The Nonlinear Graviton as an Integrable System

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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 1999

Abstract

The curved twistor theory is studied from the point of view of integrable systems.

A twistor construction of the hierarchy associated with the anti-self-dual Einstein vacuum equations (ASDVE) is given. The recursion operator R is constructed and used to build an infinite-dimensional symmetry algebra of ASDVE. It is proven that R acts on twistor functions by multiplication. The recursion operator is used to construct Killing spinors. The method is illustrated on the example of the Sparling-Tod solution.

An infinite number of commuting flows on extended space-time is constructed. It is proven that a moduli space of rational curves, with normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$ in twistor space, is canonically equipped with a Lax distribution for ASDVE hierarchies. It is demonstrated that the isomonodromy problem can, in the Fuchsian case, be understood in terms of curved twistor spaces. The solutions to the $SL(2, \mathbb{C})$ Schlesinger equation are related to the flows of the heavenly hierarchy.

The Lagrangian, Hamiltonian and bi-Hamiltonian formulations of heavenly equations are given. The symplectic form on the moduli space of solutions to heavenly equations is derived, and is proven to be compatible with the recursion operator.

It is proven that a family of rational curves in the twistor space may be found by integrating the Hamiltonian system which has the second heavenly potential as its Hamiltonian. An alternative view of heavenly potentials as generating functions on the spin bundle is given.

The potentials for linear fields on ASD vacuum backgrounds are constructed. It is shown that generalised zero–rest–mass field equations can

be solved by means of functions on $\mathcal{O}(n) \oplus \mathcal{O}(n)$ twistor spaces. The moduli space of deformed $\mathcal{O}(n) \oplus \mathcal{O}(n)$ curves is shown to be foliated by four dimensional hyper-Kähler slices.

The twistor theory of four-dimensional hyper-Hermitian manifolds is formulated as a combination of the Nonlinear Graviton Construction with the Ward transform for anti-self-dual Maxwell fields. The Lax formulation is found and used to derive a pair of potentials for a hyper-Hermitian metric. A class of examples of hyper-Hermitian metrics which depend on two arbitrary functions of two complex variables is given.

The ASDV metrics with a conformal, non-triholomorphic Killing vector are considered. The symmetric solutions to the first heavenly equation are shown to give rise to a new integrable system in three dimensions, and to a new class of Einstein–Weyl geometries. The Lax representation, Lie point symmetries, hidden symmetries and the recursion operator associated with the reduced 3D system are found, and some group invariant solutions are considered.

It is proven that if an Einstein–Weyl space admits a solution of a generalised monopole equation, which yields four dimensional ASD vacuum, or Einstein metrics, then the four-dimensional correspondence space is equipped with a closed and simple two-form. A class of Einstein–Weyl structures is given in terms of solutions to the dispersion-less Kadomtsev–Petviashvili equation.

It is explained how to construct ASDVE metrics from solutions of various 2D integrable systems by exploiting the fact that the Lax formulations of both systems can be embedded in that of the anti-self-dual Yang–Mills equations. The explicit ASDVE metrics are constructed on $\mathbb{R}^2 \times \Sigma$, where Σ is a homogeneous space for a real subgroup of $SL(2, \mathbb{C})$ associated with the two-dimensional system. The twistor interpretation of the construction is given.

Acknowledgements

Above all I wish to thank my supervisor Dr Lionel Mason for his help and patience. His influence on almost every chapter of this thesis has been enormous.

I owe a great debt to Dr George Sparling for his help in Sections 3.3.2, 3.4 and 4.1 and for teaching me all I know about splitting formulae.

I thank Dr Paul Tod who introduced me to the Einstein–Weyl geometry and to whom the crucial ideas of Chapter 8 are due.

My thanks also go to Prof. Boris Dubrovin, Prof. Nigel Hitchin, Dr Pawel Nurowski, Prof. Roger Penrose, Prof. Maciej Przanowski, Prof. David Robinson, Dr Nick Woodhouse and others for helpful clarification on various topics, and to my in-college tutor, Dr Ulrike Tillmann, for her interest in my work.

I thank my friends, especially Celia, Daniel, David, Lucy, and Sam, for their help with proof reading.

I am grateful to Merton College for the Palmer Senior Scholarship, and to St Peter's College and the SOROS Foundation for their financial support during my first year in Oxford.

Finally I would like to thank my mother and my wife Asia. I dedicate this thesis to both of them with gratitude.

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Chapter 1

Introduction

One of the most remarkable achievements of the twistor program is the link it provides between integrable differential equations and unconstrained holomorphic geometry. What lies at the heart of the twistor approach to integrability is the existence of the Lax pair which enables one to express a given nonlinear equation as the compatibility condition (usually in the form of a zero curvature representation) for a system of linear first order partial differential equations (PDEs). The two most prominent systems of nonlinear equations which fit into the program are the anti-self-dual vacuum Einstein equations (ASDVE) [56] and the anti-self-dual Yang–Mills equations (ASDYM) [78]. The basic features of the twistor approach are already visible in the following linear example.

Let (w, z, x, y) be the coordinates on $\mathbb{C}^4$ which are null with respect to the metric $2\mathrm{d}w\mathrm{d}x + 2\mathrm{d}z\mathrm{d}y$. Long before twistor theory was introduced, it was known [4] that solutions to the complex wave equation

$$\Theta_{xw} + \Theta_{yz} = 0 \tag{1.1}$$

are given by contour integral formulae

$$\Theta(w, z, x, y) = \frac{1}{2\pi i} \oint_\Gamma f(w + \lambda y, z - \lambda x, \lambda)\mathrm{d}\lambda. \tag{1.2}$$

Here $\lambda \in \mathbb{CP}^1$ and the contour Γ separates poles of the integrand. Let us make a few remarks about the last formula.

- The function f is an arbitrary holomorphic function of three variables. It is not constrained by any equations.

- As stated, the correspondence between the solutions to (1.1) and integrands (1.2) is certainly not one to one; we may change f by adding a function which is singular on one side of the contour Γ but is holomorphic on the other. We may also move the contour Γ without touching the poles of f. Both changes will not affect a corresponding solution to the wave equation. The precise relation between Θ and the pairs (f, Γ) is described in twistor theory by using sheaf cohomology [89].

- The geometric reasons for the appearance of $\lambda \in \mathbb{CP}^1$ are not clear from the formula (1.2). In the twistor approach to integrable systems λ plays the role of a spectral parameter and parametrises certain null planes passing through each point of $\mathbb{C}^4$.

From the applied mathematics point of view, the formula (1.2) only gives an alternative to other methods of solving the wave equation. The usefulness of the twistor approach is better illustrated by examples of nonlinear equations.

Modify (1.1) by adding a nonlinear term of the Monge-Ampere type

$$\Theta_{xw} + \Theta_{yz} + \Theta_{xx}\Theta_{yy} - {\Theta_{xy}}^2 = 0. \tag{1.3}$$

A major part of this thesis will be concerned with the twistor analysis of this equation, its hierarchies, reductions and generalisations. The motivation for studying the second heavenly equation[1] (1.3) comes from the work of Plebański [62]. He showed that if Θ is a solution of (1.3) then

$$\mathrm{d}s^2 = 2\mathrm{d}w\mathrm{d}x + 2\mathrm{d}z\mathrm{d}y + 2\Theta_{xx}\mathrm{d}z^2 + 2\Theta_{yy}\mathrm{d}w^2 - 4\Theta_{xy}\mathrm{d}w\mathrm{d}z \tag{1.4}$$

is a complexified hyper-Kähler metric on an open ball in $\mathbb{C}^4$. Each hyper-Kähler metric on a complex four-manifold can locally be put in the form (1.4). In four (real or complex) dimensions, hyper-Kähler metrics are solutions to anti-self-dual Einstein

[1]This terminology originates in the work of Newman [50], who studied asymptotic properties of space-times. In Minkowski space the set of asymptotically shear–free light cones can be used to reconstruct the space-time points by solving the 'good-cut equation'. This procedure does not generalise to real curved space-times, which in general do not have asymptotically shear-free null surfaces. However, if the space-time is allowed to be complex, then complex asymptotically shear-free null surfaces do exist. The set of all such surfaces is a complex Riemannian four-manifold called $\mathcal{H}$-space. The $\mathcal{H}$ stands for 'heaven - where good Cohens (cones) go'.

vacuum equations. This makes (1.3) worth studying both from the geometry and the general relativity perspectives.

A natural question which arises is whether one can generalise the formula (1.2) to solve the equation (1.3). In general such an explicit description will not be possible, but nevertheless the twistor approach assures the integrability of (1.3). This follows from Penrose's Nonlinear Graviton construction [56], in which ASDV metrics locally correspond to certain three dimensional complex manifolds - twistor spaces. The manifold structure of a twistor space is given by a set of patching functions. The process of recovering an ASDV metric on $\mathbb{C}^4$ from the patching functions involves building the holomorphic family of embedded rational curves. This usually comes down to solving a non-linear Riemann–Hilbert factorisation problem. In the case of the wave equation the analogous Riemann–Hilbert problem is linear and a solution can be given explicitly.

1.1 Outline of the Thesis

In Chapter 2 I shall summarise the twistor correspondences for flat and curved spaces. I shall establish the spinor notation, and recall basic facts about the ASD conformal condition, the geometry of the spin bundle, the ASD Yang–Mills equations, Einstein–Weyl spaces, and the isomonodromic deformations.

In Chapter 3 (following a suggestion of Dr Lionel Mason) the recursion operator R for ASDVE will be constructed by looking at ways of generating sequences of solutions to the linearised heavenly equations [17]. I shall then consider a corresponding twistor picture by using R to build a family of foliations by twistor surfaces. It will be proven that R acts on twistor functions by multiplication [19]. The general ASD linear fields on ASD vacuum backgrounds will be considered. Then I shall analyse the hidden symmetry algebra of ASDVE, and use the recursion operator to construct Killing spinors. I shall illustrate the method on the example of the Sparling–Tod solution and show how R can be used to construct $\mathcal{O}(1)\oplus\mathcal{O}(1)$ rational curves.

In Chapter **??** I shall give a twistor-geometric construction of ASDVE hierarchies. An infinite number of commuting flows on extended space-time, together with twistor description will be constructed. I shall prove that a moduli space of rational curves with normal bundle $\mathcal{O}(n)\oplus\mathcal{O}(n)$ in twistor space, is canonically equipped with the Lax distribution for ASDVE hierarchies, and conversely that truncated hierarchies imply such a twistor theory. The Lax distribution will be interpreted as a connecting map in a long exact sequence of sheafs [18] (I acknowledge the assistance of Dr Lionel Mason in the proof of Proposition **??**). In Section 6 I shall demonstrate that the isomonodromy problem in Fuchsian case can also be understood in terms of curved twistor spaces. The solutions to the $SL(2, \mathbb{C})$ Schlesinger equation will be related to the flows of the heavenly hierarchy. In Section **??** I shall investigate the Lagrangian and Hamiltonian formulations of heavenly equations. The symplectic form on the moduli space of solutions to heavenly equations will be derived, and proven to be compatible with a recursion operator.

In Chapter 4 I shall use the second heavenly equation to build the twistor space. I shall make the second heavenly equation (3.6) λ-dependent and show that a family of rational curves may be found by integrating the Hamiltonian system which has Θ as its Hamiltonian. I shall also give an alternative view on heavenly potentials as generating functions on the spin bundle.

In Chapter 5 I shall show that generalised ZRM field equations can be solved by means of functions on $\mathcal{O}(n) \oplus \mathcal{O}(n)$ twistor spaces. The fields associated with twistor functions of positive homogeneity will be shown to have both primed and unprimed symmetric indices. I shall consider a foliation of the moduli space of deformed $\mathcal{O}(n) \oplus \mathcal{O}(n)$ curves by four dimensional hyper-Kähler slices.

In Chapter 7 the twistor theory of four-dimensional hyper-Hermitian manifolds will be formulated as a combination of the Nonlinear Graviton Construction with the Ward transform for anti-self-dual Maxwell fields. The Lax formulation of the hyper-Hermiticity condition in four dimensions will be used to derive a pair of potentials for hyper-Hermitian metrics. A class of examples of hyper-Hermitian metrics which depend on two arbitrary functions of two complex variables will be given.

In Chapter 8 I shall consider ASD vacuum spaces with conformal symmetries. In Section 8.1 (which I wrote following a crucial suggestion of Dr Paul Tod and which extends [76]) I shall give the canonical form of a general conformal Killing vector. Then I shall look at conformally invariant solutions to the first heavenly equation. This will give rise to a new integrable system in three dimensions and to the corresponding Einstein–Weyl (EW) geometries. In Section 8.2 I shall give the Lax representation of the reduced equations. I shall also look at the spinor formulation of the EW condition. In Section 8.6 I shall find and classify the Lie point symmetries, and the Killing vectors of the field equations in three dimensions, and consider some group invariant solutions. In Section 8.7 I shall find hidden symmetries and the recursion operator associated to the 3D system. The conformally invariant wave equations in Weyl geometries will be analysed.

In Chapter 9 I shall reformulate EW equations in terms of a certain two-form on the spin bundle. I shall prove that if an EW space admits a solution of a generalised monopole equation, which yields ASD vacuum or Einstein metrics, then the four-dimensional correspondence space is equipped with a closed and simple two-form. In Section 9.3 I shall construct a class of EW metrics from solutions to dKP equation.

In Chapter 10 I shall explain how to construct solutions to the ASDVE from solutions of various two-dimensional integrable systems by exploiting the fact that the Lax formulations of both systems can be embedded in that of the ASD Yang–Mills equations. I shall illustrate this by constructing explicit ASDV metrics on $\mathbb{R}^2 \times \Sigma$, where Σ is a homogeneous space for a real subgroup of $SL(2, \mathbb{C})$ associated with the two-dimensional system. I shall also outline the twistor interpretation of the construction. This chapter extends my Polish MSc thesis [14]. Much of the material, derived independently by me [15], has appeared in a joint paper [20].

In Chapter 11 I shall list some open problems related to what has been done in this thesis. In particular I shall indicate the possible links between twistor theory, finite-gap solutions, and Whitham equations.

The Appendices A, B, and C are intended to record for easy reference the important theorems and formulae which underlie twistor theory.

Chapter 2

Preliminaries

2.1 The twistor correspondence for flat space–times

In this section we shall give a brief outline of the flat twistor correspondence. For more detailed expositions see [89] or [49]. We shall use the double null coordinates on $\mathbb{C}^4$ in which the metric and the volume element are

$$\mathrm{d}s^2 = 2\mathrm{d}w\mathrm{d}x + 2\mathrm{d}z\mathrm{d}y, \qquad \nu = \mathrm{d}w \wedge \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

A two-plane in $\mathbb{C}^4$ is null if $\mathrm{d}s^2(X, Y) = 0$ for every pair (X, Y) of vectors tangent to it. The null planes can be self-dual (SD) or anti self-dual (ASD), depending on whether the tangent bivector $X \wedge Y$ is SD or ASD. The SD null planes are called α-planes. The α-planes passing through a point in $\mathbb{C}^4$ are parametrised by $\lambda \in \mathbb{CP}^1$. Tangents to α-planes are spanned by two vectors

$$L_0 = \partial_y - \lambda\partial_w, \quad L_1 = \partial_x + \lambda\partial_z \tag{2.1}$$

or (∂_z, ∂_w) if $\lambda = \infty$. The set of all α-planes is called a projective twistor space and denoted $\mathcal{PT}$. It is a three-dimensional complex manifold biholomorphic to $\mathbb{CP}^3 - \mathbb{CP}^1$.

We will make use of a double fibration picture

$$\mathbb{C}^4 \xleftarrow{p} \mathcal{F} \xrightarrow{q} \mathcal{PT}.$$

The five complex dimensional correspondence space $\mathcal{F} := \mathbb{C}^4 \times \mathbb{CP}^1$ fibres over $\mathbb{C}^4$ by

$$(w, z, x, y, \lambda) = (w, z, x, y).$$

The functions on $\mathcal{F}$ which are constant on α-planes, or equivalently satisfy $L_A f = 0;\ A = 0, 1$, push down to $\mathcal{PT}$. They are called twistor functions. An example of a twistor function was used in the formula (1.2). The twistor space $\mathcal{PT}$ is a factor space of $\mathcal{F}$ by the two-dimensional distribution spanned by L_A. It can be covered by two coordinate patches U and $\widetilde{U}$, where U is a complement of $\lambda = \infty$ and $\widetilde{U}$ is a compliment of $\lambda = 0$. If (μ^0, μ^1, λ) are coordinates on U and $(\tilde{\mu}^0, \tilde{\mu}^1, \tilde{\lambda})$ are coordinates on $\widetilde{U}$ then on the overlap

$$\tilde{\mu}^0 = \mu^0/\lambda, \quad \tilde{\mu}^1 = \mu^1/\lambda, \quad \tilde{\lambda} = 1/\lambda.$$

The local coordinates (μ^0, μ^1, λ) on $\mathcal{PT}$ pulled back to $\mathcal{F}$ are

$$\mu^0 = w + \lambda y, \quad \mu^1 = z - \lambda x, \quad \lambda. \tag{2.2}$$

Before the curved twistor theory is considered, we shall review the two-spinor notation for complex Riemannian four-manifolds.

2.2 Spinor notation

We shall work in the holomorphic category with complexified space-times: thus space-time $\mathcal{M}$ is a complex four-manifold equipped with a holomorphic metric g and volume form ν.

In four complex dimensions orthogonal transformations decompose into products of ASD and SD rotations

$$SO(4, \mathbb{C}) = (SL(2, \mathbb{C}) \times \widetilde{SL}(2, \mathbb{C}))/\mathbb{Z}_2. \tag{2.3}$$

The spinor calculus in four dimensions is based on the isomorphism (2.3). We shall use the conventions of Penrose and Rindler [58] (see also [89]). Indices will be assumed concrete if we work in any of the heavenly frames and otherwise abstract: $a, b, ...$ are four-dimensional space-time indices and $A, B, ..., A', B', ...$ are two-dimensional spinor indices. The tangent space at each point of $\mathcal{M}$ is isomorphic to a tensor product of two spin spaces

$$T^a\mathcal{M} = S^A \otimes S^{A'}. \tag{2.4}$$

The complex Lorentz transformation $V^a \longrightarrow \Lambda^a{}_b V^b$ is equivalent to the composition of the SD and the ASD rotation

$$V^{AA'} \longrightarrow \lambda^A{}_B V^{BB'} \lambda^{A'}{}_{B'},$$

where $\lambda^A{}_B$ and $\lambda^{A'}{}_{B'}$ are elements of $SL(2,\mathbb{C})$ and $\widetilde{SL}(2,\mathbb{C})$.

Spin dyads (o^A, ι^A) and $(o^{A'}, \iota^{A'})$ span S^A and $S^{A'}$ respectively. The spin spaces S^A and $S^{A'}$ are equipped with symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ such that $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These anti-symmetric objects are used to raise and lower the spinor indices. We shall use the normalised spin frames, which implies that

$$o^B \iota^C - \iota^B o^C = \varepsilon^{BC}, \quad o^{B'} \iota^{C'} - \iota^{B'} o^{C'} = \varepsilon^{B'C'}.$$

Let $e^{AA'}$ be the null tetrad of one forms on $\mathcal{M}$ and let $\nabla_{AA'}$ be the frame of dual vector fields. The orientation is fixed by setting

$$\nu = e^{01'} \wedge e^{10'} \wedge e^{11'} \wedge e^{00'}.$$

Apart from orientability, $\mathcal{M}$ must satisfy some other topological restrictions for the global spinor fields to exist [89]. We shall not take them into account as we work locally in $\mathcal{M}$.

The local basis Σ^{AB} and $\Sigma^{A'B'}$ of spaces of ASD and SD two-forms are defined by

$$e^{AA'} \wedge e^{BB'} = \varepsilon^{AB} \Sigma^{A'B'} + \varepsilon^{A'B'} \Sigma^{AB}.$$

The Weyl tensor decomposes into ASD and SD part

$$C_{abcd} = \varepsilon_{A'B'} \varepsilon_{C'D'} C_{ABCD} + \varepsilon_{AB} \varepsilon_{CD} C_{A'B'C'D'}.$$

The first Cartan structure equations are

$$\mathrm{d}e^{AA'} = e^{BA'} \wedge \Gamma^A{}_B + e^{AB'} \wedge \Gamma^{A'}{}_{B'},$$

where Γ_{AB} and $\Gamma_{A'B'}$ are the $SL(2,\mathbb{C})$ and $\widetilde{SL}(2,\mathbb{C})$ spin connection one-forms symmetric in their indices, and

$$\Gamma_{AB} = \Gamma_{CC'AB} e^{CC'}, \quad \Gamma_{A'B'} = \Gamma_{CC'A'B'} e^{CC'}, \quad \Gamma_{CC'A'B'} = o_{A'} \nabla_{CC'} \iota_{B'} - \iota_{A'} \nabla_{CC'} o_{B'}.$$

The curvature of the spin connection

$$R^A{}_B = \mathrm{d}\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B$$

decomposes as

$$R^A{}_B = C^A{}_{BCD}\Sigma^{CD} + (1/12)R\Sigma^A{}_B + \Phi^A{}_{BC'D'}\Sigma^{C'D'},$$

and similarly for $R^{A'}{}_{B'}$. Here R is the Ricci scalar and $\Phi_{ABA'B'}$ is the trace-free part of the Ricci tensor R_{ab}.

Now we can rephrase the flat twistor correspondence discussed in Section 2.1 in the spinor language: A point in $\mathbb{C}^4$ is represented by its position vector (w, z, x, y). The isomorphism (2.4) is realised by

$$x^{AA'} := \begin{pmatrix} y & w \\ -x & z \end{pmatrix}, \quad \text{so that } \mathrm{d}s^2 = \varepsilon_{AB}\varepsilon_{A'B'}\mathrm{d}x^{AA'}\mathrm{d}x^{BB'}.$$

The homogeneous coordinates on the twistor space are

$$(\omega^0, \omega^1, \pi_{0'}, \pi_{1'}) = (\omega^A, \pi_{A'}).$$

They are related to (μ^0, μ^1, λ) by

$$\omega^0/\pi_{1'} = \mu^0, \quad \omega^1/\pi_{1'} = \mu^1, \quad \pi_{0'}/\pi_{1'} = \lambda.$$

For $\lambda \neq \infty$ the twistor distribution may be rewritten as

$$L_A = (\pi_{1'})^{-1}\pi^{A'}\frac{\partial}{\partial x^{AA'}}.$$

The relations between various structures on $\mathbb{C}^4$ and $\mathcal{PT}$ can be read off from the equation

$$\omega^A = x^{AA'}\pi_{A'}. \tag{2.5}$$

Assume that $\pi_{A'} \neq 0$ and consider $(\omega^A, \pi_{A'})$ to be fixed. Then (2.5) has as its solution a complex two plane spanned by vectors of the form $\pi^{A'}v^A$ for some v^A. The other way of interpreting (2.5) is fixing $x^{AA'}$ and solving for $(\omega^A, \pi_{A'})$. The solution, when factored out by the relation $(\omega^A, \pi_{A'}) \sim (k\omega^A, k\pi_{A'})$, becomes a rational curve $\mathbb{CP}^1$ with a normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This condition guarantees that the family of rational curves in $\mathcal{PT}$ is four complex dimensional, and that the conformal structure $\mathrm{d}s^2$ on $\mathbb{C}^4$ is quadratic. Here $\mathcal{O}(n)$ denotes the line bundle over $\mathbb{CP}^1$ with transition functions λ^{-n} (see Appendix A). The points p and q are null separated in $\mathbb{C}^4$ iff the corresponding rational curves l_p and l_q intersect at one point in $\mathcal{PT}$.

2.3 Curved twistor spaces and the geometry of the primed spin bundle.

Assume that g is a curved metric on some complex four-dimensional manifold $\mathcal{M}$. The notion of an α-plane must be replaced by an α-surface - a null two dimensional surface such that its tangent space at each point is an α plane. Let X and Y be two vectors tangent to an α-surface. The Frobenius integrability condition yields

$$[X, Y] = aX + bY$$

for some a and b. The last formula implies that $C_{A'B'C'D'}$ vanishes. Thus given $C_{A'B'C'D'} = 0$ we can define a twistor space $\mathcal{PT}$ to be a three complex dimensional manifold of α-surfaces in $\mathcal{M}$. If g is also Ricci flat then $\mathcal{PT}$ has more structures which are listed in the Nonlinear Graviton Theorem C.4.

The correspondence space $\mathcal{F}$ is a set of pairs (x, λ) where $x \in \mathcal{M}$ and $\lambda \in \mathbb{CP}^1$ parametrises α-surfaces through x in $\mathcal{M}$. We represent $\mathcal{F}$ as the quotient of the primed-spin bundle $S^{A'}$ with fibre coordinates $\pi_{A'}$ by the Euler vector field $\Upsilon = \pi^{A'}/\partial\pi^{A'}$ so that the fibre coordinates are related by $\lambda = \pi_{0'}/\pi_{1'}$. A homogeneous form α on non-projective spin bundle descends to $\mathcal{F}$ if

$$\Upsilon \lrcorner \alpha = 0.$$

In that case $\mathcal{L}_\Upsilon \alpha = n\alpha$ where n determines a line bundle $\mathcal{O}(n)$ in which α takes its values. The space $\mathcal{F}$ possesses a natural two dimensional distribution (called the twistor distribution, or the Lax pair, to emphasise the analogy with integrable systems). The Lax pair on $\mathcal{F}$ arises as the image under the projection $TS^{A'} \longrightarrow T\mathcal{F}$ of the distribution spanned by

$$\pi^{A'}\nabla_{AA'} + \Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\frac{\partial}{\partial\pi_{C'}}$$

and is given by[1]

$$L_A = (\pi_{1'}^{-1})(\pi^{A'}\nabla_{AA'} + f_A\partial_\lambda), \qquad \text{where} \quad f_A = (\pi_{1'}^{-2})\Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\pi^{C'}. \tag{2.7}$$

[1]Various powers of $\pi_{1'}$ in formulae like (2.7) guarantee the correct homogeneity. We usually shall omit them when working on the projective spin bundle. In a projection $S^{A'} \longrightarrow \mathcal{F}$ we shall use the replacement formula

$$\frac{\partial}{\partial\pi^{A'}} \longrightarrow \frac{\pi_{A'}}{{\pi_{1'}}^2}\partial_\lambda. \tag{2.6}$$

The integrability of the twistor distribution is equivalent to $C_{A'B'C'D'} = 0$, the vanishing of the self-dual Weyl spinor. The projective twistor space $\mathcal{PT}$ arises as a factor space of $\mathcal{F}$ by the twistor distribution. It can be covered by two sets, U and $\tilde{U}$ with $|\lambda| < 1+\epsilon$ on U and $|\lambda| > 1-\epsilon$ on $\tilde{U}$ with $(\omega^A, \pi_{A'} \neq \iota_{A'})$ being the homogeneous coordinates on U and $(\tilde{\omega}^A, \pi_{A'} \neq o_{A'})$ on $\tilde{U}$. The twistor space $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = \tilde{\omega}^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$. Let l_x be the line in $\mathcal{PT}$ that corresponds to $x \in \mathcal{M}$ and let $Z \in \mathcal{PT}$ lie on l_x. The correspondence space is

$$\mathcal{F} = \mathcal{PT} \times \mathcal{M}|_{Z \in l_x} = \mathcal{M} \times \mathbb{CP}^1.$$

This leads to a double fibration

$$\mathcal{M} \xleftarrow{p} \mathcal{F} \xrightarrow{q} \mathcal{PT}. \tag{2.8}$$

The existence of L_A can also be deduced directly from $\mathcal{PT}$. The basic twistor correspondence [56] states that points in $\mathcal{M}$ correspond in $\mathcal{PT}$ to rational curves with normal bundle $\mathcal{O}^A(1) := \mathcal{O}(1) \oplus \mathcal{O}(1)$. The normal bundle to l_x consists of vectors tangent to x (horizontally lifted to $T_{(x,\lambda)}\mathcal{F}$) modulo the twistor distribution. Therefore we have a sequence of sheaves over $\mathbb{CP}^1$

$$0 \longrightarrow D \longrightarrow \mathbb{C}^4 \longrightarrow \mathcal{O}^A(1) \longrightarrow 0.$$

The map $\mathbb{C}^4 \longrightarrow \mathcal{O}^A(1)$ is given by $V^{AA'} \longrightarrow V^{AA'}\pi_{A'}$. Its kernel consists of vectors of the form $\pi^{A'}\lambda^A$ with λ^A varying. The twistor distribution is therefore $D = O(-1) \otimes S^A$ and so $L_A \in \Gamma(D \otimes \mathcal{O}(1) \otimes S_A)$, has the form (2.7).

2.4 Some formulations of the ASD vacuum condition

The ASD vacuum condition

$$C_{A'B'C'D'} = 0, \qquad \Phi_{ABA'B'} = 0 \tag{2.9}$$

This is because (on functions of λ)

$$\frac{\partial}{\partial \pi_{A'}}\left(\frac{\pi_{0'}}{\pi_{1'}}\right) = \frac{\pi_{1'}o^{A'} - \pi_{0'}\iota^{A'}}{{\pi_{1'}}^2} = \frac{\pi^{A'}}{{\pi_{1'}}^2}.$$

implies the existence of a normalised, covariantly constant frame on $S^{A'}$

$$o^{A'} = (1,0), \qquad \iota^{A'} = (0,1),$$

such that $\Gamma_{AA'B'C'} = 0$. In this frame the Lax pair (2.7) consists of volume-preserving vector fields on $\mathcal{M}$. This fact was first observed in [1] in the context of $(3+1)$ decomposition. In the next chapters we shall use the covariant generalisation:

Proposition 2.1 (Mason & Newman [46].) *Let* $\widehat{\nabla}_{AA'} = (\widehat{\nabla}_{00'}, \widehat{\nabla}_{01'}\widehat{\nabla}_{10'}\widehat{\nabla}_{11'})$ *be four independent holomorphic vector fields on a four-dimensional complex manifold* $\mathcal{M}$ *and let* ν *be a nonzero holomorphic four-form. Put*

$$L_0 = \widehat{\nabla}_{00'} - \lambda\widehat{\nabla}_{01'}, \qquad L_1 = \widehat{\nabla}_{10'} - \lambda\widehat{\nabla}_{11'}. \tag{2.10}$$

Suppose that for every $\lambda \in \mathbb{CP}^1$

$$[L_0, L_1] = 0, \qquad \mathcal{L}_{L_A}\nu = 0. \tag{2.11}$$

Here $\mathcal{L}_V$ *denotes a Lie derivative. Then*

$$\nabla_{AA'} = f^{-1}\widehat{\nabla}_{AA'}, \qquad \textit{where} \qquad f^2 := \nu(\widehat{\nabla}_{00'}, \widehat{\nabla}_{01'}\widehat{\nabla}_{10'}\widehat{\nabla}_{11'}),$$

is a normalised null-tetrad for an ASD vacuum metric. Every such metric locally arises in this way.

Lemma 7.3 generalises the last proposition to the hyper-Hermitian case. The transformation to $f^2 = 1$ is always possible if the metric is non-degenerate. In this thesis (except Chapter 10) we shall use the heavenly frames in which $f^2 = 1$ and $\nabla_{AA'} = \widehat{\nabla}_{AA'}$. For easy reference we rewrite the field equations (2.11) in full

$$[\nabla_{A0'}, \nabla_{B0'}] = 0, \tag{2.12}$$

$$[\nabla_{A0'}, \nabla_{B1'}] + [\nabla_{A1'}, \nabla_{B0'}] = 0, \tag{2.13}$$

$$[\nabla_{A1'}, \nabla_{B1'}] = 0. \tag{2.14}$$

Let $\Sigma^{A'B'}$ be the usual basis of SD two forms. Define (on a correspondence space)

$$\Sigma(\lambda) := \Sigma^{A'B'}\pi_{A'}\pi_{B'}. \tag{2.15}$$

The formulation of the ASDVE condition dual to (2.11) is:

Proposition 2.2 (Plebański [62], Gindikin [24]) *If a two form*

$$\Sigma(\lambda) := \Sigma^{A'B'}\pi_{A'}\pi_{B'}$$

on a correspondence space satisfies

$$\mathrm{d}_h \frac{\Sigma(\lambda)}{(\pi\cdot\iota)^2} = 0, \qquad \Sigma(\lambda)\wedge\Sigma(\lambda) = 0 \tag{2.16}$$

then one forms $e^{AA'}$ *give an ASD vacuum tetrad.*

Note that the simplicity condition in (2.16) guarantees that $\Sigma^{A'B'}$ comes from a tetrad. Here d_h is a horizontal lift of d to $\mathcal{F}$ and so λ is regarded as a parameter and is not differentiated. To construct Gindikin's two form starting from the twistor space, one must fix a fibre of $\mathcal{PT} \longrightarrow \mathbb{CP}^1$ and pull the symplectic structure back to the projective spin bundle. The resulting two form is $\mathcal{O}(2)$ valued. To obtain Gindikin's two form one should divide it by a constant section of $\mathcal{O}(2)$.

Put $\Sigma^{0'0'} = -\tilde{\alpha},\ \Sigma^{0'1'} = \omega,\ \Sigma^{1'1'} = \alpha$. The second equation in (2.16) becomes

$$\omega\wedge\omega = 2\alpha\wedge\tilde{\alpha} = -2\nu, \quad \alpha\wedge\omega = \tilde{\alpha}\wedge\omega = \alpha\wedge\alpha = \tilde{\alpha}\wedge\tilde{\alpha} = 0.$$

Equivalence of (2.11) and (2.16) follows if one notices that L_A can be defined as a two-dimensional distribution which annihilates $\Sigma(\lambda)$, or alternatively

$$\varepsilon_{AB}\Sigma(\lambda) = \nu(L_A, L_B, ..., ...).$$

Two one-forms $e^A := \pi_{A'}e^{AA'}$ by definition annihilate the twistor distribution. Define $(1,1)$ tensors $\partial_{A'}^{B'} := e^{AB'}\otimes\nabla_{AA'}$ so that

$$e^A\otimes L_A = \pi_{B'}\pi^{A'}\partial_{A'}^{B'} = \partial_0 + \lambda(\partial - \tilde{\partial}) - \lambda^2\partial_2$$

where $(\partial_{0'}^{0'}, \partial_{1'}^{0'}, \partial_{0'}^{1'}, \partial_{1'}^{1'}) = (\tilde{\partial}, \partial_0, \partial_2, \partial)$. If the field equations are satisfied then the Euclidean slice of $\mathcal{M}$ is equipped with three integrable complex structures given by $J_a := \{i(\partial_2 - \partial_0),\ (\partial - \tilde{\partial}),\ (\partial_2 + \partial_0)\}$ and three symplectic structures $\{(i(\alpha - \tilde{\alpha}), iw, (\alpha + \tilde{\alpha})\}$ compatible with J_a. It is therefore a hyper-Kähler manifold.

2.5 ASD Yang–Mills equations

Consider a Yang–Mills vector bundle over a four-dimensional manifold $\mathbb{M}$ (taken here to be $\mathbb{C}^4$ in general, or $\mathbb{R}^4$ when reality conditions are imposed) with connection one-form $A = A_a(x^b)\mathrm{d}x^a \in T^*\mathbb{M} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of some gauge group G. The corresponding curvature $F = F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$ is given by

$$F_{ab} = [D_a, D_b] = \partial_b A_a - \partial_a A_b + [A_a, A_b], \tag{2.17}$$

where $D_a = \partial_a - A_a$ is the covariant derivative. The curvature decomposes as

$$F_{ab} = \Phi_{AB}\varepsilon_{A'B'} + \widetilde{\Phi}_{A'B'}\varepsilon_{AB}.$$

The ASDYM equations on a connection A are the anti-self-duality conditions on the curvature under the Hodge star operation:

$$F = - * F, \quad \text{or} \quad \widetilde{\Phi}_{A'B'} = 0. \tag{2.18}$$

They are conformally invariant and are also preserved by the gauge transformations

$$A \to g^{-1}Ag - g^{-1}\mathrm{d}g, \qquad g \in Map(\mathbb{M}, G). \tag{2.19}$$

The condition (2.18) is equivalent to the vanishing of the Yang–Mills curvature on every α plane. This observation underlies the Ward construction [78].

The ASDYM equations are the compatibility conditions $[L_0, L_1] = 0$ for the linear system of equations $L_0 F = 0$, $L_1 F = 0$ where the 'Lax pair' is given by

$$L_A = \pi^{A'} D_{AA'}, \tag{2.20}$$

and $F := F(x^{AA'}\pi_{A'}, \pi^{A'})$ is an n-component column vector.

In Chapter 10 we shall need a coordinate description of YM anti-self-duality condition. Let us introduce double-null coordinates $x^{AA'} = (w, \tilde{w}, z, \tilde{z})$, in which the metric on $\mathbb{M}$ is $\mathrm{d}s^2 = 2\mathrm{d}w\mathrm{d}\tilde{w} - 2\mathrm{d}z\mathrm{d}\tilde{z}$. In these coordinates $D_w = \partial_w - A_w, ...$ and the ASDYM equations may be rewritten as

$$F_{wz} = 0 \tag{2.21}$$

$$F_{\tilde{w}\tilde{z}} = 0 \tag{2.22}$$

$$F_{w\tilde{w}} - F_{z\tilde{z}} = 0. \tag{2.23}$$

2.6 Three-dimensional Einstein–Weyl spaces

Let $\mathcal{W}$ be a n-dimensional complex manifold, with a torsion-free connection D and a conformal metric $[h]$. We shall call $\mathcal{W}$ a Weyl space if the null geodesics of $[h]$ are also geodesics for D. This condition implies

$$D_i h_{jk} = \nu_i h_{jk} \tag{2.24}$$

for some one form ν. Here h_{jk} is a representative metric in the conformal class. The indices[2] $i, j, k, ...$ go from 1 to n. If we change this representative by $h \longrightarrow \phi^2 h$, then $\nu \longrightarrow \nu + 2\mathrm{d}\ln\phi$. The one form ν 'measures' the difference between D and the Levi-Civita connection of h:

$$(D_i - \nabla_i)V^j = \gamma^j_{ik} V^k,$$

where (as a consequence of (2.24))

$$\gamma^i_{jk} := -\delta^i_{(j}\nu_{k)} + \frac{1}{2}h_{jk}h^{im}\nu_m.$$

The Ricci tensor W_{ij} of D is related to the Ricci tensor R_{ij} of ∇ by

$$W_{ij} = R_{ij} + \frac{n-1}{2}\nabla_i\nu_j - \frac{1}{2}\nabla_j\nu_i + \frac{n-2}{4}\nu_i\nu_j + h_{ij}\Big(-\frac{n-2}{4}\nu_k\nu^k + \frac{1}{2}\nabla_k\nu^k\Big).$$

The relation between the curvature scalars is

$$W := h^{ij}W_{ij} = R + (n-1)\nabla^k\nu_k - \frac{(n-2)(n-1)}{4}\nu^k\nu_k.$$

The conformally invariant Einstein–Weyl (EW) condition on $(\mathcal{W}, h, \omega)$ is

$$W_{(ij)} = \frac{1}{n}Wh_{ij}.$$

From now on we shall assume that $dim\mathcal{W} = 3$. The Einstein–Weyl equations are

$$R_{ij} + \frac{1}{2}\nabla_{(i}\nu_{j)} + \frac{1}{4}\nu_i\nu_j = \frac{1}{3}\Big(R + \frac{1}{2}\nabla^k\nu_k + \frac{1}{4}\nu^k\nu_k\Big)h_{ij}. \tag{2.25}$$

All three-dimensional EW spaces can be obtained as spaces of trajectories of conformal Killing vectors in four-dimensional ASD manifolds:

[2] In Chapters 8 and 9, when we consider the three-dimensional case, we shall use the spinor notation, and the abstract index convention $V^i = V^{(A'B')} = v^{(A'}\pi^{B')}$ based on an isomorphism $T^i\mathcal{W} = S^{(A'} \otimes S^{B')}$. The reason for using primed spinors will be explained in Chapter 8.

Proposition 2.3 (Jones & Tod [37]) *Let $(\mathcal{M}, g)$ be an ASD four manifold with a conformal Killing vector K. The EW structure on the space $\mathcal{W}$ of trajectories of K (which is assumed to be non-pathological) is defined by*

$$h := |K|^{-2}g - |K|^{-4}\mathbf{K} \odot \mathbf{K}, \quad \nu := 2|K|^{-2} *_g (\mathbf{K} \wedge \mathrm{d}\mathbf{K}), \tag{2.26}$$

where $|K|^2 := g_{ab}K^aK^b$, $\mathbf{K}$ is a one form dual to K and $*_g$ is taken with respect to g. All EW structures arise in this way. Conversely, let (h, ν) be a three–dimensional EW structure on $\mathcal{W}$, and let (V, α) be a function and a one-form on $\mathcal{W}$ which satisfy the generalised monopole equation

$$*_h(\mathrm{d}V + (1/2)\nu V) = \mathrm{d}\alpha, \tag{2.27}$$

where $*_h$ is taken with respect to h. Then

$$g = V^2h + (\mathrm{d}t + \alpha)^2 \tag{2.28}$$

is an ASD metric with an isometry $K = \partial_t$.

The twistor construction of 3D EW spaces is given by the following proposition

Proposition 2.4 (Hitchin [30]) *Any solution to the EW equations* (2.25) *is equivalent to a complex surface $\mathcal{Z}$ (called a mini-twistor space) with a family of rational curves with a normal bundle $\mathcal{O}(2)$.*

Points of $\mathcal{W}$ correspond to curves in $\mathcal{Z}$ with self-intersection number 2. The Kodaira theorem (A.4) guarantees the existence of a three-dimensional complex family of such curves. Points of $\mathcal{Z}$ correspond to totally geodesic hyper-surfaces in $\mathcal{W}$. Non-null geodesics in $\mathcal{W}$ consists of all the curves in $\mathcal{Z}$ which intersect at two fixed points in $\mathcal{Z}$. Null geodesics correspond to curves passing through one point with a given tangent direction.

It follows from [36, 37] that the mini-twistor space $\mathcal{Z}$ corresponding to $\mathcal{W}$ is a factor space $\mathcal{PT}/\mathcal{K}$ where $\mathcal{PT}$ is the twistor space of $(\mathcal{M}, g)$ and $\mathcal{K}$ is a holomorphic vector field on $\mathcal{PT}$ corresponding to conformal Killing vector K.

In three dimensions the general solution of (2.24)-(2.25) depends on four arbitrary functions of two variables [10]. This result is a direct consequence of twistor theory: The patching data of a general twistor space $\mathcal{PT}$ depends on three complex functions of three variables. The factorisation process reduces the number of variables by one, but keeps the number of functions fixed. One function determines a solution to a linear Bogomonly equation, and remaining two determine the characteristic data for a solution to EW equations. The Cauchy data is (in terms of free functions) twice as large as the characteristic data, which yields the desired result.

In Chapters 8 and 9 we shall consider a class of solutions to EW equations which depend on two arbitrary functions of two variables.

2.7 The Schlesinger equation and isomonodromy

Consider the system of ODEs

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \Lambda\Big)\Psi(\lambda) = 0, \qquad \Lambda = \sum_{a=1}^{n+3} \frac{A_a}{\lambda - t_a} \tag{2.29}$$

where $\mathbf{t} = t_1, ..., t_{n+3} \in \mathbb{C}$ are constants and A_i are constant $N \times N$ matrices in some complex Lie algebra $\mathfrak{g}$ (which we take to be $sl(N, \mathbb{C})$) and $\lambda \in \mathbb{CP}^1$.

Here Ψ is a fundamental matrix solution to (6.1). Assume that there is no extra pole at ∞, i.e. $\sum_{a=1}^{n+3} A_a = 0$, and that eigenvalues of A_a have no integer difference for each a. Here $a, b, c, = 1...n$ are vector indices on $\mathbb{C}^n$, and $i, j = 1...dim\mathfrak{g} = k$ are indices on $\mathfrak{g}$. Let

$$\Sigma_{\mathbf{t}} := \mathbb{CP}^1/\{t_1, ..., t_{n+3}\}$$

be a punctured sphere with $n + 3$ points removed. And let $\pi : \widetilde{\Sigma}_{\mathbf{t}} \longrightarrow \Sigma_{\mathbf{t}}$ be the universal covering. Let γ be a path in $\widetilde{\Sigma}_{\mathbf{t}}$ starting at λ and ending at λ_γ such that $\pi(\lambda) = \lambda_\gamma$. The function

$$\Psi(\lambda_\gamma) = \Psi(\lambda)g_\gamma$$

is a solution to (6.1). Here g_γ is a nonsingular constant matrix depending on the homotopy class $[\gamma]$ of γ. The mapping $[\gamma] \longrightarrow g_\gamma$ defines the monodromy representation of the fundamental group of $\Sigma_{\mathbf{t}}$

$$\pi_1(\Sigma_{\mathbf{t}}) \longrightarrow SL(N, \mathbb{C})$$

The monodromy group Γ is in general the infinite discrete subgroup (with $n + 3$ generators) of $SL(N, \mathbb{C})$.

The fundamental matrix solution $\Psi(\lambda)$ is a multi valued function with branch points at t_a. If λ moves around a singular point t_a then the fundamental solution undergoes a transformation by an element of the monodromy group

$$\Psi(\lambda) \longrightarrow \Psi(t_a + (\lambda - t_a)e^{2\pi i}) = \Psi(\lambda) g_a,$$

where $g_a \in \Gamma$. The transformation g_a is conjugated to $\exp(-2\pi A_a)$.

When the poles t_a move the monodromy representation of (6.1) remains fixed if matrices $A_a(\mathbf{t})$ satisfy the Schlesinger equation

$$\mathrm{d}A_a = \sum_{a \neq b} [A_b, A_a] \mathrm{d}\ln(t_a - t_b). \tag{2.30}$$

The common geometric interpretations are:

- Take a connection
$$\nabla = \mathrm{d} - \sum_{a=1}^{n+3} \frac{A_a \mathrm{d}\lambda}{\lambda - t_a}$$
on the vector bundle with fibres $\mathbb{C}^N$ over $\Sigma_{\mathbf{t}}$. Since A is holomorphic it is a flat connection (there are no holomorphic two forms in one dimension). Equations (6.2) imply that the holonomy of ∇ is fixed

- Treat ∇ as a connection over $\mathbb{C}^{n+3} \times \Sigma_{\mathbf{t}}$ with logarithmic singularity. Equations (6.2) imply the flatness of ∇.

Chapter 3

The recursion operator

In this chapter the recursion operator R for the anti-self-dual Einstein vacuum equations is constructed [17]. It is proven that R acts on twistor functions by multiplication. The recursion operator is then used to construct the hidden symmetry algebra of heavenly equations, and Killing spinors. The general ASD linear fields on ASD vacuum backgrounds are discussed [19].

3.1 The ASD condition and heavenly equations

Part of the residual gauge freedom in (2.11) is fixed by selecting one of Plebański's null coordinate systems.

1. Equations (2.13) and (2.14) imply the existence of a coordinate system

   $$(w, z, \tilde{w}, \tilde{z}) =: (w^A, \tilde{w}^A)$$

   and a complex-valued function Ω such that

   $$\nabla_{AA'} = \begin{pmatrix} \Omega_{w\tilde{w}}\partial_{\tilde{z}} - \Omega_{w\tilde{z}}\partial_{\tilde{w}} & \partial_w \\ \Omega_{z\tilde{w}}\partial_{\tilde{z}} - \Omega_{z\tilde{z}}\partial_{\tilde{w}} & \partial_z \end{pmatrix} = \Big(\frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}_B} \quad \frac{\partial}{\partial w^A} \Big). \tag{3.1}$$

   Equation (2.12) yields the first heavenly equation

   $$\Omega_{w\tilde{z}}\Omega_{z\tilde{w}} - \Omega_{w\tilde{w}}\Omega_{z\tilde{z}} = 1 \ \text{ or } \ \frac{1}{2} \frac{\partial^2 \Omega}{\partial w_A \partial \tilde{w}_B} \frac{\partial^2 \Omega}{\partial w^A \partial \tilde{w}^B} = 1. \tag{3.2}$$

   The dual tetrad is

   $$e^{A1'} = \mathrm{d}w^A, \quad e^{A0'} = \frac{\partial^2 \Omega}{\partial w_A \partial \tilde{w}_B} \mathrm{d}\tilde{w}_B \tag{3.3}$$

with the flat solution $\Omega = w^A \tilde{w}_A$. The only nontrivial part of $\Sigma^{A'B'}$ is $\Sigma^{0'1'} = \partial\tilde{\partial}\Omega$ so that Ω is a Kähler scalar. The Lax pair for the first heavenly equation is

$$\begin{aligned} L_0 : &= \Omega_{w\tilde{w}}\partial_{\tilde{z}} - \Omega_{w\tilde{z}}\partial_{\tilde{w}} - \lambda\partial_w, \\ L_1 : &= \Omega_{z\tilde{w}}\partial_{\tilde{z}} - \Omega_{z\tilde{z}}\partial_{\tilde{w}} - \lambda\partial_z. \end{aligned} \tag{3.4}$$

Equations $L_0\Psi = L_1\Psi = 0$ have solutions provided that Ω satisfies the first heavenly equation (3.2). Here Ψ is a function on $\mathcal{F}$.

2. Alternatively we can adopt $(w, z, x, y) =: (w^A, x_A)$ as a coordinate system. Equations (2.12) and (2.13) imply the existence of a complex-valued function Θ such that

$$\nabla_{AA'} = \begin{pmatrix} \partial_y & \partial_w + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y \\ -\partial_x & \partial_z - \Theta_{xy}\partial_x + \Theta_{xx}\partial_y \end{pmatrix} = \left(\frac{\partial}{\partial x^A} \quad \frac{\partial}{\partial w^A} + \frac{\partial^2\Theta}{\partial x^A \partial x^B}\frac{\partial}{\partial x_B} \right). \tag{3.5}$$

As a consequence of (2.14) Θ satisfies second heavenly equation

$$\Theta_{xw} + \Theta_{yz} + \Theta_{xx}\Theta_{yy} - {\Theta_{xy}}^2 = 0 \quad \text{or} \quad \frac{\partial^2\Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^B \partial x^A}\frac{\partial^2\Theta}{\partial x_B \partial x_A} = 0. \tag{3.6}$$

The dual frame is given by

$$e^{A0'} = \mathrm{d}x^A + \frac{\partial^2\Theta}{\partial x^B \partial x_A}\mathrm{d}w^B, \quad e^{A1'} = \mathrm{d}w^A \tag{3.7}$$

with $\Theta = 0$ defining the flat metric. The Lax pair corresponding to (3.6) is

$$\begin{aligned} L_0 &= \partial_y - \lambda(\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x), \\ L_1 &= \partial_x + \lambda(\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x). \end{aligned} \tag{3.8}$$

Both heavenly equations were originally derived by Plebański [62] from the formulation (2.16). The closure condition is used, via Darboux's theorem, to introduce ω^A, canonical coordinates on the spin bundle, holomorphic around $\lambda = 0$ such that the two form (2.15) is $\Sigma(\lambda) = \mathrm{d}_h\omega^A \wedge \mathrm{d}_h\omega_A$. The various forms of the heavenly equations can be obtained by adapting different coordinates and gauges to these forms. Significant progress towards understanding the symmetry structure of the heavenly equations was achieved by Boyer and Plebański [7, 8] who obtained an infinite number of conservation laws for the ASDVE equations and established some connections with nonlinear graviton construction.

3.2 The recursion operator

The recursion operator R is a map from the space of linearised perturbations of the ASDVE equations to itself. This can be used to construct the ASDVE hierarchy by generating new flows acting on one of the coordinate flows with the recursion operator R.

We will identify the space of linearised perturbations to the ASDVE equations with solutions to the background coupled wave equations as follows.

Lemma 3.1 *Let $\Box_\Omega$ and $\Box_\Theta$ denote wave operators on the ASD background determined by Ω and Θ respectively. Linearised solutions to* (3.2) *and* (3.6) *satisfy*

$$\Box_\Omega \delta\Omega = 0, \qquad \Box_\Theta \delta\Theta = 0. \tag{3.9}$$

Proof. In both cases $\Box_g = \nabla_{A1'}\nabla^A{}_{0'}$ since

$$\Box_g = \frac{1}{\sqrt{g}}\partial_a(g^{ab}\sqrt{g}\partial_b) = g^{ab}\partial_a\partial_b + (\partial_a g^{ab})\partial_b$$

but $\partial_a g^{ab} = 0$ for both heavenly equations. For the first equation $(\partial\tilde{\partial}(\Omega+\delta\Omega))^2 = \nu$ implies

$$0 = (\partial\tilde{\partial}\Omega \wedge \partial\tilde{\partial})\delta\Omega = \mathrm{d}(\partial\tilde{\partial}\Omega \wedge (\partial - \tilde{\partial})\delta\Omega) = \mathrm{d} * \mathrm{d}\delta\Omega.$$

Here $*$ is the Hodge star operator corresponding to g. For the second equation we make use of the tetrad (3.5) and perform coordinate calculations.

$\Box$

From now on we identify tangent spaces to the spaces of solutions to (3.2) and (3.6) with the space of solutions to the curved background wave equation, $\mathcal{W}_g$. We will define the recursion operator on the space $\mathcal{W}_g$.

The above lemma shows that we can consider a linearised perturbation as an element of $\mathcal{W}_g$ in two ways. These two will be related by the square of the recursion operator. The linearised vacuum metrics corresponding to $\delta\Omega$ and $\delta\Theta$ are

$$h^I{}_{AA'BB'} = \iota_{(A'}o_{B')}\nabla_{(A1'}\nabla_{B)0'}\delta\Omega, \qquad h^{II}{}_{AA'BB'} = o_{A'}o_{B'}\nabla_{A0'}\nabla_{B0'}\delta\Theta.$$

where $o^{A'} = (1,0)$ and $\iota^{A'} = (0,1)$ span the constant spin frame. Given $\phi \in \mathcal{W}_g$ we use the first of these equations to find h^I. If we put the perturbation obtained in this way on the LHS of the second equation and add an appropriate gauge term we obtain ϕ' - the new element of $\mathcal{W}_g$ that provides the $\delta\Theta$ which gives rise to

$$h^{II}_{ab} = h^{I}_{ab} + \nabla_{(a}V_{b)}. \tag{3.10}$$

To extract the recursion relations we must find V such that $h^I{}_{AA'BB'} - \nabla_{(AA'}V_{BB')} = o_{A'}o_{B'}\chi_{AB}$. Take $V_{BB'} = o_{B'}\nabla_{B1'}\delta\Omega$, which gives

$$\nabla_{(AA'}V_{BB')} = -\iota_{(A'}o_{B')}\nabla_{(A0'}\nabla_{B)1'}\delta\Omega + o_{A'}o_{B'}\nabla_{A1'}\nabla_{B1'}\delta\Omega.$$

This reduces (3.10) to

$$\nabla_{A1'}\nabla_{B1'}\phi = \nabla_{A0'}\nabla_{B0'}\phi'. \tag{3.11}$$

Definition 3.2 *Define a recursion operator $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ by*

$$\iota^{A'}\nabla_{AA'}\phi = o^{A'}\nabla_{AA'}R\phi, \tag{3.12}$$

so formally $R = (\nabla_{A0'})^{-1} \circ \nabla_{A1'}$ (no summation over the index A).

From (3.12) and from (2.11) it follows that if ϕ belongs to $\mathcal{W}_g$ then so does $R\phi$. We also have $R^2\delta\Omega = \delta\Theta$. Note that the operator $\phi \mapsto \nabla_{A0'}\phi$ is over-determined, and its consistency follows from the wave equation on ϕ. Furthermore, this definition is formal in that in order to invert the operator $\phi \mapsto \nabla_{A0'}\phi$ we need to specify boundary conditions. To summarise:

Proposition 3.3 *Let $\mathcal{W}_g$ be the space of solutions of the wave equation on the curved ASD background given by g.*

(i) Elements of $\mathcal{W}_g$ can be identified with linearised perturbations of the heavenly equations.

(ii) There exists a (formal) map $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ given by (3.12) which generates new elements of $\mathcal{W}_g$ from old.

3.3 Connections with the Nonlinear Graviton

This section links the construction of the recursion operator with twistor theory. First we use R to build a family of foliations by twistor surfaces starting from a given one. In Subsection 3.3.2 we give the method for constructing the hierarchy of curved twistor spaces. In Section 3.5 the algebra of hidden symmetries of the second heavenly equation is constructed.

3.3.1 The recursion operator and twistor functions

A function f on the correspondence space F descends to twistor space if $L_A f = 0$. Given $\phi \in \mathcal{W}_g$, define, for $i \in \mathbb{Z}$, a hierarchy of linear fields, $\phi_i \equiv R^i \phi_0$. Put $\Psi = \sum_{-\infty}^{\infty} \phi_i \lambda^i$ and observe that the recursion equations are equivalent to $L_A \Psi = 0$. Thus Ψ is a function on the twistor space $\mathcal{PT}$. Conversely every solution of $L_A \Psi = 0$ defined on a neighbourhood of $|\lambda| = 1$ can be expanded in a Laurent series in λ with the coefficients forming a series of elements of $\mathcal{W}_g$ related by the recursion operator. The function Ψ can be thought of as a Čech representative of the element of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ that corresponds to the solution of the wave equation ϕ.

It is clear that a series corresponding to $R\phi$ is the function $\lambda^{-1}\Psi$. Note that R is not completely well defined when acting on $\mathcal{W}_g$ because of the ambiguity in the inversion of $\nabla_{A0'}$. This means that if one treats $\Psi(\lambda)$ as a twistor function on $\mathcal{PT}$, pure gauge elements of the first sheaf cohomology group $H^1(\mathcal{PT}, \mathcal{O}(-2))$ of the twistor space corresponding to $\mathcal{M}$ are mapped to nontrivial terms. Note, however, that the action of R is well defined on twistor functions. By iterating R on functions and then taking the corresponding cohomology classes we generate an infinite sequence of elements of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ belonging to different classes.

Put $\omega_0^A = w^A = (w, z)$; the surfaces of constant ω_0^A are twistor surfaces. We have that $\nabla^A{}_{0'}\omega_0^B = 0$ so that in particular $\nabla_{A1'}\nabla^A{}_{0'}\omega_0^B = 0$ and if we define $\omega_i^A = R^i \omega_0^A$ then we can choose $\omega_i^A = 0$ for negative i. We define

$$\omega^A = \omega_0^A + \sum_{i=1}^{\infty} \omega_i^A \lambda^i. \tag{3.13}$$

We can similarly define $\tilde{\omega}^A$ by $\tilde{\omega}_0^A = \tilde{w}^A$ and choose $\tilde{\omega}_i^A = 0$ for $i > 0$. Note that ω^A and $\tilde{\omega}^A$ are solutions of L_A holomorphic around $\lambda = 0$ and $\lambda = \infty$ respectively and they can be chosen so that they extend to a neighbourhood of the unit disc and a neighbourhood of the complement of the unit disc.

3.3.2 Twistor construction of the recursion operator

The recursion operator acts on linearised perturbations of the ASDVE equations. Under the twistor correspondence, these correspond to linearised holomorphic deformations of (part of) $\mathcal{PT}$.

Cover $\mathcal{PT}$ by two sets, U and $\tilde{U}$ with $|\lambda| < 1 + \epsilon$ on U and $|\lambda| > 1 - \epsilon$ on $\tilde{U}$ with (ω^A, λ) coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. The twistor space $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = \tilde{\omega}^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$.

It is well known that infinitesimal deformations are given by elements of

$$H^1(\mathcal{PT}, \mathbf{\Theta}),$$

where $\mathbf{\Theta}$ denotes a sheaf of germs of holomorphic vector fields. Let

$$Y = f^A(\omega^B, \pi_{B'})\frac{\partial}{\partial \omega^A} \in H^1(\mathcal{PT}, \mathbf{\Theta})$$

be defined on the overlap $U \cap \tilde{U}$. Infinitesimal deformation is given by

$$\tilde{\omega}^A = (1 + tY)(\omega^A) + O(t^2). \tag{3.14}$$

From the globality of $\Sigma(\lambda) = \mathrm{d}\omega^A \wedge \mathrm{d}\omega_A$ it follows that Y is a Hamiltonian vector field with a Hamiltonian $f \in H^1(\mathcal{PT}, \mathcal{O}(2))$ with respect to the symplectic structure Σ. The finite version of (3.14) is given by integrating

$$\frac{\mathrm{d}\tilde{\omega}^B}{\mathrm{d}t} = \varepsilon^{BA}\frac{\partial f}{\partial \tilde{\omega}^A}.$$

from $t = 0$ to 1 with $\tilde{\omega}^A(0) = \omega^A$ to obtain $\tilde{\omega}^A = \tilde{\omega}^A(1)$. We are interested in the linearised version of the last formula

$$\delta\tilde{\omega}^A = \frac{\partial \delta f}{\partial \tilde{\omega}_A}. \tag{3.15}$$

This should be understood as follows: $\tilde{\omega}^A$ is the patching function obtained by exponentiating the Hamiltonian vector field of f and corresponds to the ASD metric determined by Θ and $\delta f^A = \varepsilon^{BA}\partial\delta f/\partial\omega^B$ (or more simply δf) is a linearised deformation corresponding to $\delta\Theta \in \mathcal{W}_g$.

The recursion operator acts on linearised deformations as follows

Proposition 3.4 *Let R be the recursion operator defined by* (3.12). *Its twistor counterpart is the multiplication operator*

$$R\,\delta f = \frac{\pi_{1'}}{\pi_{0'}}\delta f = \lambda^{-1}\delta f. \tag{3.16}$$

Note that R acts on δf without ambiguity (alternatively, the ambiguity in boundary condition for the definition of R on space-time is absorbed into the choice of explicit representative for the cohomology class determined by δf).

Proof. We work on the primed spin bundle. Restrict δf to the section of μ and represent it as a coboundary

$$\delta f(\pi_{A'}, x^a) = h(\pi_{A'}, x^a) - \tilde{h}(\pi_{A'}, x^a) \tag{3.17}$$

where h and $\tilde{h}$ are holomorphic on U and $\tilde{U}$ respectively (here we abuse notation and denote by U and $\tilde{U}$ the open sets on the spin bundle that are the preimage of U and $\tilde{U}$ on twistor space). Splitting (3.17) is given by

$$\begin{aligned} h &= \frac{1}{2\pi i}\oint_{\Gamma} \frac{(\pi^{A'}o_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^3}\delta f(\rho_{E'})\rho_{D'}\mathrm{d}\rho^{D'}, \\ \tilde{h} &= \frac{1}{2\pi i}\oint_{\tilde{\Gamma}} \frac{(\pi^{A'}\iota_{A'})^3}{(\rho^{C'}\pi_{C'})(\rho^{B'}\iota_{B'})^3}\delta f(\rho_{E'})\rho_{D'}\mathrm{d}\rho^{D'}. \end{aligned} \tag{3.18}$$

Here $\iota_{A'}$ is a constant spinor satisfying $o_{A'}\iota^{A'} = 1$ and $\rho_{A'}$ are homogeneous coordinates of $\mathbb{CP}^1$ pulled back to the spin bundle. The contours Γ and $\tilde{\Gamma}$ are homologous to the equator of $\mathbb{CP}^1$ in $U \cap \tilde{U}$ and are such that $\Gamma - \tilde{\Gamma}$ surrounds the point $\rho_{A'} = \pi_{A'}$.

The functions h and $\tilde{h}$ do not descend to $\mathcal{PT}$. They are global and homogeneous of degree 2 in $\pi_{A'}$ therefore

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde{h} = \pi^{A'}\pi^{B'}\pi^{C'}\Sigma_{AA'B'C'} \tag{3.19}$$

where $\Sigma_{AA'B'C'}$ is the third potential for a linearised ASD Weyl spinor. $\Sigma_{AA'B'C'}$ is defined modulo terms of the form $\nabla_{A(A'}\gamma_{B'C')}$ but a part of this gauge freedom is fixed by choosing the Plebański coordinate system (there is also a freedom in $\delta\Theta$ which we shall describe in the next subsection) in which $\Sigma_{AA'B'C'} = o_{A'}o_{B'}o_{C'}\nabla_{A0'}\delta\Theta$. The condition $\nabla_{A(D'}\Sigma^{A}{}_{A'B'C')} = 0$ follows from equation (3.19) which, with the Plebański gauge choice, implies $\delta\Theta \in \mathcal{W}_g$. Define δf_A by $\nabla_{AA'}\delta f = \rho_{A'}\delta f_A$. Equation (3.19) becomes

$$\oint_{\Gamma} \frac{\delta f_A(\rho_{E'})}{(\rho^{B'}o_{B'})^3}\rho_{D'}\mathrm{d}\rho^{D'} = 2\pi i \nabla_{A0'}\delta\Theta. \tag{3.20}$$

The twistor function δf is not constrained by the RHS of (3.20) being a gradient. To see this define δf_{AB} by $\nabla_{AA'}(\delta f_B \rho_{B'}) = \delta f_{AB}\rho_{A'}\rho_{B'}$ and note that in the ASD vacuum δf_{AB} is symmetric which implies $\nabla^{A}{}_{A'}\delta f_A = 0$. Therefore the RHS of (3.20) is also a solution of a neutrino equation so (in the ASD vacuum) it must be given by $\alpha^{A'}\nabla_{AA'}\phi$ where $\alpha^{A'}$ is a constant spinor and $\phi \in \mathcal{W}_g$. Equation (3.20) gives the formula for a linearisation of the second heavenly equation

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^4}\rho_{D'}\mathrm{d}\rho^{D'}. \tag{3.21}$$

Now recall formula (3.12) defining R. Let $R\delta f$ be the twistor function corresponding to $R\delta\Theta$ by (3.21). The recursion relations yield

$$\oint_\Gamma \frac{R\delta f_A}{(\rho^{B'}o_{B'})^3}\rho_{D'}\mathrm{d}\rho^{D'} = \oint_\Gamma \frac{\delta f_A}{(\rho^{B'}o_{B'})^2(\rho^{B'}\iota_{B'})}\rho_{D'}\mathrm{d}\rho^{D'}$$

so $R\delta f = \lambda^{-1}\delta f$.

$\square$

Let $\delta\Omega$ be the linearisation of the first heavenly potential. From $R^2\delta\Omega = \delta\Theta$ it follows that

$$\delta\Omega = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho_{A'}o^{A'})^2(\rho_{B'}\iota^{B'})^2}\rho_{C'}\mathrm{d}\rho^{C'}.$$

3.4 Z.R.M Fields on heavenly backgrounds

Now consider a general situation of linear fields on ASD vacuum backgrounds [18].

Let δf be a function on a curved twistor space homogeneous of degree n. Then contour integrals that give a splitting on the spin bundle can be chosen to be

$$h = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi^{A'}o_{A'})^{n+1}}{(\rho^{C'}\pi_{C'})(\rho^{B'}o_{B'})^{n+1}}\delta f(\rho_{E'})\rho\cdot\mathrm{d}\rho$$

and similarly for $\tilde{h}$. The equality $\pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\tilde{h}$ defines a global, homogeneity $n+1$ function

$$\pi^{A'}\nabla_{AA'}h = \pi^{A'_1}\pi^{A'_2}...\pi^{A'_{n+1}}\Sigma_{AA'_1A'_2...A'_{n+1}}.$$

With the chosen splitting formulae, $\Sigma_{AA'_1A'_2...A'_{n+1}} = o_{A'_1}o_{A'_2}...o_{A'_{n+1}}\nabla_{A0'}\delta\Theta$ which can be thought of as a potential for the spin $(n+2)/2$ field (the field itself is well defined only in flat space)

$$\psi_{A_1A_2...A_{n+2}} = \nabla_{A_10'}\nabla_{A_20'}...\nabla_{A_{n+2}0'}\delta\Theta$$

where

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{\delta f}{(\rho^{B'}o_{B'})^{n+2}}\rho\cdot \mathrm{d}\rho.$$

Differentiating under the integral one shows that $\psi_{A_1A_2\ldots A_{n+2}}$ satisfies

$$\nabla^{A_{n+2}A'_{n+2}}\psi_{A_1A_2\ldots A_{n+2}} = C_{BCA_1(A_2}\nabla^{BA'_1}\nabla^{CA'_2}\nabla^{A'_3}{}_{A_3}\ldots\nabla^{A'_n}{}_{A_n}\Sigma_{A_{n+1})A'_1A'_2\ldots A'_n}{}^{A'_{n+2}}. \tag{3.22}$$

The last formula generalises the one given in [57] for a left-handed Rarita–Schwinger field. The Weyl spinor C_{ABCD} is present because one needs to use expressions like $\nabla_{CC'}\delta f_{AB}$ (compare the proof of the Proposition 3.16). Note that the Buchdahl constraints do not appear. This can be seen by operating on (3.22) with $\nabla^{A_{n+1}}{}_{C'}$. The usual algebraic expression will cancel out with the RHS. (Note, however, that the definition of the field is not independent of the gauge choices as it would be in flat space.)

The notion of the recursion operator generalises to solutions of equations of type (3.22). We restrict ourselves to the case of ASD neutrino and Maxwell fields on an ASD background. For these two cases the RHS of equation (3.22) vanishes and fields are gauge invariant.

Define the recursion relations

$$\mathcal{R}\psi_A := \nabla_{A0'}R\delta\Theta. \tag{3.23}$$

for a neutrino field, and

$$\mathcal{R}\psi_{AB} := \nabla_{A0'}\nabla_{B0'}R\delta\Theta$$

for a Maxwell field. It is easy to see that $\mathcal{R}$ maps solutions into solutions, although again the definition is formal in that boundary conditions are required to eliminate the ambiguities. A conjugate recursion operator $\mathcal{R}$ will in Section **??** play a role in the Hamiltonian formulation.

3.5 Hidden symmetry algebra

The ASDVE equations in the Plebański forms have a residual coordinate symmetry. This consists of area preserving diffeomorphisms in the w^A coordinates together with some extra transformations that depend on whether one is reducing to the first or second form. By regarding the infinitesimal forms of these transformations as linearised perturbations and acting on them using the recursion operator, the coordinate (passive) symmetries can be extended to give hidden (active) symmetries of the heavenly equations. Formulae (3.21) and (3.16) can be used to recover the known relations (see for example [71]) of the hidden symmetry algebra of the heavenly equations. We deal with the second equation as the case of the first equation was investigated by other methods [54].

Let M be a volume preserving vector field on $\mathcal{M}$. Define $\delta^0_M \nabla_{AA'} := [M, \nabla_{AA'}]$. This is a pure gauge transformation corresponding to addition of $\mathcal{L}_M g$ to the space-time metric. Define also

$$[\delta^0_M, \delta^0_N]\nabla_{AA'} := \delta^0_{[M,N]}\nabla_{AA'}.$$

Once a Plebański coordinate system and reduced equations have been selected, the field equation will not be invariant under all the SDiff($\mathcal{M}$) transformations. We restrict ourselves to transformations which preserve the SD two-forms $\Sigma^{1'1'} = \mathrm{d}w_A \wedge \mathrm{d}w^A$ and $\Sigma^{0'1'} = \mathrm{d}x_A \wedge \mathrm{d}w^A$. The conditions $\mathcal{L}_M \Sigma^{0'0'} = \mathcal{L}_M \Sigma^{0'1'} = 0$ imply that M is given by

$$M = \frac{\partial h}{\partial w_A}\frac{\partial}{\partial w^A} + \Big(\frac{\partial g}{\partial w_A} - x^B \frac{\partial^2 h}{\partial w_A \partial w^B}\Big)\frac{\partial}{\partial x^A}$$

where $h = h(w^A)$ and $g = g(w^A)$. The space-time is now viewed as a cotangent bundle $\mathcal{M} = T^*\mathcal{N}^2$ with w^A being coordinates on a two-dimensional complex manifold $\mathcal{N}^2$. The full SDiff($\mathcal{M}$) symmetry breaks down to the semi-direct product of SDiff($\mathcal{N}^2$), which acts on $\mathcal{M}$ by a Lie lift, with $\Gamma(\mathcal{N}^2, \mathcal{O})$ which acts on $\mathcal{M}$ by translations of the zero section by the exterior derivatives of functions on $\mathcal{N}^2$. Let $\delta^0_M \Theta$ correspond to $\delta^0_M \nabla_{AA'}$ by

$$\delta^0_M \nabla_{A1'} = \frac{\partial^2 \delta^0_M \Theta}{\partial x^A \partial x^B}\frac{\partial}{\partial x_B}.$$

The 'pure gauge' elements of $\mathcal{W}_g$ are

$$\begin{aligned}\delta^0_M \Theta \;=\;& F + x_A G^A + x_A x_B \frac{\partial^2 g}{\partial w_A \partial w_B} + x_A x_B x_C \frac{\partial^3 h}{\partial w_A \partial w_B \partial w_C} \\ &+ \frac{\partial g}{\partial w_A}\frac{\partial \Theta}{\partial x^A} + \frac{\partial h}{\partial w_A}\frac{\partial \Theta}{\partial w^A} - x^B \frac{\partial^2 h}{\partial w_A \partial w^B}\frac{\partial \Theta}{\partial x^A}\end{aligned} \tag{3.24}$$

where F, G^A, g, h are functions of w^B only[1]. The above symmetry can be seen to arise from symmetries on twistor space as follows. We have the symplectic form $\Sigma = d\omega^A \wedge d\omega_A$ on the fibres of $\mu : \mathcal{PT} \longrightarrow \mathbb{C}P^1$. Consider a canonical transformation of each fibre of μ leaving Σ invariant on a neighbourhood of $\lambda = 0$. Let $H = H(x^a, \lambda) = \sum_{i=0}^{\infty} h_i \lambda^i$ be the Hamiltonian for this transformation pulled back to the projective spin bundle. Functions h_i depend on space time coordinates only. In particular h_0 and h_1 are identified with h and g from the previous construction (3.24). This can be seen by calculating how Θ transforms if $\omega^A = w^A + \lambda x^A + \lambda^2 \partial\Theta/\partial x_A + ... \longrightarrow \hat{\omega}^A$. Now Θ is treated as an object on the first jet bundle of a fixed fibre of $\mathcal{PT}$ and it determines the structure of the second jet.

Let ${\delta_M}^i\Theta := R^i \delta_M \Theta \in \mathcal{W}_g$ and let $\delta^i_M f$ be the corresponding twistor function (by (3.21)) treated as an element of $\Gamma(U \cap \tilde{U}, \mathcal{O}(2))$ rather than $H^1(\mathcal{PT}, \mathcal{O}(2))$. Define $[\delta^i_M, \delta^j_N]$ by

$$[\delta^i_M, \delta^j_N]\Theta := \frac{1}{2\pi i} \oint \frac{\{\delta^i_M f, \delta^j_N f\}}{(\pi_{0'})^4} \pi_{A'} d\pi^{A'}$$

where the Poisson bracket is calculated with respect to a canonical Poisson structure on $\mathcal{PT}$. From Proposition (3.16) it follows that

$$[\delta^i_M, \delta^j_N]\Theta = \frac{1}{2\pi i} \oint \lambda^{-i-j} \frac{\{\delta_M f, \delta_N f\}}{(\pi_{0'})^4} \pi_{A'} d\pi^{A'} = R^{i+j} \delta_{[M,N]}\Theta$$

[1]A similar result could be obtained for the first equation by demanding

$$\mathcal{L}_M(dw^A \wedge dw_A) = \mathcal{L}_M(d\tilde{w}^A \wedge d\tilde{w}_A) = 0.$$

However, we can present a different derivation based on gauge freedom for corresponding ASDYM equations. Consider ASDYM with gauge group $G =$SDiff(Σ^2) where Σ^2 is a symplectic manifold with the symplectic form $\Sigma^{0'0'}$.

$$D_{AA'} = \frac{\partial}{\partial\, y^{AA'}} + A_{AA'} = \frac{\partial}{\partial y^{AA'}} + \frac{\partial h_{AA'}}{\partial \tilde{w}^B} \frac{\partial}{\partial \tilde{w}_B}.$$

Here $y^{AA'}$ are space time coordinates and $\tilde{w}^B$ are coordinates on Σ^2. The infinitesimal gauge transformation of the Hamiltonian is given by

$$\delta h_{AA'} = \{f(y, \tilde{w}), h_{AA'}\} + \partial_{AA'} f + g_{AA'}(y).$$

The Poisson bracket is evaluated with respect to $\Sigma^{0'0'}$. Perform the reduction by $\partial/\partial y^{A1'}$ and use the gauge freedom to set

$$h_{A0'} = h_{A0'}(y^{B1'}), \quad h_{A1'} = \partial\Omega/\partial y^{A1'}$$

where Ω is a function of $(\tilde{w}^A, y^{A1'}) = (\tilde{w}^A, w^A) = x^{AA'}$. With this choice ASDYM are equivalent to the first heavenly equation. The ASD tetrad is $\nabla_{AA'} = D_{AA'}$. The residual gauge freedom yields

$$\delta\partial\Omega/\partial w^A = \{f, \partial\Omega/\partial w^A\} + F_A(w^B).$$

so finally we have

Proposition 3.5 *Generators of the hidden symmetry algebra of the second heavenly equation satisfy the relation*

$$[\delta_M{}^i, \delta_N{}^j] = \delta_{[M,N]}{}^{i+j}. \tag{3.25}$$

3.6 Recursion procedure for Killing spinors

Let $(\mathcal{M}, g)$ be an ASD vacuum space. We say that $L_{A'_1 \dots A'_n}$ is a Killing spinor of type $(0, n)$ if

$$\nabla^A{}_{(A'} L_{B'_1 \dots B'_n)} = 0. \tag{3.26}$$

Killing spinors of type $(0, n)$ give rise to Killing spinors of type $(1, n-1)$ by

$$\nabla^A{}_{A'} L_{B'_1 \dots B'_n} = \varepsilon_{A'(B'_1} K^A{}_{B'_2 \dots B'_n)}.$$

In the ASD vacuum $K^{BB'_2 \dots B'_n}$ is also a Killing spinor [22]

$$\nabla^{(A}{}_{(A'} K^{B)}{}_{B'_1 \dots B'_n)} = 0.$$

Put (for $i = 0, \dots, n$)

$$L_i := \iota^{B'_1} \dots \iota^{B'_i} o^{B'_{i+1}} \dots o^{B'_n} L_{B'_1 \dots B'_n},$$

and contract (3.26) with $\iota^{B'_1} \dots \iota^{B'_i} o^{B'_{i+1}} \dots o^{B'_{n+1}}$ to obtain

$$i \nabla_{A1'} L_{i-1} = -(n - i + 1) \nabla_{A0'} L_i, \qquad i = 0, \dots, n-1.$$

We make use of the recursion relations (3.12):

$$\frac{-i}{n+1-i} R(L_{i-1}) = L_i.$$

This leads to a general formula for Killing spinors (with $\nabla_{A0'} L_0 = 0$)

$$L_i = (-1)^i \binom{n}{i}^{-1} R^i(L_0), \qquad L_{B'_1 B'_2 \dots B'_n} = \sum_{i=0}^{n} o_{(B'_1} \dots o_{B'_i} \iota_{B'_{i+1}} \dots \iota_{B'_n)} L_i. \tag{3.27}$$

3.7 Example

Now we shall illustrate the Propositions 3.3 and 3.4 with the example of the Sparling–Tod solution [70]. The calculations involved in this subsection where performed on MAPLE. I would like to thank David Liebowitz for introducing me to the use of computers in mathematics. In this section we shall not use the spinor notation. The coordinate formulae for the pull back of twistor functions are:

$$\begin{aligned} \mu^0 &= w + \lambda y - \lambda^2\Theta_x + \lambda^3\Theta_z + \ldots\ , \\ \mu^1 &= z - \lambda x - \lambda^2\Theta_y - \lambda^3\Theta_w + \ldots\ . \end{aligned} \tag{3.28}$$

Consider

$$\Theta = \frac{\sigma}{wx + zy}, \tag{3.29}$$

where $\sigma = const.$ It satisfies both (1.1) and (3.6).

3.7.1 The flat case

First we shall treat (3.29), with $\sigma = 1$, as a solution ϕ_0 to the wave equation on the flat background (1.1). Recursion relations are

$$(R\phi_0)_x = \frac{y}{(wx + zy)^2}, \quad (R\phi_0)_y = \frac{-x}{(wx + zy)^2}.$$

They have a solution $\phi_1 := R\phi_0 = (-y/w)\phi_0$. More generally we find that

$$\phi_n := R^n\phi_0 = \Big(-\frac{y}{w}\Big)^n \frac{1}{wx + zy}. \tag{3.30}$$

The last formula can be also found using the twistor methods. The twistor function corresponding to ϕ_0 is $1/(\mu^0\mu^1)$, where $\mu_0 = w + \lambda y$ and $\mu_1 = z - \lambda x$. By Proposition 3.16 the twistor function corresponding to ϕ_n is $\lambda^{-n}/(\mu^0\mu^1)$. This can be seen by applying the formula (3.21) and computing the residue at the pole $\lambda = -w/y$. It is interesting to ask whether any ϕ_n (apart from ϕ_0) is a solution to the heavenly equation. Inserting $\Theta = \phi_n$ to (3.6) yields $n = 0$ or $n = 2$. We parenthetically mention that ϕ_2 yields (by formula (1.4)) a metric of type D which is conformal to the Eguchi-Hanson solution.

3.7.2 The curved case

Now let Θ given by (3.29) determine the curved metric

$$\mathrm{d}s^2 = 2\mathrm{d}w\mathrm{d}x + 2\mathrm{d}z\mathrm{d}y + 4\sigma(wx+zy)^{-3}(w\mathrm{d}z - z\mathrm{d}w)^2. \tag{3.31}$$

The recursion relations

$$\partial_y(R\phi) = (\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x)\phi, \quad -\partial_x(R\phi) = (\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x)\phi$$

are

$$\begin{aligned} -\partial_x(R\psi) &= (\partial_z + 2\sigma w(wx+zy)^{-3}(w\partial_x - z\partial_y))\psi, \\ \partial_y(R\psi) &= (\partial_w + 2\sigma z(wx+zy)^{-3}(w\partial_x - z\partial_y))\psi, \end{aligned}$$

where ψ satisfies

$$\Box_\Theta\psi = 2(\partial_x\partial_w + \partial_y\partial_z + 2\sigma(wx+zy)^{-3}(z^2{\partial_x}^2 + w^2{\partial_y}^2 - 2wz\partial_x\partial_y))\psi = 0. \tag{3.32}$$

One solution to the last equation is $\psi_1 = (wx+zy)^{-1}$. We apply the recursion relations to find the sequence of linearised solutions

$$\begin{aligned} \psi_2 &= \Big(-\frac{y}{w}\Big)\frac{1}{wx+zy}, \ \psi_3 = -\frac{2}{3}\frac{\sigma}{(wx+zy)^3} + \Big(-\frac{y}{w}\Big)^2\frac{1}{wx+zy}, ..., \\ \psi_n &= \sum_{k=0}^{n} A^k_{(n)}\Big(-\frac{y}{w}\Big)^k(wx+zy)^{k-n}. \end{aligned}$$

To find $A^k_{(n)}$ note that the recursion relations imply

$$\begin{aligned} R &\ \Big(\Big(-\frac{y}{w}\Big)^k(wx+zy)^j\Big) = \\ &= \Big(\Big(-\frac{y}{w}\Big) - \sigma\Big(-\frac{y}{w}\Big)^{-1}(wx+zy)^{-2}\frac{k}{j+2}\Big)\Big(-\frac{y}{w}\Big)^k(wx+zy)^j\Big). \end{aligned}$$

This yields a recursive formula

$$A^k_{(n+1)} = A^{k-1}_{(n)} - 2\sigma\frac{k+1}{n-k+1}A^{k+1}_{(n)}, \qquad A^0_{(1)} = 1, \ \ A^1_{(1)} = 0, \ \ A^{-1}_{(n)} = 0, \ \ k = 0...n, \tag{3.33}$$

which determines the algebraic (as opposed to the differential) recursion relations between ψ_n and ψ_{n+1}. It can be checked that functions ψ_n indeed satisfy (3.32).

Notice that if $\sigma = 0$ (flat background) then we recover (3.30). We can also find the inhomogeneous twistor coordinates pulled back to $\mathcal{F}$

$$\begin{aligned}
\mu^0 &= w + \lambda y + \sum_{n=0}^{\infty} \sigma \lambda^{n+2} \sum_{k=0}^{n} B^k_{(n)} w \Big(-\frac{y}{w} \Big)^k (wx + zy)^{k-n-1}, \\
\mu^1 &= z - \lambda x + \sum_{n=0}^{\infty} \sigma \lambda^{n+2} \sum_{k=0}^{n} B^k_{(n)} z \Big(\frac{x}{z} \Big)^k (wx + zy)^{k-n-1}.
\end{aligned}$$

where

$$B^k_{(n+1)} = B^{k-1}_{(n)} - 2\sigma \frac{k+1}{n-k+2} B^{k+1}_{(n)}, \qquad B^0_{(1)} = 1, \;\; B^1_{(1)} = 0, \;\; B^{-1}_{(n)} = 0, \;\; k = 0...n.$$

The polynomials μ^A solve $L_A(\mu^B) = 0$, where now

$$\begin{aligned}
L_0 &= -\lambda \partial_w - 2\lambda \sigma z^2 (wz + zy)^{-3} \partial_x + (1 + 2\lambda \sigma wz (wz + zy)^{-3}) \partial_y, \\
L_1 &= \lambda \partial_z + (1 - 2\lambda \sigma wz (wz + zy)^{-3}) \partial_x + 2\lambda \sigma w^2 (wz + zy)^{-3}) \partial_y.
\end{aligned}$$

Chapter 4

The Hamiltonian description of twistor lines and generating functions on the spin bundle

In this chapter we shall give an alternative view on heavenly potentials as generating functions on the spin bundle. The second potential Θ will be used to construct the twistor space, and then all heavenly potentials will be reinterpreted as the generating functions on the spin bundle.

4.1 The Hamiltonian interpretation of the second heavenly potential

Newman et. al. [51] make the first heavenly equation (3.2) λ-dependent and show that ω^A may be found by integrating the Hamiltonian system which has Ω as its Hamiltonian. In their treatment λ plays the role of time. We give an analogous interpretation of the second equation.

Choose a spinor say $o_{A'} = (0, 1)$ in the base space of the fibration $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ and parametrise a section of μ by the coordinates

$$x^{AA'} := \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=o_{A'}}, \; x^{A1'} = \omega_0^A = (w, z), \; x^{A0'} = x^A = (y, -x).$$

Here $x^{A1'}$ gives the initial point on the curve, while $x^{A0'}$ is a tangent vector to the curve. To proceed further, ie to find higher terms in (3.13) we do one of the following:

(a) Insert the second heavenly tetrad into the recursion relations (3.12) and solve for ω_3^A

$$\omega^A = x^{A1'} + \lambda x^{A0'} + \lambda^2 \varepsilon^{BA} \frac{\partial \Theta}{\partial x^{B0'}} + \lambda^3 \varepsilon^{BA} \frac{\partial \Theta}{\partial x^{B1'}} + \dots \ . \tag{4.1}$$

Note that (3.12) is used to find the fourth term in the series, since the third one is the definition of Θ. This is because $\Sigma^{0'1'} \wedge \Sigma^{0'0'} = 0$ implies $\partial_{A0'} \omega_2^A = 0$ which gives integrability conditions for the existence of Θ.

(b) Make the second equation $\pi_{A'}$ (i.e. λ)-dependent. Define $X^{AA'} = \partial \omega^A / \partial \pi_{A'}$. Continue the curve to another order in λ (Figure 1), so that to order λ^2

$$X^{A1'} = x^{A1'} + \lambda x^{A0'}, \qquad X^{A0'} = x^{A0'} + \lambda \varepsilon^{BA} \frac{\partial \Theta}{\partial x^{B0'}}.$$

We then put the space-time metric into a standard, second heavenly form with respect to the coordinates $X^{AA'}$

$$\mathrm{d}s^2 = 2\varepsilon_{AB} \mathrm{d}_h X^{A1'} \mathrm{d}_h X^{B0'} + 2 \frac{\partial^2 \Theta'}{\partial X^{A0'} \partial X^{B0'}} \mathrm{d}_h X^{A1'} \mathrm{d}_h X^{B1'}$$

which forces us to introduce Θ', differing from Θ by terms of order λ

$$\Theta'(X^{AA'}, \pi_{A'}) = \Theta(x^{AA'}) + \lambda \tau(x^{AA'}).$$

We find Θ' and can then iterate the process[1] to obtain the subsequent orders in $X^{AA'}$. The parameter λ plays the role of time and Θ' plays the role of a time dependent Hamiltonian. In homogeneous coordinates, Θ is homogeneous of degree -4 in $\pi_{A'}$.

Proposition 4.1 *The construction of a compact holomorphic curve in $\mathcal{PT}$ is equivalent to the integration of the Hamiltonian system*

$$\dot{X}^{A1'} = X^{A0'}, \quad \dot{X}^{A0'} = \varepsilon^{BA} \frac{\partial \Theta'}{\partial X^{B0'}}, \quad \dot{\Theta}' = \tau, \quad \frac{\partial \tau}{\partial X_{A0'}} = \frac{\partial \Theta'}{\partial X_{A1'}} \tag{4.2}$$

with a 'time' dependent Hamiltonian $\Theta' \in \Gamma(U \cap \tilde{U}, \mathcal{O}(-4))$.

The dot means differentiation with respect to λ. The last equation (which gives the recursion relations) is valid up to the addition of $f(X^{A1'})$.

[1] The difference between approaches (a) and (b) is clear. If we understand the problem of constructing a holomorphic curve in $\mathcal{PT}$ as the formal exponentiation of the operator λR then (4.1) corresponds to the Picard method, while the process described in (b) resembles the method of Euler lines.

Figure 4.1: Construction of a holomorphic curve.

Define $\eth$ - a differential operator on the sphere - by

$$\pi^{A'_1}...\pi^{A'_k}\eth^k f = (-1)^k \frac{\partial^k}{\partial \pi_{A_{1'}}...\partial \pi_{A_{k'}}} f.$$

The first two equations can be written as

$$\eth^2 \omega^A = \pi^{A'} \nabla^A{}_{A'} \Theta, \quad \text{or} \quad \eth X^{AA'} = \{X^{AA'}, \Theta\}_\Pi, \tag{4.3}$$

where $\Pi = \pi^{A'}\pi^{B'}\nabla_{AA'} \wedge \nabla^A{}_{B'}$ is a (homogeneous) Poisson structure defined on the spin bundle tangent to the α-planes (note that it projects down to zero by the twistor fibration). The third one implies that Θ' satisfies the anti-twistor equation.

$$\frac{\partial}{\partial X_{AA'}} \frac{\partial}{\partial \pi^{A'}} \Theta' = 0. \tag{4.4}$$

4.2 Heavenly potentials as generating functions

This section provides another geometric interpretation of the various heavenly potentials, viz. as generating functions for canonical transformations of the spin bundle.

This approach was (implicitly) suggested in [56] and used in connection with the first equation in [11]. Here we apply it also to the second equation and to other forms of the first equation. First introduce another parametrisation of a curve in $\tilde{U}$ given by

$$\tilde{x}^{AA'} = -\frac{\partial \tilde{\omega}^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=\iota_{A'}}, \ \tilde{x}^{A0'} = \tilde{x}^A, \ \tilde{x}^{A1'} = \tilde{w}^A.$$

Lift $\Sigma = \mathrm{d}\omega^A \wedge \mathrm{d}\omega_A$ and $\widetilde{\Sigma} = \mathrm{d}\tilde{\omega}^A \wedge \mathrm{d}\tilde{\omega}_A$ to the spin bundle and expand them around $\lambda = 0$ and $\lambda = \infty$ respectively. Since the relations

$$\Sigma^{0'1'} = \mathrm{d}x_A \wedge \mathrm{d}w^A = \mathrm{d}(x_A \mathrm{d}w^A) = \mathrm{d}\theta, \ \widetilde{\Sigma}^{0'1'} = \mathrm{d}\tilde{w}_A \wedge \mathrm{d}\tilde{x}^A = -\mathrm{d}(\tilde{x}^A \mathrm{d}\tilde{w}_A) = \mathrm{d}\tilde{\theta}$$

define the same symplectic form we conclude that (w^A, x_A) and $(\tilde{w}^A, \tilde{x}_A)$ are related by a canonical transformation. Let S defined by $\mathrm{d}S = \theta - \tilde{\theta}$ be a generating function corresponding to this transformation. We define **finite heavenly equations** as those which are satisfied by S as a consequence of algebraic identity $\Sigma^{0'1'} \wedge \widetilde{\Sigma}^{0'1'} = -2\nu$. We list three possibilities

- Let $S = \Omega(w^A, \tilde{w}_A)$ so that $x_A = \partial\Omega/\partial w^A$ and $\tilde{x}^A = \partial\Omega/\partial\tilde{w}_A$. The relation $\Sigma^{0'1'} \wedge \widetilde{\Sigma}^{0'1'} = -2\nu$ written in $(w^A, \tilde{w}_A)$ coordinates yields the first heavenly equation for Ω. The flat solution then corresponds to the identity transformation $\Omega = w^A \tilde{w}_A$.

- By a Legendre transformation we can define another generating function

  $$M(x_A, \tilde{w}_A) = \Omega - w^A x_A$$

  which satisfies

  $$\frac{\partial M_A}{\partial \tilde{w}^B}\frac{\partial M^A}{\partial \tilde{w}_B} + \frac{\partial M_A}{\partial x^B}\frac{\partial M^A}{\partial x_B} = 0, \ \ \text{where } M_A = \frac{\partial M}{\partial x^A}. \tag{4.5}$$

  This can be considered as a new form of (3.2). Note however, that now

  $$\nu(e^{00'}, e^{01'}, e^{10'}, e^{11'}) \neq 1$$

  since the corresponding tetrad

  $$e^{A1'} = \mathrm{d}x^A + \varepsilon^{AB} e_B{}^{0'}, \quad e^{A0'} = (M_{CD}M^{CD})^{-1}\frac{\partial M^A}{\partial \tilde{w}_B}\mathrm{d}\tilde{w}_B$$

  becomes degenerate for an identity transformation.

- One can also perform 'half' of the Legendre transformation which led from Ω to M. This choice will produce the evolution form of first the heavenly equation [26], which originally was derived from the Ashtekar-Jacobson-Smolin formulation of the anti-self-duality condition. Indeed, defining $h(x, z, \tilde{w}^A)$ by $\mathrm{d}h = \mathrm{d}(\Omega - w^0 x_0)$ we obtain by the usual method

  $$h_{xx} = h_{x\tilde{w}} h_{z\tilde{z}} - h_{x\tilde{z}} h_{z\tilde{w}}. \tag{4.6}$$

  The tetrad is (now $x^{A1'} = (x, -z)$)

  $$e^{A0'} = \frac{1}{h_{xx}} \frac{\partial^2 h}{\partial x_{A1'} \partial \tilde{w}_B} \mathrm{d}\tilde{w}_B, \; e^{11'} = \mathrm{d}x^{11'}, \; e^{01'} = \mathrm{d}x^{01'} + \frac{1}{h_{xx}} \frac{\partial^2 h}{\partial x_{01'} \partial \tilde{w}_B} \mathrm{d}\tilde{w}_B.$$

  The function h, similarly to M, is degenerate for identity transformations. However the evolution form of (4.6) enabled Grant to write down a formal solution. The symmetry structure of (4.6) was investigated by Strachan in [69]. His results can be recovered by inserting Grant's tetrad into the recursion relations (3.12) and finding higher infinitesimal symmetries.

The method of generating functions can be also applied to the second heavenly equation, which according to our terminology is a representative of **infinitesimal heavenly equations**. It can be obtained as an infinitesimal form of the first equation. Consider an infinitesimal generating function

$$S = w^A \tilde{w}_A + \epsilon \Theta(w^B, \tilde{w}_B). \tag{4.7}$$

In the flat case $\tilde{w}_A = x_A$. Replace $w^A = const$ by $w^A + \epsilon x^A$. This, when inserted to the first heavenly equation, gives the second heavenly equation for Θ.

Chapter 5

Zero–Rest–Mass fields from $\mathcal{O}(n) \oplus \mathcal{O}(n)$ twistor spaces

5.1 Preliminaries

So far we have been looking at $\mathcal{O}(n) \oplus \mathcal{O}(n)$ twistor curves from the point of view of integrable systems. Now we shall elaborate on an associated paraconformal geometry. We shall see that the zero-rest-mass field equations on the moduli space of $\mathcal{O}(n) \oplus \mathcal{O}(n)$ curves can be solved by means of twistor functions.

We start by describing the flat case. The moduli space of twistor lines is $\mathbb{C}^{2n+2}$ (A.1). Its tangent space has an inner product

$$\eta^a{}_b = \varepsilon^A{}_B \varepsilon^{(A'_1}{}_{(B'_1} \dots \varepsilon^{A'_n)}{}_{B'_n)},$$

where, according to the abstract index notation, $v^a = v^{A(A'_1 \dots A'_n)}$. The incidence relation between a point and a twistor is

$$\omega^A = x^{AA'_1 \dots A'_n} \pi_{A'_1} \dots \pi_{A'_n}. \tag{5.1}$$

Each pair of spinors $Z = (\omega^A, \pi_{A'})$ determines the $2n$ dimensional null surface spanned by vectors of the form $\lambda^{A(A'_1 \dots A'_{n-1}} \pi^{A'_n)}$, with $\lambda^{AA'_1 \dots A'_{n-1}}$ varying. We shall call these kind of surfaces α planes. Points of the α plane given by Z are solutions to (5.1). The other way of interpreting (5.1) is fixing $x^{AA'_1 \dots A'_n}$ and solving for $(\omega^A, \pi_{A'})$. The solution, when factored out by the relation $(\omega^A, \pi_{A'}) \sim (k^n \omega^A, k\pi_{A'})$, is a rational curve with normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$. Normal vectors to the curve are

$$V = V^{AA'_1 \dots A'_n} \pi_{A'_1} \dots \pi_{A'_n} \frac{\partial}{\partial \omega^A}.$$

The $2n+3$ dimensional projective primed spin bundle $\mathcal{F} = \mathbb{C}^{2n+2} \times \mathbb{CP}^1$ is equipped with $2n$–dimensional distribution $L_{AA'_2...A'_n} = \pi^{A'_1}\partial_{AA'_1...A'_n}$. Lifts of objects constant on α–planes are Lie derived along $L_{AA'_2...A'_n}$. The other type of null surfaces in $\mathbb{C}^{2n+2}$ are two-dimensional β planes spanned by $\lambda^A\rho^{A'_1...A'_n}$ with λ^A fixed. The space of all β-planes is the dual twistor space $\mathcal{PT}^*$. It is an open subset in $\mathbb{CP}^{n+2}$. The incidence relation between a point and a β plane (dual twistor) is

$$\rho^{A'_1...A'_n} = -x^{AA'_1...A'_n}\lambda_A. \tag{5.2}$$

The dual twistor space arises as a factor space of a projective unprimed spin bundle by the $n+1$ dimensional dual twistor distribution

$$M_{A'_1...A'_n} = \lambda^A\partial_{AA'_1...A'_n}.$$

This leads to a dual double fibration picture[1]

$$\mathbb{C}^{2n+2} \longleftarrow \mathbb{P}S^{A\,2n+3} \longrightarrow \mathcal{PT}^{*n+2}.$$

Let $Z = (\omega^A, \pi_{A'})$ be a twistor and $W = (\lambda_A, \rho^{A'_1...A'_n})$ be a dual twistor. Define the scalar product

$$(Z, W) := \omega^A\lambda_A + \pi_{A'_1}...\pi_{A'_n}\rho^{A'_1...A'_n}.$$

We say that Z is incident to W iff $(Z, W) = 0$. The incidence relation holds at points of $\mathbb{C}^{2n+2}$ which are incident to both W and Z.

Introduce the reality conditions on $\sigma : \mathcal{N} \longrightarrow \mathcal{N}$ by $\sigma(\lambda^A) = \sigma(\lambda^0, \lambda^1) = (\overline{\lambda^1}, -\overline{\lambda^0}), \sigma(\pi^{A'}) = \sigma(\pi^{0'}, \pi^{1'}) = (-\overline{\pi^{1'}}, \overline{\pi^{0'}})$ (which are the usual definitions of the Euclidean structure on S^A and $S^{A'}$). The Atiyah–Hitchin–Singer picture (C.6) could be reproduced for β planes. The projective unprimed spin bundle is now viewed as a $2n+4$ dimensional real manifold. The $n+2$ dimensional distribution

$$M_{A'_1...A'_n},\ \partial_{\overline{\lambda}}$$

[1] Our definition of β planes resembles the approach to α planes given in [81, 49], which we shall briefly describe: Let $A = 0...n, A' = 0, 1$. The α planes are defined as

$$x^{AA'}\pi_{A'} = const$$

so that α plane is $n+1$ dimensional. The twistor space is a subset in $\mathbb{CP}^{n+2}$. Coordinates transform as

$$x^{AA'} \longrightarrow \Lambda x^{AA'}\widetilde{\Lambda}$$

where $\Lambda \in GL(n+1, \mathbb{C})$ and $\widetilde{\Lambda} \in GL(2, \mathbb{C})$. The twistor distribution is $n+1$ dimensional.

is an integrable almost complex structure (C.5), which takes $(\mathbb{P}S^A)^{2n+4}$ to an $n+2$ dimensional complex manifold. It can be identified with the dual twistor space, on which

$$\sigma(\lambda_A, \rho^{A'_1 \ldots A'_n}) = (-\overline{\lambda_A}, -\overline{\rho_{A'_1 \ldots A'_n}}).$$

For odd n σ does not have fixed points on $\mathcal{PT}^*$, so one can define lines joining W to $\sigma(W)$. This gives rise to a non-holomorphic fibration of $\mathcal{PT}^*$ over $\mathbb{C}^{2n+2}$.

5.2 ZRM fields

In this section we shall study the ZRM fields on $\mathbb{C}^{2n+2}$ and the associated integral formulae. Fields with primed indices appear in the usual way (this is a concrete realisation of the Serre duality (A.2) and $\pi \cdot \mathrm{d}\pi$ is the section of a bundle which does not depend on n). The construction for the negative helicity fields is more elaborate. In particular the 'fields' associated with twistor functions of positive homogeneity have both primed and unprimed symmetric indices. Their potentials have two sets of separately symmetric indices.

From now on the upper numerical index denotes the homogeneity in $\pi_{A'}$. Let $\mathcal{Z}_r(k, l')$ denote the subspace space of sections of $S^{(A_1 \ldots A_k)(A'_1 \ldots A'_l)}$ homogeneous of degree r in $\pi_{A'}$ which satisfy

$$\nabla^{A_1 B'_1 \ldots B'_n} \Psi_{A_1 \ldots A_k A'_1 \ldots A'_l} = 0, \quad \nabla^{B A'_1 B'_2 \ldots B'_n} \Psi_{A_1 \ldots A_k A'_1 \ldots A'_l} = 0.$$

Proposition 5.1 *There is a one to one isomorphism*

$$H^1(\mathcal{PT}, \mathcal{O}(kn-2-l)) \simeq \mathcal{Z}_0(k, l').$$

Proof. Let f^r be a twistor function homogeneous in $\pi_{A'}$ of some non-negative degree $r = pn + q$ where $p \geqslant 0$ and $0 \leqslant q < n$. Define an element of $H^1(\mathcal{PT}, \mathcal{O}(-1))$ by

$$f_{A_1 \ldots A_{p+1} A'_1 \ldots A'_{n-q-1}} = \pi_{A'_1} \ldots \pi_{A'_{n-q-1}} \frac{\partial^{p+1} f^r}{\partial \omega^{A_1} \ldots \partial \omega^{A_{p+1}}}.$$

This can be split uniquely using the Sparling formula (B.1)

$$f_{A_1 \ldots A_{p+1} A'_1 \ldots A'_{n-q-1}} = \mathcal{F}_{A_1 \ldots A_{p+1} A'_1 \ldots A'_{n-q-1}} - \widetilde{\mathcal{F}}_{A_1 \ldots A_{p+1} A'_1 \ldots A'_{n-q-1}}.$$

Define the field on $\mathbb{C}^{2n+2}$ by

$$L_{B B'_2 \ldots B'_n} \mathcal{F}_{A_1 \ldots A_{p+1} A'_1 \ldots A'_{n-q-1}} = \Psi_{A_1 \ldots A_{p+1} B B'_2 \ldots B'_n A'_1 \ldots A'_{n-q-1}}. \tag{5.3}$$

It has $p+2$ unprimed and $2n-2-q$ primed indices. Using (B.1) and making the replacement

$$\frac{\partial}{\partial x^{AA'_1\ldots A'_n}} \longrightarrow \rho_{A'_1}\ldots\rho_{A'_n}\frac{\partial}{\partial\omega^A} \tag{5.4}$$

under the integral sign yields the integral formula

$$\Psi_{A_1\ldots A_{p+1}BB'_2\ldots B'_nA'_1\ldots A'_{n-q-1}} = \frac{1}{2\pi i}\oint_\Gamma \rho_{A'_1}\ldots\rho_{A'_{n-q-1}}\rho_{B'_2}\ldots\rho_{B'_n}\frac{\partial^{p+2}f}{\partial\omega^{A_1}\ldots\partial\omega^{A_{p+1}}\partial\omega^B}\rho\cdot\mathrm{d}\rho. \tag{5.5}$$

It satisfies

$$\nabla^{A_1C'_1\ldots C'_n}\Psi_{A_1\ldots A_{p+1}BB'_2\ldots B'_nA'_1\ldots A'_{n-q-1}} = 0, \quad \nabla^{CB'_2C'_2\ldots C'_n}\Psi_{A_1\ldots A_{p+1}BB'_2\ldots B'_nA'_1\ldots A'_{n-q-1}} = 0. \tag{5.6}$$

Solutions to equation (5.6) are therefore given by $H^1(\mathcal{PT},\mathcal{O}(pn+q))$. To obtain the statement of the proposition set $2\leqslant k=p+2,\quad n-1\leqslant l=2n-2-q$. We treat the case of f^{-1} (or $p=-1, q=n-1$, or $k=1, l=n-1$) separately. The standard arguments give

$$f^{-1}=\mathcal{F}^{-1}-\tilde{\mathcal{F}}^{-1},\quad \Psi_{AA'_2\ldots A'_n}=L_{AA'_2\ldots A'_n}\mathcal{F}^{-1}=\frac{1}{2\pi i}\oint_\Gamma\rho_{A'_2}\ldots\rho_{A'_n}\frac{\partial f^{-1}}{\partial\omega^A}\rho\cdot\mathrm{d}\rho.$$

Conversely given a solution to (5.6) we can find two solutions $\mathcal{F}_{\mathcal{A}_{p+1}\mathcal{A}'_{n-q-1}}$ and $\widetilde{\mathcal{F}}_{\mathcal{A}_{p+1}\mathcal{A}'_{n-q-1}}$ to (5.3) which are holomorphic on $U_{\mathcal{F}}$ and $\widetilde{U}_{\mathcal{F}}$ respectively. Their difference descends to $\mathcal{PT}$. As a consequence of (5.6) it vanishes when contracted (on the left) with $\pi^{A'}$ or $\partial/\partial\omega^A$. This gives rise to an element of $H^1(\mathcal{PT},\mathcal{O}(r))$.

□

The fields obtained in the above proposition are symmetric in all the primed and unprimed indices. Now we shall see that (if $n>1$) the potentials are not symmetric in the primed indices.

5.3 Contracted potentials

We shall start from constructing the kth potential for the field (5.5). Let $f\in H^1(\mathcal{PT},\mathcal{O}(r))$ and let $r\geqslant kn-2$ for some integer k. We split the derivative

$$\frac{\partial^{k-1}f^r}{\partial\omega^{A_1}\ldots\partial\omega^{A_{k-1}}}=\tilde{\mathcal{F}}_{A_1\ldots A_{k-1}}-\mathcal{F}_{A_1\ldots A_{k-1}},$$

where, from (B.4),

$$\mathcal{F}_{A_1\ldots A_{k-1}} = \frac{1}{2\pi i}\oint_\Gamma \frac{(\pi\cdot o)^{r-(k-1)n+1}}{(\rho\cdot\pi)(\rho\cdot o)^{r-(k-1)n+1}}\frac{\partial^{k-1} f^r}{\partial\omega^{A_1}\ldots\partial\omega^{A_{k-1}}}\rho\cdot \mathrm{d}\rho.$$

This gives rise to a global polynomial of degree $r-(k-1)n+1$ in $\pi_{A'}$ on $\mathcal{F}$

$$L_{AA'_2\ldots A'_n}\mathcal{F}_{A_1\ldots A_{k-1}} = \Phi_{A_1\ldots A_k A'_2\ldots A'_n B'_1\ldots B'_{r-(k-1)n+1}}\pi^{B'_1}\ldots\pi^{B'_{r-(k-1)n+1}},$$

which produces the kth potential

$$\Phi_{A_1\ldots A_k A'_2\ldots A'_n B'_1\ldots B'_{r-(k-1)n+1}} =$$

$$o_{B'_1}\ldots o_{B'_{r-(k-1)n+1}}\frac{1}{2\pi i}\oint_\Gamma \frac{\rho_{A'_2}\ldots\rho_{A'_n}}{(\rho\cdot o)^{r-(k-1)n+1}}\frac{\partial^k f^r}{\partial\omega^{A_1}\ldots\partial\omega^{A_k}}\rho\cdot\mathrm{d}\rho. \tag{5.7}$$

The homogeneity of the integrand is

$$r-kn+2+n-1-r+kn-n-1=0.$$

Contracting $n-1$ primed in indices in the last formula yields

$$\widehat{\Phi}_{A_1\ldots A_k B'_1 B'_{r-kn+2}} = o_{B'_1}\ldots o_{B'_{r-kn+2}}\frac{1}{2\pi i}\oint_\Gamma \frac{1}{(\rho\cdot o)^{r-kn+2}}\frac{\partial^k f^r}{\partial\omega^{A_1}\ldots\partial\omega^{A_k}}\rho\cdot\mathrm{d}\rho.$$

This is a symmetric object in primed and unprimed indices. It is given by

$$H^1(\mathcal{O}(r),\mathcal{PT})$$

and it can be compared with the field (5.5) obtained in the previous section. Assume that $r=kn-2$. This yields an object with purely unprimed indices

$$\widehat{\Phi}_{A_1\ldots A_k} = \frac{1}{2\pi i}\oint_\Gamma \frac{\partial^k f^{kn-2}}{\partial\omega^{A_1}\ldots\partial\omega^{A_k}}\rho\cdot\mathrm{d}\rho \tag{5.8}$$

which satisfy

$$\nabla^{A_1 A'_1\ldots A'_n}\widehat{\Phi}_{A_1\ldots A_k}=0.$$

Thus, solutions to (5.8) are given by elements of $H^1(\mathcal{PT},\mathcal{O}(kn-2))$. If $n=1$ then (5.8) is the usual contour integral formula for right-handed fields.

One can also introduce the Hertz Potential, which (for a homogeneity r twistor function) has $r+2$ primed indices. Let $\Phi_{AA'_2\ldots A'_n B'_1\ldots B'_{r+1}}$ be the first potential given by the $k=1$ case of the formula (5.7). It can be written as

$$\Phi_{AA'_2\ldots A'_n B'_1\ldots B'_{r+1}} = o_{B'_1}\ldots o_{B'_{r+1}} o^{A'_1}\nabla_{AA'_1\ldots A'_n}\frac{1}{2\pi i}\oint_\Gamma \frac{f^r}{(\rho\cdot o)^{r+2}}\rho\cdot\mathrm{d}\rho, \tag{5.9}$$

where

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{f^r}{(\rho\cdot o)^{r+2}}\rho\cdot \mathrm{d}\rho \tag{5.10}$$

satisfies

$$\nabla^{AB'_1\ldots B'_n}\nabla_{A0'A'_2\ldots A'_n}\delta\Theta = 0. \tag{5.11}$$

The chain of potentials is formed by the relation (for $r > n$),

$$\Phi_{AB\mathcal{A}'_{n-1}\mathcal{B}'_{r-n+1}} = \nabla_B^{\mathcal{B}'_n}\Phi_{A\mathcal{A}'_{n-1}\mathcal{B}'_{r+1}}.$$

Here $\mathcal{A}'$ and $\mathcal{B}'$ are multi-indices (see Appendix B). The $(p+1)$th potential is the field (5.5), and the 0th potential is the Hertz potential

$$\Phi_{A'_1\ldots A'_{r+2}} = o_{A'_1}\ldots o_{A'_{r+2}}\delta\Theta.$$

The first potential (5.9) can be used to find the inverse twistor function corresponding to the field (5.5)

$$\begin{aligned} f^r(\pi_{C'},\ x^{CC'_1\ldots C'_n}\pi_{C'_1}\ldots\pi_{C'_n}) &= \oint_C \Phi_{AA'_2\ldots A'_nB'_1\ldots B'_{r+1}}\pi^{B'_1}\ldots\pi^{B'_r}\mathrm{d}x^{AA'_2\ldots A'_nB'_{r+1}} \\ &= \oint_C (\pi\cdot o)^r\nabla_{A0'A'_2\ldots A-n'}\delta\Theta\mathrm{d}x^{A1'A'_2\ldots A'_n} \end{aligned}$$

where C is a contour in $\mathbb{C}^{2n+2}$, coming from the Ward triangle.

5.3.1 $\mathcal{O}(2n)$ twistor functions

A special role is played by $f^{2n} \in H^1(\mathcal{PT},\mathcal{O}(2n))$ as it can be used for active deformations of the twistor space. Let $\tilde{\omega}^A = \omega^A$ be the standard patching relation for $\mathcal{PT}$ and let $f^A \in S^A\otimes H^1(\mathcal{PT},\mathcal{O}(n))$ give the infinitesimal deformation

$$\tilde{\omega}^A = \omega^A + tf^A + O(t^2).$$

The globality of a symplectic structure $\mathrm{d}\tilde{\omega}_A\wedge\mathrm{d}\tilde{\omega}^A = \mathrm{d}\omega_A\wedge\mathrm{d}\omega^A$ implies $f^A = \varepsilon^{AB}\frac{\partial f}{\partial\omega^B}$. We form an indexed element of $H^1(\mathcal{PT},\mathcal{O}(-1))$ (which can be split uniquely)

$$\pi_{A'_2}\ldots\pi_{A'_n}\frac{\partial^3 f^{2n}}{\partial\omega^A\partial\omega^B\partial\omega^C} = f_{ABCA'_2\ldots A'_n} = \tilde{\mathcal{F}}_{ABCA'_2\ldots A'_n} - \mathcal{F}_{ABCA'_2\ldots A'_n}.$$

From Sparlings formula (B.1)

$$\mathcal{F}_{ABCA'_2\ldots A'_n} = \frac{1}{2\pi i}\oint_\Gamma \frac{f_{ABCA'_2\ldots A'_n}}{\rho_{A'}\pi^{A'}}\rho\cdot\mathrm{d}\rho.$$

This gives rise to a global (symmetric) object

$$C_{ABCDA'_2...A'_nD'_2...D'_n} = L_{DD'_2...D'_n}\mathcal{F}_{ABCA'_2...A'_n}$$

which is given by the integral (which is the special case $p=2, q=0$ of (5.5))

$$C_{ABCDA'_2...A'_nD'_2...D'_n} = \frac{1}{2\pi i}\oint_\Gamma \rho_{A'_2}...\rho_{A'_n}\rho_{D'_2}...\rho_{D'_n}\frac{\partial^4 f^{2n}}{\partial\omega^A\partial\omega^B\partial\omega^C\partial\omega^D}\rho\cdot \mathrm{d}\rho.$$

Another way of seeing this is by potentials. Use the non-unique splitting $f^{2n} = \mathcal{F}^{2n} - \widetilde{\mathcal{F}}^{2n}$ and define a global object of degree $2n+1$ by

$$L_{AA'_2...A'_n}\mathcal{F}^{2n} = \Sigma_{AA'_2...A'_nB'_1...B'_nC'_1...C'_nD'_1}\pi^{B'_1}...\pi^{B'_n}\pi^{D'_1}\pi^{C'_1}...\pi^{C'_n}.$$

It is easy to see that

$$\nabla^{AE'_1...E'_n}\Sigma_{AA'_2...A'_nB'_1...B'_nC'_1...C'_nD'_1} = 0,$$

and $\Sigma_{AA'_2...A'_nB'_1...B'_nC'_1...C'_nD'_1}$ is one of the potentials (5.7), related to the field by

$$C_{ABCDA'_2...A'_nD'_2...D'_n} = \nabla^{D'_1}_{DD'_2...D'_n}\nabla^{C'_1...C'_n}_{C}\nabla^{B'_1...B'_n}_{B}\Sigma_{AA'_2...A'_nB'_1...B'_nC'_1...C'_nD'_1}.$$

The chain of potentials is

$$\begin{aligned}
\delta\Theta_{A'_1B'_1...B'_nC'_1...C'_nD'_1} &= o_{A'_1}o_{B'_1}...o_{B'_n}o_{C'_1}...o_{C'_n}o_{D'_1}\delta\Theta \\
\Sigma_{AA'_2...A'_nB'_1...B'_nC'_1...C'_nD'_1} &= o_{B'_1}...o_{B'_n}o_{C'_1}...o_{C'_n}o_{D'_1}\nabla_{A0'A'_2...A'_n}\delta\Theta \\
H_{ABA'_2...A'_nB'_1...B'_nD_{1'}} &= o_{B'_1}...o_{B'_n}o_{D'_1}\nabla_{B0'}\nabla_{A0'A'_2...A'_n}\delta\Theta \\
\Gamma_{ABCA'_2...A'_nD'_1} &= o_{D'_1}\nabla_{C0'}\nabla_{B0'}\nabla_{A0'A'_2...A'_n}\delta\Theta \\
C_{ABCDA'_2...A'_nD'_2...D'_n} &= \nabla_{C0'}\nabla_{B0'}\nabla_{A0'A'_2...A'_n}\nabla_{D0'D'_2...D'_n}\delta\Theta.
\end{aligned}$$

This can be compared with the corresponding chain for $n=1$ [39].

We exemplify the 'abstract multi-index notation' (Appendix B) by the chain of potentials associated with a twistor function homogeneous of degree $2n$.

In a 'weighted calculation' we first sum homogeneity indices, and then convert what is left to primed multi-index. The symbol $\rightsquigarrow$ means 'gives rise to a field'.

$$\begin{aligned}
f^{2n} &= \mathcal{F}^{2n} - \widetilde{\mathcal{F}}^{2n},\;\; L^1_{A\mathcal{A}'_{n-1}}\mathcal{F}^{2n} \rightsquigarrow \Sigma_{A\mathcal{A}'_{n-1}\mathcal{B}'_{2n+1}} \longrightarrow \widehat{\Sigma}_{A\mathcal{A}'_{n+1}} \\
\frac{\partial f^{2n}}{\partial\omega^B} &= \mathcal{F}^n_B - \widetilde{\mathcal{F}}^n_B,\;\; L^1_{A\mathcal{A}'_{n-1}}\mathcal{F}^n_B \rightsquigarrow H_{AB\mathcal{A}'_{n-1}\mathcal{B}'_{n+1}} \longrightarrow \widehat{H}_{ABA'B'} \\
\frac{\partial^2 f^{2n}}{\partial\omega^B\partial\omega^C} &= \mathcal{F}^0_{BC} - \widetilde{\mathcal{F}}^0_{BC},\;\; L^1_{A\mathcal{A}'_{n-1}}\mathcal{F}^0_{BC} \rightsquigarrow \Gamma_{ABC\mathcal{A}'_{n-1}B'} \longrightarrow \widehat{\Gamma}_{ABC\mathcal{A}'_{n-2}}
\end{aligned}$$

and

$$
\begin{aligned}
\pi_{A'_1}...\pi_{A'_{n-1}} \frac{\partial^3 f^{2n}}{\partial\omega^B \partial\omega^C \partial\omega^D} &= \mathcal{F}^{-1}_{BCD\mathcal{A}'_{n-1}} - \widetilde{\mathcal{F}}^{-1}_{BCD\mathcal{A}'_{n-1}}, \\
L^1_{A\mathcal{A}'_{n-1}} \mathcal{F}^{-1}_{BCD\mathcal{A}'_{n-1}} &\rightsquigarrow C_{ABCD\mathcal{A}'_{2n-2}}.
\end{aligned}
$$

The *hatted* objects denote contracted potentials. The relation between $\Gamma_{ABC\mathcal{A}'_n}$ and $C_{ABCD\mathcal{A}'_{2n-2}}$ is

$$
\nabla_D{}^{A'_1 A'_2 ... A'_n} \Gamma_{ABCA'_1}{}^{B'_2 ... B'_n} = C^{A'_2 ... A'_n B'_2 ... B'_n}_{ABCD}.
$$

5.3.2 The Sparling distribution

Consider a $2n+5$ dimensional space with coordinates $x^a, \pi_{A'}, \eta_{A'}$ constrained by $\pi_{A'}\eta^{A'} = 1,\ \eta \sim \eta + k\pi$. The $2n+2$ dimensional distribution spanned by

$$
\{\pi^{A'} \frac{\partial}{\partial \eta^{A'}},\ \ \pi^{A'} \nabla_{AA'_1 ... A'_n}, \Upsilon\}
$$

is integrable. In calculations we use

$$
\eta^{A'_1} ... \eta^{A'_n} \nabla_{AA'_1 ... A'_n} = \frac{\partial}{\partial \omega^A}.
$$

The pullback of the twistor co-cycle

$$
f(\pi_{A'}, x^{AA'_1 ... A'_n}) = \mathcal{F}(x^a, \pi_{A'}, \eta_{A'}) - \widetilde{\mathcal{F}}(x^a, \pi_{A'}, \eta_{A'})
$$

gives rise to potentials:

$$
\pi^{A'} \frac{\partial}{\partial \eta^{A'}} \mathcal{F} = \pi^{A'} \frac{\partial}{\partial \eta^{A'}} \widetilde{\mathcal{F}} = \phi(x^a, \pi_{A'}, \eta_{A'}),
$$

and

$$
L_{AA'_2 ... A'_n} \mathcal{F} = L_{AA'_2 ... A'_n} \widetilde{\mathcal{F}} = \phi_{AA'_2 ... A'_n}(x^a, \pi_{A'}, \eta_{A'}).
$$

The relation between them is

$$
L_{AA'_2 ... A'_n} \phi = \pi^{A'} \frac{\partial}{\partial \eta^{A'}} \phi_{AA'_2 ... A'_n}.
$$

5.4 Relations to space time geometry

In this section we shall study the geometry of the moduli space of sections from Proposition (**??**). For n odd $T\mathcal{N}$ is equipped with a metric with holonomy $\mathrm{SL}(2,\mathbb{C})$. For n even, $T\mathcal{N}$ is endowed with a skew form. They are both given by

$$G(U,W) = \varepsilon_{AB}\varepsilon_{A_1'B_1'}...\varepsilon_{A_n'B_n'}U^{AA_1'...A_n'}W^{BB_1'...B_n'}. \tag{5.12}$$

The bundle $S_p^{A'} = \Gamma(\mathcal{O}, L_p)$ is canonically trivial. The bundle S^A on space-time is the Ward transform of $\mathcal{O}(-n)\otimes T_V\mathcal{PT}$ where the subscript V denotes the subbundle of the tangent bundle consisting of vectors up the fibres of the projection to $\mathbb{CP}^1$, so that $S_p^A = \Gamma(\mathcal{O}(-n)\otimes T_V\mathcal{PT}, L_p)$. The reducible torsion-free connection is given by

$$\mathrm{d}e^{AA_1'...A_n'} = \Gamma^{AA_1'...A_n'}_{BB_1'...B_n'}\wedge e^{BB_1'...B_n'},$$

together with the decomposition

$$\Gamma^{AA_1'...A_n'}_{BB_1'...B_n'} = \Gamma^A_B\varepsilon^{(A_1'}_{(B_1'}...\varepsilon^{A_n')}_{B_n')} + \sum_{i=1}^{n}\varepsilon^A_B\varepsilon^{(A_1'}_{(B_1'}...\varepsilon^{A_{i-1}'}_{B_{i-1}'}\varepsilon^{A_{i+1}'}_{B_{i+1}'}...\varepsilon^{A_n'}_{B_n'}\Gamma^{A_i')}_{B_i')}.$$

The twistor space from Proposition **??** is fibered over $\mathbb{CP}^1$. This (and other assumptions of this Proposition) implies the existence of a covariantly constant primed spinors $\nabla_{AA_1'...A_n'}\pi_{B'} = 0$. We shall adopt a gauge in which $\Gamma^{A_i'}_{B_i'} = 0$. The $SL(2,\mathbb{C})$ connection on S^A is $\Gamma_{AB} = \Gamma_{ABCC_1'...C_n'}e^{CC_1'...C_n'}$.
We shall use the Θ formalism. From proposition (**??**)

$$g_{ABA_1'...A_n'B_1'...B_n'} = \varepsilon_{AB}\varepsilon_{A_1'B_1'}...\varepsilon_{A_n'B_n'} + o_{B_1'}...o_{B_n'}o_{A_1'}\nabla_{A0'A_2'...A_n'}\nabla_{B0'}\Theta. \tag{5.13}$$

To obtain the paraconformal structure g_{ab} we symmetrise over A_i's and B_i's. The duals to ∇_a are

$$e^{A1'A_2'...A_n'} = \mathrm{d}x^{A1'A_2'...A_n'}, \quad e^{A0'A_2'...A_n'} = \mathrm{d}x^{A0'A_2'...A_n'} + \frac{\partial^2\Theta}{\partial x_A{}^{0'}{}_{A_2'...A_n'}\partial x^{B0'}}\mathrm{d}x^{B1'1'...1'}.$$

The symmetric part is

$$e^{A(A_1'...A_n')} = \mathrm{d}x^{AA_1'...A_n'} + o^{(A_1'}\varepsilon^{A_2}_{C_2'}...\varepsilon^{A_n')}_{C_n'}\frac{\partial^2\Theta}{\partial x_A{}^{0'}{}_{C_2'...C_n'}\partial x^{B0'}}\mathrm{d}x^{B1'...1'},$$

or

$$e^{AA_1'...A_n'} = e^{AA_1'...A_n'}_{BB_1'...B_n'}\mathrm{d}x^{BB_1'...B_n'}$$

where

$$e^{AA'_1\dots A'_n}_{BB'_1\dots B'_n} := \Big(\varepsilon^A{}_B\varepsilon^{(A'_1}_{B'_1}\dots\varepsilon^{A'_n)}_{B'_n} + o^{A'_1}\varepsilon^{(A'_2}_{C'_2}\dots\varepsilon^{A'_n)}_{C'_n}o_{B'_1}\dots o_{B'_n}\frac{\partial^2\Theta}{\partial x_A{}^{0'}{}_{C'_2\dots C'_n}\partial x^{B0'}}\Big).$$

In the Newman-Penrose notation

$$e^{Ai} = \mathrm{d}x^{Ai} + \frac{n-i}{n}\frac{\partial^2\Theta}{\partial x_A{}^{n-i-1}\partial x^B}\mathrm{d}x^{Bn}, \quad e^{A(A'_1\dots A'_n)} = \sum_{i=0}^{n}\iota^{(A'_1}\dots\iota^{A'_i}o^{A'_{i+1}}\dots o^{A'_n)}e^{Ai}.$$

The Cartan equations yield

$$\Gamma_{ABCC'_1\dots C'_n} = o_{C'_1}\frac{\partial^3\Theta}{\partial x^A\partial x^B\partial x^{C0'C'_2\dots C'_n}},$$

and it follows that

$$C_{ABCDC'_2\dots C'_nD'_2\dots D'_n} = \frac{\partial^4\Theta}{\partial x^A\partial x^B\partial x^{C0'C'_2\dots C'_n}\partial x^{D0'D'_2\dots D'_n}}.$$

We rewrite the heavenly hierarchy (**??**) as

$$\partial_{A1'A'_2\dots A'_n}\Theta_{BB'_2\dots B'_n} - \partial_{B1'B'_2\dots B'_n}\Theta_{AA'_2\dots A'_n} + \{\Theta_{AA'_2\dots A'_n}, \Theta_{BB'_2\dots B'_n}\}_{xy} = 0,$$

where $\Theta_{AA'_2\dots A'_n} = \nabla_{A0'A'_1\dots A'_n}\Theta$. To find some nontrivial examples we shall look for a solution satisfying both the linear and the nonlinear parts of the hierarchy. It satisfies the linear part, so

$$\delta\Theta = \frac{1}{2\pi i}\oint_\Gamma \frac{f^{2n}}{(\rho\cdot o)^{2n+2}}\rho\cdot\mathrm{d}\rho.$$

To find an example generalising the one of Sparling and Tod we should find a solution

$$\delta\Theta_{AA'_2\dots A'_n} := o^{A'_1}\partial_{AA'_1\dots A'_n}\delta\Theta$$

to (5.11) with $r = 2n$

$$\delta\Theta_{AA'_2\dots A'_n} = \frac{1}{2\pi i}\oint_\Gamma (\rho\cdot o)\rho_{A'_2}\dots\rho_{A'_n}\frac{\partial f^{-2}}{\partial\omega^A}\rho\cdot\mathrm{d}\rho$$

where $f^{-2} = f^{2n}(\pi\cdot o)^{-2n-2}$. Take $f^{-2} = (\pi\cdot o)^{(2n-2)}/(\omega^0\omega^1)$ and write the integral as

$$\delta\Theta(x^a) = \frac{1}{2\pi i}\oint_\Gamma \frac{\lambda^{2n-2}}{(x^{0n}+\lambda x^{0n-1}+\dots+\lambda^n x^{00})(x^{1n}+\lambda x^{1n-1}+\dots+\lambda^n x^{10})}.$$

The poles are separated by demanding that for $x^{Ai} = const$ where $i > 1$, the function $\delta\Theta$ is the solution for the ordinary wave equation. We shall take the poles of ω^0 to lie inside the contour. For $n = 2$ this yields

$$\delta\Theta = \frac{x^{11}x^{02} - x^{12}x^{01}}{(x^{00}x^{11} - x^{10}x^{01})(x^{11}x^{02} - x^{12}x^{01}) + (x^{10}x^{02} - x^{12}x^{00})^2}$$

which satisfies (5.11). It was a disappointment to see that this solution (and analogous solutions for higher ns) fails to satisfy the nonlinear part of the hierarchy. It seems that one needs to look at the deformation theory. This will be done in the next subsection.

5.5 Deformation theory

Take $f^{2n} = \frac{(\pi_{0'})^{4n}}{\omega^0\omega^1}$. The deformation equations

$$\tilde{\omega}^A = \omega^A + t\varepsilon^{AB}\frac{\partial f^{2n}}{\partial \omega^B} + O(t^2)$$

are

$$\tilde{\omega}^0 = \omega^0 + t\frac{(\pi_0)^{4n}}{\omega^0(\omega^1)^2}, \quad \tilde{\omega}^1 = \omega^1 - t\frac{(\pi_0)^{4n}}{(\omega^0)^2\omega^1}.$$

They imply that $Q = \omega^0\omega^1 = \tilde{\omega}^0\tilde{\omega}^1$ is a global twistor function (up to the linear terms in t). Put $Q = \alpha^{A_1'}...\beta^{B_n'}\pi_{A_1'}...\pi_{B_n'}$. The deformation equations integrate to

$$\tilde{\omega}^0 = \exp(t(\pi_{0'})^{4n}Q^{-2})\omega^0, \quad \tilde{\omega}^1 = \exp(-t(\pi_{0'})^{4n}Q^{-2})\omega^1. \tag{5.14}$$

Restrict the exponents in (5.14) to the twistor line and perform the splitting (the method is given in Appendix B)

$$Q^{-2}(o\cdot\pi)^{4n} = \tilde{g} - g.$$

This give rise to global objects homogeneous of degree n on $\mathcal{F}$

$$\tilde{\omega}^0 e^{-\tilde{g}t} = \omega^0 e^{-gt} = \alpha_{A_1'...A_n'}\pi^{A_1'}...\pi^{A_n'}, \qquad \tilde{\omega}^1 e^{\tilde{g}t} = \omega^1 e^{gt} = \beta_{A_1'...A_n'}\pi^{A_1'}...\pi^{A_n'}.$$

We compare the following expressions to (**??**)

$$\begin{aligned} \omega^0 &= \alpha_{A_1'...A_n'}\pi^{A_1'}...\pi^{A_n'}e^{gt} \\ &= (\pi_{1'})^n(x^{0n} + \lambda x^{0n-1} + ... + \lambda^n x^{00} + \lambda^{n+1}\frac{\partial\Theta}{\partial x^{10}} + +\lambda^{n+2}\frac{\partial\Theta}{\partial x^{11}} + \end{aligned}$$

$$
\begin{aligned}
& \quad ... + \lambda^{2n}\frac{\partial\Theta}{\partial x^{1n}} + ...) \\
\omega^1 &= \beta_{A'_1...A'_n}\pi^{A'_1}...\pi^{A'_n}e^{-gt} \\
&= (\pi_{1'})^n(x^{1n} + \lambda x^{1n-1} + ... + \lambda^n x^{10} + \lambda^{n+1}\frac{\partial\Theta}{\partial x^{00}} + +\lambda^{n+2}\frac{\partial\Theta}{\partial x^{01}} + \\
& \quad ... + \lambda^{2n}\frac{\partial\Theta}{\partial x^{0n}} + ...) \ .
\end{aligned}
$$

Expanding the pull back ω^A to $\mathcal{F}$ of the above formulae in λ and identifying various terms we recover the potential and so the paraconformal structure (5.13).

5.6 The foliation picture

If one considers $\mathcal{N} = \mathcal{M} \times \mathbb{X}$ as being foliated by four dimensional slices $t^{Ai} = const$ then structures (1)-(3) on $\mathcal{PT}$ induce anti-self-dual vacuum metrics on the leaves of the foliation. Consider $\Theta(x^{AA'}, \mathbf{t})$ where $\mathbf{t} = \{t^{Ai}, i = 2...n\}$. For each fixed $\mathbf{t}$ the function Θ satisfies the second heavenly equation. The ASD metric on a corresponding four-dimensional slice $\mathcal{N}_{\mathbf{t}=\mathbf{t_0}}$ is given by

$$
\mathrm{d}s^2 = 2\varepsilon_{AB}\mathrm{d}x^{A1'}\mathrm{d}x^{B0'} + 2\frac{\partial^2\Theta}{\partial x^{A0'}\partial x^{B0'}}\mathrm{d}x^{A1'}\mathrm{d}x^{B1'}.
$$

One would like to determine this metric from the structure of the $\mathcal{O}(n) \oplus \mathcal{O}(n)$ twistor space.

If we fix $2n-2$ parameters in the expansion (**??**) then the normal vector $W = W^A\partial/\partial\omega^A$ is given by

$$
W^A = \delta\omega^A = \lambda^{n-1}W^{A1'} + \lambda^n W^{A0'} + \lambda^{n+1}\frac{\partial\delta\Theta}{\partial x^{0'}_A} + ...
$$

where $\delta\Theta = W^{AA'}\partial\Theta/\partial x^{AA'}$. The metric is

$$
g(U,W) = \frac{\alpha^{C'}\beta_{C'}}{(\pi_{0'})^{2n-2}\alpha^{A'}\pi_{A'}\beta^{B'}\pi_{B'}}\Sigma(U(\pi^{D'}), W(\pi^{D'})). \tag{5.15}
$$

Here $\alpha_{A'}$ and $\beta_{A'}$ are zeros of U and W. The last formula follows also from (5.12) if one puts

$$
W^{AA'_1...A'_n} = W^{A(A'_1}o^{A'_2}...o^{A'_n)}
$$

for W tangent to $t^{Ai} = const$. Note that it is sufficient to consider the slice $\mathbf{t} = 0$. This is because an appropriate (canonical) coordinate transformation of $\mathcal{PT}$, $\omega^A \to \hat{\omega}^A(\omega^B, \lambda)$ induces the transformation of parameters $\{\mathbf{t} = \mathbf{t_0}\} \to \{\hat{\mathbf{t}} = 0\}$.

Chapter 6

The Schlesinger equation and curved twistor spaces

In this chapter we analyse the general curved twistor spaces with a 'maximal symmetry condition'. Consider the system of ODEs

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} - \Lambda\Big)\Psi(\lambda) = 0, \qquad \Lambda = \sum_{a=1}^{n+3} \frac{A_a}{\lambda - t_a} \tag{6.1}$$

where $\mathbf{t} = t_1, ..., t_{n+3} \in \mathbb{C}$ are constants and A_i are constant $N \times N$ matrices in some complex Lie algebra $\mathfrak{g}$ (which we take to be $sl(N, \mathbb{C})$) and $\lambda \in \mathbb{CP}^1$.

Here Ψ is a fundamental matrix solution to (6.1). Assume that there is no extra pole at ∞, i.e. $\sum_{a=1}^{n+3} A_a = 0$, and that eigenvalues of A_a have no integer difference for each a.

We need some notation; $a, b, c, = 1...n$ are vector indices on $\mathbb{C}^n$, and $i, j = 1...dim\mathfrak{g} = k$ are indices on $\mathfrak{g}$. Let

$$\Sigma_{\mathbf{t}} := \mathbb{CP}^1/\{t_1, ..., t_{n+3}\}$$

be a punctured sphere with $n+3$ points removed. And let $\pi : \widetilde{\Sigma}_{\mathbf{t}} \longrightarrow \Sigma_{\mathbf{t}}$ be the universal covering. Let γ be a path in $\widetilde{\Sigma}_{\mathbf{t}}$ starting at λ and ending at λ_γ such that $\pi(\lambda) = \lambda_\gamma$. The function

$$\Psi(\lambda_\gamma) = \Psi(\lambda) g_\gamma$$

is a solution to (6.1). Here g_γ is a nonsingular constant matrix depending on the homotopy class $[\gamma]$ of γ. The mapping $[\gamma] \longrightarrow g_\gamma$ defines the monodromy representation of the fundamental group of $\Sigma_{\mathbf{t}}$

$$\pi_1(\Sigma_{\mathbf{t}}) \longrightarrow SL(N, \mathbb{C})$$

The monodromy group Γ is general the infinite discreet subgroup (with $n+3$ generators) of $SL(N,\mathbb{C})$.

The fundamental matrix solution $\Psi(\lambda)$ is a multi valued function with branch points at t_a. If λ moves around a singular point t_a then the fundamental solution undergoes a transformation by an element of the monodromy group.

$$\Psi(\lambda) \longrightarrow \Psi(t_a + (\lambda - t_a)e^{2\pi i}) = \Psi(\lambda) g_a$$

where $g_a \in \Gamma$. The transformation g_a is conjugated to $\exp(-2\pi A_a)$.

When the poles t_a move the monodromy representation of (6.1) remains fixed if matrices $A_a(\mathbf{t})$ satisfy the Schlesinger equation

$$\mathrm{d}A_a = \sum_{a\neq b} [A_b, A_a] \mathrm{d}\ln(t_a - t_b). \tag{6.2}$$

The usual geometric interpretation is one of the following

- Take a connection
$$\nabla = \mathrm{d} - \sum_{a=1}^{n+3} \frac{A_a \mathrm{d}\lambda}{\lambda - t_a}$$
on the vector bundle with fibres $\mathbb{C}^N$ over $\Sigma_{\mathbf{t}}$. Since A is holomorphic it is a flat connection (there are no holomorphic two forms in one dimension). Equations (6.2) imply the holonomy of ∇ is fixed

- Treat ∇ as a connection over $\mathbb{C}^{n+3} \times \Sigma_{\mathbf{t}}$ with logarithmic singularity. Equations (6.2) imply that ∇ is flat.

6.1 Twistor Construction

In [48, 49] Mason and Woodhouse established a connection between the Schlesinger equation and twistor theory. In their construction solutions to (6.2) were parametrised by $GL(n+3,\mathbb{C})$ invariant holomorphic vector bundles over rational curves in $\mathbb{CP}^{n+2}$. We shall demonstrate that the isomonodromy problem can in the Fuchsian case also be understood in terms of curved twistor spaces.

By a projective transformation fix the position of 3 poles to $0, 1$ and ∞. Equation (6.2) arises as the commutativity condition for $n+1$ operators

$$M_0 = \frac{\partial}{\partial \lambda} - \Lambda, \quad M_a = \frac{\partial}{\partial t_a} + \frac{A_a}{\lambda - t_a}. \tag{6.3}$$

Let $\mathcal{F}$ be the principal G bundle associated to $\mathbb{C}^N \otimes \mathbb{C}^n \otimes \mathbb{CP}^1$ by the representation of A_a by left invariant vector fields on G. The geometry involved in (6.2) is that of flat G connection on the principal bundle $\mathcal{F}$ over $\mathbb{C}^n \times \mathbb{CP}^1$. The vectors $M = (M_0, M_a) \in T\mathcal{F}$ span the $(n+1)$-dimensional integrable horizontal distribution. Factorise $\mathcal{F}$ by distribution M and call the resulting quotient three dimensional manifold $\mathcal{Z}$ - *the twistor space of the Schlesinger equation*

$$p : \mathcal{F} = G \times \mathbb{C}^n \times \mathbb{CP}^1 \longrightarrow \mathcal{Z}_t := \mathcal{F}/\{M\}$$

$\mathcal{Z}_t$ is a (possibly non-Hausdorf) manifold of dimension equal to the dimension of G. Let ν_G be the volume form on G. We define

$$\nu_{\mathcal{F}} := \nu_G \wedge \mathrm{d}t^1 \wedge ... \wedge \mathrm{d}t^n \wedge \mathrm{d}\lambda$$

where $\mathrm{d}\lambda \in \mathcal{O}(2) \otimes \Omega^1$ is a canonical section of $\mathbb{CP}^1$. The volume form on $\mathcal{Z}_t$ is given by

$$\nu_{\mathcal{Z}_t} := \nu_{\mathcal{F}}(M_0, M_1, ..., M_n).$$

Let $X_1, ..., X_k$ be the basis of the right (so they commute with A_k) invariant vector fields on $\mathcal{F}$, and let $X'_i := p_* X_i$ be the set of holomorphic vector fields on $\mathcal{Z}_t$. They are independent outside a divisor Q defined by

$$Q := \{(z_1, ..., z_k) \in \mathcal{Z}_t | \nu_{\mathcal{Z}_t}({X'}_1, ..., {X'}_k) = 0\} \in \Gamma(\mathcal{O}(n+3)).$$

Let

$$\alpha : \mathcal{O} \otimes \mathfrak{g} \longrightarrow T\mathcal{Z}_t \tag{6.4}$$

be a vector bundles homomorphism. In general $\Lambda^k(\alpha)$ vanishes at $n+3$ points on a line. Let

$$A := \alpha^{-1} : T\mathcal{Z}_t \longrightarrow \mathcal{O} \otimes \mathfrak{g} \tag{6.5}$$

be a meromorphic connection on $\mathcal{Z}_t$. The $\mathfrak{g}$ valued one form A is holomorphic on the complement of Q. The connection A is flat on the complement of Q and has logarithmic singularities on Q. The connection on the twistor lines is

$$A = \sum_{a=1}^{n} \frac{A_a \mathrm{d}\lambda}{\lambda - t_a}.$$

The holonomy representation of A restricted to each twistor line (which does not lie on the divisor) is the same. This can be seen as follows: Let $\pi_1(\mathcal{Z}_t/Q) \longrightarrow G$ be a fixed holonomy representation. Restrict it to a line l

$$\pi_1(\mathbb{CP}^1/\{z_1, ..., z_k\}) \longrightarrow \pi_1(\mathcal{Z}_t/Q) \longrightarrow G.$$

The holonomy is unchanged since the second homomorphism is fixed.

Let l be a twistor line (a copy of the $\mathbb{CP}^1$ under the projection p). Its normal bundle N_l is a rank $(k-1)$ vector bundle, so, by the Grothendieck theorem,

$$N_l = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus ... \oplus \mathcal{O}(m_{k-1}),$$

but dim $H^0(\mathbb{CP}^1, N_l) = n + k$ (dimension of the moduli space), so

$$m_1 + ...m_{k-1} = n + 1, \qquad m_i > -1.$$

Let $\mathcal{N}$ be a moduli space of rational curves in $\mathcal{Z}$. We consider vectors in $T(\mathcal{N})$ which correspond to sections of N vanishing outside Y at (how many?) points. If the m_is are all equal, this defines a paraconformal structure on $\mathcal{N}$. The point is that a paraconformal structure is a decomposition of the tangent space into a tensor product of two vector bundles. This requires $N = O^B(m)$ for $m = (n+1)/(k-1)$ (which should be integer), and $B = 1, ..., k-1$ so that the tangent space is S^B tensor $S^{B'_1 ... B'_n}$. One also requires a torsion free condition.

The manifold $\mathcal{N}$ admits an action of G (with k dimensional orbits) preserving the paraconformal structure. The paraconformal structure is given by a left invariant metric on each orbit. Take a set of unit vectors ∂_a, normal to the orbits. Then

$$g = \sum_{a,b=1}^{n} B_{ab}(\mathbf{t}) \mathrm{d}t^a \mathrm{d}t^b + \sum_{i,j=1}^{k} C_{ij}(\mathbf{t}) \sigma_i \sigma_j. \tag{6.6}$$

Here σ_i is an orthonormal basis of $\mathfrak{g}^*$

6.2 Examples

Let us give three examples in which the number of movable poles n and the dimension of G are not independent. The first example retains analogy with the Nonlinear Graviton construction, since the twistor space is three dimensional. Solutions to

(6.2) are parametrised by flows of the ASD hierarchy. In the second example we look at 'special solutions' to the Schlesinger equation. It follows from the work of Dubrovin that they correspond to semi-simple Frobenius manifolds. The third example we take the paraconformal structure on $\mathcal{N}$ to be a metric on the cotangent bundle to the group G.

Example 1($G = SL(2,\mathbb{C})$**).** We restrict ourselves to the Garnier system which corresponds to the $N = 2$ case of (6.2).

$$G = SL(2,\mathbb{C}), \quad N_l = \mathcal{O}(m) \oplus \mathcal{O}(m), \quad \mathcal{N} = \mathbb{C}^{2m-1} \times SL(2,\mathbb{C}).$$

A linear transformation $\{M\} \longrightarrow \{L_{Ai}\}$ where $i = 1...m$ gives ASD hierarchies.

Inverse construction: Let

$$\widetilde{L}_{AA'_2...A'_m} = \pi^{A'_1} D_{AA'_1A'_2...A'_m} + \pi^{A'_1}\pi^{B'}\gamma_{AA'_1...A'_mB'C'}\frac{\partial}{\partial \pi_{C'}}$$

be the twistor distribution. Write it as

$$\widetilde{L}_{Ap} = \widetilde{L}_{A\underbrace{1'...1'}_{p}\underbrace{0'...0'}_{m-p}} = L_{Ap} + f_{Ap}\frac{\partial}{\partial\lambda}$$

where $f_{Ap} = (\pi_{1'})^{-3}\gamma_{AA'B'C'p}\pi^{A'}\pi^{B'}\pi^{C'}$, and $p = 1...m$. Let $\mathcal{N} = C^{2m-1} \times SL(2,\mathbb{C})$ has a Bianchi IX type symmetry. Let $X_i = (X_1, X_2, X_3)$ be generators of the left action of $SL(2,\mathbb{C})$, and let $T_a = \partial/\partial t^a$ be the $2m-1$ vectors orthogonal to the orbits. We now use the local (t^a, ξ_i) coordinate system on $\mathcal{N}$, where ξ_i are coordinates on $SL(2,\mathbb{C})$. The vector M_0 is tangent to the orbits, and $(T_a)^{AA'_1...A'_m}$ are orthogonal to the orbits.

We have

$$\pi^{A'_1}...\pi^{A'_m}\pi^{B'_2}...\pi^{B'_m}(T_a)^{A}{}_{A'_1...A'_m}\widetilde{L}_{AB'_2...B'_m}(t^a) = 0.$$

Therefore

$$\begin{aligned} M_0 &= \frac{\sum_{a=1}^{2m-1}(\beta_a)\pi^{A'_1}...\pi^{A'_m}\pi^{B'_1}...\pi^{B'_m}(T_a)^{A}{}_{A'_1...A'_n}}{\sum_{b=1}^{2m-1}(\beta_b)\pi^{C'_1}...\pi^{C'_m}\pi^{D'_1}...\pi^{D'_m}\pi^{E'}\pi^{F'}(T_b)^{C}{}_{C'_1...C'_n}\gamma_{AD'_1...D'_nE'F'}}\nabla_{AB'_1...B'_m} \\ &\quad +\frac{\partial}{\partial\lambda} \\ &= \frac{\partial}{\partial\lambda} + \frac{k^iX_i}{Q}, \end{aligned}$$

where β_a and k^i are some functions to be specified, and

$$Q = \pi^{C'_1} ... \pi^{C'_m} \pi^{D'_1} ... \pi^{D'_m} \pi^{E'} \pi^{F'} (T_b)^C{}_{C'_1 ... C'_m} \gamma_{AD'_1 ... D'_m E' F'} \in \Gamma(\mathcal{O}(2m+2))$$

is a polynomial in λ of degree $2m+2$. We shall assume that its zeros are distinct. A Mobiüs transformation moves three of them to $0, 1$ and ∞. Rescale the coordinates such that the remaining $2m-1 = n$ roots are in t_a. Now define

$$M_a := \sum_{i+j=a} P_a(f_{1j}L_{0i} - f_{0i}L_{1j}) Q_a(f_{1j}L_{1i} - f_{1i}L_{1j}) P_a(f_{0j}L_{0i} - f_{0i}L_{0j})$$

where P_a, Q_a, R_a are some functions. The conditions

$$[M_a, M_b] = [M_a, M_0] = 0, \quad \sum_{a=1}^{2m-1} M_a = \frac{k^i X_i}{Q} + \sum_{a=1}^{2m-1} \frac{\partial}{\partial t^a}$$

should specify functions $\beta_a, P_a, Q_a, R_a, k_i$, and we are left with the Lax system for equations (6.2).

Example 2 ($dim\mathfrak{g} = n^2 + 2n$)**.** Frobenius manifolds. Dubrovin considers [13]

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda} - U - \frac{1}{\lambda} V\right)\Psi = 0. \tag{6.7}$$

Where U and V are (respectively, diagonal and skew-symmetric) $(N+1) \times (N+1)$ matrices. The equation (6.7) has one single pole at $\lambda = 0$ and one double pole at $\lambda = \infty$. If we pass to the dual system [28], then we obtain an ODE of the form (6.1) with A_a being $(N+1) \times (N+1)$ matrices (in $sl(N+1, \mathbb{C})$) and $N+2$ simple poles (plus a simple pole at ∞ already fixed). Fix two more poles at 0 and 1. This yields $n = N$ (no Mobiüs freedom left). Now :

$$dim\mathcal{Z}_t = n^2 + 2n, \quad dim\mathcal{N} = n^2 + 3n, \quad Q \in \mathcal{O}(n+3), \quad rkN_l = n^2 + 2n - 1,$$

so that the moduli space of sections is even-dimensional.

Example 3 ($dim\mathfrak{g} = n$) Now we take $n = dimG = k$, therefore $\mathcal{N} = G \times \mathbb{C}^k = T^*G$. The paraconformal structure may be the metric on the cotangent (or tangent) bundle to the group.

Chapter 7

The Twisted Photon Associated to Hyper–Hermitian Four–Manifolds

In this chapter the twistor theory of four-dimensional hyper-Hermitian manifolds is formulated as a combination of the Nonlinear Graviton Construction with the Ward transform for anti-self-dual Maxwell fields.

The Lax formulation of the hyper-Hermiticity condition in four dimensions is used to generalise the second heavenly equation to hyper-Hermitian four-manifolds.

A class of examples of hyper-Hermitian metrics which depend on two arbitrary functions of two complex variables is given [16].

7.1 Complexified hyper-Hermitian manifolds

A smooth manifold $\mathcal{M}$ equipped with three almost complex structures (I, J, K) satisfying the algebra of quaternions is called hypercomplex iff the almost complex structure

$$\mathcal{J}_\lambda = aI + bJ + cK$$

is integrable for any $(a, b, c) \in S^2$. We shall use a stereographic coordinate $\lambda = (a+ib)/(c-1)$ on S^2 which we will view as a complex projective line $\mathbb{CP}^1$. Let g be a Riemannian metric on $\mathcal{M}$. If $(\mathcal{M}, \mathcal{J}_\lambda)$ is hypercomplex and $g(\mathcal{J}_\lambda X, \mathcal{J}_\lambda Y) = g(X, Y)$ for all vectors X, Y on $\mathcal{M}$ then the triple $(\mathcal{M},\ \mathcal{J}_\lambda, g)$ is called a hyper-Hermitian structure. From now on we shall restrict ourselves to oriented four manifolds. In four dimensions a hyper-complex structure defines a conformal structure, which in explicit terms is represented by a conformal frame of vector fields (X, IX, JX, KX), for any $X \in T\mathcal{M}$.

It is well known that this conformal structure is ASD with the orientation determined by the complex structures. Let g be a representative of the conformal structure defined by $\mathcal{J}_\lambda$, and let $\Sigma^{A'B'} = (\Sigma^{00'}, \Sigma^{01'}, \Sigma^{11'})$ be a basis of the space of SD two forms ${\Lambda^2}_+(\mathcal{M})$. The following holds

Proposition 7.1 ([5]) *The Riemannian four manifold $(\mathcal{M}, g)$ is hyper-Hermitian if there exists a one form A (called the Lee form) depending only on g such that*

$$\mathrm{d}\Sigma^{A'B'} = -A \wedge \Sigma^{A'B'}. \tag{7.1}$$

Moreover if A is exact, then g is conformally hyper-Kähler.

In Section 7.2 we establish the twistor correspondence for the hyper-Hermitian four-manifolds. In Section 7.3 we shall express the hyper-Hermiticity condition on the metric in four dimensions in terms of Lax pairs of vector fields on $\mathcal{M}$. The Lax formulation will be used to encode the hyper-Hermitian geometry in a generalisation of Plebański's formalisms [62]. Some examples of hyper-Hermitian metrics are given in Section 7.4. In the last two Sections we make further remarks about the hyper-Hermitian equation, its symmetries and hierarchies.

7.2 The twistor construction

If $\mathcal{M}$ is real then the associated twistor space is identified with a sphere bundle of almost-complex structures and the resulting twistor theory is well-known [5, 60]. We shall work with the complexified correspondence and assume that $\mathcal{M}$ is a complex four-manifold. The integrability conditions under which (7.1) can hold are $\mathrm{d}A \in {\Lambda^2}_-(\mathcal{M})$ so $\mathrm{d}A$ can formally be identified with an ASD Maxwell field on an ASD background. This will enable us to formulate the twistor theory of hyper-Hermitian manifolds as a non-linear graviton construction 'coupled' to a Twisted Photon Construction [79].

In this section we shall establish the following result:

Proposition 7.2 *Let $\mathcal{PT}$ be a three-dimensional complex manifold with the following structures*

(A) a projection $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$,

(B) a four complex dimensional family of sections with a normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Then the moduli space $\mathcal{M}$ of sections of μ is equipped with hyper-Hermitian structure. Conversely given a hyper-Hermitian four-manifold there will always exist a corresponding twistor space satisfying conditions (A) and (B).

Remarks

(i) Let $K = \Lambda^3(\mathcal{PT})$ be the canonical line bundle. Proposition 4 is different from the original Nonlinear Graviton construction because the line bundle $L := K^* \otimes \mathcal{O}(-4)$, where $\mathcal{O}(-4) = \mu^*(T^*\mathbb{CP}^1)^{\otimes 2}$, is in general nontrivial over $\mathcal{PT}$. It is the twisted photon line bundle associated with $\mathrm{d}A$.

(ii) If $\mathcal{M}$ is compact then it follows from Hodge theory that $\mathrm{d}A = 0$ and the hyper-Hermitian structure is locally conformally hyper-Kähler. We focus on the non-compact case.

(iii) If $\mathcal{M}$ is real then $\mathcal{PT}$ is equipped with an antiholomorphic involution preserving (A) and we recover a result closely related to one of Petersen and Swann [60] who constructed a twistor space corresponding to a real four-dimensional ASD Einstein–Weyl metric with vanishing scalar curvature.

(iv) The correspondence is preserved under holomorphic deformations of $\mathcal{PT}$ which preserve (A).

Proof. Consider the line bundle

$$L = K^* \otimes \mathcal{O}(-4)$$

over $\mathcal{PT}$ given by the transition function $f = \det(\partial\tilde{\omega}^A/\partial\omega^B)$. When pulled back to $\mathcal{F}$ it satisfies

$$L_A f = 0.$$

Since $H^1(\mathcal{F}, \mathcal{O}) = 0$, we can perform the splitting $f = h_0 h_\infty^{-1}$. By the standard Liouville arguments (see [79]) we deduce that

$$h_0^{-1} L_A(h_0) = h_\infty^{-1} L_A(h_\infty) = -(1/2)A_A \tag{7.2}$$

where $A_A = A_{AB'}\pi^{B'}$ is global on $\mathcal{F}$. The integrability conditions imply that $F_{AB} = \nabla_{A'(A}A^{A'}_{B)}$ is an ASD Maxwell field on the ASD background. The one-form $A = A_{AA'}e^{AA'}$ is a Maxwell potential. The canonical line bundle of $\mathcal{PT}$ is $K = \mathcal{O}(-4) \otimes L^*$. To obtain a global, line bundle valued three-form on $\mathcal{PT}$ one must tensor the last equation with $\mathcal{O}(4) \otimes L$. We pick a global section $\xi \in \Gamma(K \otimes \mathcal{O}(4) \otimes L)$ and restrict ξ to l

$$\xi|_l = \Sigma_\lambda \wedge \pi_{A'}\mathrm{d}\pi^{A'} \tag{7.3}$$

where $\pi_{A'}\mathrm{d}\pi^{A'} \in \Omega^1 \otimes \mathcal{O}(2)$. A two-form

$$\Sigma_\lambda \in \Gamma(\Lambda^2(\mu^{-1}(\lambda)) \otimes \mathcal{O}(2) \otimes L) \tag{7.4}$$

is defined on vectors vertical with respect to μ by $\Sigma_\lambda(U,V)\pi_{A'}\mathrm{d}\pi^{A'} = \xi(U,V,...)$. Let $p^*\Sigma_\lambda$ be the pullback of Σ_λ to $\mathcal{F}$. Note that if

$$A \longrightarrow A - \mathrm{d}\phi \;\; \text{(gauge transformation on } L) \qquad \text{then} \qquad p^*\Sigma_\lambda \longrightarrow e^\phi p^*\Sigma_\lambda.$$

Let $p^*\Sigma_\lambda$ be defined over U and $p^*\widetilde{\Sigma}_\lambda$ over $\tilde{U}$. We have $f(p^*\Sigma_\lambda) = p^*\widetilde{\Sigma}_\lambda$. By definition, $p^*\Sigma_\lambda$ descends to the twistor space, i.e.,

$$\mathcal{L}_{L_A}(p^*\Sigma_\lambda) = 0. \tag{7.5}$$

We make use of the splitting formula, and define (on $\mathcal{F}$) $\Sigma_0 = h_0(p^*\Sigma_\lambda)$. The line bundle valued two-form Σ_0 is a globally defined object on $\mathcal{F}$, and therefore it is equal to $\pi_{A'}\pi_{B'}\Sigma^{A'B'}$. Note that Σ_0 does not descend to $\mathcal{PT}$. Fix $\lambda \in \mathbb{CP}^1$ (which gives a copy $\mathcal{M}_\lambda$ of $\mathcal{M}$ in $\mathcal{F}$) and apply (7.5). This yields

$$\mathcal{L}_{L_A}\Sigma_0 = h_0^{-1}L_A(h_0)\Sigma_0.$$

After some work we obtain formula (7.1):

$$\mathrm{d}\Sigma^{A'B'} = -A \wedge \Sigma^{A'B'}. \tag{7.6}$$

The integrability conditions for the last equation are guaranteed by the existence of solutions to (7.2). Equation (7.6) and the forward part of Proposition 7.1 imply that $\mathcal{M}$ is equipped with hyper-Hermitian structure. If the line bundle L over $\mathcal{PT}$ is trivial, then $\mathcal{M}$ is conformally hyper-Kähler.

Now we discuss the converse problem of recovering various structures on $\mathcal{PT}$ from the geometry of $\mathcal{M}$. Let $\mathcal{M}$ by a hyper-Hermitian four-manifold. Therefore $C_{A'B'C'D'} = 0$ and there exists a twistor space satisfying Condition (A). Equation (7.6) implies that $F = \mathrm{d}A$ is an ASD Maxwell field, and we can solve

$$\pi^{A'}(\nabla_{AA'} + (1/2)A_{AA'})\rho = 0$$

on each α-surface (self-dual, two dimensional null surface in $\mathcal{M}$). We define fibres of L as one-dimensional spaces of solutions to the last equation. The solutions on α-surfaces intersecting at $p \in \mathcal{M}$ can be compared at one point, so L restricted to a line l_x in $\mathcal{PT}$ is trivial. In order to prove that $\mathcal{PT}$ is fibred over $\mathbb{CP}^1$ notice that equation $\pi^{A'}(\nabla_{AA'}+(1/2)A_{AA'})\pi_{B'} = 0$ implies $\pi^{A'}\nabla_{AA'}\lambda = 0$, so λ and $1/\lambda$ descend to give meromorphic functions on twistor space and defines the map $\mathcal{PT} \rightarrow \mathbb{CP}^1$.

$\square$

7.3 Hyper-Hermiticity condition as an integrable system

The hyper-Hermiticity condition on a metric g can be reduced to a system of second order PDEs for a pair of functions[1]. The Lax representation for such an equation will be a consequence of the integrability of the twistor distribution. We shall need the following lemma:

Lemma 7.3 *Let $\nabla_{AA'}$ be four independent holomorphic vector fields on a four-dimensional complex manifold $\mathcal{M}$, and let*

$$L_0 = \nabla_{00'} - \lambda\nabla_{01'}, \qquad L_1 = \nabla_{10'} - \lambda\nabla_{11'}, \quad \textit{where} \;\; \lambda \in \mathbb{CP}^1.$$

If

$$[L_0, L_1] = 0 \tag{7.7}$$

for every λ, then $\nabla_{AA'}$ is a null tetrad for a hyper-Hermitian metric on $\mathcal{M}$. Every hyper-Hermitian metric arises in this way.

[1]K. P. Tod has given a generalisation the first heavenly equation to the case of real hyper-Hermitian four-manifolds [75].

Proof. Let $\nabla_{AA'}$ be a tetrad of holomorphic vector fields on $\mathcal{M}$. It determines an anti self-dual conformal structure if and only if the distribution on the primed spin bundle $S^{A'}$ spanned by the vectors

$$L_A = \pi^{A'}\nabla_{AA'} + \Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\frac{\partial}{\partial \pi_{C'}}$$

is integrable. This then implies that the spin bundle is foliated by the horizontal lifts of α-surfaces. Here $\pi^{A'} = \pi^{0'}o^{A'} + \pi^{1'}\iota^{A'}$ is the spinor determining an α-surface and is related to $\lambda = (-\pi^{1'}/\pi^{0'})$. From the general formula

$$\mathrm{d}\Sigma^{A'B'} + 2\Gamma^{(A'}_{C'} \wedge \Sigma^{B')C'} = 0,$$

we conclude that $\Gamma_{AA'B'C'} = -A_{A(C'}\varepsilon_{B')A'}$ for some $A_{AA'}$ and

$$L_A = \pi^{A'}\nabla_{AA'} + (1/2)\pi^{A'}A_{AA'}\Upsilon,$$

where $\Upsilon = \pi^{A'}/\partial\pi^{A'}$ is the Euler vector field. We have

$$\begin{aligned}
[L_A, L_B] &= \pi^{A'}\pi^{B'}([\nabla_{AA'}, \nabla_{BB'}] + 1/2([\nabla_{BB'}, A_{AA'}\Upsilon] - [\nabla_{AA'}, A_{BB'}\Upsilon])) \\
&= \pi^{A'}\pi^{B'}([\nabla_{AA'}, \nabla_{BB'}] + (1/2)\varepsilon_{AB}\nabla^{C}{}_{(A'}A_{B')C}\Upsilon) \\
&= \pi^{A'}\pi^{B'}[\nabla_{AA'}, \nabla_{BB'}] \qquad \text{since } \mathrm{d}A \text{ is ASD.} \qquad (7.8)
\end{aligned}$$

We shall introduce the rotation coefficient C^c_{ab} defined by

$$[\nabla_a, \nabla_b] = C^c_{ab}\nabla_c,$$

They satisfy $C_{abc} = \Gamma_{acb} - \Gamma_{bca}$. From the last formula we can find a spinor decomposition of C_{abc},

$$C_{abc} = C_{ABCC'}\varepsilon_{A'B'} + C_{A'B'CC'}\varepsilon_{AB}$$

where

$$C_{A'B'CC'} = \Gamma_{C(A'B')C'} + \varepsilon_{C'(B'}\Gamma_{A')AC}{}^{A}. \qquad (7.9)$$

Collecting (7.8), and (7.9) we obtain

$$[L_A, L_B] = \varepsilon_{AB}\pi^{A'}\pi^{B'}((1/2)A^C_{B'}\varepsilon_{A'}{}^{C'} + \varepsilon_{A'}{}^{C'}\Gamma_{B'D}{}^{CD})\nabla_{CC'}.$$

We choose a spin frame (o^A, ι^A) constructed from two independent solutions to the charged neutrino equation

$$(\nabla_{AA'} + (1/2)A_{AA'})o^A = (\nabla_{AA'} + (1/2)A_{AA'})\iota^A = 0.$$

In this frame $\Gamma_{AA'}{}^{BA} = -(1/2)A^B_{A'}$. To obtain the equation (7.7) we project L_A to the projective prime spin bundle $\mathcal{F} = \mathbb{P}S_{A'}$. In terms of the tetrad

$$[\nabla_{A0'}, \nabla_{B0'}] = 0, \tag{7.10}$$

$$[\nabla_{A0'}, \nabla_{B1'}] + [\nabla_{A1'}, \nabla_{B0'}] = 0, \tag{7.11}$$

$$[\nabla_{A1'}, \nabla_{B1'}] = 0. \tag{7.12}$$

The formulation of the hyper-complex condition in formulae (7.10-7.12) was in the Riemannian case given in [38] and used in [33]. The Lax equation (7.7) can be interpreted as the anti-self-dual Yang-Mills equations on $\mathbb{C}^4$ with the gauge group G =Diff($\mathcal{M}$), reduced by four translations in $\mathbb{C}^4$.

$\square$

Define $(1,1)$ tensors $\mathcal{J}^{A'}_{B'} := e^{AA'} \otimes \nabla_{AB'}$. As a consequence of (7.10-7.12) the Nijenhuis tensors

$$N^{A'}_{B'}(X,Y) := (\mathcal{J}^{A'}_{B'})^2[X,Y] - \mathcal{J}^{A'}_{B'}[\mathcal{J}^{A'}_{B'}X,Y] - \mathcal{J}^{A'}_{B'}[X,\mathcal{J}^{A'}_{B'}Y] + [\mathcal{J}^{A'}_{B'}X,\mathcal{J}^{A'}_{B'}Y] \tag{7.13}$$

vanish for arbitrary vectors X and Y. Tensors $\mathcal{J}^{B'}_{A'}$ can be treated as 'complexified complex structures' on $\mathcal{M}$. The complex structure $\mathcal{J}_\lambda$ on $S^{A'}$ can be conveniently expressed as

$$\mathcal{J}_\lambda = \pi_{A'}\tilde{\pi}^{B'}\mathcal{J}^{A'}_{B'}, \quad \text{where} \quad \pi_{A'}\tilde{\pi}^{A'} = 1.$$

Now we shall fix some remaining gauge and coordinate freedom. Equations (7.10-7.12) will be reduced to a coupled system of nonlinear differential equations for a pair of functions.

Proposition 7.4 *Let $x^{AA'} = (x^A, w^A)$ be local null coordinates on $\mathcal{M}$ and let Θ^A be a pair of complex valued functions on $\mathcal{M}$ which satisfy*

$$\frac{\partial^2\Theta_C}{\partial x_A \partial w^A} + \frac{\partial\Theta_B}{\partial x^A}\frac{\partial^2\Theta_C}{\partial x_A \partial x_B} = 0. \tag{7.14}$$

Then

$$ds^2 = 2\mathrm{d}x_A \otimes \mathrm{d}w^A + 2\frac{\partial\Theta_A}{\partial x^B}\mathrm{d}w^B \otimes \mathrm{d}w^A \tag{7.15}$$

is a hyper-Hermitian metric on $\mathcal{M}$. Conversely every hyper-Hermitian metric locally arises by this construction.

Equation (7.1) and its connection with a scalar form of (7.14) was investigated by different methods in [21] in the context of weak heavenly spaces. Other integrable equations associated to Hyper-Hermitian manifolds have been studied in [27].

Proof. Choose a conformal factor such that $A_{AA'} = o_{A'}A_A$ for some $o_{A'}$ and A_A. This can be done since the two form $\Sigma^{1'1'}$ is simple and therefore equation (7.1) together with the Frobenius theorem imply the existence of the conformal factor such that $\mathrm{d}\Sigma^{1'1'} = 0$. Hence, using the Darboux's theorem, one can introduce canonical coordinates w^A such that

$$\Sigma^{1'1'} = (1/2)\varepsilon_{AB}\mathrm{d}w^A \wedge \mathrm{d}w^B,$$

and choose un unprimed spin frame so that $o_{A'}e^{AA'} = \mathrm{d}w^A$. Coordinates w^A parametrise the space of null surfaces tangent to $o^{A'}$, i.e. $o^{A'}\nabla_{AA'}w^B = 0$. Consider

$$\mathcal{J}_{0'}^{1'} = o^{B'}\mathrm{d}w^A \otimes \nabla_{AB'}$$

The tensor $\mathcal{J}_{0'}^{1'}$ is a degenerate complex structure. Therefore $(\mathcal{J}_{0'}^{1'})^2 = 0$ where $\mathcal{J}_{0'}^{1'}$ is now thought of as a differential operator acting on forms. Let h be a function on $\mathcal{M}$. Then

$$\mathcal{J}_{0'}^{1'} \lrcorner\, \mathrm{d}(\mathcal{J}_{0'}^{1'}(\mathrm{d}h)) = 0 \quad \text{implies that} \;\; [\nabla_{A0'}, \nabla_{B0'}] = 0,$$

and our choice of the spin frame is consistent with (7.10-7.12). By applying the Frobenius theorem we can find coordinates x^A such that

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} - \Theta_A{}^B\frac{\partial}{\partial x^B}.$$

Using equation (7.11), we deduce the existence of a potential Θ_A such that $\Theta_A{}^B = \nabla_{A0'}\Theta^B$. Now (7.12) gives the field equations (7.14)

$$\frac{\partial^2\Theta_C}{\partial x_A \partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial^2\Theta_C}{\partial x_A \partial x_B} = 0.$$

The dual frame is

$$e^{A0'} = \mathrm{d}x^A + \frac{\partial\Theta^A}{\partial x^B}\mathrm{d}w^B, \qquad e^{A1'} = \mathrm{d}w^A,$$

which justifies formula (7.15).

$\square$

In the adopted gauge, the Maxwell potential is

$$A = \frac{\partial^2 \Theta^B}{\partial x^A \partial x^B} \mathrm{d}w^A$$

and $\nabla^a A_a = 0$ i.e. this is a 'Gauduchon gauge'. Electromagnetic gauge transformations on A correspond to conformal rescalings of the tetrad (which preserve the hypercomplex structure). The second heavenly equation (and therefore the hyper-Kähler condition) follows from (7.14) if in addition $\nabla_{A0'}\Theta^A = 0$. This condition guarantees the existence of a scalar function Θ, such that $\Theta^A = \nabla^A{}_{0'}\Theta$, which satisfies the second Plebański equation (3.6). In this case A is exact so can be gauged away by a conformal rescaling.

For convenience we express various spinor objects on $\mathcal{M}$ in terms of Θ_A.

$$\begin{aligned}
\text{Tetrad} \quad & e^{A0'} = \mathrm{d}x^A + \frac{\partial \Theta^A}{\partial x^B} dw^B, \ \ e^{A1'} = \mathrm{d}w^A, \\
\text{dual tetrad} \quad & \nabla_{A0'} = \frac{\partial}{\partial x^A}, \quad \nabla_{A1'} = \frac{\partial}{\partial w^A} - \frac{\partial \Theta^B}{\partial x^A}\frac{\partial}{\partial x^B}, \\
\text{metric determinant} \quad & det(g) = 1 \\
\text{Weyl spinors} \quad & C_{A'B'D'E'} = 0, \ \ C_{ABCD} = \nabla_{(A0'}\nabla_{B0'}\nabla_{C0'}\Theta_{D)}, \\
\text{spin connections} \quad & \Gamma_{AA'BC} = -\frac{1}{2} o_{A'}(\nabla_{(B0'}\nabla_{C0'}\Theta_{A)} + \nabla_{B0'}\nabla_{C0'}\Theta_A), \\
& \Gamma_{AA'B'C'} = -\frac{\partial^2 \Theta^B}{\partial x^B \partial x^A} o_{(B'}\varepsilon_{C')A'}, \\
\text{Lee form} \quad & A = \frac{\partial^2 \Theta^B}{\partial x^B \partial x^A} dw^A, \\
\text{wave operator} \quad & \Box_g = A^a \partial_a + \nabla^A_{1'} \nabla_{A0'} \\
& \quad = \frac{\partial^2}{\partial x_A \partial w^A} + \frac{\partial^2 \Theta_B}{\partial x_A \partial x_B}\frac{\partial}{\partial x^A} + \frac{\partial \Theta^A}{\partial x_B}\frac{\partial}{\partial x^A}\frac{\partial}{\partial x^B}, \\
\text{Ricci scalar} \quad & R = 1/12(\nabla^a A_a + A_a A^a) = 0.
\end{aligned}$$

The last formula follows because A is null and satisfies the Gauduchon gauge.

7.4 Examples

We look for solutions to (7.14) for which the linear and nonlinear terms vanish separately, ie.

$$\frac{\partial^2 \Theta_C}{\partial x_A \partial w^A} = \frac{\partial \Theta_B}{\partial x^A}\frac{\partial^2 \Theta_C}{\partial x_A \partial x_B} = 0. \tag{7.16}$$

Put $w^A = (w, z), x^A = (y, -x)$. A simple class of solutions to (7.16) is provided by

$$\Theta_0 = ax^l, \quad \Theta_1 = by^k, \qquad\qquad k, l \in \mathbb{Z}, \quad a, b \in \mathbb{C}.$$

The corresponding metric and the Lee form are

$$\mathrm{d}s^2 = 2\mathrm{d}w \otimes \mathrm{d}x + 2\mathrm{d}z \otimes \mathrm{d}y + 2(alx^{l-1} + bky^{k-1})\mathrm{d}w \otimes \mathrm{d}z, \tag{7.17}$$

$$A = b\,(k-1)\,ky^{k-2}\mathrm{d}w - a\,(l-1)\,lx^{l-1}\mathrm{d}z.$$

From calculating the invariant

$$C_{ABCD}C^{ABCD} = (3/2)abk(k-1)(k-2)l(l-1)(l-2)x^{l-3}y^{k-3}$$

we conclude that the metric (7.17) is in general of type I or D (or type III or N if a or b vanish, or $k < 3$ or $l < 3$).

7.4.1 Hyper-Hermitian elementary states

A more interesting class of solutions (which generalise the metric of Sparling and Tod (3.31) to the hyper-Hermitian case) is given by

$$\Theta_C = \frac{1}{x_A w^A} F_C(W^A), \tag{7.18}$$

where $W^A = w^A/(x_B w^B)$ and F_C are two arbitrary complex functions of two complex variables. The corresponding metric is

$$\mathrm{d}s^2 = 2\mathrm{d}x_A \otimes \mathrm{d}w^A + \frac{2}{(x_A w^A)^2}\Big(F_C + \frac{w^B}{(x_A w^A)}\frac{\partial F_C}{\partial W^B}\Big)\mathrm{d}w^C \otimes (w_A \mathrm{d}w^A).$$

This metric is singular at the light-cone of the origin. The singularity may be moved to infinity if we introduce new coordinates $X^A = x^A/(x_B w^B), W^A = w^A/(x_B w^B)$ and rescale the metric by $(X_A W^A)^2$. This yields

$$\hat{\mathrm{d}s}^2 = 2\mathrm{d}X_A \otimes \mathrm{d}W^A + 2\Big(F_B + W^C\frac{\partial F_B}{\partial W^C}\Big)\Big(X_A W^A \mathrm{d}W^B - W^B \mathrm{d}(X_A W^A)\Big) \otimes W_A \mathrm{d}W^A \tag{7.19}$$

and

$$A = -\Big(3W^A F_A + 5W^A W^B \frac{\partial F_A}{\partial W^B} + W^A W^B W^C \frac{\partial^2 F_A}{\partial W^B \partial W^C}\Big) W_D \mathrm{d}W^D.$$

The metric of Sparling and Tod corresponds to setting $F_A = W_A$.

Let us consider the particular case $F_A = (aW^kZ^l, bW^mZ^n)$. The metric is

$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}w \otimes \mathrm{d}x + 2\mathrm{d}z \otimes \mathrm{d}y \\ &\quad +2\Big(\frac{a(k+l+1)w^k z^l}{(wx+zy)^{k+l+2}}\mathrm{d}w + \frac{b(m+n+1)w^m z^n}{(wx+zy)^{m+n+2}}\mathrm{d}z\Big) \otimes (w\mathrm{d}z - z\mathrm{d}w). \end{aligned} \tag{7.20}$$

If

$$a = -b, l = n+1, k = m-1 \qquad \text{then} \qquad \Theta_A = \nabla_{A0'}\Theta$$

where $\Theta = -aw^k z^{l-1}(wx+zy)^{-(k+l)}$. For these values of parameters the metric is hyper-Kähler and of type N.

Some solutions to (7.16) have real Euclidean slices. For example

$$\Theta_0 = -\frac{y\,(2\,wx+zy)}{w^2\,(wx+zy)^2}, \quad \Theta_1 = -\frac{y^2}{w\,(wx+zy)^2}$$

with $w = \bar{x}, z = \bar{y}$ yield a solution of type D, which is equivalent to the Eguchi–Hanson metric.

7.4.2 Twistor description

In this Subsection we shall give the twistor correspondence for the family of hyper-Hermitian metrics (7.20). First we shall look at the passive twistor constructions of Θ_C by the contour integral formulae. It will turn out that Θ_C are examples of Penrose's elementary states. Then we explain how the cohomology classes corresponding to Θ_C can be used to deform a patching description of $\mathcal{PT}$. The deformed twistor space will, by Proposition 7.2, give rise to the metric (7.20). Both passive and active constructions in this subsection use methods developed by Sparling in his twistorial treatment of the Sparling-Tod metric.

Parametrise a section of $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ by the coordinates

$$x^{AA'} := \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=o_{A'}} = \begin{pmatrix} y & w \\ -x & z \end{pmatrix},$$

so that $x^{A1'} = w^A = (w, z),\ x^{A0'} = x^A = (y, -x)$.

Let us consider the particular case $F_A = (aW^kZ^l, bW^mZ^n)$ discussed in Subsection 3.1. We work on the non-deformed twistor space $\mathcal{PT}$ with homogeneous coordinates $(\omega^A, \pi_{A'})$. On the primed spin bundle $\omega^0 = \pi_{1'}(w+\lambda y), \omega^1 = \pi_{1'}(z-\lambda x)$. Consider two twistor functions (sections of $H^1(\mathbb{CP}^1, \mathcal{O}(-2))$

$$h_0 = (-1)^k a\frac{(\pi_{0'})^{k+l}}{(\omega^0)^{l+1}(\omega^1)^{k+1}}, \quad h_1 = (-1)^m b\frac{(\pi_{0'})^{m+n}}{(\omega^0)^{n+1}(\omega^1)^{m+1}}.$$

where $a, b \in \mathbb{C}$ and $k, l, m, n \in \mathbb{Z}$ are constant parameters. Then

$$\Theta_A(w, z, x, y) = \frac{1}{2\pi i} \oint_\Gamma h_A(\omega^B, \pi_{B'})\pi_{A'} \mathrm{d}\pi^{A'}.$$

Here Γ is a contour in l_x, the $\mathbb{CP}^1$ that corresponds to $(w, z, x, y) \in \mathcal{M}$. It separates the two poles of the integrand. To find Θ^A we compute the residue at one of these poles, which gives

$$\Theta_0 = a\frac{w^k z^l}{(wx + zy)^{k+l+1}}, \quad \Theta_1 = b\frac{w^m z^n}{(wx + zy)^{m+n+1}}, \tag{7.21}$$

and hence the metric (7.20).

Now we shall use h_A to deform the complex structure of $\mathcal{PT}$. We change the standard patching relations by setting

$$\tilde{\omega}^A = f^A(\omega^A, t)$$

where t is a deformation parameter and f^A is determined by the deformation equations

$$\frac{\mathrm{d}f^0}{\mathrm{d}t} = \frac{b\pi_{0'}^{m+n+3}}{(\tilde{\omega}^0)^{n+1}(\tilde{\omega}^1)^{m+1}}(-1)^m, \quad \frac{\mathrm{d}f^1}{\mathrm{d}t} = \frac{a\pi_{0'}^{k+l+3}}{(\tilde{\omega}^0)^{l+1}(\tilde{\omega}^1)^{k+1}}(-1)^{k+1}. \tag{7.22}$$

This equation has a first integral. If $a = -b, l = n + 1, k = m - 1$ then (7.22) imply that $\omega^0\omega^1 = \tilde{\omega}^0\tilde{\omega}^1$ is a global twistor function. When pulled back to the spin bundle this can be expressed as $P_{A'B'}\pi^{A'}\pi^{B'}$, and the corresponding metric admits a null Killing vector $K_{AA'}$ given by

$$\nabla_{AC'} P_{A'B'} = K_{A(A'}\varepsilon_{B')C'}.$$

Assume that $n + 1 \neq l$, and $k + 1 \neq m$. Then the first integral of (7.22)

$$Q = \frac{a(\pi^{0'})^{k+l+3}(-1)^{k+1}}{n+1-l}(\omega^0)^{n+1-l} + \frac{b(\pi^{0'})^{m+n+3}(-1)^{m+1}}{k+1-m}(\omega^1)^{k+1-m}$$

is given by a function homogeneous of degree $k + n + 4$. Its pull backs to $\mathcal{F}$ (which we also denote Q) satisfies $L_A(Q) = 0$. This implies the existence of a Killing spinor of valence $(0, k + n + 4)$ on $\mathcal{M}$.

7.5 Symmetries

The equation (7.14) has the obvious first integral given by functions Λ_C which satisfy

$$\frac{\partial \Theta_C}{\partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial \Theta_C}{\partial x^B} = \frac{\partial \Lambda_C}{\partial x^A}.$$

It is implicit from the twistor construction that equation (7.14) has infinitely many first integrals given by hidden symmetries. They should give rise to a hierarchy of equations. Here we give a description of those symmetries that correspond to the pure gauge transformations.

Let M be a vector field on $\mathcal{M}$. Define $\delta^0_M \nabla_{AA'} := [M, \nabla_{AA'}]$. This is a pure gauge transformation corresponding to the addition of $\mathcal{L}_M g$ to the space-time metric.

Once a coordinate system leading to equation (7.14) has been selected, the field equations will not be invariant under all the diff($\mathcal{M}$) transformations. We restrict ourselves to transformations that preserve the canonical structures on $\mathcal{M}$, namely

$$\Sigma^{1'1'} = (1/2)\mathrm{d}w_A \wedge \mathrm{d}w^A, \quad \text{and} \quad \mathcal{J}^{1'}_{0'} = \mathrm{d}w^A \otimes \frac{\partial}{\partial x^A}.$$

The condition $\mathcal{L}_M \Sigma^{0'0'} = \mathcal{L}_M \mathcal{J}^{1'}_{0'} = 0$ implies that M is given by

$$M = \frac{\partial h}{\partial w_A}\frac{\partial}{\partial w^A} + \Big(g^A - x^B \frac{\partial^2 h}{\partial w_A \partial w^B}\Big)\frac{\partial}{\partial x^A}$$

where $h = h(w^A)$ and $g^A = g^A(w^B)$. Space-time is now viewed as a tangent bundle $\mathcal{M} = T\mathcal{N}^2$ with w^A being coordinates on the two-dimensional complex manifold $\mathcal{N}^2$. The full diff($\mathcal{M}$) symmetry breaks down to sdiff($\mathcal{N}^2$) which acts on $\mathcal{M}$ by Lie lift. Let $\delta^0_M\Theta$ corresponds to $\delta^0_M\nabla_{AA'}$ by

$$\delta^0_M \nabla_{A1'} = \frac{\partial \delta^0_M \Theta^B}{\partial x^A}\frac{\partial}{\partial x^B}.$$

The 'pure gauge' elements are

$$\delta^0_M \Theta^B = \mathcal{L}_M(\Theta^B) + F^B - x^A \frac{\partial g^B}{\partial w^A} + x^A x^C \frac{\partial^2 h}{\partial w^A \partial w^C \partial w_B}$$

where F^B, g^A and h are functions of w^B only.

7.6 $gl(2,\mathbb{C})$ connection

A natural connection which arises in hyper-Hermitian geometry is the Obata connection [53]. In this section we discuss other possible choices of connections associated with hyper-Hermitian geometry. We shall motivate our choices by considering the conformal rescalings of the null tetrad. The first Cartan structure equations are

$$\mathrm{d}e^{AA'} = e^{BA'} \wedge \Gamma^A{}_B + e^{AB'} \wedge \Gamma^{A'}{}_{B'}.$$

Rescaling $e^{AA'} \longrightarrow \hat{e}^{AA'} = e^{\phi} e^{AA'}$ yields

$$\mathrm{d}\hat{e}^{AA'} = \hat{e}^{BA'} \wedge \Gamma^A{}_B + \hat{e}^{AB'} \wedge \Gamma^{A'}{}_{B'} + \mathrm{d}\phi \wedge \hat{e}^{AA'}.$$

The last equation can be interpreted in (at least) three different ways;

(a) Introduce the torsion three-form by $T = *(\mathrm{d}\phi) = T_{abc}\hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c$. Then

$$\mathrm{d}\hat{e}^a + \Gamma^a{}_b \wedge \hat{e}^b = T^a$$

where $T^a = (1/2)T^a_{bc}\hat{e}^b \wedge \hat{e}^c$.

(b) Use the torsion-free $sl(2,\mathbb{C}) \oplus \widetilde{sl}(2,\mathbb{C})$ spin connection

$$\Gamma_{AB} \longrightarrow \Gamma_{AB} + (1/4) * (\mathrm{d}\phi \wedge \Sigma_{AB}), \quad \Gamma_{A'B'} \longrightarrow \Gamma_{A'B'} + (1/4) * (\mathrm{d}\phi \wedge \Sigma_{A'B'}),$$

(c) Work with the torsion-free $gl(2,\mathbb{C}) \oplus \widetilde{gl}(2,\mathbb{C})$ connection

$$\mathbb{G}_{AB} = \Gamma_{AB} + a\varepsilon_{AB}\mathrm{d}\phi, \quad \mathbb{G}_{A'B'} = \Gamma_{A'B'} + (1-a)\varepsilon_{A'B'}\mathrm{d}\phi$$

with $\Gamma_{AB} = \Gamma_{(AB)} \in sl(2,\mathbb{C}) \otimes \Lambda^1(T^*\mathcal{M})$, $\Gamma_{A'B'} = \Gamma_{(A'B')} \in \widetilde{sl}(2,\mathbb{C}) \otimes \Lambda^1(T^*\mathcal{M})$ and $a \in \mathbb{C}$. This leads to

$$\mathrm{d}\hat{e}^a + \Gamma^a{}_b \wedge \hat{e}^b = 0$$

where $\mathbb{G}_{ab} = \Gamma_{ab} + \varepsilon_{A'B'}\varepsilon_{AB}\mathrm{d}\phi$. The structure group reduces to

$$sl(2,\mathbb{C}) \oplus \widetilde{sl}(2,\mathbb{C}) \oplus u(1) \subset gl(2,\mathbb{C}) \oplus \widetilde{gl}(2,\mathbb{C}).$$

For (complexified) hyper-Hermitian four-manifolds $\mathrm{d}\phi$ is replaced by the Lee form $-A$ in the above formulae. The possibility (a) would then correspond to the heteretic geometries studied by physicists in connection with $(4,0)$ supersymmetric σ-models (see [12] and references therein). Choice (b) is what we have used in this chapter. Let us make a few remarks about the possibility (c).

Equation (7.1) implies that $a = 1/2$ and

$$\mathbb{G}_{AB} = \Gamma_{AB} - (1/2)\varepsilon_{AB}A, \quad \mathbb{G}_{A'B'} = -(1/2)\varepsilon_{A'B'}A$$

with $\Gamma_{AB} = \Gamma_{(AB)} \in sl(2,\mathbb{C})$. In the adopted coordinate system

$$\Gamma_{AA'BC} = -o_{A'}\Big(\nabla_{(A0'}\nabla_{B0'}\Theta_{C)} + \frac{1}{2}\varepsilon_{BC}\frac{\partial\Theta^D}{\partial x^A \partial x^D}\Big), \quad \Gamma_{AA'B'C'} = -\frac{1}{2}o_{A'}\varepsilon_{B'C'}\frac{\partial\Theta^D}{\partial x^A \partial x^D}.$$

The curvatures of $\mathbb{G}_{AB}$ and $\mathbb{G}_{A'B'}$ are

$$\mathbb{R}^A{}_B = \mathrm{d}\mathbb{G}^A{}_B + \mathbb{G}^A{}_C \wedge \mathbb{G}^C{}_B = R^A{}_B - (1/2)\varepsilon^A{}_B F, \quad \mathbb{R}^{A'}{}_{B'} = -(1/2)\varepsilon^{A'}{}_{B'}F$$

where $F = \mathrm{d}A$ is an ASD two form. It would be interesting to investigate this possibility with connection to $gl(2,\mathbb{C})$ formulation of Einstein–Maxwell equations [61], and its Lagrangian description [66].

Chapter 8

Einstein–Weyl metrics from conformal Killing vectors

In this Chapter we shall consider ASD vacuum spaces with a conformal symmetry[1]. By the general construction [36, 37] such spaces will give rise to Einstein–Weyl structures on the space of trajectories of a given conformal symmetry K. The cases where K is a pure or a conformal tri-holomorphic Killing vector have been extensively studied [6, 86, 12, 43]. Therefore we shall consider the most general case of K being a conformal, non-triholomorphic Killing vector. In the next section we shall give the canonical form of an allowed conformal Killing vector. Then we shall look at solutions to the first heavenly equation (3.2) which admit the symmetry K. This will give rise to a new integrable system in three dimensions and to the corresponding EW geometries. In Section 8.2 we shall give the Lax representation of the reduced equations. When Euclidean reality conditions are imposed we shall recover some known results [6, 86] as limiting cases of our construction. In Section 8.6 we shall find and classify the Lie point symmetries (and so the Killing vectors) of the field equations in three dimensions, and consider some group invariant solutions. In Section 8.7 we shall study hidden symmetries and the recursion operator associated to the 3D system. In Section 8.8 we shall consider a tri-holomorphic conformal reduction of the second heavenly equation.

[1]The results of this chapter have now appeared in [93]

8.1 Heavenly spaces with conformal Killing vectors

Let g be a compexified hyper-Kähler metric on a complex four-manifold $\mathcal{M}$ and $x^{AA'} = (w, z, \tilde{w}, \tilde{z})$ be a null coordinate system on $\mathcal{M}$. Locally g is given by

$$\mathrm{d}s^2 = 2(\Omega_{w\tilde{w}}\mathrm{d}w\mathrm{d}\tilde{w} + \Omega_{w\tilde{z}}\mathrm{d}w\mathrm{d}\tilde{z} + \Omega_{z\tilde{w}}\mathrm{d}z\mathrm{d}\tilde{w} + \Omega_{z\tilde{z}}\mathrm{d}z\mathrm{d}\tilde{z}) \tag{8.1}$$

where $\Omega = \Omega(w, z, \tilde{w}, \tilde{z})$ is a solution to the first heavenly equation (3.2). Assume that g admits a conformal Killing vector K;

$$\mathcal{L}_K g = \eta g, \qquad \text{or equivalently } \nabla_a K_b = \phi_{A'B'}\varepsilon_{AB} + \psi_{AB}\varepsilon_{A'B'} + (1/2)\varepsilon_{A'B'}\varepsilon_{AB}\eta$$

where symmetric spinors $\phi_{A'B'}$ and ψ_{AB} are respectively SD and ASD parts of the covariant derivative of K. In vacuum the following integrability conditions hold

$$\nabla_{AA'}\phi_{B'C'} = 2C_{A'B'C'D'}K_A{}^{D'} - 2\varepsilon_{A'(B'}\nabla_{C')A}\eta, \;\; \nabla_{AA'}\nabla_{BB'}\eta = 0, \;\; C_{ABCD}\nabla^A{}_{A'}\eta = 0$$

(similar for ψ_{AB}). In particular in an ASD vacuum $\phi_{A'B'} = const$ and $\eta = const$ (or the space time is of type N).

In this chapter we shall analyse a situation where K is not hyper–surface orthogonal and $det(\phi_{A'B'}) \neq 0, \eta \neq 0$. Reductions by pure ($\eta = 0$) Killing vectors were considered in [6, 23]. See [64, 73] (and Subsection 9.1.2) for the case of the non-vanishing cosmological constant.

Lemma 8.1 *In an ASD vacuum the most general conformal Killing vector with* $det(\phi_{A'B'}) \neq 0$ *can be transformed to the form*

$$K = \eta(z\partial_z - \tilde{z}\partial_{\tilde{z}}) + \rho(z\partial_z + \tilde{z}\partial_{\tilde{z}}). \tag{8.2}$$

Proof. The proof is along the lines of the derivation of the canonical form of a pure Killing vector given in [6]. In the adopted coordinate system

$$\Sigma^{0'0'} = \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z}, \;\; \Sigma^{1'1'} = \mathrm{d}w \wedge \mathrm{d}z,$$

$$\Sigma^{1'0'} = \Omega_{w\tilde{w}}\mathrm{d}w \wedge \mathrm{d}\tilde{w} + \Omega_{w\tilde{z}}\mathrm{d}w \wedge \mathrm{d}\tilde{z} + \Omega_{z\tilde{w}}\mathrm{d}z \wedge \mathrm{d}\tilde{w} + \Omega_{z\tilde{z}}\mathrm{d}z \wedge \mathrm{d}\tilde{z}.$$

Let $K = K^A\partial/\partial w^A + \widetilde{K}^A\partial/\partial\tilde{w}^A$, where $w^A = (w, z)$ and $\tilde{w}^A = (\tilde{w}, \tilde{z})$. The action of K on self-dual two forms is determined by

$$\begin{aligned}
\mathcal{L}_K\Sigma^{0'0'} &= m\Sigma^{0'0'} + n\Sigma^{1'1'},\\
\mathcal{L}_K\Sigma^{1'1'} &= \tilde{n}\Sigma^{0'0'} + \tilde{m}\Sigma^{1'1'},\\
\mathcal{L}_K\Sigma^{0'1'} &= \eta\Sigma^{0'1'}
\end{aligned}$$

for some constants $m, \tilde{m}, n, \tilde{n}$. This is because for non-degenerated $\phi_{A'B'}$ the Kähler structure can be identified with $\mathrm{d}K_+ = \phi_{A'B'}\Sigma^{A'B'}$. It follows that $n = \tilde{n} = 0$, and $K^A = mw^A$, $\widetilde{K}^A = \tilde{m}\tilde{w}^A$. From $2\Sigma^{0'0'} \wedge \Sigma^{1'1'} = \Sigma^{0'1'} \wedge \Sigma^{0'1'}$ we find that $\eta := (m + \tilde{m})/2$. Define $\rho := (m - \tilde{m})/2$. We have the freedom to transform $w^A \to W^A(w^B)$ and $\tilde{w}^A \to \widetilde{W}^A(\tilde{w}^B)$ in a way which preserves $\Sigma^{0'0'}$ and $\Sigma^{1'1'}$. Put $Z = z^2/2, W = w/z, \widetilde{Z} = \tilde{z}^2/2, \widetilde{W} = \tilde{w}/\tilde{z}$. This yields (coming back to $(w^A, \tilde{w}^A)$) (8.2). Now

$$\nabla_{AA'}K^A{}_{B'} = \begin{pmatrix} 0 & \rho + \eta \\ \rho - \eta & 0 \end{pmatrix}.$$

□

The real form of the Killing vector (8.2) also appears in the list of Lie point symmetries of (3.2) given in [9].

8.1.1 Symmetry reduction

In this section we shall look at the heavenly equation (3.2) with the additional constraint $\mathcal{L}_K g = \eta g$. This will lead to a new integrable equation describing a class of three–dimensional Einstein–Weyl geometries.

Proposition 8.2 *Every ASD vacuum metric with conformal symmetry is locally given by*

$$\begin{aligned}
\mathrm{d}s^2 &= e^{\eta t}(V^{-1}h + V(\mathrm{d}t + \omega)^2) \quad \text{where} && (8.3)\\
h &= -e^{2\rho u}\mathrm{d}w\mathrm{d}\tilde{w} - \frac{1}{16}(\eta^2 F\mathrm{d}u + \eta(F_w\mathrm{d}w - F_{\tilde{w}}\mathrm{d}\tilde{w}) - \mathrm{d}F_u)^2 && (8.4)\\
\omega &= \frac{\eta(\mathrm{d}F - F_u\mathrm{d}u) + (\mathrm{d}\tilde{w}\partial_{\tilde{w}} - \mathrm{d}w\partial_w)F_u}{\eta^2 F - F_{uu}}, \qquad V = \frac{1}{4}(\eta^2 F - F_{uu}), && (8.5)
\end{aligned}$$

and $F = F(w, \tilde{w}, u)$ is a holomorphic function on an open set $\mathcal{W} \subset \mathbb{C}^3$ which satisfies

$$(\eta F_{\tilde{w}} + F_{u\tilde{w}})(\eta F_w - F_{uw}) - (\eta^2 F - F_{uu})F_{w\tilde{w}} = 4e^{2\rho u} \qquad (8.6)$$

for constants $\eta, \rho \in \mathbb{C}$.

Corollary 8.3 *The metric h is defined on the space $\mathcal{W}$ of trajectories of K in $\mathcal{M}$. From Proposition* 2.3 *it follows that h is the most general Einstein–Weyl (EW) metric which arises as a reduction of ASD vacuum solutions. Equations* (8.6) *are therefore equivalent to the Einstein–Weyl equations* (2.25).

Proof. The general ASDV metric can locally be given by (8.1). From Lemma 8.1 it follows that we can take K as in (8.2). Perform the coordinate transformation $(z, \tilde{z}) \to (t, u)$ given by

$$2t := \ln(z^{1/m}\tilde{z}^{1/\tilde{m}}), \quad 2u := \ln(z^{1/m}\tilde{z}^{-1/\tilde{m}}).$$

In these coordinates $K = \partial_t$ and so $\Omega(t, u, w, \tilde{w}) = e^{\eta t}F(u, w, \tilde{w})$. The first heavenly equation is equivalent to (8.6). Rewriting the metric (8.1) in the new coordinate system yields (8.3) and $det(h) = -(1/4)V^2 e^{4\rho u}$.

The dual to K is $\mathbf{K} = e^{\eta t}V(\mathrm{d}t + \omega)$. From Proposition 2.3 we find the EW one-form to be

$$\begin{aligned}
\nu &= 2 *_g \frac{\mathbf{K}\wedge \mathrm{d}\mathbf{K}}{|K|^2} = 2e^{\eta t}V *_g ((\mathrm{d}t+\omega)\wedge \mathrm{d}\omega) \\
&= 2\eta\omega - \frac{\rho}{V}\frac{1}{2}(\eta^2 F\mathrm{d}u + \eta(F_w\mathrm{d}w - F_{\tilde{w}}\mathrm{d}\tilde{w}) - \mathrm{d}F_u) \\
&= 2\frac{(\eta-\rho)(\eta F_w - F_{uw})\mathrm{d}w + (\eta+\rho)(\eta F_{\tilde{w}} + F_{u\tilde{w}})\mathrm{d}\tilde{w} - \rho(\eta^2 F - F_{uu})\mathrm{d}u}{\eta^2 F - F_{uu}}
\end{aligned}$$

where $*_g$ and $*_h$ are the Hodge operators determined by g and h respectively.

□

8.2 Lax representation

In this section we shall represent equation (8.6) as the integrability condition for a linear system of equations. We shall interpret the Lax pair as a (mini twistor) distribution on a reduced projective spin bundle. The local coordinates on a projective primed spin bundle $\mathcal{F}$ are $(w, \tilde{w}, z, \tilde{z}, \lambda)$. Define the Lie lift of a Killing vector K to $\mathcal{F}$ by

$$\widetilde{K} := K + Q\partial_\lambda, \quad \text{where} \quad Q := \pi_{A'}\pi_{B'}\phi^{A'B'}/(\pi_{1'})^2. \tag{8.7}$$

The flow of $\widetilde{K}$ in $\mathcal{F}$ determines the behaviour of α-planes under the action of K in $\mathcal{M}$. The linear system L_A for equation (3.2) is given by (3.4). The vector fields $(L_0, L_1, \widetilde{K})$ span an integrable distribution. This can be seen as follows:

$$\begin{aligned} [K, L_A] &= -\pi^{A'}({\phi_{A'}}^{B'}{\epsilon_A}^B + {\psi_A}^B{\epsilon_{A'}}^{B'} + (1/2)\eta{\epsilon_A}^B{\epsilon_{A'}}^{B'})\nabla_{BB'} \\ &= -\pi^{C'}{\phi_{C'}}^{A'}\nabla_{AA'} + ({\psi_A}^B + \eta{\epsilon_A}^B)L_B. \end{aligned}$$

The Lie lift of K to $S^{A'}$ is

$$\widetilde{K} = K + \pi_{A'}\phi^{A'B'}\frac{\partial}{\partial\pi^{B'}} + \frac{1}{2}\eta\pi^{A'}\frac{\partial}{\partial\pi^{A'}}. \tag{8.8}$$

Now $[\widetilde{K}, L_A] = 0$ modulo L_A.

The projection of $\widetilde{K}$ to $\mathcal{F}$ is given by (8.7), where the factor $\pi_{1'}^2$ is used to dehomogenise a section of $\mathcal{O}(2)$. If K is given by (8.2) then $\widetilde{K} = K + \rho\lambda\partial_\lambda$. Introduce an invariant spectral parameter $\tilde{\lambda}$ (which is constant along $\widetilde{K}$) by $(\lambda, t) \longrightarrow (\tilde{\lambda} := \lambda e^{-\rho t}, \hat{t} := t)$. In the new coordinates

$$\partial_t = \partial_{\hat{t}} - \rho\tilde{\lambda}\partial_{\tilde{\lambda}}, \;\; \partial_\lambda = e^{-\rho\hat{t}}\partial_{\tilde{\lambda}}, \qquad \text{so that} \;\; \widetilde{K} = \partial_{\hat{t}}.$$

The linear system for the reduced equation is obtained from (3.4) by rewriting it in $(w, \tilde{w}, u, \hat{t}, \tilde{\lambda})$ coordinates and ignoring $\partial_{\hat{t}}$. This yields (after rescaling)

$$\begin{aligned} L_{0'} &= me^{\tilde{m}u}\Big(F_{w\tilde{w}}\Big(\frac{\partial}{\partial u} + \rho\tilde{\lambda}\frac{\partial}{\partial\tilde{\lambda}}\Big) + (\eta F_w - F_{uw})\frac{\partial}{\partial\tilde{w}}\Big) + 2\tilde{\lambda}\frac{\partial}{\partial w} \qquad (8.9) \\ L_{1'} &= \tilde{m}e^{\tilde{m}u}\Big((\eta F_{\tilde{w}} + F_{u\tilde{w}})\Big(\frac{\partial}{\partial u} + \rho\tilde{\lambda}\frac{\partial}{\partial\tilde{\lambda}}\Big) + (\eta^2 F - F_{uu})\frac{\partial}{\partial\tilde{w}}\Big) + 2\tilde{\lambda}\Big(\frac{\partial}{\partial u} - \rho\tilde{\lambda}\frac{\partial}{\partial\tilde{\lambda}}\Big). \end{aligned}$$

The mini-twistor space $\mathcal{Z}$ corresponding to solutions of (8.6) is the quotient of $\mathcal{F}$ by the integrable distribution $(L_{0'}, L_{1'}, \widetilde{K})$. This gives rise to a double fibration picture

$$\mathcal{W} \xleftarrow{p} \mathcal{F}_\mathcal{W} \xrightarrow{q} \mathcal{Z}. \tag{8.10}$$

where $\mathcal{F}_\mathcal{W} = \mathcal{F}/\widetilde{K}$ is coordinatised by $(w, \tilde{w}, u, \tilde{\lambda})$. The volume form on $\mathcal{F}_\mathcal{W}$ is

$$\nu_{F_W} = \mathrm{d}\tilde{\lambda} \wedge \mathrm{d}\hat{t} \wedge \mathrm{d}w \wedge \mathrm{d}\tilde{w}.$$

In Chapter 9 we shall study an interesting two form $\nu_{F_W}(L_{0'}, L_{1'}, ..., ...)$ on a reduced correspondence space.

8.3 Spinor formulation

The reason for using primed spinor indices for a minitwistor distribution comes from Hitchin's construction [30]; the basic mini-twistor correspondence states that points in $\mathcal{W}$ correspond in $\mathcal{Z}$ to rational curves with normal bundle $\mathcal{O}(2)$. Let l_x be the line in $\mathcal{Z}$ that corresponds to $x \in \mathcal{W}$. The normal bundle to l_x consists of tangent vectors at x (horizontally lifted to $T_{(x,\lambda)}\mathcal{F}_W$) modulo the twistor distribution. Therefore we have a sequence of sheaves over $\mathbb{C}P^1$

$$0 \longrightarrow D_W \longrightarrow \mathbb{C}^3 \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

The map $\mathbb{C}^3 \longrightarrow \mathcal{O}(2)$ is given by $V^{A'B'} \longrightarrow V^{A'B'}\pi_{A'}\pi_{B'}$. Its kernel consists of vectors of the form $\pi^{(A'}v^{B')}$ with $v^{B'}$ varying. The twistor distribution is therefore $D_W = O(-1) \otimes S^{A'}$ and so $L_{A'}$ is the global section of $\Gamma(D_W \otimes \mathcal{O}(1) \otimes S_{A'})$. Let Z be a totally geodesic two-plane corresponding to a point Z of a mini-twistor space. This two plane is spanned by vectors of the form $V^a = \pi^{(A'}v^{B')}$ with $\pi^{A'}$ fixed. Let $W^a = \pi^{(A'}w^{B')}$ be another vector tangent to Z. The Frobenius theorem implies that the Lie bracket $[V, W]$ must be tangent to some geodesic in Z, i.e.

$$[V, W] = aV + bW$$

for some a, b. The last equation determines the mini-twistor distribution. Consider

$$\pi^{B'}D_{A'B'}\pi_{C'} = -\Gamma_{A'B'C'D'}\pi^{B'}\pi^{D'} + \frac{1}{4}\pi_{A'}\nu_{B'C'}\pi^{B'} - \frac{1}{2}\varepsilon_{C'A'}\nu_{B'D'}\pi^{B'}\pi^{D'} + \frac{1}{2}\nu_{A'D'}\pi^{D'}\pi_{C'},$$

where $\Gamma_{A'B'C'D'}$ is a spinor Levi-Civita connection. Define $L_{A'}$ to be a horizontal lift of $\pi^{B'}D_{A'B'}$ to the weighted spin bundle by demanding $L_{A'}\pi_{C'} = 0$. This yields

$$\begin{aligned} L_{A'} &= \pi^{B'}D_{A'B'} + \Gamma_{A'B'C'D'}\pi^{B'}\pi^{D'}\frac{\partial}{\partial \pi_{C'}} \\ &+ \frac{1}{2}\nu_{B'D'}\pi^{B'}\Big(\pi^{D'}\frac{\partial}{\partial \pi^{A'}} - \frac{1}{2}\pi_{A'}\frac{\partial}{\partial \pi_{D'}} - \varepsilon_{A'}{}^{D'}\pi \cdot \frac{\partial}{\partial \pi}\Big). \end{aligned} \tag{8.11}$$

The integrability conditions imply

$$[L_{A'}, L_{B'}] = 0 \pmod{L_{A'}}.$$

In fact if one picks two independent solutions of a 'neutrino' equation on the EW background, say $\rho^{A'}$ and $\lambda^{A'}$, then $\widehat{L}_{0'} := \rho^{A'}L_{A'}$, and $\widehat{L}_{1'} := \lambda^{A'}L_{A'}$ commute exactly:

$$[\widehat{L}_{0'}, \widehat{L}_{1'}] = 0.$$

Let us now formulate Einstein–Weyl equations in terms of spinors. Let ϵ^{ijk} be the totally antisymmetric object on $\mathcal{W}$. We shall use the abstract index notation, and put $i = (A'B'), j = (C'D'), k = (E'F')$ etc. The metric h and the volume form vol_h are

$$h_{ij} = \varepsilon_{A'(C'}\varepsilon_{D')B'}, \qquad \epsilon_{ijk} = \varepsilon_{A'(C'}\varepsilon_{D')(E'}\varepsilon_{F')B'}.$$

In three dimensions the Ricci tensor of the Levi-Civita connection determines the Riemann tensor. However it is also true that the the Riemann tensor W_{ijkl} of the Weyl connection is determined by its Ricci tensor:

$$W_{ijkl} = \epsilon_{kl}^{m}\epsilon_{ij}^{n}(\frac{1}{2}h_{mn}W + W_{mn} - \frac{2}{3}W_{[mn]}) + \frac{2}{3}h_{ij}W_{[kl]}.$$

Therefore it is enough to study the spinor decomposition of the Ricci tensor $W_{ij} = W_{A'B'C'D'}$ into irreducible symmetric spinors

$$W_{ij} = \chi_{A'B'C'D'} + \psi_{A'(C'}\varepsilon_{D')B'} + \frac{1}{3}W\varepsilon_{A'(C'}\varepsilon_{D')B'},$$

where

$$\chi_{A'B'C'D'} = W_{(A'B'C'D')} = W_{(ij)} - \frac{1}{3}Wh_{ij},\ \psi_{A'B'} = \psi_i = \frac{1}{2}\epsilon_i{}^{jk}W_{jk},\ W = h^{ij}W_{ij}.$$

The spinor Levi–Civita connection

$$\nabla_{A'B'}\pi_{C'} = -\Gamma_{A'B'C'D'}\pi^{D'}$$

decomposes according to

$$\Gamma_{A'B'C'D'} = \phi_{A'B'C'D'} + \rho_{A'(C'}\varepsilon_{D')B'}$$

where

$$\phi_{A'B'C'D'} = \Gamma_{(A'B'C'D')}, \qquad \rho_{A'B'} = \rho_i = \Gamma_{jk}^{m}\epsilon^{jkn}\epsilon_{mni}.$$

The Einstein–Weyl equations are

$$\chi_{A'B'C'D'} = 0,$$

or

$$\Phi_{A'B'C'D'} + \frac{1}{2}\nabla_{(A'B'}\nu_{C'D')} + \frac{1}{4}\nu_{(A'B'}\nu_{C'D')} = 0,$$

where $\Phi_{A'B'C'D'}$ is the symmetric part of Ricci spinor of the Levi-Civita connection.

8.4 Reality conditions

To obtain real Einstein–Weyl metrics we have to impose reality conditions on coordinates $(w, \tilde{w}, z, \tilde{z})$

- The reduction from the Euclidean slice ($\overline{z} = \tilde{z}, \overline{w} = -\tilde{w}$) yields positive definite EW metrics with $u := iv$ for $v \in \mathbb{R}$. We can impose the condition $m\tilde{m} = 1$, so $\eta = \cos\alpha$ and $\rho = i\sin\alpha$. The Euclidean version of (8.6) is

$$(F\cos^2\alpha + F_{vv})F_{w\overline{w}} - (F_{\overline{w}}\cos\alpha - iF_{v\overline{w}})(F_w\cos\alpha + iF_{vw}) = 4e^{-2v\sin\alpha}. \quad (8.12)$$

  To obtain another form introduce G by $G = e^{v\sin\alpha}F$. The transformed equation, the metric (rescaled by $e^{2v\sin\alpha}$) and the EW one-form are:

$$(G + G_{vv} - 2G_v\sin\alpha)G_{w\overline{w}} - (e^{i\alpha}G_{\overline{w}} - iG_{v\overline{w}})(e^{-i\alpha}G_w + iG_{vw}) = 4, \quad (8.13)$$

$$\begin{aligned} h &= \mathrm{d}w\mathrm{d}\overline{w} + \frac{1}{16}(G\mathrm{d}v + \mathrm{d}G_v - 2G_v\sin\alpha\mathrm{d}v + ie^{i\alpha}G_w\mathrm{d}w - ie^{-i\alpha}G_{\overline{w}}\mathrm{d}\overline{w})^2 \\ \nu &= 2\frac{(e^{-i\alpha}G_w + iG_{vw})\mathrm{d}w + (e^{i\alpha}G_{\overline{w}} - iG_{v\overline{w}})\mathrm{d}\overline{w}}{G + G_{vv} - 2G_v\sin\alpha}. \end{aligned} \quad (8.14)$$

- On an ultra-hyperbolic slice we have $\overline{z} = \tilde{z}, \overline{w} = \tilde{w}$ which again implies $u = iv$. The metric (8.3) has signature $(+ + -)$. Another possibility is to take all coordinates as real. This gives a different real metric of signature $(+ + -)$. The function F is real and $\eta = \sinh\alpha, \rho = \cosh\alpha$.

The analogous reality conditions are imposed on the linear system (8.9). From now on we shall be mostly concerned with the positive definite case. The correspondence space is now viewed as a real six-dimensional manifold. The real lift of a Killing vector is

$$\widetilde{K} = \partial_t + i\sin\alpha(\lambda\partial_\lambda - \overline{\lambda}\partial_{\overline{\lambda}}).$$

8.5 Special cases

Solutions to (8.12) describe the most general EW metrics which arise as reductions of hyper-Kähler structures. In this section we look at limiting cases and recover special EW spaces [12], and LeBrun-Ward EW spaces which come from the $SU(\infty)$

Toda equation. The real form of the Killing vector (8.2) is a linear combination of a rotation and a dilation;

$$K = K_D \cos\alpha + K_R \sin\alpha, \qquad \alpha \in [-\pi/2, 0],\ K_D := z\partial_z + \overline{z}\partial_{\overline{z}},\ K_R := i(z\partial_z - \overline{z}\partial_{\overline{z}}).$$

8.5.1 LeBrun–Ward spaces

Take $\alpha = -\pi/2$. This is a pure Killing vector which does not preserve the complex structures on $\mathcal{M}$. This case was studied in [6, 87, 43]. Put $F_v = j, F_{\overline{w}} = \overline{p}$ and rewrite equation (8.12) as

$$\begin{aligned} \mathrm{d}\overline{p} \wedge \mathrm{d}j \wedge \mathrm{d}\overline{w} &= 4e^{2v}\mathrm{d}w \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}v \\ \mathrm{d}j \wedge \mathrm{d}w \wedge \mathrm{d}v &= \mathrm{d}\overline{p} \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}w. \end{aligned} \tag{8.15}$$

Use $(j, w, \overline{w})$ as coordinates and eliminate $\overline{p}$ to obtain

$$v_{w\overline{w}} - 2(e^{2v})_{jj} = 0 \tag{8.16}$$

which is the Boyer–Finley equation [6]. The metric (8.3) reduces to

$$h = e^{2v}\mathrm{d}w\mathrm{d}\overline{w} + \frac{1}{16}\mathrm{d}j^2, \qquad \nu = 2v_j\mathrm{d}j.$$

Let us come back to complex coordinates and put $w = e^{s+\theta}, \tilde{w} = e^{s-\theta}$ and $M = 2u + 2s$. In the (s, u, θ) coordinates equation (8.16) and the metric become

$$M_{ss} - M_{\theta\theta} - 8(e^M)_{jj} = 0, \qquad h = -e^M(\mathrm{d}s^2 - \mathrm{d}\theta^2) - \frac{1}{16}\mathrm{d}j^2.$$

Imposing a symmetry in $\theta = \ln(\sqrt{w/\tilde{w}})$ direction we arrive at

$$M_{ss} - 8(e^M)_{jj} = 0,$$

which was solved by Ward [87] who transformed it to a linear equation. The conclusion is that LeBrun–Ward EW metrics with $w\partial_w - \tilde{w}\partial_{\tilde{w}}$ symmetry are solved by the same anzatz as those with $\partial_w - \partial_{\tilde{w}}$ symmetry. In Subsection 8.6.1 it will be shown that imposing $\partial_w - \partial_{\tilde{w}}$ symmetry leads to a linear equation even if α is arbitrary.

8.5.2 Gauduchon–Tod spaces

Put $\alpha = 0$. This is a triholomorphic conformal symmetry. The corresponding EW metrics were in [25] called 'special'. The equation (8.12) reduces to

$$F_{w\overline{w}}(F + F_{vv}) - (F_w + iF_{vw})(F_{\overline{w}} - iF_{v\overline{w}}) = 4, \tag{8.17}$$

which is the form given in [76]. The corresponding Lax pair is

$$\begin{aligned} L_0 &= e^{iv}\Big(iF_{w\overline{w}}\frac{\partial}{\partial v} - (F_w + iF_{uw})\frac{\partial}{\partial \overline{w}}\Big) + 2\lambda\frac{\partial}{\partial w} \\ L_1 &= e^{iv}\Big((F_{v\overline{w}} + iF_{\overline{w}})\frac{\partial}{\partial v} - (F + F_{vv})\frac{\partial}{\partial \overline{w}}\Big) - 2i\lambda\frac{\partial}{\partial v}. \end{aligned}$$

8.6 Lie point symmetries

In order to find the Lie algebra of infinitesimal symmetries of (8.13) we shall convert it to system of differential forms [29]. Introduce Q and J by $J := G_{\overline{w}}, \quad Q := (e^{i\alpha}G - iG_v)$

$$\begin{aligned} \omega_1 &:= i\mathrm{d}Q \wedge \mathrm{d}J \wedge \mathrm{d}\overline{w} + e^{-i\alpha}(Q\mathrm{d}J - J\mathrm{d}Q) \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}v \\ &\quad +\mathrm{d}Q \wedge \mathrm{d}\overline{Q} \wedge \mathrm{d}v - 4\mathrm{d}w \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}v, \\ \omega_2 &:= \mathrm{d}Q \wedge \mathrm{d}w \wedge \mathrm{d}v + e^{i\alpha}J\mathrm{d}w \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}v - i\mathrm{d}J \wedge \mathrm{d}w \wedge \mathrm{d}\overline{w}. \end{aligned} \tag{8.18}$$

This system forms a closed differential ideal. Its integral manifold is a subspace of $\mathbb{R}^6$ on which $\omega_\mu = 0$. This integral manifold represents a solution to (8.13).

Let X be a vector field on $\mathbb{R}^6$. The action of X does not change the integral manifold if

$$\mathcal{L}_X\omega_\mu = \Lambda_\mu^\nu\omega_\nu$$

where Λ_μ^ν is a matrix of differential forms. The general solution is

$$\begin{aligned} X &= (Aw + B)\frac{\partial}{\partial w} + (\overline{A}\overline{w} + \overline{B})\frac{\partial}{\partial \overline{w}} + C\frac{\partial}{\partial v} + \frac{1}{2}(A + \overline{A})G\frac{\partial}{\partial G} \\ &+ D_1 e^{v\sin\alpha}\cos(v\cos\alpha)\frac{\partial}{\partial G} + D_2 e^{v\sin\alpha}\sin(v\cos\alpha)\frac{\partial}{\partial G}, \end{aligned}$$

where $A, B \in \mathbb{C}$, and $C, D_1, D_2 \in \mathbb{R}$ are constants[2]. Real generators are

$$X_1 = \partial_w + \partial_{\overline{w}}, \quad X_2 = i(\partial_w - \partial_{\overline{w}}), \quad X_3 = i(w\partial_w - \overline{w}\partial_{\overline{w}}) \tag{8.19}$$

[2]Note that a corresponding algebra of Lie point symmetries for the heavenly equation (3.2) is infinite dimensional [9]. In order to obtain a finite dimensional algebra one needs to factorize it by the infinite dimensional gauge algebra corresponding to the freedom in the definition of Ω. In our case the gauge freedom in Ω was already used to find the canonical form of the Killing vector. There is no residual gauge freedom in F.

$$
\begin{aligned}
X_4 &= \partial_v, \quad X_5 = w\partial_w + \overline{w}\partial_{\overline{w}} + G\partial_G, \\
X_6 &= e^{v\sin\alpha}\sin(v\cos\alpha)\partial_G, \quad X_7 = e^{v\sin\alpha}\cos(v\cos\alpha)\partial_G.
\end{aligned}
$$

The commutation relations between these vector fields are given by the following table, the entry in row i and column j representing $[X_i, X_j]$.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	X_2	0	X_1	0	0
X_2	0	0	$-X_1$	0	X_2	0	0
X_3	$-X_2$	X_1	0	0	0	0	0
X_4	0	0	0	0	0	$\sin\alpha X_6 + \cos\alpha X_7$	$\sin\alpha X_7 - \cos\alpha X_6$
X_5	$-X_1$	$-X_2$	0	0	0	$-X_6$	$-X_7$
X_6	0	0	0	$-\sin\alpha X_6 - \cos\alpha X_7$	X_6	0	0
X_7	0	0	0	$-\sin\alpha X_7 + \cos\alpha X_6$	X_7	0	0

This list of symmetries may seem disappointingly small (as equation (8.13) is an integrable PDE). Further symmetry properties reflecting the existence of infinitely many conservation laws will require the recursive procedure of constructing 'hidden symmetries'. This will be developed in Section 8.7.

8.6.1 Group invariant solutions

We can simplify equation (8.13) by looking at group invariant solutions. The finite transformation generated by X_7 does not change the metric. The one by X_5 rescales it by a constant factor. All transformations are conformal Killing vectors for h.

- $X_3 = i(w\partial_w - \overline{w}\partial_{\overline{w}})$ and the corresponding solutions depend on $(v, R := \ln(w\overline{w}))$. This will lead to a new 2D integrable system (8.21). Multiplying (8.13) by e^R yields

  $$(G + G_{vv} - 2\alpha G_v)G_{RR} - (e^{i\alpha}G_R - iG_{vR})(e^{-i\alpha}G_R + iG_{vR}) = 4e^R.$$

  The ideal (8.18) reduces to

  $$
  \begin{aligned}
  0 &= i\mathrm{d}Q \wedge \mathrm{d}J + e^{-i\alpha}(J\mathrm{d}Q \wedge \mathrm{d}v - Q\mathrm{d}J \wedge \mathrm{d}v) - 4\mathrm{d}(e^R) \wedge \mathrm{d}v, \\
  0 &= \mathrm{d}Q \wedge \mathrm{d}v - e^{i\alpha}J\mathrm{d}R \wedge \mathrm{d}v - i\mathrm{d}J \wedge \mathrm{d}R,
  \end{aligned}
  $$

  where $J = G_R$, $Q = (e^{i\alpha}G - iG_v)$. Eliminate Q and use (J, v) as coordinates to obtain an equation for $R(J, v)$

  $$4(e^R)_{JJ} + R_{vv} + 2(JR_J)_v \sin\alpha + J(JR_J)_J = 0. \tag{8.20}$$

With the definition $\xi := \ln J, M := M(v,\xi) = R - 2\xi$ we have

$$M_{vv} + 2M_{v\xi}\sin\alpha + M_{\xi\xi} + 4e^M(M_{\xi\xi} + {M_\xi}^2 + 3M_\xi + 2) = 0. \tag{8.21}$$

A simple solution is

$$G = e^{v\sin\alpha}\frac{w\overline{w}}{b} + 4e^{-v\sin\alpha}\frac{b}{1+3\sin^2\alpha}. \tag{8.22}$$

With the definition $w = \sqrt{r}e^{i\phi}$ the Einstein–Weyl metric is

$$h = (e^{2v\sin\alpha}\cos^2\alpha + 1)r^2\mathrm{d}\phi^2 + (\frac{1}{4}e^{2v\sin\alpha}r^2\cos^4\alpha + \frac{1}{2}r^2\cos^2\alpha + \frac{1}{4}e^{-2v\sin\alpha})\mathrm{d}v^2$$
$$+\frac{1}{4r}\mathrm{d}r^2 + (r\cos\alpha + e^{2v\sin\alpha}r^2\cos^3\alpha)\mathrm{d}v\mathrm{d}\phi.$$

It has

$$\Omega(w,z,\overline{w},\overline{z}) = (z\overline{z})^{(\cos^2\alpha)/2}\left(\frac{\overline{z}}{z}\right)^{(i\sin\alpha\cos\alpha)/2}\frac{w\overline{w}}{b} + (z\overline{z})^{1+(\cos\alpha)/2}\frac{4b}{1+3\sin^2\alpha}.$$

Calculation of curvature components shows it describes a flat metric on $\mathbb{R}^4$.

- $X_4 = \partial_v$. Equation (8.13) reduces to

$$GG_{w\overline{w}} - G_wG_{\overline{w}} = 4. \tag{8.23}$$

With no loss of generality[3] we can take

$$G = 4b + \frac{w\overline{w}}{b} \tag{8.24}$$

which (with the definition $w = re^{i\phi}$) gives

$$h = 4(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2) + \frac{1}{16}((4+r^2)\mathrm{d}v - 2r^2\cos\alpha\mathrm{d}\phi - 2r\sin\alpha\mathrm{d}r)^2. \tag{8.25}$$

Calculating the curvature components shows that the corresponding hyper-Kähler metric is flat. K.P. Tod [75] shows that (8.25) is conformal to the metric

[3] With the definition $\ln G = g$, (8.23) becomes the the Liouville equation

$$g_{w\overline{w}} = 4e^{-2g}.$$

Let $P := P(w)$ be an arbitrary holomorphic function of w. The general solution

$$e^g = \frac{i(P-\overline{P})}{4\sqrt{P_w\overline{P}_{\overline{w}}}}$$

can be generated from (8.24) by a Backlund transformation.

on the Berger sphere. The transformation of solution (8.24) corresponding to Lie point symmetries

$$G(w,\overline{w}) \longrightarrow \widetilde{G}(w,\overline{w},v) = Be^{-v\sin\alpha}(G(w,\overline{w})+g(v)),$$

where

$$g(v) = -4b + \frac{be^{2v\sin\alpha}}{B\cos^2\alpha} + Ce^{iv\cos\alpha} + \overline{C}e^{-iv\cos\alpha}$$

gives a new solution. In particular (8.22) can be obtained in this way. Therefore the metric corresponding to (8.22) also describes a Berger sphere. If $\alpha = 0$ then (8.24) and (8.22) coincide and give the standard metric on S^3. Put $b = 2$. We have $\nu = \mathrm{d}\ln(1+w\overline{w}/2)$. Rescale $\hat{h} = V^{-2}h$ to have $\hat{\nu} = 0$ and put

$$w = \tan\frac{\theta}{2}e^{i\phi}, \ \ \mathrm{d}v = \mathrm{d}\psi - \mathrm{d}\phi$$

to obtain

$$\hat{h} = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 + (\mathrm{d}\psi - \cos\theta\mathrm{d}\phi)^2$$

which is S^3.

- $X_2 = i(\partial_w - \partial_{\overline{w}})$ (or X_1). This reduction leads to a linear equation. Put $w + \overline{w} = f$ to obtain

$$(G + G_{vv} - 2\alpha G_v)G_{ff} - (e^{i\alpha}G_R - iG_{vf})(e^{-i\alpha}G_f + iG_{vf}) = 4.$$

With the definition $J := G_f, \ \ Q := (e^{i\alpha}G - iG_v)$ this yields

$$\begin{aligned} 0 &= i\mathrm{d}Q\wedge\mathrm{d}J + e^{-i\alpha}(J\mathrm{d}Q\wedge\mathrm{d}v - Q\mathrm{d}J\wedge\mathrm{d}v) - 4\mathrm{d}f\wedge\mathrm{d}v,\\ 0 &= \mathrm{d}Q\wedge\mathrm{d}v - e^{i\alpha}J\mathrm{d}f\wedge\mathrm{d}v - i\mathrm{d}J\wedge\mathrm{d}f. \end{aligned}$$

Now eliminate Q and use $(v, \xi = \ln J)$ as coordinates to obtain a linear equation for $f(\xi, v)$

$$4e^{-2\xi}(f_{\xi\xi} - f_\xi) + f_{vv} + 2\sin\alpha f_{\xi v} + f_{\xi\xi} = 0. \qquad (8.26)$$

8.7 Hidden symmetries

In this section we shall find a recursion procedure for generating 'hidden symmetries' of (8.12). We start with discussing the general conformally invariant wave equation in Einstein–Weyl background.

A tensor object T which transforms as

$$T \longrightarrow \phi^m T \qquad \text{when} \qquad h_{ij} \longrightarrow \phi^2 h_{ij}$$

is said to be conformally invariant of weight m. Let β be a p-form of weight m. The covariant derivative

$$D\beta := \mathrm{d}\beta - \frac{m}{2}\nu \wedge \beta$$

is a well defined $p+1$ form of weight m. Its Hodge dual, $*_h D\beta$, is a $(2-p)$-form of weight $m+1-p$. Therefore we can write the weighted Weyl wave operator which takes p-forms of weight m to $(3-p)$-forms of weight $m+1-p$

$$D *_h D = \Big(\mathrm{d} - \frac{m+1-p}{2}\nu\wedge\Big) *_h \Big(\mathrm{d} - \frac{m}{2}\nu\wedge\Big).$$

Consider the case $p = 0$. Let ϕ be a function of weight m. The most general wave equation is

$$D *_h D\phi = kW\phi \; vol_h$$

where k is some constant. The RHS has weight $m+1$ so the whole expression is conformally invariant. Adopting the index notation we obtain

$$\nabla^i\nabla_i\phi - \Big(m+\frac{1}{2}\Big)\nu^i\nabla_i\phi + \frac{1}{4}\Big(m(m+1)\nu^i\nu_i - 2m\nabla^i\nu_i\Big)\phi = k\Big(R + 2\nabla^i\nu_i - \frac{1}{2}\nu_i\nu^i\Big)\phi. \tag{8.27}$$

At this stage one can make some choices concerning the values of m and k. One can also fix the gauge freedom. In [12] it was assumed that $k = 0, m = -1$ and $\nabla_i\nu^i = 0$ (the Gauduchon gauge) which led to the derivative of the generalised monopole equation (2.27):

$$\nabla^i\nabla_i\phi + \frac{1}{2}\nu^i\nabla_i\phi = 0.$$

Another possibility is to set $m = -(1/2), k = 1/8$. With this choice equation (8.27) simplifies to

$$\nabla^i\nabla_i\phi = \frac{1}{8}R\phi,$$

which is the well known conformally invariant wave equation in the 3D Riemannian geometry. Note that the gauge freedom was not fixed to derive the last equation. All we did was to get rid of the 'non-Riemannian' data.

8.7.1 The recursion procedure

Let δF be a linearised solution to (8.6) (i.e. $F + \delta F$ satisfies (8.6) up to the linear terms in δF). Then

$$\Big([(\eta F_w - F_{uw})\frac{\partial^2}{\partial u \partial \tilde{w}} - (\eta F_{\tilde{w}} + F_{u\tilde{w}})\frac{\partial^2}{\partial u \partial w} - (\eta^2 F - F_{uu})\frac{\partial^2}{\partial \tilde{w} \partial w} + F_{w\tilde{w}}\frac{\partial^2}{\partial u^2}]$$
$$+\eta[(\eta F_w - F_{uw})\frac{\partial}{\partial \tilde{w}} + \eta(\eta F_{\tilde{w}} + F_{u\tilde{w}})\frac{\partial}{\partial w}]\Big)\delta F = F_{w\tilde{w}}\delta F. \tag{8.28}$$

This equation can be viewed more geometrically: let $\Box_\Omega$ denote the wave operator on an ASDV curved background given by Ω, let $\delta\Omega$ be the linearised solution to the first heavenly equation and let $\mathcal{W}_\Omega$ be the kernel of $\Box_\Omega$. From Lemma (3.1) we know that $\delta\Omega \in \mathcal{W}_\Omega$. Impose the additional constrain $\mathcal{L}_K \delta\Omega = \eta\delta\Omega$. This implies $\delta\Omega = e^{\eta t}\delta F$. This yields

$$\begin{aligned} 0 &= \mathrm{d} *_g \mathrm{d}(e^{\eta t}\delta F) \\ &= e^{\eta t}((\eta^2(\mathrm{d}t \wedge *_g \mathrm{d}t) + \eta \mathrm{d} *_g \mathrm{d}t)\delta F + \eta \mathrm{d}t \wedge *_g \mathrm{d}\delta F + \eta \mathrm{d}\delta F \wedge *_g \mathrm{d}t + \mathrm{d} *_g \mathrm{d}\delta F). \end{aligned}$$

But $\mathrm{d} *_g \mathrm{d}t = \Box_\Omega t = 0$ and

$$\mathrm{d}t \wedge *_g \mathrm{d}t = |\mathrm{d}t|^2 \nu_g = \frac{1}{4}\Omega_{w\tilde{w}} z^{-1/m} \tilde{z}^{-1/\tilde{m}} \nu_g = \frac{2}{V} e^{\eta t + 2\rho v} F_{w\tilde{w}} \mathrm{d}t \wedge vol_h.$$

Therefore (8.28) is equivalent to

$$\Box_\Omega \delta F + \eta^2 |\mathrm{d}t|^2 \delta F = 0.$$

It seems likely (although we have not proven it) that there exist a choice of m and k which, in the appropriate gauge, reduces equation (8.27) down to (8.28).

Let $\mathcal{W}_F$ be the space of solutions to (8.28) around a given solution F. We shall construct a map $R : \mathcal{W}_F \longrightarrow \mathcal{W}_F$. Recall that $\mathcal{W}_\Omega$ denotes the kernel of the curved wave operator determined by the solution Ω to the first heavenly equation.

Let $\phi \in \mathcal{W}_\Omega$. In Section 3.2 we constructed a recursion operator $R : \mathcal{W}_\Omega \longrightarrow \mathcal{W}_\Omega$ given by (3.12), where in coordinates $(w, \tilde{w}, t, u)$,

$$\begin{aligned}
\nabla_{00'} &= \frac{m}{2} e^{\rho t + \tilde{m} u}\Big(F_{w\tilde{w}}\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\Big) - (\eta F_w - F_{uw})\frac{\partial}{\partial \tilde{w}}\Big),\\
\nabla_{10'} &= \frac{1}{4} e^{-\eta t - 2\rho u}\Big((\eta F_{\tilde{w}} + F_{u\tilde{w}})\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\Big) - (\eta^2 F - F_{uu})\frac{\partial}{\partial \tilde{w}}\Big),\\
\nabla_{01'} &= \frac{\partial}{\partial w},\\
\nabla_{11'} &= \frac{1}{2m} e^{-m(t+u)}\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\Big).
\end{aligned}$$

To construct a reduced recursion operator we should be able to Lie derive (3.12) along K. In order to do so we introduce an invariant spin frame

$$\hat{o}^{A'} := e^{-(1/2)\rho t} o^{A'}, \quad \hat{\iota}^{A'} := e^{(1/2)\rho t} \iota^{A'},$$

in which $\tilde{\lambda} = (\pi_{A'}\hat{o}^{A'})/(\pi_{A'}\hat{\iota}^{A'})$. Note that now $\Gamma_{A'B'} \neq 0$. Recursion relations are

$$e^{-\rho t}\nabla_{A0'}(e^{\eta t} R\delta F) = \nabla_{A1'} e^{\eta t}\delta F.$$

This yields the following result

Proposition 8.4 *The map $R : \mathcal{W}_F \longrightarrow \mathcal{W}_F$ defined by*

$$\begin{aligned}
m e^{\tilde{m} u}(F_{w\tilde{w}}(\eta - \partial_u) - (\eta F_w - F_{uw})\partial_{\tilde{w}})R\delta F &= 2\partial_w \delta F \qquad (8.29)\\
\tilde{m} e^{\tilde{m} u}((\eta F_{\tilde{w}} + F_{u\tilde{w}})(\eta - \partial_u) - (\eta^2 F - F_{uu})\partial_{\tilde{w}})R\delta F &= 2(\eta + \partial_u)\delta F.
\end{aligned}$$

generates new elements of $\mathcal{W}_F$ from the old ones.

By cross differentiating we verify that two equations in (8.29) are consistent as a consequence of (8.6).

We start the recursion from two solutions ($e^{-\eta u}$, $\frac{2\tilde{m}}{m+\eta} e^{mu}$) to (8.28). Equations (8.29) yield

$$e^{-\eta u} \longrightarrow -\frac{\eta F + F_u}{2m} \longrightarrow ..., \qquad \frac{2\tilde{m}}{m+\eta} e^{mu} \longrightarrow F_w \longrightarrow ... .$$

Suppose that $F = F(u, w, \tilde{w}, \mathbf{T})$ depends on three local coordinates on a complex EW space and a sequence of parameters $\mathbf{T} = (T_2, T_3, ...)$. Put

$$\frac{\partial F}{\partial T_n} := R^n\Big(\frac{2\tilde{m}}{m+\eta} e^{mu}\Big),$$

so that $T_1 = w$. The recursion relations $R(\partial_{T_n} F) = \partial_{T_{n+1}} F$ form an over-determined system of equations which involve arbitrarily many independent variables, but initial data can be specified freely only on a two dimensional surface. To look for solutions to (8.6) invariant under some combination of hidden symmetries ('finite gap' type solutions) assume that

$$\sum_{i=1}^{n} c_i \frac{\partial F}{\partial T_i} = 0, \qquad \sum_{j=1}^{m} b_j \frac{\partial F}{\partial T_j} = 0.$$

for constants $c_1, ..., c_n, b_1, ..., b_m$. This would reduce (8.6) down to an ODE solvable by the *theta*–function (see Section 11.1 for a relevant discussion).

8.8 Conformal reduction of the second heavenly equation

In this section we shall look at a conformal reduction of the second heavenly equation. Let (w, z, x, y) be the coordinates on $\mathbb{C}^4$. Let $\Theta = \Theta(x, y, w, z)$ satisfy the second heavenly equation (3.6) and let $K = z\partial_z + x\partial_x$ be a homothetic Killing vector $\mathcal{L}_K g = g$. This implies (after some work) that $\mathcal{L}_K \Theta = \Theta$. Put $\Theta = zA(w, y, t := x/z)$ which yields

$$A_{tt}A_{yy} + A_{ty}^2 + A_{tw} + A_y - tA_{ty} = 0. \tag{8.30}$$

The metric (with $e^{xi} = z$) is

$$\mathrm{d}s^2 = 2e^{\xi}(\mathrm{d}\xi\mathrm{d}w + \mathrm{d}t\mathrm{d}w + \mathrm{d}\xi\mathrm{d}y + A_{tt}\mathrm{d}\xi^2 + A_{yy}\mathrm{d}w^2 - 2A_{ty}\mathrm{d}w\mathrm{d}\xi).$$

Write the reduced equation as

$$\mathrm{d}A_t \wedge \mathrm{d}A_y \wedge \mathrm{d}w + \mathrm{d}A_t \wedge \mathrm{d}t \wedge \mathrm{d}y + \mathrm{d}(A - tA_t) \wedge \mathrm{d}w \wedge \mathrm{d}t = 0.$$

Now eliminate t; Set $B = B(u, w, y) = A - tu$, $u = A_t$, $t = -B_u$ which yields[4]

$$B_{yy} + B_{uw} + B_u B_{uy} - B_y B_{uu} = 0. \tag{8.31}$$

[4]This equation is now known as the hyper-CR integrable system [95].

8.9 Alternative formulations

Here we shall give an alternative formulation (due to K.P. Tod) of equation (8.6). Define functions $(V, S, \widetilde{S})$ by

$$4V := \eta^2 F - F_{uu}, \qquad 2S := \eta F_w - F_{uw}, \ \ 2\widetilde{S} := \eta F_{\tilde{w}} + F_{u\tilde{w}},$$

so equation (8.6) takes the form

$$V = \frac{(-e^{2\rho u} + S\widetilde{S})\eta}{S_{\tilde{w}} + \widetilde{S}_w}, \quad S_u + \eta S = 2V_w, \quad -\widetilde{S}_u + \eta\widetilde{S} = 2V_{\tilde{w}}. \tag{8.32}$$

The hyper-Kähler metric is

$$\begin{aligned} \mathrm{d}s^2 &= e^{\eta t}(V(\mathrm{d}t^2 - \mathrm{d}u^2) + V^{-1}(S\widetilde{S} - e^{2\rho u})\mathrm{d}w\mathrm{d}\tilde{w} + S(\mathrm{d}t - \mathrm{d}u)\mathrm{d}w + \widetilde{S}(\mathrm{d}t + \mathrm{d}u)\mathrm{d}\tilde{w} \\ &= e^{nt}(V^{-1}h + V(\mathrm{d}t + \omega)^2) \end{aligned}$$

where

$$h := -e^{2\rho u}\mathrm{d}w\mathrm{d}\tilde{w} - \Big(V\mathrm{d}u + \frac{S\mathrm{d}w - \tilde{S}\mathrm{d}\tilde{w}}{2}\Big)^2 \quad \omega := \frac{S\mathrm{d}w + \tilde{S}\mathrm{d}\tilde{w}}{2V}.$$

The EW one form is

$$\nu = 2\eta\omega - \frac{2\rho e^3}{V} = \frac{\tilde{m}(S\mathrm{d}w + V\mathrm{d}u) + m(\tilde{S}\mathrm{d}\tilde{w} - V\mathrm{d}u)}{V}.$$

Euclidean reality conditions force $\overline{S} = -\widetilde{S}$ and V real. On the $++--$ slice we have $\overline{S} = \widetilde{S}$, or alternatively (on a different real slice) functions $V, S, \widetilde{S}$ real and independent. The orthonormal frame on the Euclidean slice is

$$\begin{aligned} e^1 &= \frac{1}{2}(e^{imv}\mathrm{d}w + e^{-i\overline{m}v}\mathrm{d}\overline{w}), \qquad \nabla_1 = e^{-imv}\partial_w + e^{i\overline{m}v}\partial_{\overline{w}} + i\frac{Se^{-imv} - \overline{S}e^{i\overline{m}v}}{2V}\partial_v \\ e^2 &= \frac{i}{2}(e^{-i\overline{m}v}\mathrm{d}\overline{w} - e^{imv}\mathrm{d}w), \qquad \nabla_2 = i(e^{-imv}\partial_w - e^{i\overline{m}v}\partial_{\overline{w}}) - \frac{Se^{-imv} + \overline{S}e^{i\overline{m}v}}{2V}\partial_v \\ e^3 &= V\mathrm{d}v - i\frac{S\mathrm{d}w - \overline{S}\mathrm{d}\overline{w}}{2}, \qquad \nabla_3 = \frac{1}{V}\partial_v. \end{aligned}$$

The EW one form is

$$\nu = \frac{\cos\alpha(S\mathrm{d}w + \overline{S}\mathrm{d}\overline{w}) + i\sin\alpha(\overline{S}\mathrm{d}\overline{w} - S\mathrm{d}w - 2iV\mathrm{d}v)}{V}$$

The three and four dimensional volume elements are:

$$\mathrm{vol}_h = \frac{i}{2} V e^{-2v\sin\alpha} \mathrm{d}w \wedge \mathrm{d}\overline{w} \wedge \mathrm{d}v, \qquad \nu_g = \frac{2}{V} e^{2v\sin\alpha + 2t\cos\alpha} vol_h \wedge \mathrm{d}t.$$

Equations (8.32) can be rewritten in a compact form

$$\mathrm{d}e^3 = \omega \wedge e^3 \cos\alpha + \frac{\cos\alpha}{V} e^1 \wedge e^2.$$

Chapter 9

Einstein–Weyl equations as a differential system on the spin bundle

In this chapter we shall reformulate Einstein–Weyl equations in terms of a certain two-form on the reduced correspondence space. Roughly speaking, if an Einstein–Weyl space admits a solution of a generalised monopole equation, which yields four dimensional ASD vacuum or Einstein metrics, then the four-dimensional correspondence space $\mathcal{F}_W = \mathcal{W} \times \mathbb{CP}^1$ is equipped with a two-form Π which satisfies

$$\mathrm{d}\Pi = 0, \qquad \Pi \wedge \Pi = 0, \tag{9.1}$$

where d is a full exterior derivative on $\mathcal{F}_{\mathcal{W}}$. We shall establish this fact in the next two sections. In Section 9.3 we shall find Einstein–Weyl structures corresponding to solutions to the dispersion-less Kadomtsev–Petviashvili equation[1].

9.1 Construction of the two form

Let K be a Killing vector on a general ASD conformal manifold $(\mathcal{M}, [g])$, and let $\Xi = D\pi_{A'} \wedge D\pi^{A'} \wedge \nu$ be a volume form on the non-projective primed spin bundle $S^{A'}$. Here $D\pi^{A'} := \mathrm{d}\pi^{A'} + \Gamma^{A'B'}\pi_{B'}$ is a pair of forms which annihilates horizontal vectors. Define the two form on $S^{A'}$

$$\widetilde{\Sigma} := \Xi(L_0, L_1, \widetilde{K}, \Upsilon, ..., ...). \tag{9.2}$$

[1]The results of this chapter have now appeared in [94]

Here $\Upsilon = \pi^{A'}/\partial\pi^{A'}$ is the Euler vector field on $S^{A'}$, L_A is the twistor distribution, and $\widetilde{K}$ is a Lie lift of K to $S^{A'}$. From

$$\mathcal{L}_\Upsilon\widetilde{\Sigma} = 4\widetilde{\Sigma}, \quad \Upsilon \lrcorner \widetilde{\Sigma} = 0$$

it follows that $\widetilde{\Sigma}$ descends to $\mathcal{F}$ where it takes values in $\mathcal{O}(4)$. Note however that $\mathrm{d}\widetilde{\Sigma}$ does not descend as

$$\Upsilon \lrcorner \mathrm{d}\widetilde{\Sigma} = \mathcal{L}_\Upsilon\widetilde{\Sigma} \neq 0.$$

Therefore to differentiate $\widetilde{\Sigma}$ on $\mathcal{F}$ we either need a connection on $\mathcal{O}(4)$ or a nonzero section of $\mathcal{O}(4)$ which could be used to dehomogenise $\widetilde{\Sigma}$. The three-form $\mathrm{d}\widetilde{\Sigma}$ nevertheless makes sense on $\Lambda^3T^*(S^{A'})$ and we have the following result:

Proposition 9.1 *The two form $\widetilde{\Sigma}$ defined by* (9.2) *satisfies*

$$\widetilde{\Sigma}\wedge\widetilde{\Sigma} = 0, \qquad \mathrm{d}\widetilde{\Sigma} = \beta\wedge\widetilde{\Sigma} \qquad \mathcal{L}_{\widetilde{K}}\widetilde{\Sigma} = 0 \tag{9.3}$$

for some one form β homogeneous of degree 0 *in $\pi^{A'}$.*

Proof. From the definition of $\widetilde{\Sigma}$ it follows that the integrable twistor distribution L_A belongs to the kernel of $\widetilde{\Sigma}$. Therefore equations (9.3) are implied by the Frobenius theorem C.2.

□

The one-form β is defined up to the addition of $\mathrm{d}(\ln\sigma)$ where σ is a twistor function homogeneous of degree 0.

The two-form (9.2) can be equivalently constructed from the twistor space. Let $\mathcal{T}$ be a non-projective twistor space corresponding to $(\mathcal{M}, [g])$, and let ϕ be a section of the canonical bundle of $\mathcal{T}$. Let $\mathcal{K}$ be the holomorphic vector field on $\mathcal{T}$ which corresponds to the Killing vector K on $\mathcal{M}$. Define

$$\widehat{\Sigma} := \phi(\mathcal{K}, \Upsilon, ..., ...), \tag{9.4}$$

where $\Upsilon = \pi^{A'}/\partial\pi^{A'} + \omega^A/\partial\omega^A$ is the homogeneity operator on $\mathcal{T}$. The two-form $\widehat{\Sigma}$ descends to $\mathcal{PT}$ where it takes its value in the dual canonical bundle K^*. Let $q : S^{A'} \longrightarrow \mathcal{T}$ be the standard factorisation by the twistor distribution. From

$$q^*\phi = \Xi(L_0, L_1, ..., ...)$$

it follows that $q^*\widehat{\Sigma} = \widetilde{\Sigma}$. By the Frobenius theorem (C.2) $\widetilde{\Sigma}$ defines an integrable distribution on $\mathcal{F}$. This is to be identified with a Lax pair for Einstein–Weyl equations. The factor space of the reduced spin bundle by this distribution is the mini-twistor space.

9.1.1 Hyper-Kähler case

Now assume that $(\mathcal{M}, g)$ is also vacuum. Consequently $\phi_{A'B'} = const$ and the spin bundle is equipped with a canonical divisor[2] $Q := \pi_{A'}\pi_{B'}\phi^{A'B'} \in \mathcal{O}(2)$ which descends to the reduced spin bundle[3] (9.1). It is easy to prove that now

$$\begin{aligned} \widetilde{\Sigma} &= \pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}\phi^{A'B'}\Sigma^{C'D'} + \pi_{A'}\pi_{B'}\pi_{C'}\mathrm{d}\pi^{C'} \wedge (K \lrcorner \Sigma^{A'B'}), \\ \beta &= \frac{4\phi_{A'B'}\pi^{A'}\mathrm{d}\pi^{B'}}{\pi_{A'}\pi_{B'}\phi^{A'B'}} = \mathrm{d}\ln Q^2. \end{aligned} \tag{9.5}$$

On the projective spin bundle $\mathcal{F}$ define

$$\Pi := Q^{-2}\widetilde{\Sigma}.$$

We have the following result:

Proposition 9.2 *The two-form Π is well defined on the Einstein–Weyl correspondence space $\mathcal{F}_W$. It satisfies*

$$\mathrm{d}\Pi = 0, \qquad \Pi \wedge \Pi = 0, \tag{9.6}$$

where $\mathrm{d} = \mathrm{d}x^i \otimes \partial_i + \mathrm{d}\tilde{\lambda} \otimes \partial_{\tilde{\lambda}}$ is the exterior derivative on $\mathcal{F}_W$. Any two linearly independent vectors $L_{A'}$ such that $L_{A'} \lrcorner S = 0$ form a Lax pair for the EW equations.

Proof. To prove the closure use (9.3) and (9.5). The simplicity follows from $\widetilde{\Sigma} \wedge \widetilde{\Sigma} = 0$. The form S descends to $\mathcal{F}_W$ because $\widetilde{K} \lrcorner \mathrm{d}\Pi = 0$ and $\mathrm{d}(\widetilde{K} \lrcorner \Pi) = 0$.

□

[2] We assume that $\phi_{A'B'} \neq 0$. If $\phi_{A'B'} = 0$ then K is triholomorphic and a section of $\mathcal{O}(2)$ which descends to the reduced spin bundle is $(\iota \cdot \pi)^2$ where $\iota_{A'}$ is any constant spinor.

[3] By the reduced spin bundle (correspondence space) we mean the space of orbits of $\widetilde{K}$ in $S^{A'}$ (in $\mathcal{F}$).

Figure 9.1: Divisor on a mini-twistor space.

Again, the twistor construction of Π is much simpler. The twistor space fibres over $\mathbb{CP}^1$ and so $\pi \cdot \mathrm{d}\pi$ is well defined on $\mathcal{PT}$. Let $\xi \in \mathcal{O}(4) \otimes \Lambda^3(T^*\mathcal{PT})$ be a holomorphic volume form. Killing vectors on $\mathcal{M}$ correspond in the double fibration picture to holomorphic volume preserving vector fields on $\mathcal{PT}$

$$\mathcal{K} = \mathcal{K}^A \frac{\partial}{\partial \omega^A} + \mathcal{K}^{A'} \frac{\partial}{\partial \pi^{A'}}.$$

Vector field $\mathcal{K}$ is obtained from $\widetilde{K}$ by $\mathcal{K}^A := \widetilde{K}(\omega^A)$ and $\mathcal{K}^{A'} := \phi^{A'}_{B'} \pi^{B'} + (\eta/2)\pi^{A'}$ where $\phi^{A'}_{B'}$ is a symmetric spinor. In fact we have the following

Proposition 9.3 *Let Σ be an $\mathcal{O}(2)$ valued symplectic form on the fibres of $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ and let $\mathcal{K}$ be a holomorphic volume preserving vector field on $\mathcal{PT}$ such that $\mathcal{L}_{\mathcal{K}}\Sigma = \eta\Sigma$. Then $K = p_* q^* \mathcal{K}$ is a conformal Killing vector on $\mathcal{M}$.*

Proof. We have

$$0 = \mathcal{L}_{\mathcal{K}}\xi = \eta\Sigma \wedge \pi \cdot \mathrm{d}\pi + \Sigma \wedge \mathcal{L}_{\mathcal{K}}(\pi \cdot \mathrm{d}\pi)$$

so $\mathcal{L}_{\mathcal{K}}\pi \cdot \mathrm{d}\pi = -\eta\pi \cdot \mathrm{d}\pi$ which yields $\mathcal{K}^{A'} = \phi^{A'}_{B'}\pi^{B'} + (\eta/2)\pi^{A'}$. The spinor $\phi^{A'}_{B'}$ is symmetric and homogeneous of degree 0 in $\pi^{A'}$. It also doesn't depend on ω^A, therefore it is constant. Lift $\mathcal{K}$ to cal $\mathcal{F}$ and fix λ. From the assumption

$$0 = \pi^{A'} \frac{\partial}{\partial \omega^{(A}} \mathcal{K}_{B)} = \pi^{A'} \pi^{B'} \frac{\partial}{\partial \omega^{(A}} K_{B)B'} = \nabla^{(A'}_{(A} K^{B')}_{B)}.$$

Conversely $C_{A'B'C'D'} = 0, \Phi_{ab} = 0$ imply that $\phi_{A'B'} = \mathit{const}$ and $\{\tilde{K}, L_A\}$ is an integrable distribution so $\widetilde{K}$ projects to a holomorphic vector yield on $\mathcal{PT}$. The condition $\nabla^{(A'}_{(A} K^{B')}_{B)} = 0$ proves that $\mathcal{L}_{\mathcal{K}} \Sigma = \eta \Sigma$.

□

The two form

$$\widehat{\Pi} := \frac{\mathcal{K} \lrcorner \, \xi}{\mathcal{K} \lrcorner \, (\pi \cdot \mathrm{d}\pi)} \tag{9.7}$$

descends to the mini-twistor space $\mathcal{Z}$. On $\mathcal{Z}$ it is closed and simple (as it is a two form on a two-dimensional manifold). The two-form Π from Proposition 9.2 is a pull back of $\widehat{\Pi}$ to $\mathcal{F}_W$.

9.1.2 ASD Einstein case

Consider the situation when $(\mathcal{M}, g)$ is ASD Einstein ($\Lambda \neq 0$). It turns out that in this case we can also find a divisor to dehomogenise $\widetilde{\Sigma}$ and the proposition (9.2) holds. This can be best seen from the twistor construction.

Let $\mathcal{PT}_E$ be a projective twistor space corresponding to solutions of ASD Einstein equations. It is not fibred over $\mathbb{CP}^1$ but nevertheless it is equipped with a contact structure $\tau \in \Lambda^2(T^*\mathcal{PT}_E) \otimes \mathcal{O}(2)$ such that [80]

$$\tau \wedge \mathrm{d}\tau = \Lambda \xi.$$

Although $\tau \wedge \mathrm{d}\tau$ is defined on $\mathcal{PT}_E$, $\mathrm{d}\tau$ is not as

$$\Upsilon \lrcorner \, \mathrm{d}\tau = \mathcal{L}_{\Upsilon} \tau = 2\tau \neq 0.$$

Therefore we choose to work on the non-projective twistor space $\mathcal{T}_E$.

Lemma 9.4 *If K is a Killing vector on an ASD Einstein manifold then the corresponding holomorphic vector field on the non-projective twistor space is Hamiltonian with respect to the symplectic structure* $\mathrm{d}\tau$*.*

Proof. Define a section of $O(2)$ by $Q := \mathcal{K} \lrcorner \, \tau$. We have $\mathrm{d}Q = \mathcal{L}_{\mathcal{K}} \tau - \mathcal{K} \lrcorner \, \mathrm{d}\tau = -\mathcal{K} \lrcorner \, \mathrm{d}\tau$ as $\mathcal{K}$ is a symmetry.

$\square$

The pull back of Q to $S^{A'}$ is $\phi_{A'B'}\pi^{A'}\pi^{B'}$. The difference with the vacuum case is that

$$\nabla_{AA'}\phi_{B'C'} = -\frac{2}{3}\Lambda\varepsilon_{A'(B'}K_{C')A} \neq 0.$$

The two form (9.4) is

$$\widehat{\Sigma} = \mathcal{K} \lrcorner (\Lambda\tau \wedge \mathrm{d}\tau) = Q^2\Lambda \mathrm{d}(\tau/Q),$$

so that $\widehat{\Sigma} \wedge \tau = Q\xi$. The two form on $\mathcal{F}$ given by

$$\Pi = q^*\Big(\frac{\widehat{\Sigma}}{Q^2}\Big)$$

satisfies equations (9.1). Therefore Einstein–Weyl metrics which come from ASD Einstein and hyper-Kähler four manifolds give rise to the same structure on the reduced spin bundle, which is in agreement with results of Przanowski [65] and Tod [74].

9.2 Examples

We shall now illustrate the construction of Π and Proposition 9.2 with some examples. In the case of reductions from ASD vacuum we can work with the covariantly constant spin frame (given by $\Gamma_{A'B'} = 0$), or with an invariant spin frame (characterised by $K = \widetilde{K}$). We shall compare these approaches. First assume that $(\mathcal{M}, g)$ is $\mathbb{C}^4$ with a flat metric. The flat twistor distribution and the lifted symmetry are:

$$L_0 = \partial_{\tilde{w}} - \lambda\partial_z, \quad L_1 = \partial_{\tilde{z}} - \lambda\partial_w, \quad \widetilde{K} = z\partial_z - \tilde{z}\partial_{\tilde{z}} + \lambda\partial_\lambda.$$

The volume form on $\mathcal{F}$ and the Gindikin two form are given by

$$\Xi = \mathrm{d}\lambda \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z} \wedge \mathrm{d}w \wedge \mathrm{d}\tilde{w}, \;\; \Sigma(\lambda) = -\lambda^2 \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z} + \lambda(\mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z}) + \mathrm{d}w \wedge \mathrm{d}z.$$

- In the covariantly constant frame we introduce

  $$2r := \ln(z\tilde{z}), \;\; 2\phi := \ln(z/\tilde{z}), \;\; \text{so that} \;\; \widetilde{K} = \partial_\phi + \lambda\partial_\lambda.$$

  In these coordinates

  $$\Sigma(\lambda) = -\lambda^2 e^{r-\phi}\mathrm{d}\tilde{w} \wedge (\mathrm{d}r - \mathrm{d}\phi) + \lambda(\mathrm{d}w \wedge \mathrm{d}\tilde{w} + 2e^{2r}\mathrm{d}r \wedge \mathrm{d}\phi) + e^{r+\phi}\mathrm{d}w \wedge (\mathrm{d}r + \mathrm{d}\phi)$$

and

$$\begin{aligned}\Pi &= \lambda^{-2}\Xi(\widetilde{K}, L_0, L_1, ..., ...)\lambda^{-2}(\mathrm{d}\lambda \wedge (K \lrcorner \Sigma) - \lambda\Sigma) \\ &= e^r(\mathrm{d}\overline{w} \wedge \mathrm{d}\tilde{\lambda} + \tilde{\lambda}^{-2}\mathrm{d}w \wedge \mathrm{d}\tilde{\lambda} + \tilde{\lambda}\mathrm{d}\tilde{w} \wedge \mathrm{d}r - \tilde{\lambda}^{-1}\mathrm{d}w \wedge \mathrm{d}r) \\ &+ 2\tilde{\lambda}^{-1}e^{2r}\mathrm{d}r \wedge \mathrm{d}\tilde{\lambda} - \mathrm{d}w \wedge \mathrm{d}\tilde{w} \end{aligned} \tag{9.8}$$

where $\tilde{\lambda} = \lambda e^{-\phi}$ is an invariant spectral parameter.

- In the invariant frame we use the coordinate system

$$(\tilde{\lambda}, s, r, w, \tilde{w}) \qquad \text{defined by} \qquad \tilde{\lambda} = \lambda e^{-\phi},\ s = \phi + \ln\lambda.$$

In these coordinates

$$\begin{aligned}\Xi &= \tilde{\lambda}^{-1/2}e^{2r+s/2}\mathrm{d}\tilde{\lambda} \wedge \mathrm{d}s \wedge \mathrm{d}r \wedge \mathrm{d}w \wedge \mathrm{d}\tilde{w}, \\ \widetilde{K} &= \partial_s, \\ L_0 &= \partial_{\tilde{w}} - (1/2)\tilde{\lambda}e^{-r}(\partial_r - \tilde{\lambda}\partial_{\tilde{\lambda}}),\ \ \lambda^{-1}L_1 = (1/2)\tilde{\lambda}^{-1}e^{-r}(\partial_r + \tilde{\lambda}\partial_{\tilde{\lambda}}) - \partial_w.\end{aligned}$$

This gives rise to

$$\Pi = (\tilde{\lambda}e^s)^{-1}\Xi(\partial_s, L_0, L_1, ..., ...),$$

which agrees with the formula (9.8).

- The two form Π can be also obtained as a pull-back from $\mathcal{PT}$.

Let (λ, μ^0, μ^1) be inhomogeneous coordinates on $\mathcal{PT}$. On the spin bundle they are given by

$$\lambda,\ \ \mu^1 = \lambda\tilde{w} + z,\ \ \mu^0 = \lambda\tilde{z} + w.$$

The holomorphic vector field on $\mathcal{PT}$ is $\mathcal{K} = \mu^0\partial_{\mu^0} + \lambda\partial_\lambda$. From (9.7) we have

$$\widetilde{\Sigma} = q^*(\mathcal{K} \lrcorner (\mathrm{d}\lambda \wedge \mathrm{d}\mu^0 \wedge \mathrm{d}\mu^1) = (\mu^0\mathrm{d}\lambda - \lambda\mathrm{d}\mu^1) \wedge \mathrm{d}\mu^1 = \lambda^2\mathrm{d}\mu^1 \wedge \mathrm{d}(\mu^0/\lambda).$$

Thus

$$\Pi = \lambda^{-2}\widetilde{\Sigma} = \mathrm{d}\mu^1 \wedge \mathrm{d}(\mu^0/\lambda) = \mathrm{d}\mathcal{P} \wedge \mathrm{d}\mathcal{Q}$$

where $\mathcal{P} = \tilde{w} + \tilde{\lambda}^{-1}e^r$ and $\mathcal{Q} = \tilde{\lambda}e^r + w$ are coordinates on mini-twistor space pulled back to the reduced spin bundle. Foliate reduced $\mathcal{F}$ by planes $R = const, S = const$ where $L_0 = \partial_{R_1}, L_1 = \partial_{R_2}$. This defines 2D mini-twistor space which is $\mathbb{CP}^1 \times \mathbb{CP}^1$. This can be seen from the transition relations for $(\mathcal{P}, \mathcal{Q})$.

Now analyse the curved case in the covariantly constant frame. Using the twistor distribution (3.4) we find

$$\Pi = e^{-r}(\mathrm{d}\tilde{w}\wedge \mathrm{d}\tilde{\lambda} + \frac{1}{\tilde{\lambda}^2}\mathrm{d}w\wedge \mathrm{d}\tilde{\lambda} + \tilde{\lambda}\mathrm{d}\tilde{w}\wedge \mathrm{d}r - \frac{1}{\tilde{\lambda}}\mathrm{d}w\wedge \mathrm{d}r) - \partial\tilde{\partial}\Omega - \frac{1}{2\tilde{\lambda}}\mathrm{d}\Omega_r\wedge \mathrm{d}\tilde{\lambda}, \quad (9.9)$$

where

$$\partial := \mathrm{d}w\otimes\partial_w + \frac{1}{2}\mathrm{d}r\otimes\partial_r, \quad \tilde{\partial} := \mathrm{d}\tilde{w}\otimes\partial_{\tilde{w}} + \frac{1}{2}\mathrm{d}r\otimes\partial_r.$$

It is obvious that $\mathrm{d}\Pi = 0$. The simplicity condition yields field equations:

$$\Pi\wedge\Pi = \mathrm{d}r\wedge \mathrm{d}\ln\tilde{\lambda}\wedge \mathrm{d}w\wedge \mathrm{d}\tilde{w}(4e^{2r} + \Omega_{rr}\Omega_{w\tilde{w}} - \Omega_{r\tilde{w}}\Omega_{rw}),$$

so

$$\Omega_{r\tilde{w}}\Omega_{rw} - \Omega_{rr}\Omega_{w\tilde{w}} = 4e^{2r}.$$

Put $\Omega_r = J,\ \ \Omega_{\tilde{w}} = q$ and make use of a hodograph transformation (8.15) to obtain the $SU(\infty)$ Toda equation

$$r_{w\tilde{w}} = 2e^{2r}_{JJ}.$$

If we choose to work in the invariant frame:

$$\tilde{\lambda} = \lambda e^{-\phi}, \quad s = 1/2\ln(z/\tilde{z}) + \ln\lambda,$$

then Π is obtained by the contraction of the rescaled volume form

$$\lambda^{-1}\Xi = e^{2r}\mathrm{d}\ln\tilde{\lambda}\wedge \mathrm{d}s\wedge \mathrm{d}r\wedge \mathrm{d}w\wedge \mathrm{d}\tilde{w}$$

with ∂_s together with the Lax pair for the $SU(\infty)$ Toda equation

$$\begin{aligned} L_0 &= \frac{1}{4}e^{-2r}(\Omega_{r\tilde{w}}(\partial_r + \tilde{\lambda}\partial_{\tilde{\lambda}}) - \Omega_{rr}\partial_{\tilde{w}}) - \frac{\tilde{\lambda}}{2}e^{-r}(\partial_r - \tilde{\lambda}\partial_{\tilde{\lambda}}) \\ L_1 &= e^{s/2}\sqrt{\tilde{\lambda}}(\frac{e^{-r}}{2\tilde{\lambda}}\Omega_{w\tilde{w}}(\partial_r + \tilde{\lambda}\partial_{\tilde{\lambda}}) - \frac{e^{-r}}{2\tilde{\lambda}}\Omega_{wr}\partial_{\tilde{w}} - \partial_w). \end{aligned}$$

To finalise, we shall look at the most general reduction of ASD vacuum metric. The two-form corresponding to the solution of equation (8.6) is

$$\begin{aligned} \Pi &= \mathrm{d}\lambda\wedge\frac{K\lrcorner\Sigma(\lambda)}{Q^2} - \frac{\Sigma(\lambda)}{Q} \qquad (9.10) \\ &= e^{-\eta\Lambda}((\eta-\rho)e^{\rho(\Lambda+u)-\eta u}\mathrm{d}\tilde{w}\wedge \mathrm{d}(\Lambda-u) + (\eta+\rho)e^{-\rho(\Lambda-u)+\eta u}\mathrm{d}w\wedge \mathrm{d}(\Lambda+u) \\ &+ 2V\mathrm{d}\Lambda \mathrm{d}u + S\mathrm{d}w\wedge \mathrm{d}(\Lambda-u) - \widetilde{S}\mathrm{d}\tilde{w}\wedge \mathrm{d}(\Lambda+u) - \eta^{-1}(S_{\tilde{w}} + \widetilde{S}_w)\mathrm{d}w\wedge \mathrm{d}\tilde{w}. \end{aligned}$$

Here $\Lambda := \rho^{-1} \ln \tilde{\lambda}$ is an invariant spectral parameter. In the derivation we used a covariantly constant spin frame.

Formulae (9.1) resemble Gindikin's formulation (2.16) of ASD vacuum condition. This motivates a practical method of constructing Π:

1. rewrite the two-form $\pi_{A'}\pi_{B'}\Sigma^{A'B'}$ in the coordinates in which $K = \partial_t$,
2. replace all $\mathrm{d}t$s by $\mathrm{d}\Lambda$s for a suitably defined invariant spectral parameter Λ,
3. put
$$\Pi = \frac{\pi_{A'}\pi_{B'}\Sigma^{A'B'}(\mathrm{d}t \to \mathrm{d}\Lambda)}{Q}.$$

9.3 Einstein–Weyl spaces from the dispersion-less Kadomtsev–Petviashvili equation

In this section we shall use Proposition 9.2 to construct Einstein–Weyl metrics out of solutions to the dispersion-less Kadomtsev–Petviashvili (dKP) equation.

Following Krichever [42] define

$$\Omega_2 := \frac{\lambda^2}{2} + u, \quad \Omega_3 := \frac{\lambda^3}{3} + \lambda u + w$$

for some $u = u(x, y, t)$ and $w = w(x, y, t)$. The two form

$$\Pi = \mathrm{d}x \wedge \mathrm{d}\lambda + \mathrm{d}y \wedge \mathrm{d}\Omega_2 + \mathrm{d}t \wedge \mathrm{d}\Omega_3 \tag{9.11}$$

is closed by its definition. It is also simple iff u and w satisfy

$$w_x = u_y, \qquad u_t - uu_x = w_y.$$

Eliminating w yields the dKP equation

$$(u_t - uu_x)_x = u_{yy}. \tag{9.12}$$

Therefore, by Proposition 9.2, it should correspond to some Einstein–Weyl geometry. The simplicity gives rise to

$$[\partial_y + X_{\Omega_2}, \partial_t + X_{\Omega_3}] = 0$$

where X_h stands for a vector field Hamiltonian with respect to $\mathrm{d}\lambda \wedge \mathrm{d}x$. Define a triad of vectors

$$\nabla_{0'0'} := \partial_x, \quad \nabla_{0'1'} := \partial_y, \quad \nabla_{1'1'} := \partial_t - u\partial_x$$

and introduce a Lax pair (which is a kernel of (9.11))

$$L_{0'} := \nabla_{0'1'} - \lambda\nabla_{0'0'} + (\nabla_{0'0'}u)\partial_\lambda, \qquad L_{1'} := \nabla_{1'1'} - \lambda\nabla_{1'0'} + (\nabla_{1'0'}u)\partial_\lambda,$$

or $L_{A'} = \pi^{B'}\nabla_{A'B'} + (o^{B'}\nabla_{B'A'}u)\partial_{\tilde{\lambda}}$, which has the form (8.11). The dKP equation is equivalent to

$$[L_{1'}, L_{0'}] = -u_x L_{0'}, \quad \text{or} \quad [L_{1'} - \lambda L_{0'}, L_{0'}] = 0.$$

The next proposition shows that we can find a one form ν such that $\nabla_{A'B'}$ is a contravariant triad for an EW metric:

Proposition 9.5 *Let $u := u(x, y, t)$ be a solution of the dKP equation* (9.12). *Then the metric and the one-form*

$$h = \mathrm{d}y^2 - 4\mathrm{d}x\mathrm{d}t - 4u\mathrm{d}t^2, \qquad \nu = -4u_x\mathrm{d}t \tag{9.13}$$

give an EW structure.

Proof. Let $x^1 := x, x^2 := y, x^3 := t$. Define trace-free part of the Ricci tensor of the Weyl connection

$$\chi_{ij} := R_{ij} + \frac{1}{2}\nabla_{(i}\nu_{j)} + \frac{1}{4}\nu_i\nu_j - \frac{1}{3}\Big(R + \frac{1}{2}\nabla^k\nu_k + \frac{1}{4}\nu^k\nu_k\Big)h_{ij}$$

Five (out of six) EW equations $\chi_{ij} = 0$ are satisfied identically by anzatz (9.13). The equation $\chi_{33} = 0$ is equivalent to (9.12). We also find $W = u_{xx}$.

□

Chapter 10

ASD vacuum metrics from soliton equations

In this chapter we explain how to construct solutions to the anti-self-dual Einstein vacuum equations from solutions of various two-dimensional integrable systems by exploiting the fact that the Lax formulations of both systems can be embedded in that of the anti-self-dual Yang–Mills equations. We illustrate this by constructing explicit ASDV metrics on $\mathbb{R}^2 \times \Sigma$, where Σ is a homogeneous space for a real subgroup of $SL(2, \mathbb{C})$ associated with the two-dimensional system.

Ward [82] has observed that many integrable systems in two dimensions may be obtained from the anti-self-dual Yang–Mills (ASDYM) equations reduced by two translations. On the other hand Proposition 2.1 implies, that solutions of the ASDYM equations with two translational symmetries and gauge group SDiff(Σ) for some two-manifold Σ determine solutions of the ASDVE equations (see also [86]). But $SL(2, \mathbb{C})$ $\in$SDiff(Σ), therefore the reduced $SL(2, \mathbb{C})$ ASDYM will give rise to ASDV metrics.

In the next section we review briefly the classification of two-dimensional integrable systems arising from the $SL(2, \mathbb{C})$ ASDYM equations. Section 10.2 is devoted to the construction of normalised null tetrads and hence metrics on $\mathbb{R}^2 \times \Sigma$ from the ASDYM Lax pairs for the two-dimensional integrable systems. In the last section we outline the twistor interpretation of the construction.

10.1 Anti-self-dual Yang-Mills and 2D integrable systems

For the notation and conventions see Section 2.5. We shall consider the reality conditions for real ultra-hyperbolic spaces, recovered by imposing $w = x - y$, $z = t + v$, $\tilde{w} = x + y$, $\tilde{z} = t - v$. (Reality conditions for Euclidean space are recovered by imposing $\tilde{w} = \bar{w}$ and $\tilde{z} = -\bar{z}$.) Solutions to (2.21–2.23) can be real for this choice of signature.

We fix the gauge group to be $SL(2, \mathbb{C})$ or one of its real subgroups. Conformal reduction of the ASDYM equations involves the choice of the group H of conformal isometries of $\mathbb{M}$. We shall restrict ourselves to the simplest case and suppose that a connection A is invariant under the flows of two independent translational Killing vectors X and Y. These reductions are classified partially by the signature of the metric restricted to two-plane spanned by the translations.

1) **Nondegenerate cases ($\mathbf{H_1}$)**

a) $X = \partial_w - \partial_{\tilde{w}}$, $Y = \partial_z - \partial_{\tilde{z}}$.

$$\begin{aligned}
A_w &= \frac{1}{4}\begin{pmatrix} \phi_t & -2\cos(\phi/2) \\ -2\cos(\phi/2) & -\phi_t \end{pmatrix}, \\
A_{\tilde{w}} &= \frac{1}{4}\begin{pmatrix} \phi_t & 2\cos(\phi/2) \\ 2\cos(\phi/2) & -\phi_t \end{pmatrix}, \\
A_z &= \frac{1}{4}\begin{pmatrix} -\phi_x & -2\sin(\phi/2) \\ 2\sin(\phi/2) & \phi_x \end{pmatrix}, \\
A_{\tilde{z}} &= \frac{1}{4}\begin{pmatrix} -\phi_x & 2\sin(\phi/2) \\ -2\sin(\phi/2) & \phi_x \end{pmatrix}.
\end{aligned} \tag{10.1}$$

The ASDYM equations are satisfied in ultra-hyperbolic signature if $\phi_{xx} + \phi_{tt} = \sin\phi$; the elliptic sine-Gordon equation.

b) $G = SU(2)$, $X = \partial_w$, $Y = \partial_{\tilde{w}}$.

$$A_{\tilde{z}} = 0, \qquad A_w = cos\phi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + sin\phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$A_{\tilde{w}} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad A_z = 1/2(\phi_v - \phi_t)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{10.2}$$

The ASDYM equations in ultra-hyperbolic signature yield $\phi_{tt} - \phi_{vv} = 4\sin\phi$, the hyperbolic sine-Gordon equation.

For details of these reductions, see [2]. Note also that if we reduce from Euclidean signature we obtain Hitchin's Higgs bundle equations (which can also be represented as harmonic maps from $\mathbb{R}^2$ to $SL(2,\mathbb{C})/G$ where G is $SU(2)$ or $SU(1,1)$) [49].

2) **Partially degenerate case ($\mathbf{H_2}$)**

We consider ultra-hyperbolic signature with $X = \partial_w - \partial_{\tilde{w}}$ and $Y = \partial_{\tilde{z}}$.

a)

$$A_w = \begin{pmatrix} q & 1 \\ b & -q \end{pmatrix},\ A_{\tilde{w}} = 0,\ 2A_z = \begin{pmatrix} b_x & -2q_x \\ 2w & -b_x \end{pmatrix},\ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{10.3}$$

where $4w = q_{xxx} - 4qq_x - 2{q_x}^2 + 4q^2 q_x$ and $b = q_x - q^2$. The ASDYM equations (with the definition $u = -q_x$) are equivalent to the Korteweg de Vries equation $4u_z = u_{xxx} + 12uu_x$. The reduced Lax pair (2.20) yields a zero curvature representation of KdV.

b)

$$\begin{aligned} A_w &= \begin{pmatrix} 0 & \phi \\ \mp\overline{\phi} & 0 \end{pmatrix},\ A_{\tilde{w}} = 0, \\ A_z &= i\begin{pmatrix} |\,\phi\,|^2 & \pm\phi_x \\ \overline{\phi}_x & -|\,\phi\,|^2 \end{pmatrix},\ 2A_{\tilde{z}} = \pm i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{10.4}$$

Here the upper (lower) sign corresponds to $G = SU(2)$ (or $SU(1,1)$). The ASDYM become $i\phi_z = -\phi_{xx} \mp 2|\,\phi\,|^2\phi$ which is the nonlinear Schrödinger equation with an attractive (respectively repulsive) self interaction [47].

10.2 Anti-self-dual metrics on principal bundles

We connect the anti-self-duality equations on a Yang-Mills field and those on a four-dimensional metric by considering gauge potentials that take values in a Lie algebra of vector fields on some manifold. Proposition 2.1 reveals one such connection:

$$\nabla_{AA'} = \begin{pmatrix} \widetilde{Z} & W \\ \widetilde{W} & Z \end{pmatrix}$$

and $W, \widetilde{W}, Z$ and $\widetilde{Z}$ are generators of the group of volume-preserving (holomorphic) diffeomorphisms of $(\mathcal{M},\,\nu)$. We make the identification: $W = D_w,\ \widetilde{W} = D_{\tilde{w}},\ Z =$

$D_z,\ \widetilde{Z} = D_{\tilde{z}}$. By comparing (2.11) with (2.20), we see that the half flat equation is a reduction of the ASDYM with this gauge group by translations along the four coordinate vectors $\partial_w,\ \partial_{\tilde{w}},\ \partial_z,\ \partial_{\tilde{z}}$.

In order to understand the relationship with two-dimensional integrable systems, we look at this in a slightly different way. Let $(\Sigma,\ \Omega_\Sigma)$ be a two-dimensional symplectic manifold and let SDiff(Σ) be the group of canonical transformations of Σ. Consider the ASDYM equations with the gauge group G, where G is the subgroup of SDiff(Σ). We can represent the components of the connection form of D by Hamiltonian vector fields and hence by Hamiltonians on Σ depending also on the coordinates on $\mathbb{M}$:

$$W = \partial_w - X_{H_w},\ \widetilde{W} = \partial_{\tilde{w}} - X_{H_{\tilde{w}}},\ Z = \partial_z - X_{H_z},\ \widetilde{Z} = \partial_{\tilde{z}} - X_{H_{\tilde{z}}} \tag{10.5}$$

where X_{H_μ} denotes the Hamiltonian vector field corresponding to A_μ with Hamiltonian H_μ.

Now we suppose that D is invariant under two translations. The reduced Lax pair will then descend to $\mathbb{R}^2 \times \Sigma$ and give rise to a half flat metric. This requires that the gauge group is a subgroup of the canonical transformations of Σ. Although it has been observed that SDiff$(\Sigma) \approx SL(\infty)$, it seems that $SL(n, \mathbb{C})$ is a subgroup of such defined $SL(\infty)$ only for $n = 2$. In this case we can take the linear action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$ or a Möbius action of $SU(2)$ and $SU(1, 1)$ on $\mathbb{CP}^1$ or D (the Poincaré disc) respectively. We shall restrict ourselves to real vector fields, which will imply that our ASDV metrics will have ultra-hyperbolic signature (Euclidean examples can also be obtained in a similar way).

To be more explicit we write down the Hamiltonian[1] corresponding to the matrix

$$A_\mu = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in LSL(2, \mathbb{C}).$$

In the three cases we have

$$\Sigma = R^2,\ \ \Omega_\Sigma = \mathrm{d}m \wedge \mathrm{d}n,\ \ H_\mu = (\frac{bn^2}{2} + amn - \frac{cm^2}{2}), \tag{10.6}$$

$$\Sigma = \mathbb{CP}^1,\ \ \Omega_\Sigma = \frac{i\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}}{(1 + \xi\bar{\xi})^2},\ \ H_\mu = -i\frac{\xi\bar{b} - \bar{\xi}b + 2a}{1 + \xi\bar{\xi}}, \tag{10.7}$$

[1] We only require the representation of A_μ by volume-preserving vector fields on Σ; Hamiltonians are defined up to the addition of a function of the (residual) space variables, but different choices of such functions do not change the metric.

$$\Sigma = D, \quad \Omega_\Sigma = \frac{i \mathrm{d}\xi \wedge \mathrm{d}\overline{\xi}}{(1-\xi\overline{\xi})^2}, \quad H_\mu = -i\frac{\xi\overline{b} - \overline{\xi}b - 2a}{1-\xi\overline{\xi}}. \tag{10.8}$$

The covariant metric is conveniently expressed in terms of the dual frame (see Proposition 2.1)

$$g = 2f^2(e_W \odot e_{\widetilde{W}} - e_Z \odot e_{\widetilde{Z}}), \tag{10.9}$$

where

$$\begin{aligned} e_W = f^{-2}\nu(..., \widetilde{W}, Z, \widetilde{Z}) \quad &, \quad e_{\widetilde{W}} = f^{-2}\nu(W, ..., Z, \widetilde{Z}) \\ e_Z = f^{-2}\nu(W, \widetilde{W}, ..., \widetilde{Z}) \quad &, \quad e_Z = f^{-2}\nu(W, \widetilde{W}, Z, ...). \end{aligned} \tag{10.10}$$

Self-dual two-forms on $\mathcal{M}$ are

$$\alpha = f^2 e_W \wedge e_Z, \quad \omega = f^2(e_W \wedge e_{\widetilde{W}} - e_Z \wedge e_{\widetilde{Z}}), \quad \tilde{\alpha} = f^2 e_{\widetilde{W}} \wedge e_{\widetilde{Z}}. \tag{10.11}$$

The metric and two-forms obtained after the two-dimensional reductions of ASDYM are as follows: **(i)** $\mathbf{H_1}$ $(X = \partial_w, Y = \partial_{\tilde{w}})$, $\nu = \mathrm{d}z \wedge \mathrm{d}\tilde{z} \wedge \Omega_\Sigma$

$$f^2 = \nu(W, \widetilde{W}, Z, \widetilde{Z}) = \Omega_\Sigma(W, \widetilde{W}) = \{H_w, H_{\tilde{w}}\} = F_{w\tilde{w}}. \tag{10.12}$$

In the last formula $F_{w\tilde{w}}$ is a function rather then a matrix. This follows from the identification (via (10.6)-(10.8)) of 2×2 matrices in the Lie algebra of $SL(2, \mathbb{C})$ and Hamiltonians. Let d_Σ stand for the exterior derivative on Σ.

$$\begin{aligned} e_W &= f^{-2}(\Omega_\Sigma(\widetilde{W}, Z)\mathrm{d}z + \Omega_\Sigma(\widetilde{W}, \widetilde{Z})\mathrm{d}\tilde{z} + \Omega_\Sigma(..., \widetilde{W})) \\ &= f^{-2}(\{H_{\tilde{w}}, H_z\}\mathrm{d}z + \{H_{\tilde{w}}, H_{\tilde{z}}\}\mathrm{d}\tilde{z} - \mathrm{d}_\Sigma H_{\tilde{w}}) \\ e_{\widetilde{W}} &= f^{-2}(\{H_w, H_z\}\mathrm{d}z - \{H_w, H_{\tilde{z}}\}\mathrm{d}\tilde{z} + \mathrm{d}_\Sigma H_w) \\ e_Z &= \mathrm{d}z \\ e_{\widetilde{Z}} &= \mathrm{d}\tilde{z} \end{aligned} \tag{10.13}$$

$$\begin{aligned} \Sigma^{1'1'} &= -\{H_{\tilde{w}}, H_{\tilde{z}}\}\mathrm{d}z \wedge \mathrm{d}\tilde{z} - \mathrm{d}_\Sigma H_{\tilde{w}} \wedge \mathrm{d}z \\ \Sigma^{0'1'} &= (\{H_z, H_{\tilde{z}}\} - \{H_w, H_{\tilde{w}}\})\mathrm{d}z \wedge \mathrm{d}\tilde{z} + \Omega_\Sigma + \mathrm{d}_\Sigma H_z \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde{z}} \wedge \mathrm{d}\tilde{z} \\ \Sigma^{0'0'} &= \{H_w, H_z\}\mathrm{d}z \wedge \mathrm{d}\tilde{z} + \mathrm{d}_\Sigma H_w \wedge \mathrm{d}\tilde{z}. \end{aligned}$$

The gauge freedom is used to set $A_{\tilde{z}}$ (and hence $H_{\tilde{z}}$) to 0.

$$\mathrm{d}s^2 = \frac{2}{\{H_w, H_{\tilde{w}}\}}\Big(- (\{H_{\tilde{w}}, H_z\}\{H_w, H_z\})\mathrm{d}z^2 - \{H_w, H_{\tilde{w}}\}^2 \mathrm{d}z\mathrm{d}\tilde{z}$$

$$-(\partial_\xi H_w \partial_\xi H_{\tilde w})\mathrm{d}\xi^2 - (\partial_{\overline{\xi}} H_w \partial_{\overline{\xi}} H_{\tilde w})\mathrm{d}\overline{\xi}^2 - ((\partial_\xi H_{\tilde w}\partial_{\overline{\xi}} H_w) + (\partial_{\overline{\xi}} H_{\tilde w}\partial_\xi H_w))\mathrm{d}\xi\mathrm{d}\overline{\xi}$$
$$+(\partial_\xi H_{\tilde w}\{H_w, H_z\} + \partial_\xi H_w\{H_{\tilde w}, H_z\})\mathrm{d}z\mathrm{d}\xi + (\partial_{\overline{\xi}} H_{\tilde w}\{H_w, H_z\} + \partial_{\overline{\xi}} H_w\{H_{\tilde w}, H_z\})\mathrm{d}z\mathrm{d}\overline{\xi}\Big).$$

(ii) $\mathbf{H_2}$, $(X = \partial_w - \partial_{\tilde w}, Y = \partial_{\tilde z})$, $\nu = \mathrm{d}x \wedge \mathrm{d}z \wedge \Omega_\Sigma$,

$$f^2 = \{H_w - H_{\tilde w}, H_{\tilde z}\} = F_{w\tilde z}, \tag{10.14}$$

$$\begin{aligned}
e_W &= f^{-2}(\{H_{\tilde z}, H_{\tilde w}\}\mathrm{d}x + \{H_{\tilde z}, H_z\}\mathrm{d}z - \mathrm{d}_\Sigma H_{\tilde z})\\
e_{\widetilde W} &= f^{-2}(-\{H_{\tilde z}, H_w\}\mathrm{d}x - \{H_{\tilde z}, H_z\}\mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde z})\\
e_Z &= \mathrm{d}z\\
e_{\widetilde Z} &= f^{-2}(\{H_w, H_{\tilde w}\}\mathrm{d}x + \{H_w - H_{\tilde w}, H_z\}\mathrm{d}z - \mathrm{d}_\Sigma(H_w - H_{\tilde w}))
\end{aligned} \tag{10.15}$$

$$\begin{aligned}
\Sigma^{1'1'} &= \{H_{\tilde z}, H_{\tilde w}\}\mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde z} \wedge \mathrm{d}z\\
\Sigma^{0'1'} &= (\{H_w, H_z\} - \{H_{\tilde w}, H_{\tilde z}\})\mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}_\Sigma H_{\tilde z} \wedge \mathrm{d}x - \mathrm{d}_\Sigma(H_w - H_{\tilde w}) \wedge \mathrm{d}z\\
\Sigma^{0'0'} &= \{H_w, H_z\}\mathrm{d}x \wedge \mathrm{d}z + \Omega_\Sigma + \mathrm{d}_\Sigma H_w \wedge \mathrm{d}x - \mathrm{d}_\Sigma H_z \wedge \mathrm{d}z.
\end{aligned}$$

We can perform a further gauge transformation to set $H_{\tilde w} = 0$ in which case

$$\mathrm{d}s^2 = -2\Big(\left(\frac{\{H_{\tilde z}, H_z\}^2}{\{H_w, H_{\tilde z}\}} + \{H_w, H_z\}\right)\mathrm{d}z^2 - \frac{(\partial_\xi H_{\tilde z})^2}{\{H_w, H_{\tilde z}\}}\mathrm{d}\xi^2 - \frac{(\partial_{\overline{\xi}} H_{\tilde z})^2}{\{H_w, H_{\tilde z}\}}\mathrm{d}\overline{\xi}^2$$
$$+\left(\partial_\xi H_w + 2\frac{\{H_{\tilde z}, H_z\}}{\{H_w, H_{\tilde z}\}}\partial_\xi H_{\tilde z}\right)\mathrm{d}z\ \mathrm{d}\xi + \left(\partial_{\overline{\xi}} H_w + 2\frac{\{H_{\tilde z}, H_z\}}{\{H_w, H_{\tilde z}\}}\partial_{\overline{\xi}} H_{\tilde z}\right)\mathrm{d}z\mathrm{d}\overline{\xi}$$
$$-\frac{2\partial_\xi H_{\tilde z}\partial_{\overline{\xi}} H_{\tilde z}}{\{H_w, H_{\tilde z}\}}\mathrm{d}\xi\mathrm{d}\overline{\xi} + \{H_{\tilde z}, H_z\}\mathrm{d}x\mathrm{d}z - \partial_\xi H_{\tilde z}\mathrm{d}x\mathrm{d}\xi - \partial_{\overline{\xi}} H_{\tilde z}\mathrm{d}x\mathrm{d}\overline{\xi}\Big).$$

Reductions by $X = \partial_w,\ Y = \partial_z$ are not considered because the resulting metric turns out to be degenerate everywhere as a direct consequence of the ASDYM equations. Equation (2.21) becomes now $[X_{H_w}, X_{H_z}] = 0$ which, in the case of finite dimensional sub-algebras of LSDiff(Σ), implies linear dependence of X_{H_w} and X_{H_z}.

The construction naturally applies to complex four-manifolds. We start from the ASDYM equations on $\mathbb{C}^4$ with gauge group $SL(2,\mathbb{C})$. Then we perform one of the possible reductions to $\mathbb{C}^2$. Let ${\Sigma^2}_{\mathbb{C}}$ be a two-dimensional complex manifold, for example $\mathbb{CP}^1 \times \mathbb{CP}^{1*}$. $SL(2,\mathbb{C})$ acts on one Riemann sphere by a Möbius transformation, and the other by the inverse:

$$(\xi, \tilde\xi) \longrightarrow \left(\frac{A\xi + B}{C\xi + D}, \frac{D\tilde\xi - C}{-B\tilde\xi + A}\right).$$

Here ξ and $\tilde{\xi}$ are independent complex coordinates on $\mathbb{CP}^1$ and $\mathbb{CP}^{1*}$. The action preserves the symplectic form $\Omega_{\Sigma_\mathbb{C}} = (1+\xi\tilde{\xi})^{-2}(\mathrm{d}\xi \wedge \mathrm{d}\tilde{\xi})$ defined on the complement of $1+\xi\tilde{\xi}=0$. All results of this section may be extended to the complex case by replacing $\overline{\xi}$ by the independent coordinate $\tilde{\xi}$.

10.2.1 Solitonic metrics

We can now establish the connection between the integrable systems reviewed in Section 10.1 and anti-self-dual vacuum metrics. We do so by expressing the Hamiltonians above in terms of solutions to various soliton equations. From a given solution of a two-dimensional nonlinear equation we can generate a null tetrad (10.10).

1) **NlS**

$$
\begin{aligned}
W &= \partial_x + (\overline{\phi}\xi^2 + \phi)\partial_\xi + (\phi\overline{\xi}^2 + \overline{\phi})\partial_{\overline{\xi}} \\
\widetilde{W} &= \partial_x \\
\widetilde{Z} &= -i\xi\partial_\xi + i\overline{\xi}\partial_{\overline{\xi}} \\
Z &= \partial_z + i(-\overline{\phi}_x\xi^2 + 2|\ \phi\ |^2\xi + \phi_x)\partial_\xi - i(-\phi_x\overline{\xi}^2 + 2|\ \phi\ |^2\overline{\xi} + \overline{\phi}_x)\partial_{\overline{\xi}} \\
f^2 &= \frac{2Re(\overline{\xi}\phi)}{1+|\ \xi\ |^2}
\end{aligned}
$$

2) **KdV**

$$
\begin{aligned}
W &= \partial_x + (qm+n)\partial_m + (bm-qn)\partial_n \\
\widetilde{W} &= \partial_x \\
\widetilde{Z} &= m\partial_n \\
Z &= \partial_z + (\frac{b_x}{2}m - q_x n)\partial_m + (wm - \frac{b_x}{2}n)\partial_n \\
f^2 &= -m(q+mn)
\end{aligned}
$$

where $b = q_x - q^2$ and $4w = q_{xxx} - 4qq_{xx} - 2{q_x}^2 + 4q^2q_x$.

3) **SG**; elliptic case.

$$
\begin{aligned}
W &= \partial_x + \frac{1}{4}(\phi_t m - 2\cos(\phi/2)n)\partial_m + \frac{1}{4}(-\phi_t n - 2\cos(\phi/2)m)\partial_n \\
\widetilde{W} &= \partial_x + \frac{1}{4}(\phi_t m + 2\cos(\phi/2)n)\partial_m + \frac{1}{4}(-\phi_t n + 2\cos(\phi/2)m)\partial_n
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{Z} &= \partial_t + \frac{1}{4}(-\phi_x m - 2\sin(\phi/2)n)\partial_m + \frac{1}{4}(\phi_x n - 2\sin(\phi/2)m)\partial_n \\
Z &= \partial_t + \frac{1}{4}(-\phi_x m + 2\sin(\phi/2)n)\partial_m + \frac{1}{4}(\phi_x n + 2\sin(\phi/2)m)\partial_n \\
f^2 &= (\sin\phi)mn
\end{aligned}
$$

4) **SG**; hyperbolic case

$$
\begin{aligned}
W &= (-i\xi^2 e^{-i\phi} + ie^{i\phi})\partial_\xi + (i\overline{\xi}^2 e^{i\phi} - ie^{-i\phi})\partial_{\overline{\xi}} \\
\widetilde{W} &= (-i\xi^2 + i)\partial_\xi + (i\overline{\xi}^2 - i)\partial_{\overline{\xi}} \\
\widetilde{Z} &= \partial_{\tilde{z}} \\
Z &= \partial_z - i(\partial_{\tilde{z}}\phi)\xi\partial_\xi + i(\partial_{\tilde{z}}\phi)\overline{\xi}\partial_{\overline{\xi}} \\
f^2 &= \frac{4\sin\phi(|\,\xi\,|^2 - 1)}{|\,\xi\,|^2 + 1}.
\end{aligned}
$$

Put $\mathrm{d}_A\xi = \mathrm{d}\xi + i\xi\partial_{\tilde{z}}\phi\,\mathrm{d}z$. Then we have

$$
\begin{aligned}
\mathrm{d}s^2 = \frac{2}{1+\xi\overline{\xi}}\Big(&[(1-\overline{\xi}^2)^2\cot\phi + i(1-\overline{\xi}^4)]\mathrm{d}_A\xi\otimes\mathrm{d}_A\xi + 2\sin\phi\,\mathrm{d}z\otimes\mathrm{d}\tilde{z} \\
&+(\cot\phi(1-\overline{\xi}^2)(1-\xi^2) + i[(1+\overline{\xi}^2)(1-\xi^2) \\
&-(1-\overline{\xi}^2)(1+\xi^2)])\mathrm{d}_A\xi\otimes\overline{\mathrm{d}_A\xi} + [(1-\xi^2)^2\cot\phi - i(1-\xi^4)]\overline{\mathrm{d}_A\xi}\otimes\overline{\mathrm{d}_A\xi}\Big).
\end{aligned}
\tag{10.16}
$$

If one takes a solution describing the interaction of a half kink and a half anti-kink (two topological solitons travelling in $z-\tilde{z}$ direction and increasing from 0 to π as $z+\tilde{z}$ goes from $-\infty$ to ∞) then the singularity in $\sin\phi = 0$ may be absorbed by a conformal transformation of $z+\tilde{z}$ [15].

From the Yang-Mills point of view, the solutions that we have obtained are metrics on the total space of $\mathcal{E}$, the Σ-bundle associated to the Yang-Mills bundle. Therefore it is of interest to consider the effect of gauge transformations. First notice that diffeomorphisms of $\mathbb{R}^2\times\Sigma$ given by

$$
x^a \longrightarrow x^a + \epsilon X_F(x^a) \tag{10.17}
$$

yield $H_\mu \longrightarrow H_\mu + \epsilon(\{H_\mu, F\} + \partial_\mu F)$ which is an infinitesimal form of the full gauge transformation (2.19). Here μ is an index on $\mathbb{M}$, whereas a is an index on $\mathcal{M} = \mathbb{R}^2\times\Sigma$. The vector field X_F is Hamiltonian with respect to Ω_Σ, with Hamiltonian $F = F(x^a)$.

If (10.17) preserves the Kähler structure of Σ then H_μ transforms under (a real form of) $SL(2,\mathbb{C})$ and therefore our construction remains 'invariant'.

10.2.2 The twistor correspondence

To finish, we explain how our construction ties in with the twistor correspondences for the anti-self-duality equations. We consider only the complex case of the ASDYM equations with two commuting symmetries X, Y. The $SL(2,\mathbb{C})$ ASDYM connection defines, by the Ward construction [78], a holomorphic vector bundle over the (non-deformed) twistor space, $E_W \to \mathcal{P}$. It is convenient[2] to use the bundle $\mathcal{E}^5_W$ - associated to E_W by the representation of $SL(2,\mathbb{C})$ as holomorphic canonical transformations of the complex symplectic manifold ${\Sigma^2}_{\mathbb{C}}$.

On the other hand, the ASD vacuum metric corresponds to a deformed twistor space $\mathcal{P}_{\mathcal{M}}$, [56]. In this chapter we have explained how the quotient of $\mathcal{E}$ by lifts of X, Y is, by theorem (2.1), equipped with a half-flat metric . To give a more complete picture we can obtain the deformed twistor space directly from $\mathcal{E}^5_W$ and show that this is the twistor space of $\mathcal{M}$. Consider the following chain of correspondences:

$$\begin{tikzcd}
& \mathcal{E}^5_W \arrow[dl] \arrow[dr] & & \mathcal{F} \arrow[dl] \arrow[dr] & & \mathcal{E} \arrow[dl] \arrow[dr] & & \mathcal{F}_{\mathcal{M}} \arrow[dl] \arrow[dr] & \\
\mathcal{P}_{\mathcal{M}} & & \mathcal{P} & & \mathbb{C}^4 & & \mathcal{M} & & \mathcal{P}_{\mathcal{M}}
\end{tikzcd}$$

Here $\mathcal{F}$ and $\mathcal{F}_{\mathcal{M}}$ are the standard projective spin bundles fibred over $\mathbb{C}^4$ and $\mathcal{M}$ respectively. The space $\mathcal{F}^7_{\mathcal{E}}$, the pullback of the spin bundle $\mathcal{F}$ to the total space of the bundle $\mathcal{E}$, fibres over all the spaces in the above diagram. Taking the quotient by lifts of X, Y we project $\mathcal{F}^7_{\mathcal{E}}$ to $\mathcal{F}_{\mathcal{M}}$. Taking the quotient by the twistor distribution, $\mathcal{F}^7_{\mathcal{E}}$ also projects to the Ward bundle $\mathcal{E}^5_W$. By definition it projects to $\mathcal{E}$ and it could equivalently have been defined as the pullback of $\mathcal{E}$ to $\mathcal{F}$. The compatibility of these projections is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{C}P^1 \times \mathbb{C}^4 \times \Sigma^2_{\mathbb{C}} = \mathcal{F}^7_{\mathcal{E}} \arrow[r, "(X,Y)"] \arrow[d] & \mathcal{F}^5_{\mathcal{M}} \arrow[d] \\
\mathcal{E}^5_W \arrow[r, "(X,Y)"] & \mathcal{P}_{\mathcal{M}}.
\end{tikzcd} \tag{10.18}$$

which follows from the integrability the the distribution spanned by (lifts of)

$$X,\ Y,\ L_0,\ L_1,$$

[2] The diagram (10.18) describes also the general case of $G = SDiff({\Sigma^2}_{\mathbb{C}})$. For this we work with $\mathcal{E}^5_W$ rather than the principal Ward bundle, since the latter has infinite-dimensional fibres. The notation is such that the upper index of a space stands for the complex dimension of that space.

and from the fact that (X, Y) commute with (L_0, L_1).

10.2.3 Global issues

In order to obtain a compact space one might attempt the following:

- choose the gauge group to be $SU(2)$ so that the fibre space is compact, and
- Compactify $\mathbb{R}^2$ after the reduction.

We restrict the rate of decay of A_μ by the requirement that A_μ should be smoothly extendible to S^2 in the split signature case. Other possibilities are to restrict to the class of rapidly decreasing soliton solution of corresponding integrable equation. If we have reduced from a Euclidean signature solution to the ASDYM equations, then it is more natural to compactify $\mathbb{R}^2$ in such a way as to obtain a Riemann surface of genus greater than one as it is only for such a compactifications that one can have existence of nontrivial solutions, [32].

However, we still have singularities in the metrics corresponding to (10.13) and (10.15), even if we can eliminate those from the Yang-Mills connection. We are left with singularities associated with sets on which the tetrad becomes linearly dependent. This reduces to the proportionality (or vanishing) of the Higgs fields on Σ, which generically occurs on a real co-dimension one subset of each fibre (and hence co-dimension one in the total space). In the above formulae this set is given by the vanishing of f. The ASD Weyl curvature C_{ABCD} blows up as f goes to zero. Calculation of curvature invariants show that these lead to genuine singularities that cannot be eliminated by a change of frame or coordinates. For example

$$C_{ABCD}C^{ABCD} = \sum_{i=-3}^{3} C_i f^{2i},$$

where $C_i = C_i(x^a)$ are generally non-vanishing regular functions on $\mathcal{M}$, which explicitly depend on Yang-Mills curvature $F_{\mu\nu}$ and (derivatives of) Hamiltonians (10.6-10.8). Those singularities appear (for purely topological reasons) because each vector in the tetrad $(W, \widetilde{W}, Z, \widetilde{Z})$ has at least one zero, when restricted to $\Sigma = S^2$.

One can also obtain Euclidean metrics as above by using reductions of the ASDYM equations from Euclidean space, but we will still be unable to avoid these same co-dimension one singularities.

10.2.4 Other reductions

We have focused on the familiar $1+1$ soliton equations. However, it is clear from the discussion of Section 10.2 that the construction will extend to any symmetry reduction of the ASDYM equations to systems in two dimensions with gauge group contained in $SL(2,\mathbb{C})$, in particular when the symmetry imposed consists of two translations as for the Euclidean signature examples mention previously. However, one can also use the same device to embed examples using any other two-dimensional symmetry subgroup of the conformal group. In particular, with cylindrical symmetry, one obtains the Ernst equations (the two symmetry reduction of the full, four-dimensional Einstein vacuum equations) and this can similarly be embedded into the anti-self-dual (but not vacuum) equations.

Chapter 11

Outlook

In this chapter we shall briefly discuss some open research problems which are related to what has been said in this thesis.

11.1 Towards finite gap solutions in twistor theory

This section motivates the study of solutions to heavenly equations which are invariant under some hidden symmetries[1]. This should lead to a large class of ASD vacuum metrics analogous to the celebrated 'finite gap' solutions in the soliton theory [52].

Let us give a simple example of an analogous construction for the first heavenly equation (3.2). Let $\phi_n = \partial_{t_n}\Omega$ be a linearisation of the first heavenly equation. The recursion relations (3.12) are

$$(\Omega_{w\tilde{w}}\partial_{\tilde{z}} - \Omega_{w\tilde{z}}\partial_{\tilde{w}})\partial_{t_{n+1}}\Omega = \partial_w\partial_{t_n}\Omega, \qquad (\Omega_{z\tilde{w}}\partial_{\tilde{z}} - \Omega_{z\tilde{z}}\partial_{\tilde{w}})\partial_{t_{n+1}}\Omega = \partial_z\partial_{t_n}\Omega.$$

We have $R : z \longrightarrow \Omega_w = \partial_{t_1}\Omega$. Look for solutions to (3.2) with an additional constraint $\partial_{t_2}\Omega = 0$. The recursion relations imply $\Omega_{wz} = \Omega_{ww} = 0$, therefore

$$\Omega(w, z, \tilde{w}, \tilde{z}) = wq(\tilde{w}, \tilde{z}) + P(z, \tilde{w}, \tilde{z}).$$

The heavenly equation yields $\mathrm{d}q \wedge \mathrm{d}P \wedge \mathrm{d}z = \mathrm{d}\tilde{z} \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}z$. With the definition $\partial_z P = p$ the metric is

$$\mathrm{d}s^2 = 2\mathrm{d}w\mathrm{d}q + 2\mathrm{d}z\mathrm{d}p + f\mathrm{d}z^2,$$

[1] This project has now been completed - see [92].

where $f = -2P_{zz}$. We adopt (w, z, q, p) as a new coordinate system. Heavenly equations imply that $f = f(q, z)$ is an arbitrary function of two variables. The metric is of non-expanding type N. This simple solution does not admit a conformal symmetry of the form (8.2), so nontrivial reductions of Einstein–Weyl metrics will require combinations of hidden symmetries.

It would be enlightening to develop the framework for the finite gap solutions, in twistor theory. We should study both ASDYM and heavenly hierarchies.

Imposing three independent hidden symmetries on the twistor space should lead to solutions of ASDYM/ASDVE expressible by θ-functions. Some solutions of this type are implicitly given in Chapter 10 where we expressed ASD metrics in terms of KdV potentials.

One approach could be based on the generalisation of [77]. One could look at twistor spaces which have a globally defined twistor function homogeneous of degree $n+1$. This would imply that the metric admits a Killing spinor (see example (7.20)). The canonical forms of patching functions should be derived to give explicit ASD solutions.

11.2 Einstein–Weyl hierarchies and dispersion-less integrable models

Equations (9.1) are closely related to Krichever's formulation [41] of the Whitham hierarchies. Here we shall discuss a possible relationship between Einstein–Weyl geometry, twistor theory and Whitham equations.

Let $\mathcal{Z}$ be a complex surface and let $l \subset \mathcal{Z}$ be rational curve with a normal bundle $\mathcal{O}(2n)$. By the Kodaira theorem (A.4) the moduli space $\mathcal{V}$ of l is $2n+1$ dimensional. The $2n+2$ correspondence space $\mathcal{F}_V := \mathcal{V} \times \mathbb{CP}^1$ is equipped with a $2n$ dimensional distribution $L_{A'_1 \dots A'_n} \in \Gamma(D_V \otimes \mathcal{O}(1) \times \mathbb{C}^{2n}$. The existence of this distribution follows (using arguments from the proof of Proposition (**??**)) from the double-fibration picture

$$\mathcal{V} \xleftarrow{p} \mathcal{F}_V \xrightarrow{q} \mathcal{Z},$$

together with the sequence

$$0 \longrightarrow D_V \longrightarrow \mathbb{C}^{2n+1} \longrightarrow \mathcal{O}(2n) \longrightarrow 0.$$

It is 'reasonable' to treat $\mathcal{Z}$ as a twistor space of the Einstein–Weyl hierarchy; this terminology should be justified by a closer study of symmetries and recursion relations associated to EW equations.

We shall restrict ourselves to surfaces $\mathcal{Z}$ which arise as factor spaces of the twistor spaces corresponding to heavenly hierarchies. In the following lemma we shall develop the analogy of the Jones and Tod's factoring construction.

Lemma 11.1 *Let $\mathcal{K}$ be a holomorphic volume preserving vector field on a twistor space $\mathcal{PT}$ from Proposition (??) such that $\mathcal{L}_{\mathcal{K}}\Sigma = \eta\Sigma$, for $\eta = const$. Then*

- *$K = p_* q^*\mathcal{K}$ is a 'para-conformal Killing vector' on $\mathcal{N}$, i.e.*

$$\nabla^{(A'_1\dots A'_n}_{(A} K^{B'_1\dots B'_n)}_{B)} = 0. \tag{11.1}$$

- *The factor space $\mathcal{Z} := \mathcal{PT}/\mathcal{K}$ is a complex surface with a $2n+1$ dimensional family of rational curves with self-intersection number $2n$.*

Proof. The proof of a) goes along the lines of the proof of Lemma (9.3). To prove b) choose a rational curve l in $\mathcal{PT}$ on which $\mathcal{K}$ does not vanish. The vector field $\mathcal{K}$ defines a trivial sub-bundle of the normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$ to the line l. The normal bundle of the image of l in $\mathcal{Z}$ is $\mathcal{O}(2n)$. The Kodaira theorem A.4 implies the existence of $(2n+1)$–dimensional family of such curves[2]. The projection $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ equips $\mathcal{Z}$ with a divisor

$$Q = \mathcal{K} \lrcorner\, \pi \cdot \mathrm{d}\pi \in \mathcal{O}(2). \tag{11.2}$$

In the special case $Q \equiv 0$, surface $\mathcal{Z}$ is fibred over $\mathbb{CP}^1$.

□

Let ξ be a holomorphic volume form on $\mathcal{PT}$. Define a line bundle valued two form on the correspondence space $\mathcal{F}$ by

$$\tilde{\Sigma} = q^*(\mathcal{K} \lrcorner\, \xi) \in \Lambda^2(\mathcal{F}) \otimes \mathcal{O}(2n+2).$$

This two form descends to the reduced correspondence space $\mathcal{F}_V$. Assume that $Q \neq 0$. In this case we can 'dehomogenise' $\widetilde{\Sigma}$ to give a closed and simple two form:

[2] If $n = 3$ then the mini-twistor space will have $\mathcal{O}(6)$ rational curves, and the moduli space $\mathbb{C}^7$ could have a G_2 structure.

Proposition 11.2 *Let* $\Pi := Q^{-2n}\tilde{\Sigma} \in \Lambda^2 \mathcal{F}_V$ *be a two form on a* $2n+2$ *dimensional space* $\mathcal{F}_V$*. Then*

$$\mathrm{d}\Pi = 0, \qquad \Pi \wedge \Pi = 0. \tag{11.3}$$

Proof. The two-form Π descends to $\mathcal{Z}$ where equations (11.3) are satisfied trivially.

□

Formulae (11.2) are central to the Krichever description of the Whitham hierarchy [41, 42]. To make a closer analogy with his approach, we should identify the coordinates on $\mathcal{V}$ with 'slow' time variables t_{Ai} coupled to meromorphic differentials Ω_{Ai} of the finite-gap construction, and λ with a coordinate on a holomorphic curve Γ_g. Krichever constructs (see [42] for notation) a potential $S(\lambda, t_{Ai})$ such that $\Omega_{Ai} = \partial_{Ai} S$. In the genus 0 case

$$F := \int_\Gamma \bar{\mathrm{d}} S \wedge \mathrm{d} S$$

is the prepotential for the Frobenius manifolds [13]. It should be possible to give interpretation of this formula in terms of the geometry of $\mathcal{PT}$, which should allow a twistor construction of Frobenius Manifolds. To understand the general Whitham hierarchies one would have to extend the twistor theory to the case when $\mathcal{F} = \mathcal{N} \times \Gamma_g$.

11.3 Other ASD hierarchies

It is natural to ask whether twistor methods used in Chapter 3 to construct ASDVE hierarchy could be applied to other hierarchies associated with ASD geometries. Below we discuss two cases which we think can be treated by similar methods.

11.3.1 Hyper-complex hierarchies

Studying the linearised hyper-complex equation (7.14) should result in a recursion procedure analogous to (3.12). The consistency conditions for the recursion relations will demand solvability of a linear system $L_{Ai} s = 0$, where

$$L_{Ai} = \frac{\partial}{\partial x^{Ai}} - \lambda \Big(\frac{\partial}{\partial x^{Ai+1}} + \frac{\partial \Theta^B}{\partial x^{Ai}} \frac{\partial}{\partial x^B} \Big). \tag{11.4}$$

The condition $[L_{Ai}, L_{Bj}] = 0$ gives rise to the hyper-complex hierarchy

$$\frac{\partial^2\Theta^C}{\partial x^{Ai+1}\partial x^{Bj}} - \frac{\partial^2\Theta^C}{\partial x^{Bj+1}\partial x^{Ai}} + \frac{\partial\Theta^D}{\partial x^{Ai}}\frac{\partial^2\Theta^C}{\partial x^{Bj}\partial x^D} - \frac{\partial\Theta^D}{\partial x^{Bj}}\frac{\partial^2\Theta^C}{\partial x^{Ai}\partial x^D} = 0. \qquad (11.5)$$

If $\Theta_C = \partial\Theta/\partial x^C$, then the flows (11.5) are derivatives of flows (**??**) of the heavenly hierarchy constructed in Chapter 3. The combination of Propositions 7.2 and **??** should lead to a twistor description of hyper-complex hierarchies by the following:

Conjecture 11.3 *Let $\mathcal{PT}$ be a three-dimensional complex manifold with the following structures:*

1) a projection $\mu : \mathcal{PT} \longrightarrow \mathbb{C}P^1$,

2) a $2(n+1)$-dimensional family of sections with normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$,

and let $\mathcal{N}$ be the moduli space of sections from (2). Then

a) There exist two function $\Theta^C : \mathcal{N} \longrightarrow \mathbb{C}$ which (with the appropriate choice of the coordinates) satisfy the set of equations (11.5)*.*

b) The correspondence space $\mathcal{F} = \mathcal{N} \times \mathbb{CP}^1$ is equipped with the $2n$-dimensional distribution $D \subset T(\mathcal{N} \times \mathbb{CP}^1)$ which, as a bundle on $\mathcal{F}$, has an identification with $\mathcal{O}(-1) \otimes \mathbb{C}^{2n}$ so that the linear system can be written as (11.4) *Equations em(11.5) are equivalent to $[L_{Ai}, L_{Bj}] = 0$.*

It is expected that the transition functions for the line bundle $K^* \otimes \mathcal{O}(-2n-2)$ will play an important role in the proof.

11.3.2 ASD Einstein hierarchies

The vanishing of the cosmological constant underlies the original nonlinear graviton construction. The ASD Einstein metrics with $\Lambda \neq 0$ have a natural twistor construction [80] where the extra information about a scalar curvature is encoded into a contact structure on $\mathcal{PT}$. On the other hand Przanowski [64] reduced $\Lambda \neq 0$ case to the single second order PDE for one scalar function $u(w, \tilde{w}, z, \tilde{z})$

$$u_{w\tilde{w}}u_{z\tilde{z}} - u_{w\tilde{z}}u_{z\tilde{w}} - (2u_{w\tilde{w}} + u_w u_{\tilde{w}})e^{-u} = 0. \qquad (11.6)$$

This equation could be a starting point from construction of ASD Einstein hierarchy. This is because the linearisations of (11.6) satisfy

$$(\Box_g + 4\Lambda)\delta u = 0$$

and the recursion relations (3.12) are still valid. It should be possible to derive Przanowski's result from the structure of a curved twistor space. In the above cases rational curves in $\mathcal{PT}$ have normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. We expect that replacing N by $\mathcal{O}(n) \oplus \mathcal{O}(n)$ should lead to some generalisations of Proposition **??**.

11.4 Real Einstein–Maxwell metrics

The operation of taking 'real parts' of complex vacuum metrics does not usually lead to real solutions of the Einstein equations. There are few effective constructions [67, 63] which use superpositions of half-flat metrics to produce Lorentzian Ricci flat metrics. The geometric interpretation is, however, obscure. It should be possible to clarify the geometric meaning of these procedures. In the next few lines we shall propose a construction of Lorentzian solutions of the Einstein–Maxwell (EM) equations by superposing holomorphic hyper-Hermitian metrics.

Let Θ^A be a pair of complex valued functions on $\mathcal{M}$ which satisfy (7.14). The ASD tetrad and the Lee form (which is now treated as the ASD Maxwell potential) are

$$\begin{aligned} e^{AA'} &= (\varepsilon^A{}_B\varepsilon^{A'}{}_{B'} - o^{A'}o_{B'}o^{C'}\nabla_{BC'}\Theta^A)\mathrm{d}x^{BB'} := e^{AA'}_{BB'}\mathrm{d}x^{BB'}, \\ A &= o_{A'}o_{B'}o_{C'}e^{AA'}(\nabla_A{}^{B'}\nabla^{BC'}\Theta_B). \end{aligned} \tag{11.7}$$

Robinson's construction [67] motivates pulling back $e^{AA'}$ to a real four manifold by $x^{AA'} \longrightarrow y^{AA'}$. It is hoped that the Hermitian matrix valued one form

$$\omega^{AA'} = e^{AC'}_{CB'}\bar{e}^{A'C}_{BC'}\mathrm{d}y^{BB'} \tag{11.8}$$

with the one form $\mathcal{A} := i(A - \bar{A})$ are conformal to a solution of the full EM system on a Lorentzian four-manifold.

First this conjecture may be tested by taking a simple solution to equation (7.14) (ie. a known hyper–Hermitian metric) and looking for a corresponding real solution (11.8). Further work might involve substitution of $(\omega^{AA'}, \mathcal{A})$ defined above into the conformal EM equation. The constraints yielded for Θ_A would follow naturally if the conformal EM equation could be reformulated in terms of the Sparling three form. We might gain a deeper understanding through a search for additional structures on a twistor space corresponding to hyper-Hermitian four-manifolds. These structures might shed some light on the procedure of taking the real section, and applying the nonlinear superposition rule given above. It is possible that the structures are related to those of Penrose [59].

11.5 Large n limits of ODEs and Einstein–Weyl structures

It is known [88] that equations describing four-dimensional hyper-Kähler metrics with rotational or translational symmetry arise as 'large n limits' of certain ODEs associated to Lie algebras $SU(n)$. It would be very interesting to find an ODE which, in the limiting case, yields equations (8.6) or (8.13). We claim that various Callogero models and a Darboux-Halphen equation are interesting systems of ODEs to investigate. Let Y be a two dimensional complex manifold. Explicit Einstein–Weyl structures on $Y \times \mathbb{R}$ should be determined by solutions to ODEs with $G = SL(2, \mathbb{C}) \subset SL(\infty)$. This construction should have a twistor counter-part.

It would also be interesting to look at the case of a general Killing vector taking the equation (7.14) as a starting point. One might also consider reduction of real slices with $(+ + --)$ signature to obtain an 'evolution' form of Einstein-Weyl equations for metrics of signature $(+ - -)$.

11.6 Computer methods in Twistor Theory

Many aspects of twistor theory which were used in this thesis were to some extent algorithmic. Computer programs should be written (in MATHEMATICA or MAPLE) which simplify some calculations. In particular, checking the integrability of twistor distributions, reducing Lax pairs and classifying Lie point symmetries on the twistor space can and should be computerised. Some examples of programs used in this work are listed in Appendix D.

Appendix A

Complex Analysis

Let $\pi_{A'} = (\pi_{0'}, \pi_{1'})$ be coordinates on $\mathbb{C}^2$. Remove $\pi_{A'} = 0$ and use $\pi_{A'}$ as homogeneous coordinates on $\mathbb{CP}^1$. We shall also use the affine coordinate $\lambda = \pi_{0'}/\pi_{1'}$.

Theorem A.1 (Hartog) *Holomorphic functions on* $\mathbb{C}^2 - 0$ *extend to holomorphic functions on* $\mathbb{C}^2$.

Therefore functions on $\mathbb{CP}^1$ are polynomials

$$f = f_{A'B'\ldots C'}\pi^{A'}\pi^{B'}\ldots\pi^{C'}.$$

In particular, holomorphic functions homogeneous of degree 0 are constant (Liouville theorem). This will be used to define fields by formulae (B.2, B.3).

For each $\lambda \in \mathbb{CP}^1$ there exists a one dimensional subspace of $L_\lambda \subset \mathbb{C}^2$, consisting of points $(\pi_{0'}, \pi_{1'}) \in \mathbb{C}^2$ for which $\pi_{0'}/\pi_{1'} = \lambda$. As λ varies L_λ form *the tautological line bundle* over $\mathbb{CP}^1$. This bundle is called $\mathcal{O}(-1)$. Let $\mathcal{O}(n)$ be the dual nth tensor power of $\mathcal{O}(-1)$. The $\mathcal{O}(n)$ transition function is λ^{-n}. Its sections are given by functions homogeneous of degree n in a sense that

$$f(\xi\pi_{A'}) = \xi^n f(\pi_{A'}).$$

The space of these sections is

$$H^0(\mathbb{C}P^1, \mathcal{O}(n)) = \begin{cases} 0 & \text{for } n < 0 \\ \mathbb{C}^{n+1} & \text{for } n \geq 0. \end{cases} \tag{A.1}$$

Let Σ be a compact Riemann surface, and let $K = T^*\Sigma$ be a canonical bundle of Σ. The genus of Σ is equal to the dimension of $H^0(\Sigma, K)$.

Theorem A.2 (Serre) *If L is a line bundle over Σ then*

$$H^1(\Sigma, \mathcal{O}(L)) = H^0(\Sigma, \mathcal{O}(K \otimes L^*))^*.$$

Let $\Sigma = \mathbb{CP}^1$. From the transition relations it follows that $K = \mathcal{O}(-2)$. There are no non-zero sections of K. Let $\pi \cdot \mathrm{d}\pi$ be the section of $K \otimes \mathcal{O}(2)$. The Serre duality can be rephrased as follows: let $L = \mathcal{O}(n)$, and let

$$F = F^{A'_1 \dots A'_n} \pi_{A'_1} \dots \pi_{A'_n} \in H^0(\Sigma, \mathcal{O}(n)).$$

Let $G \in H^1(\Sigma, O(-2) \otimes \mathcal{O}(-n))$. Define the scalar product

$$(F, G) = \frac{1}{2\pi i} \oint_\Gamma FG\pi \cdot \mathrm{d}\pi = G_{A'_1 \dots A'_n} F^{A'_1 \dots A'_n}.$$

The integrand is homogeneous of degree 0 and the result of integration depends only on the cohomology class of G. Here

$$G_{A'_1 \dots A'_n} = \frac{1}{2\pi i} \oint_\Gamma \pi_{A'_1} \dots \pi_{A'_n} G\pi \cdot \mathrm{d}\pi.$$

Therefore integral formulae for the negative helicity fields are concrete realisations of the Serre duality A.2.

$$H^1(\mathbb{C}P^1, \mathcal{O}(-n)) = \left\{ \begin{array}{ll} 0 & \text{for } n < 2 \\ \mathbb{C}^{n-1} & \text{for } n \geq 2. \end{array} \right. \tag{A.2}$$

The classification of holomorphic vector bundles over $\mathbb{CP}^1$ is given by the following:

Theorem A.3 (Grothendieck) *Let $E \longrightarrow \mathbb{CP}^1$ be a rank m holomorphic vector bundle. Then*

$$E = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_m),$$

where $(k_1, \dots k_m) \in \mathbb{Z}^m$.

Therefore a section of E is of the form

$$s = (\Psi^1_{A'_1 \dots A'_{k_1}} \pi^{A'_1} \dots \pi^{A'_{k_1}}, \dots, \Psi^m_{A'_1 \dots A'_{k_m}} \pi^{A'_1} \dots \pi^{A'_{k_m}}).$$

The following result of Kodaira underlies the twistor approach to curved geometries. Let $\mathcal{Z}$ be a complex manifold of dimension $d + r$. A pair $(Y, \mathcal{M})$ is called *a complete analytic family of compact sub-manifolds of $\mathcal{Z}$ of dimension d* if

- Y is a complex analytic sub-manifold of $\mathcal{Z} \times \mathcal{M}$ of codimension r with the property that for each $t \in \mathcal{M}$ the intersection $Y_t := Y \cap (\mathcal{Z} \times t)$ is a compact sub-manifold of $\mathcal{Z} \times t$ of dimension d.

- There exists an isomorphism

$$T_t\mathcal{M} \simeq H^0(Y_t, N_t)$$

  where $N_t \longrightarrow Y_t$ is the normal bundle of Y_t in $\mathcal{Z}$.

Theorem A.4 (Kodaira[40]) *Let Y be a complex compact sub-manifold of $\mathcal{Z}$ of dimension d, and let N be the normal bundle of Y in $\mathcal{Z}$. If $H^1(Y, N) = 0$ then there exists a complete analytic family $(Y, \mathcal{M})$ such that $Y = Y_{t_0}$ for some $t_0 \in \mathcal{M}$.*

We will apply the above theorem to the situation when $\mathcal{Z}$ is a projective twistor space and $Y = \mathbb{CP}^1$. Roughly speaking, the moduli space $\mathcal{M}$ is the 'arena' of differential geometry and integrable systems. One way to proceed is to consider infinitesimal deformations (given by $H^1(Y, \mathbf{\Theta})$) where $\mathbf{\Theta}$ is a sheaf of germs of holomorphic vector fields), and to integrate them. The integration process involves the splitting formulae which are summarised in the next appendix.

Appendix B

Splitting Formulae

From (A.2) it follows that cocycles in $H^1(\mathbb{CP}^1, \mathcal{O}(r))$ can be represented by coboundaries if $r \geqslant -1$. The freedom one has is measured by $H^0(\mathbb{CP}^1, \mathcal{O}(r))$. The concrete realisations, known as splitting formulae, were developed by G.A.J. Sparling. His approach is based on the fact that $H^1(\mathbb{CP}^1, \mathcal{O}(-1)) = 0$ and its cocycles can be uniquely represented as coboundaries.

We shall use the 'abstract multi–index notation'. This notation is set to avoid an orgy of spinor indices on the primed spin bundle. We define it for the primed indices. The definitions for primed indices are analogous.

Let $S^{\mathcal{A}'_n} := S^{(A'_1 \dots A'_n)}$ be a symmetric tensor product of two dimensional spin spaces. Different letters $\mathcal{A}', \mathcal{B}', \mathcal{C}', ...$ denote sets of separately symmetric spinors. The upper numerical index denotes homogeneity in $\pi^{A'}$ and $A, B, ..., A', B', ...$ are usual spinor indices. The twistor distribution is $L^1_{A\mathcal{A}'_{n-1}}$. Multi–indices undergo contraction according to: $Q^{\mathcal{A}_p} Q_{\mathcal{A}_q} = Q^{\mathcal{A}_{p-q}}$ for $p > q$.

Let U and $\widetilde{U}$ be a covering of $\mathcal{PT}$ such that $\pi_{1'} \neq 0$ on U, and $\pi_{0'} \neq 0$ on $\widetilde{U}$ and let $U_{\mathcal{F}}$ and $\widetilde{U}_{\mathcal{F}}$ denote the open sets on the correspondence space $\mathcal{F}$ that are the pre-image of U and $\widetilde{U}$ on $\mathcal{PT}$. Given an indexed object on $\mathcal{PT}$ we multiply it by $\pi_{A'}$ and differentiate with respect to ω^A to make it homogeneous of degree -1. Then we restrict it to a line and pull it back to $\mathcal{F}$. On the projective primed spin bundle we split it according to $f_{\mathcal{A}_p\mathcal{B}'_q} = \mathcal{F}_{\mathcal{A}_p\mathcal{B}'_q} - \widetilde{\mathcal{F}}_{\mathcal{A}_p\mathcal{B}'_q}$. Here $\mathcal{F}_{\mathcal{A}_p\mathcal{B}'_q}$ is holomorphic on $U_{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_{\mathcal{A}_p\mathcal{B}'_q}$ is holomorphic on $\widetilde{U}_{\mathcal{F}}$. They are given by

$$\mathcal{F}_{\mathcal{A}_p\mathcal{B}'_q}(\pi_{A'}, x^a) \quad = \quad \frac{1}{2\pi i} \oint_\Gamma \frac{f_{\mathcal{A}_p\mathcal{B}'_q}(\rho_{A'}, x^{AA'}\rho_{A'})}{\pi \cdot \rho} \rho \cdot \mathrm{d}\rho,$$

$$\widetilde{\mathcal{F}}_{\mathcal{A}_p\mathcal{B}'_q}(\pi_{A'}, x^a) = \frac{1}{2\pi i}\oint_{\widetilde{\Gamma}} \frac{f_{\mathcal{A}_p\mathcal{B}'_q}(\rho_{A'}, x^{AA'}\rho_{A'})}{\pi\cdot\rho}\rho\cdot \mathrm{d}\rho \tag{B.1}$$

where $\mathcal{A}_p\mathcal{B}'_q$ is the multi index, and $\rho_{A'}$ are homogeneous coordinates on $\mathbb{CP}^1$. The contours Γ and $\widetilde{\Gamma}$ are homologous to the equator of $\mathbb{CP}^1$ in $U\cap\widetilde{U}$ and are such that $\Gamma-\widetilde{\Gamma}$ surrounds the point $\rho_{A'}=\pi_{A'}$. We restrict ourself to the case where

$$f_{\mathcal{A}_p\mathcal{B}'_q} = \pi_{B'_1}...\pi_{B'_q}\frac{\partial^p f}{\partial\omega^{A_1}...\partial\omega^{A_p}}.$$

The field is given by

$$L_{A\mathcal{B}'_{n-1}}\mathcal{F}_{\mathcal{A}_p\mathcal{B}'_q} = L_{A\mathcal{B}'_{n-1}}\widetilde{\mathcal{F}}_{\mathcal{A}_p\mathcal{B}'_q} = \Psi_{\mathcal{A}_{p+1}\mathcal{B}'_{q+n-1}} = \frac{1}{2\pi i}\oint_{\Gamma}\frac{\partial f_{\mathcal{A}_p\mathcal{B}'_{q+n-1}}}{\partial\omega^A}\rho\cdot\mathrm{d}\rho. \tag{B.2}$$

There is always the 'most economic way' of constructing the -1 object (which in the usual case of $\mathcal{O}(1)\oplus\mathcal{O}(1)$ twistor theory does not involve multiplying by $\pi_{A'}$s). All the others can be obtained from it by acting on $f_{\mathcal{A}_p\mathcal{B}'_q}$ with

$$\pi_{A'_1}...\pi_{A'_n}\frac{\partial}{\partial\omega^A}.$$

The corresponding fields will be derivatives of $\Psi_{\mathcal{A}_{p+1}\mathcal{B}'_{q+n-1}}$.

Another method for constructing a field is

$$\pi_{A'}f^{-2}_{\mathcal{A}_k\mathcal{B}'_l} = \mathcal{F}^{-1}{}_{\mathcal{A}_k\mathcal{B}'_l A'} - \widetilde{\mathcal{F}}^{-1}_{\mathcal{A}_k\mathcal{B}'_l A'}$$

and

$$\pi^{A'}\mathcal{F}^{-1}{}_{\mathcal{A}_k\mathcal{B}'_l A'} = \Sigma_{\mathcal{A}_k\mathcal{B}'_l} = \frac{1}{2\pi i}\oint_{\Gamma} f_{\mathcal{A}_k\mathcal{B}'_l}. \tag{B.3}$$

In particular, if $k=p+1$ and $l=q+n-1$ then formulae (B.2) and (B.3) give the same field. More general splitting formulae are used to construct potentials.

If $f^r\in H^1(\mathbb{CP}^1,\mathcal{O}(r))$ then

$$f^r = \mathcal{F}^r - \widetilde{\mathcal{F}}^r,$$

where

$$\begin{aligned}\mathcal{F}^r(\pi_{A'},x^a) &= \frac{1}{2\pi i}\oint_{\Gamma}\frac{(\pi\cdot o)^{r+1}f^r}{(\rho\cdot\pi)(\rho\cdot o)^{r+1}}\rho\cdot\mathrm{d}\rho,\\ \widetilde{\mathcal{F}}^r(\pi_{A'},x^a) &= \frac{1}{2\pi i}\oint_{\widetilde{\Gamma}}\frac{(\pi\cdot\iota)^{r+1}f^r}{(\rho\cdot\pi)(\rho\cdot\iota)^{r+1}}\rho\cdot\mathrm{d}\rho\,. \end{aligned}\tag{B.4}$$

In the chosen spin frame we find (for $n=1$) that

$$\frac{o^{A'}\nabla_{AA'}\mathcal{F}^r}{\pi\cdot o}=\mathcal{F}^{r-1}_A,\quad \frac{\iota^{A'}\nabla_{AA'}\widetilde{\mathcal{F}}^r}{\pi\cdot\iota}=\widetilde{\mathcal{F}}^{r-1}_A$$

where

$$\frac{\partial f^r}{\partial\omega^A}=\mathcal{F}^{r-1}_A-\widetilde{\mathcal{F}}^{r-1}_A.$$

In $\mathcal{O}^A(n)$ theory (with $r>n$) we have

$$\frac{o^{A'_1}...o^{A'_n}\nabla_{AA'_1...A'_n}\mathcal{F}^r}{(\pi\cdot o)^n}=\mathcal{F}^{r-n}_A,\quad \frac{\iota^{A'_1}...\iota^{A'_n}\nabla_{AA'_1...A'_n}\widetilde{\mathcal{F}}^r}{(\pi\cdot\iota)^n}=\widetilde{\mathcal{F}}^{r-n}_A.$$

Let us give one of the standard methods of splitting elements of $H^1(\mathbb{CP}^1,\mathcal{O})$ of the form

$$H=\frac{\pi^{i+j}_{0'}}{(\omega^0)^i(\omega^1)^j}=h-\tilde{h}.$$

On the correspondence space $\omega^0=\pi_{A'}\alpha^{A'}:=\pi\cdot\alpha$ and $\omega^1=\pi_{A'}\beta^{A'}:=\pi\cdot\beta$. We make use of the formula

$$\frac{\pi\cdot o}{(\pi\cdot\alpha)(\pi\cdot\beta)}=\frac{1}{\alpha\cdot\beta}\Big(\frac{\alpha\cdot o}{\pi\cdot\alpha}-\frac{\beta\cdot o}{\pi\cdot\beta}\Big).\tag{B.5}$$

Put

$$\partial_\alpha:=o^{A'}\frac{\partial}{\partial\alpha^{A'}},\quad \partial_\beta:=o^{A'}\frac{\partial}{\partial\beta^{A'}}.$$

It follows that

$$(\partial_\alpha)^{i-1}(\partial_\beta)^{j-1}\frac{\pi\cdot o}{(\pi\cdot\alpha)(\pi\cdot\beta)}=(-1)^{i+j}(i-1)!(j-1)!H.$$

Therefore differentiating the RHS of (B.5) we can find the splitting of H.

Next we want to split

$$\frac{(\pi\cdot o)^{4n}}{(\alpha_1\cdot\pi)^2...(\alpha_n\cdot\pi)^2(\beta_1\cdot\pi)^2...(\beta_n\cdot\pi)^2}=g-\tilde{g}.$$

First note that

$$\int_0^\infty\frac{\mathrm{d}u}{((\alpha+\tilde{\alpha}u)\cdot\pi)^2}=\frac{1}{(\alpha\cdot\pi)(\tilde{\alpha}\cdot\pi)},$$

which generalises to

$$(\pi_{0'})^{2n-1}\underbrace{\int_0^\infty...\int_0^\infty}_{2n-2}\frac{\mathrm{d}u_2...\mathrm{d}u_n\mathrm{d}w_2...\mathrm{d}w_n}{((\alpha_1+u_2\alpha_2+...+u_n\alpha_n)\cdot\pi)^2((\beta_1+u_2\beta_2+...+u_n\beta_n)\cdot\pi)^2}$$

$$= \frac{(\pi_{0'})^{2n-1}}{(\alpha_1 \cdot \pi)...(\alpha_n \cdot \pi)(\beta_1 \cdot \pi)...(\beta_n \cdot \pi)}. \tag{B.6}$$

Differentiating (B.5) and using (B.6) we obtain

$$g - \tilde{g} = (\pi \cdot o)^{2n-2} \partial_{\alpha_1} ... \partial_{\alpha_n} \partial_{\beta_1} ... \partial_{\beta_n} \underbrace{\int_0^\infty ... \int_0^\infty}_{2n-2} \frac{\mathcal{Q}}{\alpha \cdot \beta} \mathrm{d}u_2 ... \mathrm{d}u_n \mathrm{d}w_2 ... \mathrm{d}w_n, \tag{B.7}$$

where

$$\mathcal{Q} = \Big(\frac{(\alpha \cdot o)^2 (\pi \cdot o)}{(\pi \cdot \alpha)^2} - \frac{2(\beta \cdot o)(\alpha \cdot o)^2}{(\alpha \cdot \beta)(\pi \cdot \alpha)} - \frac{(\beta \cdot o)^2 (\pi \cdot o)}{(\pi \cdot \beta)^2} + \frac{2(\alpha \cdot o)(\beta \cdot o)^2}{(\alpha \cdot \beta)(\pi \cdot \beta)} \Big)$$

and $\alpha = \alpha_1 + u_2 \alpha_2 + ... + u_n \alpha_n, \; \beta = \beta_1 + w_2 \beta_2 + ... + w_n \beta_n$.

Appendix C

Differential Geometry

Let e be a one form on a connected and simply-connected domain of some manifold $\mathcal{M}$. Let

$$e_{(2k+1)} := e \wedge (\mathrm{d}e)^{\wedge k}, \quad e_{2k} := \mathrm{d}e^{\wedge k}$$

where $\wedge k$ denotes the kth exterior power for $k \in \mathbb{Z}$. We say that e is of class c if c is the maximal integer such that $e_{(c)} \neq 0$. We shall assume that $c = const$ for each e.

Theorem C.1 (Darboux) *Let e be a one form of a constant class c. There exists a set of independent functions p_i, q^i, t where $i = 1...c$ such that*

$$e = \sum_{i=1}^{k} p_i \mathrm{d}q^i \quad \textit{if}\ \ c = 2k, \qquad \textit{or} \quad e = \mathrm{d}t + \sum_{i=1}^{k} p_i \mathrm{d}q^i \quad \textit{if}\ \ c = 2k+1.$$

Let $D = (X_1, ...X_k)$ be a C^∞ k-dimensional distribution of vector fields on an n-dimensional manifold $\mathcal{F}$. We say that D is integrable if $[X_i, X_j] \in D$ whenever $X_i, X_j \in D$. Let $\mathcal{I}(D)$ denote a set of differential forms (of any degree) such that if $\omega \in \mathcal{I}(D)$ is a l-form then

$$\omega(X_1, ..., X_l) = 0, \quad \text{if}\ \ (X_1, ..., X_l) \in D.$$

The ideal $\mathcal{I}(D)$ is locally generated by $n-k$ one forms $(e^1, ..., e^{n-k})$. A sub-manifold Y^k is called an integral sub-manifold of D if its tangent space at each point is spanned by D, or equivalently $\mathcal{I}(D) = 0$, when restricted to Y^k. Assume that the dimension of the integral sub-manifold is a constant.

Theorem C.2 (Frobenius) *The following conditions are equivalent*

- *D spans an integrable distribution;* $[D, D] \subset D$,

- *$\mathcal{I}(D)$ is a closed differential ideal;* $\mathrm{d}(\mathcal{I}(D)) \subset \mathcal{I}(D)$,

- *$\mathcal{M}$ is foliated by integral sub-manifolds of D,*

- *there exists local coordinates $(y^1, ..., y^k, f^1, ..., f^{n-k})$ such that the vectors in D are linear combinations of $\partial/\partial y^i$ for $i = 1...k$, and are tangent to surfaces of constant f^{n-i}.*

If, furthermore, $[D, D] = 0$ then $X_i = \partial/\partial y^i$ in the above coordinate system. Let us give a more explicit corollary:

Corollary C.3 *If $e^1, ..., e^r$ are r linearly independent one forms*

$$\Sigma = e^1 \wedge ... \wedge e^r \neq 0$$

on some connected and simply-connected domain of $\mathcal{M}$, and there exists a one form A which satisfies

$$\mathrm{d}\Sigma = A \wedge \Sigma, \tag{C.1}$$

then

- *there exists a set of functions $f^1, ..., f^r$ such that $e^a = C^a_b \mathrm{d} f^b$ for some non-singular C^a_b. The functions f^a are constant on an integral submanifold of $e^1, ..., e^r$,*

- $\mathrm{d}e^a \wedge \Sigma = 0$ *for $a = 1...r$,*

- $\mathrm{d}e^a = \Gamma^a_b \wedge e^b$ *for some matrix of one forms Γ^a_b such that $\Gamma^a_a = A$.*

The Frobenius theorem applies when $\mathcal{F}$ is a projective spin bundle of some complex four–manifold $\mathcal{M}$ with a conformal structure $[g]$. The two-dimensional distribution D is spanned by horizontal lifts of $\pi^{A'}\nabla_{AA'}$ to $\mathcal{F}$. This distribution is integrable iff $[g]$ is ASD or SD. According to the Frobenius theorem $\mathcal{F}$ is foliated by leaves of this distribution. The three-dimensional space of these leaves is called the projective twistor space $\mathcal{PT}$. It should be mentioned that $\mathcal{F}$ and D are considered to be auxiliary tools. What matters is the correspondence between $(\mathcal{M}, [g])$ and $\mathcal{PT}$. In the original formulation below it was assumed that $[g]$ is Ricci flat.

Theorem C.4 (Penrose[56]) *There is one to one correspondence between ASD vacuum metrics on complex four-manifolds and three dimensional complex manifolds* $\mathcal{PT}$ *such that*

- *There exists a holomorphic projection* $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$
- $\mathcal{PT}$ *is equipped with a four complex parameter family of sections of* μ *each with a normal bundle* $\mathcal{O}(1) \oplus \mathcal{O}(1)$*,*
- *Each fibre of* μ *has a symplectic structure* $\Sigma_\lambda \in \Gamma(\Lambda^2(\mu^{-1}(\lambda)) \otimes \mathcal{O}(2))$*, where* $\lambda \in \mathbb{CP}^1$*.*

Let $\mathcal{F}$ be a $2n$ dimensional real manifold. The almost complex structure $I : T\mathcal{F} \longrightarrow T\mathcal{F}$ is an endomorphism of a tangent bundle $T\mathcal{F}$ such that $I^2 = -1$. Define the torsion of I by

$$N_I(X, Y) = [IX, IY] - [X, Y] - [IX, Y] - [X, IY], \quad \text{for} \quad X, Y \in T\mathcal{F}.$$

Decompose the complexification of the tangent bundle

$$\mathbb{C} \otimes T\mathcal{F} = T^{1,0}\mathcal{F} \oplus T^{0,1}\mathcal{F},$$

where $T^{1,0}\mathcal{F}$ and $T^{0,1}\mathcal{F}$ are eigenspaces of I corresponding to eigenvalues i and $-i$.

Theorem C.5 (Newlander-Nirenberg) *The following conditions are equivalent:*

- $T^{1,0}\mathcal{F}$ *spans an integrable distribution,*
- $T^{0,1}\mathcal{F}$ *spans an integrable distribution,*
- $N_I(X, Y) = 0$ *for any* $X, Y \in T\mathcal{F}$*,*
- $\mathcal{F}$ *is a complex manifold and its complex structure induces an almost complex structure* I*.*

If any of the conditions in the last theorem is satisfied, I is called integrable.

The theorem C.5 applies in the twistor approach to the positive definite four-metrics [3, 90]. The real even-dimensional manifold in question is the projective spin bundle $\mathcal{F}$, and the basis of antiholomorphic vector fields is given by $\pi^{A'}\nabla_{AA'}, \partial_{\overline{\lambda}}$.

Let $(\mathcal{M}_\mathbb{R}, [g])$ be a real oriented four manifold.

Theorem C.6 (Atiyah-Hitchin-Singer[3]) *The almost complex structure on $\mathcal{F}$ is integrable iff $[g]$ is ASD or SD.*

The last theorem follows from (C.4) if one introduces a real structure $\sigma : \mathcal{M} \longrightarrow \mathcal{M}$. It induces an antiholomorphic involution (with no fixed points) on $\mathcal{PT}$, therefore there is a unique *real line* joining Z to $\sigma(Z)$, where $Z \in \mathcal{PT}$. Real lines do not intersect and they define a nonholomorphic fibration of $\mathcal{PT}$ over the Euclidean slice of $\mathcal{M}_{\mathbb{R}}$. As a real manifold $\mathcal{PT}$ can be identified with $\mathcal{F} = \mathcal{M}_{\mathbb{R}} \times \mathbb{CP}^1$.

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Note added in 2014: Some chapters of this thesis have since been developed and published in the following papers

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