH}}, \tag{61}$$

where the Gibbons–Hawking metric is

$$g^{\mathrm{GH}} = V^{-1}\,(\mathrm{d}t + \chi)^{2} + V\,\mathrm{d}\mathbf{x}^{2}, \tag{62}$$

with the harmonic function

$$V = \tfrac{1}{2}(U - \widetilde{U}). \tag{63}$$

The remaining functions in the metric (61) are

$$f = \frac{V}{U\widetilde{U}}, \tag{64}$$

and

$$\omega' = \boldsymbol{\omega} - \frac{1}{2f^{2}}(U^{-1} + \widetilde{U}^{-1})(\mathrm{d}t + \boldsymbol{\chi}). \tag{65}$$

Note that the hyper-Kähler base itself is in general not regular, even changing signature at points where $U = \widetilde{U}$. This is perfectly compatible with the regularity of the five-dimensional spacetime.

The case $U = \widetilde{U}$ is exceptional and cannot be written in the form (61). Instead, these metrics have null supersymmetry in five dimensions. The metric is[6]

$$g^{(5)} = \frac{2\,\mathrm{d}t\,\mathrm{d}\tau}{U} - \mathrm{d}t^{2} + U^{2}\,\mathrm{d}\mathbf{x}^{2}. \tag{66}$$

[5] Writing the spacetime in the form (61) locates the five-dimensional solution in the classification of [22]. In section 3.7 of that paper, the general supersymmetric fibration over a Gibbons–Hawking base with $\partial/\partial t$ a Killing vector is given in terms of three harmonic functions. For our solution, these correspond to $L = 2\widetilde{U}$, $K = -\widetilde{U}$ and $M = -2\widetilde{U}$.

[6] The metric (66) falls within the classification of [22] for spacetimes with null supersymmetry by setting their functions $H = -\mathcal{F} = U$ and $\mathbf{a} = 0$.

4.2. Regularity and causality

The interesting points in the five-dimensional metric are the centres where $U \to \infty$ or $\widetilde{U} \to \infty$. In the four-dimensional Riemannian Israel–Wilson–Perjés solutions, these can always be made to be regular points [6, 7] as we reviewed above. We need to re-examine the regularity of the metric around these points and also check for the possible occurrence of closed time-like curves.

Before zooming in on the centres, note the following. Firstly, that

$$g^{(5)}{}_{\tau\tau} \equiv g^{(5)}\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial\tau}\right) = -\frac{(U-\widetilde{U})^2}{(2U\widetilde{U})^2} < 0, \tag{67}$$

if $U \neq \widetilde{U}$. Therefore, to avoid closed time-like curves throughout the five-dimensional spacetime we must not identify τ. Secondly, possible candidates for the location of horizons are where the metric becomes degenerate

$$0 = g^{(5)}{}_{tt}\, g^{(5)}{}_{\tau\tau} - \left[g^{(5)}{}_{t\tau}\right]^2 = -\frac{1}{U\widetilde{U}}. \tag{68}$$

This occurs at the centres where U or $\widetilde{U}$ diverges.

In order to understand the geometry near the centres, there are three different cases we need to consider separately. The first is that $U \to \infty$ while $\widetilde{U}$ remains finite. Using polar coordinates ($r = \rho^2/4, \theta, \phi$) centred on the point $\mathbf{x}_m$ and requiring that $a_m \widetilde{U}(\mathbf{x}_m) = 1$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \frac{\rho^2}{4}\left[(\mathrm{d}\tau + \cos\theta\, \mathrm{d}\phi)^2 + \mathrm{d}\Omega^2_{S^2}\right] - (\mathrm{d}t - a_m\, \mathrm{d}\tau/2)^2 \tag{69}$$

as $\rho \to 0$, with $\mathrm{d}\Omega^2_{S^2} = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$. The metric may be made regular about this point if we identify τ with period 4π. Unfortunately, this introduces closed time-like curves as we discussed. If we choose not to identify τ, we are left with time-like naked singularities at the centres. We see that there is no horizon at these points, but rather a (singular) origin of polar coordinates. Therefore, metrics with this behaviour at the centres cannot lift to causal, regular solitons in five dimensions.

The remaining two possibilities involve coincident centres where both U and $\widetilde{U}$ go to infinity, so that $\mathbf{x}_m = \tilde{\mathbf{x}}_m$. One needs to treat separately the cases where $a_m = \tilde{a}_m$ and $a_m \neq \tilde{a}_m$. In the latter case we again find regularity at the expense of closed time-like curves going out to infinity, or alternatively naked singularities. This leaves only the former case with $a_m = \tilde{a}_m$ for all m. That is, $\widetilde{U} = U + k$, with k some constant.

By considering the asymptotic regime, one can see that in order to obtain a sensible asymptotic geometry without closed time-like curves, one requires that either both U and $\widetilde{U}$ go to a constant at infinity or both go to zero. Rescaling the harmonic functions and performing a duality rotation on the Maxwell field, as we discussed in four dimensions above, imply that without loss of generality $U = \widetilde{U}$. We consider this case in the following subsection.

4.3. Multi-solitonic strings

The only lift that leads to a globally regular and causal five-dimensional spacetime is the case $U = \widetilde{U}$, which corresponds to the Euclidean Majumdar–Papapetrou metric in four dimensions. The metric is (66), with a null Killing spinor. Away from the centres, the spacetimes approach either $\mathbb{R}^{1,4}$ or $\mathrm{AdS}_3 \times S^2$, with U going to a constant or zero at infinity, respectively.

With a rescaling of coordinates, the geometry near the centres where $U \to \infty$ may be written as

$$\mathrm{d}s^2 = a_m^2\left[\frac{\mathrm{d}r^2}{r^2} + 2r\, \mathrm{d}t\, \mathrm{d}\tau - \mathrm{d}t^2 + \mathrm{d}\Omega^2_{S^2}\right]. \tag{70}$$

Calculating the curvature shows that this metric locally describes $AdS_3 \times S^2$. One might be tempted to conclude that this represents the near horizon geometry of an extremal black string in five dimensions. However, the coordinates (70) are a little unusual, the sign of dt^2 differing from the metric of an extremal BTZ black hole [23]. In particular, the Killing vector $\partial/\partial t$ is everywhere regular and timelike. This remains true in the full spacetime (66). There is no horizon, and the degeneration of the metric at the centres is analogous to an origin of polar coordinates.

The coordinates in (70) may be mapped to Poincaré coordinates as follows:

$$Y = \frac{1}{r^{1/2}\cos\frac{t}{2}}, \qquad X = \frac{\tau}{2} - \frac{1}{2}\left[\frac{1}{r} - 1\right]\tan\frac{t}{2}, \qquad T = \frac{\tau}{2} - \frac{1}{2}\left[\frac{1}{r} + 1\right]\tan\frac{t}{2}, \tag{71}$$

so that the metric becomes

$$ds^2 = \frac{4a_m^2}{Y^2}(-dT^2 + dX^2 + dY^2) + a_m^2\, d\Omega_{S^2}^2. \tag{72}$$

There is no singularity at $t = \pm\pi$ as may be checked by writing down the embedding of AdS_3 as a quadric in $\mathbb{R}^{2,2}$ in terms of these coordinates. The map (71) is periodic in t. Taking t with an infinite range corresponds to passing to the (causal) universal cover of AdS_3. There is no need to identify τ, and therefore the spacetime is causal.

The metrics (66) give causal, regular solutions to the five-dimensional theory with an everywhere-defined time-like Killing vector. Writing the metric in the form

$$g^{(5)} = -(dt - d\tau/U)^2 + \frac{d\tau^2}{U^2} + U^2\, d\mathbf{x}^2 \tag{73}$$

suggests that the spacetimes should be thought of as containing N parallel 'solitonic strings'. The strings have world volumes in the $t - \tau$ plane. There is a plane fronted wave [22] carrying momentum along the $\partial/\partial\tau$ direction of the string. We call these plane fronted waves solitonic strings to emphasize that the fields are localized along strings and there are no horizons. The strings are magnetic sources for the 2-form field strength

$$H = -\sqrt{3}\star_3\, dU. \tag{74}$$

This is possible because of the topologically nontrivial S^2 at each centre (70).

We end this subsection by remarking that any solution to the Einstein–Maxwell-Chern–Simons theory (55) in (4 + 1) dimensions can be lifted to a solution to eleven-dimensional supergravity given by the product metric of $g^{(5)}$ and a flat metric on the six torus. The eleven-dimensional 4-form is given by $H \wedge \Omega_T$, where Ω_T is the Kähler form on the torus. We have not pursued here an M theory interpretation of these solutions.

5. Slow motion in five dimensions

An interesting feature of BPS solitons is the cancellation between forces which makes static multi-soliton configurations possible. This is clear for the (3+1)-dimensional Majumdar–Papapetrou multi black holes, where the electrostatic repulsion is balanced by gravitational attraction. These black holes are in this sense analogous to a nonrelativistic system of massive charged particles, with the charge-to-mass ratio chosen to balance the Newtonian attraction and Coulomb repulsion.

The nature of the forces in the, stationary but not static, (4 + 1)-dimensional solution (56) is presumably more complicated. We shall not study this problem here, and instead focus on the scattering of slowly moving solitons. The question we are interested in is whether there is a direct connection between the metric on the moduli space of four-dimensional instantons and the metric on the moduli space describing slow motion of the (4 + 1)-dimensional solitons. The metrics do coincide for pure gravity instantons [17].

One can follow Manton's method for truncating the infinite number of degrees of freedom of the gravitational field to the finite dimensional moduli space $\mathcal{M}$ of solitons[7]. This means that we shall be neglecting both gravitational and electromagnetic radiation, and consider only velocity-dependent forces which perturbed solitons induce on each other. As for the four-dimensional instantons, the space $\mathcal{M}$ is not the whole of $\mathbb{R}^{3(N+\tilde{N})}$. To obtain $\mathcal{M}$ we need to quotient by the permutation group $S_N \times S_{\tilde{N}}$, and the Euclidean group in three dimensions.

By considering the slow motion approximation to the initial value formulation of the (4 + 1)-dimensional Einstein–Maxwell–Chern–Simons theory, one can find the moduli space metric from the effective action where the field degrees of freedom have been integrated out. In the moduli space approximation the centres become functions of t, and the geodesic curves $\{\mathbf{x}_m(t), \tilde{\mathbf{x}}_n(t)\}$ correspond to slow motion of a multi-solitonic string configuration.

The initial data for EMCS theory (54) consist of a four-dimensional manifold Σ together with a Riemannian metric $\gamma_{\mu\nu}$, a symmetric tensor $K_{\mu\nu}$, a 2-form B and a 1-form E. Given a metric $g^{(5)}$ and a 1-form potential W on M_5, we can perform a 4 + 1 decomposition if there exists a function t whose gradient is everywhere timelike. In this case Σ is a level set of t, and we choose adapted local coordinates (t, x^a) such that the normal to Σ takes the form

$$\mathcal{N} = N^{-1}(\partial_t - N^\mu \partial_\mu), \tag{75}$$

where N and N^μ are the lapse function and the shift vector, respectively. The spatial metric $\gamma_{\mu\nu}$ and the second fundamental form $K_{\mu\nu}$ can now be read off from the formulae

$$\begin{aligned} g^{(5)} &= -N^2\,\mathrm{d}t^2 + \gamma_{\mu\nu}(\mathrm{d}x^\mu + N^\mu\,\mathrm{d}t)(\mathrm{d}x^\nu + N^\nu\,\mathrm{d}t),\\ K_{\mu\nu} &= \tfrac{1}{2}N^{-1}(\partial_t\gamma_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu), \end{aligned} \tag{76}$$

where D is the covariant derivative compatible with γ on Σ. We also decompose the 1-form W and 2-form $H = \mathrm{d}W$ as

$$W = W_0 N\,\mathrm{d}t + W_\mu\,\mathrm{d}x^\mu, \qquad H = E \wedge N\,\mathrm{d}t + B. \tag{77}$$

This last formula implies expressions for E and B as exterior derivatives of the potentials W_0 and W_μ, respectively.

The next step is to implement the 4 + 1 decomposition at the level of the action. After neglecting a total derivative term, the following action is obtained by substituting (76) and (77) into the EMCS action (54):

$$\begin{aligned} S_{4+1} = \int \mathrm{d}^4x\,\mathrm{d}t\,N\sqrt{\gamma}[R^\gamma + K_{\mu\nu}K^{\mu\nu} - K^2] + \int \mathrm{d}^4x\,\mathrm{d}t\,N\sqrt{\gamma}\\ \times\left[2E_\mu E^\mu - B_{\mu\nu}B^{\mu\nu} + 2B_{\mu\nu}B_\rho{}^\nu\frac{N^\mu N^\rho}{N^2}\right]\\ - \frac{8}{3\sqrt{3}}\int [W_0 B \wedge B - 2B \wedge E \wedge W_\mu\,\mathrm{d}x^\mu] \wedge N\,\mathrm{d}t. \end{aligned} \tag{78}$$

Here R^γ is the Ricci scalar of γ, $K = \gamma^{\mu\nu}K_{\mu\nu}$, and all contractions use the metric γ. The three lines come from the Einstein–Hilbert, Maxwell and Chern–Simons terms in the action (54), respectively. If we think of expression (78) as an action for the fields $\{\gamma_{\mu\nu}, W_\mu, W_0, N_\mu, N\}$,

[7] In this section we will refer to any of the solutions (56) as solitons, even if they are singular or contain closed time-like curves. Part of our motivation is to compare with the moduli space metric of four-dimensional gravitational instantons (2) where everything is regular, even if $U \neq \widetilde{U}$.

then we see that the last three of these appear without time derivatives. They are Lagrange multipliers and impose the constraints of conservation of energy, momentum and charge

$$\frac{\delta S_{4+1}}{\delta N} = \frac{\delta S_{4+1}}{\delta N_\mu} = \frac{\delta S_{4+1}}{\delta N W_0} = 0. \tag{79}$$

Arbitrary initial data will not evolve to a solution of the EMCS theory. One needs to impose the constraint equation (79).

To consider the slow motion dynamics of a perturbed stationary solution, we allow the moduli to become time dependent and work to first order in the velocities

$$v^J = \frac{\mathrm{d}x^J}{\mathrm{d}t}, \tag{80}$$

where we have used x^J to denote a general modulus. This induces a time dependence in the solution which to first order can be written as

$$\frac{\mathrm{d}\gamma_{\mu\nu}}{\mathrm{d}t} = \delta\gamma^J_{\mu\nu} v^J, \qquad \frac{\mathrm{d}W_\mu}{\mathrm{d}t} = \frac{\sqrt{3}}{2}\delta A^J_\mu v^J, \tag{81}$$

where $\delta\gamma^J_{\mu\nu}, \delta A^J_\mu$ is the zero mode corresponding to the modulus x^J. In general, simply allowing the moduli to depend on time will not give a spacetime that solves the constraint equations, even to first order in the velocities. Instead, it will be necessary to add extra terms linear in the velocities to the original solution. An early example of this technique in gravity is the slow motion of Majumdar–Papepetrou black holes [24].

For the case of the Kaluza–Klein monopole lift of the Gibbons–Hawking solutions, it turns out that it is sufficient to simply promote the moduli to time-dependent fields. The constraint equations are automatically solved to first order in velocities [17]. This lies behind the simple identification of the moduli space metrics in four and five dimensions. Let us see whether the constraint equations are solved in our case.

To first order in velocities, the charge conservation and momentum conservation constraints become

$$D^\mu\left(\delta A^J_\mu/N\right) = 0, \qquad D^\mu\left(\delta\gamma^J_{\mu\nu}/N\right) = D_\nu\left(\delta\gamma^{J\mu}{}_\mu/N\right). \tag{82}$$

Here we used (81). It is interesting to see that these two constraints take the form of gauge conditions. They may be imposed on the moduli fields, and no extra terms are necessary. Although these gauge conditions look similar to those encountered in section 3.1, they are quite different. The choice of time slicing is not the same. By comparing (56) and (76), we see that $\gamma_{\mu\nu} = g_{\mu\nu} - \Phi_\mu\Phi_\nu$. Working through the changes to the covariant derivative shows that the charge conservation constraint in (82), for instance, becomes

$$\nabla_\mu([(1-\Phi^2)g^{\mu\nu} + \Phi^\mu\Phi^\nu]\delta A^J_\nu) = 0. \tag{83}$$

Deriving this expression uses $1/N^2 = 1 - \Phi^2$. Here Φ^2 is contracted with $g_{\mu\nu}$. A similar expression exists for the momentum constraint. It is clear that this is not the Lorenz gauge that we used for the instanton moduli space. As discussed, the instanton moduli space metric is gauge dependent. This is the first indication that there is not a direct connection between the instanton and soliton moduli space metrics for our solutions.

A more significant problem arises from the Hamiltonian constraint. To first order in velocities, the constraint is

$$\delta g^J_{\mu\nu} D^\mu N^\nu - \delta g^{J\mu}{}_\mu D^\nu N_\nu = \frac{N}{\sqrt{\gamma}}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\delta A^J_\rho A_\sigma. \tag{84}$$

This is an algebraic relation between the various metric and Maxwell field moduli. We might hope that (84) is solved for all moduli for $\lambda = 1$. Unfortunately, it is clear that this will not work. Note that the Hamiltonian constraint (84) involves a symmetric derivative of N^{μ}. This translates into a symmetrized derivative of Φ^{μ}. However, only the antisymmetrized derivative of Φ^{μ} can be expressed in terms of the four-dimensional fields via (57). The Hamiltonian constraint will require extra modes to be turned on for a consistent time-dependent solution.

The upshot of this section is therefore that, unlike in the case of Yang–Mills instantons or pure gravitational instantons, the slow motion moduli space metric of the five-dimensional soliton cannot be directly reduced to the four-dimensional instanton moduli space metric. A full blooded computation of the backreaction of the moduli velocities onto the spacetime is necessary.

6. Discussion

In this paper, we have discussed various properties of multi-instanton solutions of the Euclidean Einstein–Maxwell theory. We have also shown how these solutions may be lifted to 'solitonic string' solutions of the five-dimensional Lorentzian Einstein–Maxwell–Chern–Simons theory. There are roughly three types of application for the solutions we have discussed. We hope that the present work has provided a solid base for future investigations.

Firstly and perhaps most interestingly, given that the instantons only involve fields that are observed to exist in nature, would be to understand the physical effects mediated by these solutions. A well-known example of the physical effect of the Euclidean Einstein–Maxwell theory is the bounce that describes the pair creation of charged black holes in a sufficiently strong electromagnetic field [25, 26]. One possible direction of study would be to ask whether the instantons tell us anything about the structure of the vacuum of the Einstein–Maxwell theory, say as a function of temperature.

Secondly, it would be of interest to understand the role of these solutions as supersymmetric building blocks within string and M theory, either as higher dimensional supergravity instantons [27] or as a component of Lorentzian compactification or brane solutions. This would be analogous to the ubiquitous appearance of the Gibbons–Hawking metrics in higher dimensions.

Thirdly, there are various mathematical aspects that we have not developed completely. Some of these have physical consequences. It is important to understand the index theory associated with the zero modes of the instantons. This will determine which correlators the instantons contribute to and also their effect on topological terms in the Lagrangian. Furthermore, we have not discussed determinants of quadratic fluctuations about the solutions. An interesting question is whether supersymmetry is sufficient in this case to force the one-loop determinants to cancel.

On a slightly different note, a completely distinct set of Einstein–Maxwell instantons may be constructed. LeBrun has found explicit multicentred scalar flat Kähler metrics [4]. These give solutions to the Einstein–Maxwell theory with the field strength given by half the Kähler form plus the Ricci form. It would be interesting to study these solutions in more depth and elucidate their relation, if any, with the solutions that we have discussed.

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Appendix A. Expressions for the potentials

An explicit formula for $\boldsymbol{\omega}$ may be obtained from integrating (3). There is a choice of gauge involved as $\boldsymbol{\omega}$ is only defined up to a gradient. We can see that the contributions to $\boldsymbol{\omega}$ will come from cross terms in the sums defining U and $\widetilde{U}$ (6). Therefore, we can write

$$\boldsymbol{\omega} = \sum_{mn} \boldsymbol{\omega}_{mn} - \frac{4\pi}{\beta} \sum_n \tilde{\boldsymbol{\omega}}_n + \frac{4\pi}{\tilde{\beta}} \sum_m \boldsymbol{\omega}_m. \tag{A.1}$$

A possible form for the first term $\boldsymbol{\omega}_{mn}$ is

$$\begin{aligned}\boldsymbol{\omega}_{mn} &= -a_m\tilde{a}_n \frac{(\mathbf{x}-\mathbf{x}_m)\cdot(\mathbf{x}-\tilde{\mathbf{x}}_n)}{|\mathbf{x}-\mathbf{x}_m||\mathbf{x}-\tilde{\mathbf{x}}_n|} \frac{(\mathbf{x}_m-\tilde{\mathbf{x}}_n)\times(\mathbf{x}-(\mathbf{x}_m+\tilde{\mathbf{x}}_n)/2)}{|(\mathbf{x}_m-\tilde{\mathbf{x}}_n)\times(\mathbf{x}-(\mathbf{x}_m+\tilde{\mathbf{x}}_n)/2)|^2} \\ &= \frac{a_m\tilde{a}_n}{|\mathbf{x}_{mn}|} \frac{\mathbf{x}_{-m}\cdot\mathbf{x}_{-n}}{|\mathbf{x}_{-m}||\mathbf{x}_{-n}|} \nabla\left\{\tan^{-1}\frac{[\mathbf{x}_{mn}\times(\mathbf{x}_{mn}\times(\mathbf{x}_{-m}+\mathbf{x}_{-n}))]\cdot\mathbf{k}}{|\mathbf{x}_{mn}|\,[\mathbf{x}_{mn}\times(\mathbf{x}_{-m}+\mathbf{x}_{-n})]\cdot\mathbf{k}}\right\}.\end{aligned} \tag{A.2}$$

In the second expression $\mathbf{x}_{mn} = \mathbf{x}_m - \tilde{\mathbf{x}}_n$, $\mathbf{x}_{-m} = \mathbf{x} - \mathbf{x}_m$, $\mathbf{x}_{-n} = \mathbf{x} - \tilde{\mathbf{x}}_n$ and $\mathbf{k}$ is an arbitrary constant vector. This breaking of symmetry is the price we need to pay for expressing part of the term as a gradient.

A possible expression for the remaining terms, writing $\boldsymbol{\omega}$ as a form for ease of notation, is

$$\boldsymbol{\omega}_m = a_m \frac{(z-z_m)\,(-(y-y_m)\,\mathrm{d}x + (x-x_m)\,\mathrm{d}y)}{|\mathbf{x}-\mathbf{x}_m|[(x-x_m)^2+(y-y_m)^2]}. \tag{A.3}$$

$\tilde{\boldsymbol{\omega}}_n$ are given by the same expression, but with $a_m \to \tilde{a}_n$ and $\mathbf{x}_m \to \tilde{\mathbf{x}}_n$. As is usual, the choice of gauge for (A.3) necessarily breaks the rotational symmetry and has Dirac strings.

Both of the previous two formulae are more naturally given in polar coordinates. However, the angles would depend on the centres or pairs of centres in question. If we want coordinates that are valid for all $\boldsymbol{\omega}_{mn}$ and $\boldsymbol{\omega}_m$ at once then we need to use Cartesian coordinates.

The gauge for $\boldsymbol{\omega}$ that we have chosen in (A.2) and (A.3) satisfies $\nabla\cdot\boldsymbol{\omega} = 0$. In fact, the expression in curly brackets in (A.2) is a harmonic function.

We may also integrate the field strength (5) to obtain an explicit potential. This is defined up to a gradient. A possible expression is

$$A = A_4(\mathrm{d}\tau + \boldsymbol{\omega}) + \mathbf{A}, \tag{A.4}$$

with

$$A_4 = \frac{U-\widetilde{U}}{2U\widetilde{U}} \qquad \text{and} \qquad \mathbf{A} = -\frac{1}{2}\left[\sum_m \boldsymbol{\omega}_m + \sum_n \tilde{\boldsymbol{\omega}}_n\right], \tag{A.5}$$

where $\boldsymbol{\omega}_m, \tilde{\boldsymbol{\omega}}_n$ are as given in (A.3). More invariantly, $\nabla\times\mathbf{A} = -\frac{1}{2}(U+\widetilde{U})$. Note that with this choice of gauge, $\nabla\cdot\mathbf{A} = 0$.

Appendix B. Gamma matrix conventions and spin connection

We work with a chiral representation of the Euclidean gamma matrices

$$\Gamma^a = \begin{pmatrix} 0 & -\mathrm{i}\sigma^a \\ \mathrm{i}\tilde{\sigma}^a & 0 \end{pmatrix}, \tag{B.1}$$

where $\sigma^a = (i, \boldsymbol{\tau})$ and $\tilde{\sigma}^a = (-i, \boldsymbol{\tau})$. Here, $\boldsymbol{\tau}$ are the Pauli matrices. The gamma matrices satisfy $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$. We define

$$\Gamma^{ab} \equiv \frac{1}{2}[\Gamma^a, \Gamma^b] = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \tilde{\sigma}^{ab} \end{pmatrix}, \tag{B.2}$$

where $\sigma^{ab} = \frac{1}{2}[\sigma^a\tilde{\sigma}^b - \sigma^b\tilde{\sigma}^a]$ and $\tilde{\sigma}^{ab} = \frac{1}{2}[\tilde{\sigma}^a\sigma^b - \tilde{\sigma}^b\sigma^a]$. As 2-forms, σ^{ab} is anti-self-dual whilst $\tilde{\sigma}^{ab}$ is self-dual. Finally, let $\Gamma^5 = \Gamma^1\Gamma^2\Gamma^3\Gamma^4$.

In computing the Killing spinor, one needs to know the self-dual and anti-self-dual parts of the spin connection and field strength. For the field strength, these are

$$F^{ab}\sigma_{ab} = \frac{-2\mathrm{i}\boldsymbol{\tau}\cdot\nabla U}{U^2}, \qquad F^{ab}\tilde{\sigma}_{ab} = \frac{-2\mathrm{i}\boldsymbol{\tau}\cdot\nabla \widetilde{U}}{\widetilde{U}^2}, \tag{B.3}$$

whilst for the spin connection, we have

$$\begin{aligned} \omega^{ab}\sigma_{ab} &= \frac{-2\mathrm{i}}{(U\widetilde{U})^{1/2}}\left[\frac{\boldsymbol{\tau}\cdot\nabla U e^0}{U} + \frac{(\boldsymbol{\tau}\times\nabla U)\cdot e}{U}\right], \\ \omega^{ab}\tilde{\sigma}_{ab} &= \frac{-2\mathrm{i}}{(U\widetilde{U})^{1/2}}\left[-\frac{\boldsymbol{\tau}\cdot\nabla \widetilde{U} e^0}{\widetilde{U}} + \frac{(\boldsymbol{\tau}\times\nabla \widetilde{U})\cdot e}{\widetilde{U}}\right]. \end{aligned} \tag{B.4}$$

Appendix C. Two-component spinor conventions

We can use the matrices of appendix B to relate vectors in four-component notation to two-component spinor notation

$$X^{AA'} = \frac{-\mathrm{i}\sigma_a^{AA'}}{\sqrt{2}}X^a. \tag{C.1}$$

Because a is a Euclidean signature tangent space index, raising and lowering this index does not have any effect. The inverse to this relation is

$$X^a = \frac{\mathrm{i}\sigma^a_{B'B}}{\sqrt{2}}\tilde{X}^{BB'}, \tag{C.2}$$

which works because

$$\sigma_a^{AA'}\tilde{\sigma}^a_{B'B} = 2\delta^A_B\delta^{A'}_{B'}. \tag{C.3}$$

Another useful relation is

$$\tilde{\sigma}^a_{A'A}\tilde{\sigma}_{aB'B} = -2\epsilon_{AB}\epsilon_{A'B'}, \tag{C.4}$$

and similarly for $\sigma_a^{AA'}$.

Appendix D. Equivalence and inequivalence of inner products

Suppose a metric perturbation satisfies the gauge condition

$$\nabla^\nu\delta g_{\mu\nu} = \lambda\nabla_\mu\delta g^\nu{}_\nu. \tag{D.1}$$

Consider the same perturbation in a different gauge,

$$\nabla^\nu\delta\hat{g}_{\mu\nu} = \hat{\lambda}\nabla_\mu\delta\hat{g}^\nu{}_\nu. \tag{D.2}$$

The two perturbations are thus related by

$$\delta\hat{g}_{\mu\nu} = \delta g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{D.3}$$

and the two gauge conditions (D.1) and (D.2) require that ξ satisfy

$$2\nabla^\mu \nabla_{(\mu}\xi_{\nu)} = (\hat{\lambda} - \lambda)\nabla_\nu \delta g^\mu{}_\mu + 2\hat{\lambda}\nabla_\nu \nabla^\mu \xi_\mu. \tag{D.4}$$

Using this relation, it is straightforward to show that

$$\langle \delta\hat{g}, \delta\hat{g}\rangle_{\hat{\lambda}} = \langle \delta g, \delta g\rangle_\lambda + 2(\lambda - \hat{\lambda}) \int \mathrm{d}^4x \sqrt{g}\delta g^\mu{}_\mu \delta\hat{g}^\nu{}_\nu. \tag{D.5}$$

Therefore, if $\lambda \neq \hat{\lambda}$ the two inner products are generically inequivalent. They are equivalent if all the modes are trace free in one of the gauges. This is consistent with the observation in the main text that the inner products are manifestly equivalent on trace-free modes. We also noted in the text that on non-compact gravitational instantons all normalizable zero modes are indeed trace free, as follows from the fact that these satisfy $\nabla^2 \delta g^\mu{}_\mu = 0$. However, even in this non-compact pure gravity case, non-zero modes will of course generically have a trace component.

References

[1] Hawking S W 1977 Gravitational instantons *Phys. Lett.* A **60** 81
[2] Gibbons G W and Hawking S W 1978 Gravitational multi-instantons *Phys. Lett.* B **78** 430
[3] Strominger A, Horowitz G T and Perry M J 1984 Instantons in conformal gravity *Nucl. Phys.* B **238** 653
[4] LeBrun C R 1991 Explicit self-dual metrics on $\mathbf{CP}^2\#\cdots\#\mathbf{CP}^2$ *J. Diff. Geom.* **34** 233
[5] Hartnoll S A and Policastro G 2004 Spacetime foam in twistor string theory *Preprint* hep-th/0412044
[6] Whitt B 1985 Israel–Wilson metrics *Ann. Phys.* **161** 241
[7] Yuille A L 1987 Israel–Wilson metrics in the Euclidean regime *Class. Quantum Grav.* **4** 1409
[8] Israel W and Wilson G A 1972 A class of stationary electromagnetic vacuum fields *J. Math. Phys.* **13** 865
[9] Perjés Z 1971 Solutions of the coupled Einstein–Maxwell equations representing the fields of spinning sources *Phys. Rev. Lett.* **27** 1668
[10] Hartle J B and Hawking S W 1972 Solutions of the Einstein–Maxwell equations with many black holes *Commun. Math. Phys.* **26** 87
[11] Chrusciel P T, Reall H S and Tod P 2006 On Israel–Wilson–Perjes black holes *Class. Quantum Grav.* **23** 2519 (*Preprint* gr-qc/0512116)
[12] Gibbons G W and Hull C M 1982 A bogomolny bound for general relativity and solitons in $N = 2$ supergravity *Phys. Lett.* B **109** 190
[13] Ferrara S and van Nieuwenhuizen P 1976 Consistent supergravity with complex spin 3/2 gauge fields *Phys. Rev. Lett.* **37** 1669
[14] Tod K P 1983 All metrics admitting supercovariantly constant spinors *Phys. Lett.* B **121** 241
[15] Chrusciel P T and Nadirashvili N S 1995 All electrovacuum Majumdar–Papapetrou spacetimes with non-singular black holes *Class. Quantum Grav.* **12** L17 (*Preprint* gr-qc/9412044)
[16] Braden H W, Brown J D, Whiting B F and York J W 1990 Charged black hole in a grand canonical ensemble *Phys. Rev.* D **42** 3376
[17] Ruback P J 1986 The motion of Kaluza–Klein monopoles *Commun. Math. Phys.* **107** 93
[18] Headrick M and Wiseman T 2006 Ricci flow and black holes *Preprint* hep-th/0606086
[19] Manton N S 1982 A remark on the scattering of Bps monopoles *Phys. Lett.* B **110** 54
[20] Chamseddine A H and Nicolai H 1980 Coupling the SO(2) supergravity through dimensional reduction *Phys. Lett.* B **96** 89
[21] Lozano-Tellechea E, Meessen P and Ortin T 2002 On $d = 4, 5, 6$ vacua with 8 supercharges *Class. Quantum Grav.* **19** 5921 (*Preprint* hep-th/0206200)
[22] Gauntlett J P, Gutowski J B, Hull C M, Pakis S and Reall H S 2003 All supersymmetric solutions of minimal supergravity in five dimensions *Class. Quantum Grav.* **20** 4587 (*Preprint* hep-th/0209114)
[23] Banados M, Henneaux M, Teitelboim C and Zanelli J 1993 Geometry of the (2 + 1) black hole *Phys. Rev.* D **48** 1506 (*Preprint* gr-qc/9302012)

[24] Ferrell R C and Eardley D M 1987 Slow motion scattering and coalescence of maximally charged black holes *Phys. Rev. Lett.* **59** 1617
[25] Gibbons G W 1986 Quantized flux tubes in Einstein–Maxwell theory and noncompact internal spaces, Print-86-0411 (CAMBRIDGE), *Presented at 22nd Karpacz Winter School of Theoretical Physics: Fields and Geometry (Karpacz, Poland, 17 February–1 March)*
[26] Garfinkle D and Strominger A 1991 Semiclassical wheeler wormhole production *Phys. Lett.* B **256** 146
[27] de Vroome M and Vandoren S 2006 Supergravity description of spacetime instantons *Preprint* hep-th/0607055