An interpolating dispersionless integrable system

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Abstract
We introduce a dispersionless integrable system which interpolates between the
dispersionless Kadomtsev–Petviashvili equation and the hyper-CR equation.
The interpolating system arises as a symmetry reduction of the anti-self-dual
Einstein equations in (2, 2) signature by a conformal Killing vector whose self-
dual derivative is null. It also arises as a special case of the Manakov–Santini
integrable system. We discuss the corresponding Einstein–Weyl structures.

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1. Introduction

It has been known for more than 20 years that many integrable systems admitting soliton
solutions arise as symmetry reductions of anti-self-dual Yang–Mills (ASDYM) equations in
four dimensions [25]. The Riemann–Hilbert factorization problem underlies this approach
to integrability: it appears in classical solution generating techniques such as dressing
transformations [18], as well as in the twistor treatment of ASDYM [24].

The dispersionless integrable systems in 2+1 dimensions do not fit into this framework:
they do not admit soliton solutions and there is no associated Riemann–Hilbert problem where
the corresponding Lie group is finite dimensional. These systems can nevertheless be described
in terms of anti-self-duality (ASD) conditions on a four dimensional conformal structure. In
this case the 'unknown' in the equations is not a gauge field, but rather a metric (up to scale)
on some four-manifold [21]. This makes the dispersionless systems more geometric than
their solitonic cousins. This point of view may be of deep significance in the description
of shock formations: a recent beautiful analysis by Manakov and Santini [16] deduced the
gradient catastrophe of the localized solutions to the dKP equation using the inverse scattering
transform. It may be however, that this catastrophe is only an artefact of a chosen coordinate
system, and the underlying conformal structure is regular, but needs to be covered by more
than one coordinate patch. It remains to be seen whether this is indeed the case.

To classify existing 2+1 dispersionless integrable systems, and perhaps discover some
new ones one needs to classify the symmetry reductions of the conformal ASD equations. If

the Ricci-flat condition is imposed on top of the anti-self-duality, the work of Plebański [22] implies the existence of a local coordinate system (X, Y, W, Z) and a function Θ on an open set $\mathcal{M} \subset \mathbb{R}^4$ such that any ASD Ricci-flat metric is locally of the form

$$g = 2(dZ\,dY + dW\,dX - \Theta_{XX}\,dZ^2 - \Theta_{YY}\,dW^2 + 2\Theta_{XY}\,dW\,dZ), \qquad (1.1)$$

where $\Theta_{XY} = \partial_X \partial_Y \Theta$ etc, and Θ satisfies the second heavenly equation

$$\Theta_{ZY} + \Theta_{WX} + \Theta_{XX}\Theta_{YY} - \Theta_{XY}^2 = 0. \qquad (1.2)$$

This metric has the signature $(+ + - -)$ but this is precisely what it needs: given a non-null symmetry, the conformal structure on a space of orbits will have the signature $(2, 1)$ and will be described by a hyperbolic integrable equation. We are thus led to study (1.1), (1.2) subject to the existence of a conformal Killing vector K

$$\mathcal{L}_K g = cg, \qquad g(K, K) \neq 0,$$

where c is some function. The classification of reductions is based on studying the action of K on the bundle of self-dual 2-forms over $\mathcal{M}$. Assume that $\mathcal{M}$ is oriented and recall that given a metric of $(+ + - -)$ signature on $\mathcal{M}$, the Hodge $*$ operator is an involution on 2-forms and induces a decomposition

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$$

of 2-forms into self-dual and anti-self-dual components. Moreover, there exist real two-dimensional vector bundles $\mathbb{S}$ and $\mathbb{S}'$ (called spin bundles) over $\mathcal{M}$ such that $T\mathcal{M} \cong \mathbb{S} \otimes \mathbb{S}'$ and $\Lambda_+^2 \cong \mathbb{S} \odot \mathbb{S}$. Therefore the self-dual derivative of (the 1-form metric dual to) K

$$dK_+ := \tfrac{1}{2}(dK + *dK)$$

corresponds to a symmetric 2 by 2 matrix ϕ explicitly given by

$$dK_+ = \phi_{AB}\Sigma^{AB}, \qquad A, B = 0, 1,$$

where the self-dual 2-forms[1] $(\Sigma^{00}, \Sigma^{01}, \Sigma^{11})$ span Λ_+^2. The rank of the matrix ϕ does not depend on the choice of the basis Σ^{AB} and the classification is based on this rank. In the Riemannian signature this can only be 0 or 2 (the former case is called the tri-holomorphic symmetry), but in the $(+ + - -)$ signature we can also have rank$(\phi) = 1$, in which case $(dK)_+ \wedge (dK)_+ = 0$ so the self-dual derivative of K is null.

This classification programme has almost been completed and one aim of this paper is to remove the question mark from the following table summarizing the reductions of the second heavenly equation (1.2)

	$c = 0$	$c \neq 0$
rank$(\phi) = 0$	2+1 Linear wave equation [9, 11]	Hyper-CR equation [3, 4]
rank$(\phi) = 1$	Dispersionless KP equation [5]	?
rank$(\phi) = 2$	SU(∞) Toda equation [9]	Integrable equation studied in [6]

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[1] The ASD Ricci-flat equations are equivalent to

$$d\Sigma^{AB} = 0, \qquad \Sigma^{(AB} \wedge \Sigma^{CD)} = 0, \qquad A, B, C, D = 0, 1$$

and the second heavenly equation (1.2) arises by using the Darboux theorem to introduce coordinates such that

$$\Sigma^{00} = dW \wedge dZ, \qquad \Sigma^{01} = dW \wedge dX + dZ \wedge dY,$$

and deducing the existence of the function Θ from the remaining conditions on Σ^{AB}.

The case where $\mathrm{rank}(\phi) = 1$ and $c \neq 0$ has not yet been investigated, and the resulting integrable system is the subject of the present paper. This system is given by

$$u_y + w_x = 0, \qquad u_t + w_y - c(uw_x - wu_x) + buu_x = 0, \tag{1.3}$$

where b and c are constants and u, w are smooth functions of (x, y, t). We propose to call (1.3) an interpolating system as it contains two well-known dispersionless equations as the limiting cases: setting $b = 0, c = -1$ gives the hyper-CR equation [3, 4, 8, 14, 17, 19, 20] and setting $c = 0, b = 1$ gives the dKP equation. In fact one constant can always be eliminated from (1.3) by redefining the coordinates and it is only the ratio of b/c which remains[2]. We prefer to keep both constants as it makes various limits more transparent.

In the following section we shall give the dispersionless Lax pair for (1.3). The Lie group underlying the Lax formulation is $\mathrm{Diff}(\Sigma^2)$—an infinite-dimensional group of diffeomorphisms of some two-dimensional manifold Σ^2. The interpolating system corresponds to Lorentzian Einstein–Weyl structures in three dimensions. This gives an intrinsic geometric interpretation without the need of going to four dimensions. This will be described in section 3. In section 4 we shall show that (1.3) is a special case of the Manakov–Santini integrable system [15, 14] and find the Manakov–Santini Einstein–Weyl structure.

The explicit reduction of the second heavenly equation (1.2) to the interpolating system (1.3) will be presented in the appendix. In particular, we shall show that the most general $(+ + - -)$ ASD Ricci-flat metric with a conformal Killing vector whose self-dual derivative is null is of the form

$$g = \mathrm{e}^{c\phi}(Vh - V^{-1}(\mathrm{d}\phi + A)^2), \tag{1.4}$$

where ϕ parametrizes the orbits of the conformal Killing vector $K = \partial/\partial\phi$ and

$$h = (\mathrm{d}y + cu\,\mathrm{d}t)^2 - 4(\mathrm{d}x + cu\,\mathrm{d}y - (cw + bu)\,\mathrm{d}t)\,\mathrm{d}t,$$

$$A = -\frac{1}{2}u_x\,\mathrm{d}y + \left(\frac{c}{2}uu_x - u_y\right)\mathrm{d}t, \qquad V = \frac{1}{2}u_x.$$

This reduction explains the origin of the two parameters (b, c) in (1.3) as in this case the conformal symmetry is

$$K = c \times (\text{dilatation}) + b \times (\text{rotation with a null SD derivative}).$$

2. Lax pair and Diff(Σ^2) hierarchies

The system (1.3) admits a dispersionless Lax pair

$$
\begin{aligned}
L_0 &= \frac{\partial}{\partial t} + (cw + bu - \lambda cu - \lambda^2)\frac{\partial}{\partial x} + b(w_x - \lambda u_x)\frac{\partial}{\partial \lambda}, \\
L_1 &= \frac{\partial}{\partial y} - (cu + \lambda)\frac{\partial}{\partial x} - bu_x\frac{\partial}{\partial \lambda}
\end{aligned}
\tag{2.1}
$$

with a spectral parameter $\lambda \in \mathbb{CP}^1$. The overdetermined system of linear equations $L_0\Phi = L_1\Phi = 0$, where $\Phi = \Phi(x, y, t, \lambda)$ admits solutions because equations (1.3) are equivalent to $[L_0, L_1] = 0$.

In general, consider the vector fields of the form

$$L_i = \frac{\partial}{\partial t_i} + A_i\frac{\partial}{\partial x} + B_i\frac{\partial}{\partial \lambda}, \tag{2.2}$$

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[2] Ferapontov has pointed out that if $b \neq 0$ the hydrodynamic reductions of (1.3) coincide with the hydrodynamic reductions of the dKP equation, despite the fact that dKP and (1.3) are not point or contact equivalent unless $c = 0$.

where A_i, B_i are polynomials in λ with coefficients depending on $(t^0 = x, t^i)$. The flows of the Diff(Σ^2) hierarchy are defined by

$$[L_i, L_j] = 0, \qquad i, j = 1, \dots, n. \qquad (2.3)$$

To achieve a dual formulation, generalizing Krichever's approach to dispersionless integrable systems [13], complexify the hierarchy (so that $(t^0, t^i) \in \mathbb{C}^{n+1}$) and define a 2-form Ω on $\mathbb{C}^{n+1} \times \mathbb{CP}^1$ by

$$\Omega(X, Y) = dt_1 \wedge \cdots \wedge dt_n \wedge dx \wedge d\lambda(L_1, \dots, L_n, X, Y)$$

so that

$$\Omega = dx \wedge d\lambda + \sum_i (A_i \, d\lambda - B_i \, dx) \wedge dt_i + \sum_{i,j} (A_i B_j - B_j A_i) \, dt_i \wedge dt_j.$$

The 2-form Ω is simple and satisfies the Frobenius integrability conditions

$$\Omega \wedge \Omega = 0, \qquad d\Omega = \Omega \wedge \beta$$

for some 1-form β. We recover various dispersionless hierarchies as special cases of this formulation

- *SDiff*(Σ^2) *hierarchy.* The group Diff(Σ^2) reduces to SDiff(Σ^2) generated by Hamiltonian vector fields. The corresponding Lie algebra is homomorphic with the Poisson bracket algebra. The vector fields L_i preserve the 2-form $dx \wedge d\lambda$ and

$$A_i = \frac{\partial H_i}{\partial \lambda}, \qquad B_i = -\frac{\partial H_i}{\partial x}.$$

  The 2-form is given by $\Omega = dx \wedge d\lambda + \sum_i dH_i \wedge dt_i$ (where we have used (2.3)), and the SDiff(Σ^2) hierarchy is given by

$$\Omega \wedge \Omega = 0, \qquad d\Omega = 0,$$

  which is the original Krichever's formulation [13]. The Darboux theorem implies the existence of functions P, Q such that $\Omega = dP \wedge dQ$. These two functions are local coordinates on the twistor space, which is a quotient of $\mathbb{C}^{n+1} \times \mathbb{CP}^1$ by the integrable distribution $\{L_i\}$. Both the dKP and SU(∞) Toda hierarchies fit into this category [5, 13, 23].

- *Diff*(S^1) *hierarchy.* The group Diff(Σ^2) reduces to Diff(S^1), where $\Sigma^2 = TS^1$. This case corresponds to

$$B_i = 0.$$

  The underlying Lie algebra is that of a Wronskian with a Lie bracket $\langle f, g \rangle = f_x g - f g_x$. In this case $\Omega = e \wedge d\lambda$, where $e = dx - \sum_i A_i \, dt_i$. This 1-form is integrable in the Frobenius sense

$$e \wedge de = 0,$$

  where d keeps $\lambda = $ const. The twistor space fibres holomorphically over $\mathbb{CP}^1$. The hyper-CR hierarchy [4] and the universal hierarchy [17] are of this type.

3. Interpolating Einstein–Weyl structure

A three dimensional Lorentzian Weyl structure $(M, D, [h])$ consists of a 3-manifold M, a torsion-free connection D and a conformal metric $[h]$ of Lorentzian signature such that the null geodesics of $[h]$ are also geodesics for D. This condition is equivalent to

$$Dh = \omega \otimes h$$

for some 1-form ω. Here h is a representative metric in the conformal class. If we change this representative by $h \longrightarrow \gamma^2 h$, then $\omega \longrightarrow \omega + 2\mathrm{d}\ln\gamma$. A Weyl structure is called Einstein–Weyl if the conformally invariant equations

$$R_{(ab)} = \Lambda h_{ab}, \qquad a, b, = 1, \ldots, 3 \tag{3.1}$$

hold for function Λ. Here $R_{(ab)}$ is the symmetrized Ricci tensor of D and h_{ab} is a representative metric in a conformal class $[h]$. In practice the Einstein–Weyl structure is given by specifying the metric $h \in [h]$, and the 1-form ω which measures the difference between the Weyl connection D and the Levi-Civita connection of h.

The three dimensional Einstein–Weyl condition is a dispersionless integrable system [5]: let Z, W, $\widetilde{W}$ be independent vector fields on M such that a contravariant metric in $[h]$ is

$$h = h^{ab}\frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} = Z \otimes Z - 2(W \otimes \widetilde{W} + \widetilde{W} \otimes W).$$

Then there exists a connection D such that $(M, [h], D)$ is Einstein–Weyl if the dispersionless Lax pair

$$L_0 = W - \lambda Z + f_0\frac{\partial}{\partial\lambda}, \qquad L_1 = Z - \lambda\widetilde{W} + f_1\frac{\partial}{\partial\lambda} \tag{3.2}$$

satisfies the integrability condition

$$[L_0, L_1] = 0 \quad \text{modulo} \quad L_0, L_1$$

for functions (f_0, f_1) which are cubic polynomials in $\lambda \in \mathbb{CP}^1$. Conversely, every Einstein–Weyl structure arises from some Lax pair (3.2). The corresponding 1-form ω can be read off from the Levi-Civita connection of h and the coefficients of (f_0, f_1). This Lax formulation has a geometric origin which goes back to Cartan [2]

- Einstein–Weyl condition is equivalent to the existence of a two-parameter family of totally geodesic null surfaces in M.

Comparing this with the Lax pair (2.1) for the interpolating system, and taking linear combinations to put (2.1) in the form (3.2) we find the corresponding Einstein–Weyl structure to be

$$\begin{aligned}
h &= (\mathrm{d}y + cu\,\mathrm{d}t)^2 - 4(\mathrm{d}x + cu\,\mathrm{d}y - (cw + bu)\,\mathrm{d}t)\,\mathrm{d}t, \\
\omega &= -cu_x\mathrm{d}y + (4bu_x + c^2uu_x - 2cu_y)\,\mathrm{d}t.
\end{aligned} \tag{3.3}$$

Taking the limits we recover various known Einstein–Weyl structures from (3.3). These structures can be characterized by the properties null shear-free and geodesic congruence $\mathrm{d}t$ (this elegant framework is described, in the Riemannian case, in [1])

- Setting $b = 0, c = -1$ gives the hyper-CR Einstein–Weyl structure [4]. The congruence is divergence-free. This is a Lorentzian analogue Einstein–Weyl structure studied in [1, 6, 10].
- Setting $c = 0, b = 1$ gives the dKP Einstein–Weyl spaces [5]. The congruence $\mathrm{d}t$ is now twist-free, and the dual vector $\partial/\partial x$ is parallel with a weight $-1/2$ with respect to the Weyl connection.

We remark that the Einstein–Weyl (3.3) structure can also be read off from the ASD Ricci-flat metric (1.4) using the Jones–Tod correspondence [12].

4. The Manakov–Santini system

The system (1.3) is a special case of the Manakov–Santini system [,14, 15]

$$U_{xt} - U_{yy} + (UU_x)_x + V_x U_{xy} - V_y U_{xx} = 0$$

$$V_{xt} - V_{yy} + UV_{xx} + V_x V_{xy} - V_y V_{xx} = 0,$$

(4.1)

where $U = U(x, y, t)$, $V = V(x, y, t)$. To see it, note that the first equation in (1.3) implies the existence of $v(x, y)$ such that $u = v_x$, $w = -v_y$, and v satisfies

$$v_{xt} - v_{yy} + c(v_x v_{xy} - v_y v_{xx}) + buv_{xx} = 0.$$

(4.2)

Differentiating the second equation in (1.3), and eliminating w yields

$$u_{xt} - u_{yy} - c(v_y u_{xx} - v_x u_{xy}) + b(uu_x)_x = 0.$$

(4.3)

Now assume the generic case when the constants c, b are non zero, and set $U = bu$, $V = cv$. Then the system (4.2) and (4.3) is equivalent to (4.1) with an additional constraint

$$cU - bV_x = 0.$$

(4.4)

The Manakov–Santini system is more general than (1.3): Regarding the second equation in (4.1) as the definition of U, and substituting U to the first equation in (4.1) yields a fourth-order scalar PDE for V. Thus the naive counting suggests that the general solution to (4.1) depends on four functions of two variables (a caution is needed as the resulting PDE is not in the Cauchy–Kowalewska form). In the special case when the constraint (4.4) holds, both equations in (4.1) reduce to a single second-order PDE for V. The solution depends on two functions of two variables, and a constant (the ratio b/c).

The Manakov–Santini system also corresponds to an Einstein–Weyl structure

$$h = (dy - V_x \, dt)^2 - 4(dx - (U - V_y) \, dt) \, dt,$$

$$\omega = -V_{xx} \, dy + (4U_x - 2V_{xy} + V_x V_{xx}) \, dt.$$

(4.5)

To verify it set $x^a = (y, x, t)$. The (11), (12), (22), (23) components of the Einstein–Weyl equations hold identically. The (13) component vanishes if the second equation in (4.1) holds, and finally the (33) component vanishes if both equations in (4.1) are satisfied.

Conversely, consider the general conformal structure in (2+1) dimensions given in local coordinates x^a by a representative metric

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix}.$$

Using the diffeomorphism freedom, and the conformal rescaling we can impose four constraints on six functions $h_{ab}(x^c)$ as long as the resulting quadratic form is non-degenerate. We choose to set $h_{11} = h_{12} = 0$, $h_{13} = -2$, $h_{23} = -A$, $h_{33} = A^2 + 4B$, where A and B are some functions of (x, y, t), so that[3]

$$h = (dy - A \, dt)^2 - 4(dx - B \, dt) \, dt.$$

Now given A, B we can always find two functions U, V such that $A = V_x$, $B = U - V_y$ so that the metric is in the form (4.5)

Now we find the corresponding dual basis, and construct the Lax pair (3.2). Before imposing the integrability conditions it is convenient to take a linear combination of the vectors in this distribution, so that the resulting pair of vectors commutes exactly. This yields

$$L_0 = \partial_y - (\lambda + A)\partial_x + f_0 \partial_\lambda, \qquad L_1 = \partial_t - (\lambda^2 + \lambda A - B)\partial_x + f_1 \partial_\lambda,$$

[3] This is analogous to the existence of orthogonal coordinates in three dimensions. The proof is relatively straightforward in the real-analytic category, and more subtle in the smooth category.

where the polynomials f_0 and f_1 are respectively cubic and quartic in λ. There is some additional freedom which will preserve the above form of the Lax pair. We can translate the fibres by $\lambda \rightarrow \lambda + \kappa(x, y, t)$. We use this freedom to set the linear term in f_0 to zero. It is possible that some further coordinate freedom can be used to set the quadratic term in f_0 to zero which would imply that ω is given by (4.5). This would imply that every Einstein–Weyl structure is equivalent to the Manakov–Santini EW structure (4.5). So far we have been unable to find the right transformation, and we need to impose $\partial_\lambda^2 f_0 = 0$ as an additional condition. Then the integrability condition $[L_0, L_1] = 0$ implies (4.1).

Acknowledgments

Appendix. Reduction of the second heavenly equation

We shall prove that the system (1.3) arises as the most general symmetry reduction of the second heavenly equation (1.2) by a conformal Killing vector with a null self-dual derivative.

Let $\Theta = \Theta(W, Z, X, Y)$ satisfy the second heavenly equation and let the corresponding metric be given by (1.1). Let K be a conformal Killing vector for (1.1). Using Penrose's two-component spinor formalism we can show that the conformal Killing equations and the Ricci identity imply that $\nabla_{AA'} K^A{}_{B'}$ is covariantly constant, or otherwise g is of Petrov–Penrose type N and can be found explicitly. In the spin frame of the heavenly metric (1.1) the connection on the spin bundle $\mathbb{S}$ vanishes, so $\nabla_{A'A} K^{A'}{}_B$ is in fact constant. We are interested in the case where the self-dual derivative $\phi_{AB} = \nabla_{A'(A} K^{A'}{}_{B)}$ is of rank 1. Therefore we need to integrate the linear system

$$\nabla_{A'A} K^{A'}{}_B = \begin{pmatrix} 0 & c \\ -c & b \end{pmatrix}.$$

The constant c appears because K is a conformal Killing vector.

Following the method of Finley and Plebański [9] and using a freedom in the heavenly potential Θ as well as in the choice of coordinates we find the general solution to be

$$K = (cZ + b)\partial_Z + (cX - 2bZ)\partial_X.$$

The conformal Killing equations $\mathcal{L}_K g = cg$ now yield

$$\mathcal{L}_K(\Theta_{XX}) = -c\Theta_{XX}, \qquad \mathcal{L}_K(\Theta_{XY}) = b, \qquad \mathcal{L}_K(\Theta_{YY}) = c\Theta_{XX}.$$

Let U and T be functions such that $K = \partial_T$ and $\mathcal{L}_K(U) = 0$. For $c \neq 0$ we can take

$$T = \frac{\ln(cZ + b)}{c}, \qquad U = \frac{2b}{c}T + \frac{2b^2 + Xc^2}{c^2(cZ + b)}.$$

The compatibility conditions for the Killing equations imply the existence of $\tilde{G} = \tilde{G}(Y, W, U)$ such that

$$\Theta_{XX} = e^{-cT}\tilde{G}_{UU}, \qquad \Theta_{XY} = \tilde{G}_{YU} + bT, \qquad \Theta_{YY} = e^{cT}\tilde{G}_{YY}.$$

The heavenly equation (1.2) becomes

$$bU + \frac{2b}{c}\tilde{G}_{YU} + c(\tilde{G}_Y - U\tilde{G}_{YU}) + \tilde{G}_{UW} + \tilde{G}_{YY}\tilde{G}_{UU} - \tilde{G}_{YU}^2 = 0.$$

To obtain a simplified form define

$$G(Y, W, U) = \tilde{G}(Y, W, U) + \frac{b}{c}UY + \frac{b^2}{c^2}UW$$

so that

$$bU + c(G_Y - UG_{YU}) + G_{UW} + G_{YY}G_{UU} - G_{YU}^2 = 0. \tag{A.1}$$

Now rewrite (A.1) in terms of differential forms

$$bu\, dY \wedge dU \wedge dW + c(G_Y\, dY \wedge dU \wedge dW - U\, dG_U \wedge dU \wedge dW)$$
$$+ dG_U \wedge dY \wedge dU + dG_Y \wedge dG_U \wedge dW = 0. \tag{A.2}$$

Define

$$x = G_U, \qquad y = Y, \qquad t = -W,$$
$$H(x, y, t) = xU(x, y, t) - G(Y, W, U(x, y, t)),$$

and preform a Legendre transform

$$dH = d(xU - G) = U\, dx - G_Y\, dY - G_W\, dW$$
$$= H_x\, dx + H_y\, dy + H_t\, dt.$$

Therefore

$$U = H_x, \qquad G_Y = -H_y, \qquad G_W = H_t.$$

Differentiating these relations we find

$$G_{UU} = \frac{1}{H_{xx}}, \qquad G_{YU} = -\frac{H_{xy}}{H_{xx}}, \qquad G_{YY} = -H_{yy} + \frac{H_{xy}^2}{H_{xx}}.$$

The differential equation for $H(x, y, t)$ is obtained from (A.2),

$$H_{xt} + bH_xH_{xx} + c(H_{xy}H_x - H_yH_{xx}) = H_{yy}. \tag{A.3}$$

Setting $u = H_x$, $w = -H_y$ we recover (1.4), where (u, w) solve (1.3).

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