Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Solution of the Hermite equation by integral representation

In this example, the Hermite equation is solved using the Laplace representation. There is a parameter in the equation and the nature of the singular points of the integrand of the representation depends on this parameter (is the origin a pole or a branch point or neither, for example). This means that the paths of integration have to be chosen for the different cases that arise.

We will solve the Hermite equation[1]

$$w'' - 2zw' + 2\nu w = 0$$

using the Laplace kernel $K(z,t) = e^{zt}$.

First we write $w(z)$ in its *Laplace representation*:

$$w(z) = \int_\gamma e^{zt} f(t)\,\mathrm{d}t, \qquad (*)$$

where γ and $f(t)$ are to be determined. Substituting this integral into the differential equation gives

$$\begin{aligned} 0 &= \int_\gamma \big(t^2 - 2tz + 2\nu\big) f(t) e^{zt}\,\mathrm{d}t \\ &= \int_\gamma \big(t^2 + 2\nu\big) f(t) e^{zt}\,\mathrm{d}t - \int_\gamma 2tf(t) z e^{zt}\,\mathrm{d}t \\ &= \int_\gamma \big(t^2 + 2\nu\big) f(t) e^{zt}\,\mathrm{d}t - \int_\gamma 2tf(t) \frac{\partial e^{zt}}{\partial t}\,\mathrm{d}t \\ &= \int_\gamma \big[(t^2 + 2\nu + 2) f(t) + 2tf'(t)\big] e^{zt}\,\mathrm{d}t - \Big[\, 2tf(t)e^{zt}\,\Big]_\gamma . \end{aligned}$$

The integrand can be made to vanish by choosing $f(t)$ to satisfy the first order differential equation:

$$(t^2 + 2\nu + 2)f(t) + 2tf'(t) = 0.$$

Ignoring the constant of integration, which is just an overall multiple, the solution is

$$f(t) = t^{-\nu-1} e^{-\frac{1}{4}t^2}.$$

Hermite's equation is therefore satisfied provided γ is chosen such that

$$\Big[\, e^{zt} t f(t)\,\Big]_\gamma = 0 \quad \text{i.e.} \quad \Big[\, t^{-\nu} e^{-\frac{1}{4}t^2 + zt}\,\Big]_\gamma \equiv \Big[\, g(z,t)\,\Big]_\gamma = 0.$$

[1] The Hermite equation is a transformation of the Schrödinger equation with a simple harmonic oscillator potential $-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2 x^2\psi = E\psi$. The new (dimensionless) independent variable is given by $z = x(m\omega/\hbar)^{1/2}$ and the new dependent variable w given by $\psi = we^{-z^2/2}$. The new constant ν is defined by $E = \hbar\omega(\nu + \frac{1}{2})$. The Hermite equation is a special case of the confluent hypergeometric equation with $c = \nu/2$ and $z_{\text{hermite}} = z_{\text{chge}}^{1/2}$.

Next, we find the zeros of $g(z,t)$, regarded as a function of t. $g(z,t)$ vanishes in the sectors of infinity for which

$$Re\ \big(-\frac{1}{4}t^2 + zt\big) < 0,$$

i.e. when $|\arg t| < \pi/4$ or $|\pi - \arg t| < \pi/4$. If $Re\ \nu < 0$, $g(z,t)$ also vanishes at $t = 0$.

Finally, we investigate the singularities of the integrand of $(*)$:

$$e^{zt} f(t) = t^{-\nu-1} e^{-\frac{1}{4}t^2 + zt}.$$

If ν is not an integer, then the t-plane must have a branch cut from 0 to ∞ along (say) the negative real axis. If ν is a positive integer or zero, then there is a pole at $t = 0$, which can be encircled by a closed contour.

Putting this together gives various possibilities, which cover all values of ν.

(i) $\nu \neq$integer.
In this case, there is a branch cut along the negative real axis in the t-plane. One path can wrap round this cut and the other can run parallel to the real axis from $-\infty$ to ∞. If $Re\ \nu < 0$, $g(z,t)$ has a zero at the origin, so instead of wrapping round the cut, the first path could terminate at the origin.

(ii) $\nu = 0, 1, 2 \ldots$.
Now $f(t)$ has pole instead of a branch point at the origin. One path can run parallel to the real axis and the other can simply circle the origin since the branch cut is not now necessary.

(iii) $\nu = -1, -2, \ldots$.
Now there is no singularity of $f(t)$ at the origin but $g(z,0) = 0$. One path can run parallel to the real axis and the other can begin at $-\infty$ or $+\infty$ and terminate at the origin.

Remarks

1) The choice of paths is by no means unique. In fact a simplified choice for is given by a path parallel to the real axis (in all cases) and either a path wrapping round the negative real axis if $Re\ \nu \geq 0$ or a path along the negative real axis terminating at the origin if $Re\ \nu < 0$.

2) We have not demonstrated that the two solutions are linearly independent. Two paths which are not *obviously* equivalent will generally give linearly independent solutions, but we have to be cautious. For example, for case (iii), paths which run above and below the branch cut in the case $Re\ \nu < 0$ and terminate at the origin do not give linearly independent solutions: the two integral representations just differ by an overall factor of $e^{-2\pi i(\nu+1)}$.

3) The choice of the branch in the integrand was arbitrary: any other choice (for example, a cut along the positive t-axis) would have served equally well. However, it is dangerous to use one branch for one of the solutions and a different branch for the other. If it is difficult to find a second suitable path, it is tempting to move the cut, but this can lead to a second solution which is not linearly independent.