Non-relativistic twistor theory and Newton–Cartan Geometry

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

- MD, James Gundry. arXiv:1502.03034 .

Outline

- Twistor theory (Penrose 1967).
  - Non–local theory of space–time.
  - Light rays more fundamental than events.
  - Non–perturbative physics constrained by self–duality.
  - Impact on pure mathematics (differential geometry, integrability, . . .).
- Newton-Cartan theory (Cartan 1923, Trautman 1963, . . .).
  - Space–time description of Newtonian (non–relativistic) gravity.
  - Baroque mathematical structure (connection, degenerate metric, one–form, . . .).
  - A degenerate limit of General Relativity when $c \to \infty$.
  - Recently resurrected in non–relativistic AdS–CFT correspondence.
- Non–relativistic limit of twistor theory.
  - Jumping lines in twistor space.
  - Unstable under holomorphic deformations.
  - Describes all Newtonian space–times (not constrained by self–duality).
  - Non–relativistic limits of gravitational instantons.

- Integral formula

$$\phi(x^\mu) = \oint_{\Gamma\subset\mathbb{CP}^1} f((x+iy)+2\lambda z-\lambda^2(x-iy), t-\frac{1}{c}(z-\lambda(x-iy)), \lambda)d\lambda.$$

Differentiate inside the integral

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}=0.$$

- Complex 3–fold PT_c. Covering $U=\{(Q,T,\lambda)\}$, $\widetilde{U}=\{(\widetilde{Q},\widetilde{T},\tilde{\lambda})\}$.

$$\tilde{\lambda}=\frac{1}{\lambda}, \quad \begin{pmatrix}\widetilde{T}\\ \widetilde{Q}\end{pmatrix}=\begin{pmatrix}1 & -(c\lambda)^{-1}\\ 0 & \lambda^{-2}\end{pmatrix}\begin{pmatrix}T\\ Q\end{pmatrix} \quad \text{on } U\cap\widetilde{U}.$$

- 4–parameter family $M_{\mathbb{C}}$ of $\mathbb{CP}^1$s in PT_c.

$$Q=-(x+iy)-2\lambda z+\lambda^2(x-iy), \quad T=t-\frac{1}{c}(z-\lambda(x-iy)).$$

Vector bundles over $\mathbb{CP}^1$ and jumping lines

- Vector bundle $\mu : PT_c \to \mathbb{CP}^1$ with a patching matrix

$$F_c = \begin{pmatrix} 1 & -(c\lambda)^{-1} \\ 0 & \lambda^{-2} \end{pmatrix}.$$

- Grothendieck (1928-2014): $PT_c = \mathcal{O}(m) \oplus \mathcal{O}(n)$

- $H(\lambda), \widetilde{H}(\tilde{\lambda}) \in GL(2, \mathbb{C})$, such that $F_c = \widetilde{H}\ \text{diag}(\lambda^{-m}, \lambda^{-n})\ H^{-1}$.
  - $c = \infty$. $PT_c = \mathcal{O} \oplus \mathcal{O}(2)$.
  - $c \neq \infty$. $PT_c = \mathcal{O}(1) \oplus \mathcal{O}(1)$.
- Real structure $\sigma : PT_c \to PT_c$

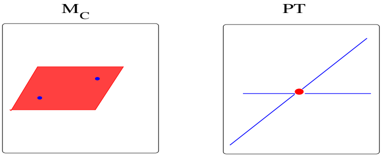
$$\sigma(Q, \lambda, T) = \Big(-\overline{\lambda}^{-2}\overline{Q}, -\overline{\lambda}^{-1}, -\overline{T} + (c\overline{\lambda})^{-1}\overline{Q} \Big).$$

σ–invariant curves: (x, y, z) real, $t = i\tau$ imaginary.

Twistor correspondence

- Twistor correspondence

$$
\begin{array}{rcl}
\text{Complexified space-time } M_{\mathbb{C}} & \longleftrightarrow & \text{Twistor space } PT_c \\
\text{Point } p & \longleftrightarrow & \text{Complex line } \; L_p = \mathbb{CP}^1 \\
\text{null self-dual } (=\alpha) \text{ two-plane} & \longleftrightarrow & \text{Point.}
\end{array}
$$

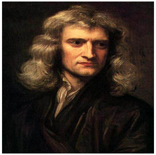


- p_1, p_2 null separated iff L_1 and L_2 intersect at one point.

$$\delta Q(x^\mu, \lambda) = 0, \quad \delta T(x^\mu, \lambda) = 0.$$

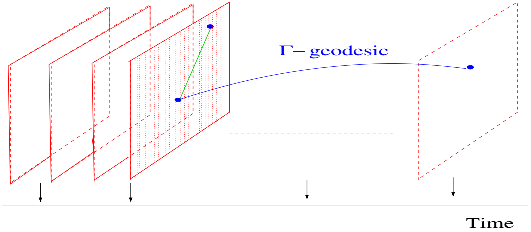
- $c \neq \infty$ conformal structure.
- $c = \infty$. $T = \widetilde{T}$ global twistor function
  - Fibration $M \to M/\text{ker}(\theta) = \mathbb{R}$, where $\theta = d\tau$ (clock).
  - Degenerate metric $h = (\partial/\partial x)^2 + (\partial/\partial y)^2 + (\partial/\partial z)^2$.

Newton–Cartan theory

- $\frac{d^2\mathbf{x}}{dt^2} = -\nabla V, \quad V : \mathbb{R}^3 \to \mathbb{R}.$
- Geometric perspective: $\ddot{x}^a + \Gamma^a_{bc}\, \dot{x}^b \dot{x}^c = 0, \;\; \Gamma^i_{00} = \delta^{ij}\partial_j V.$
- Newton–Cartan structure (∇, h, θ)
  - $h \in \Gamma(\mathrm{Sym}^2(TM)), \quad$ signature $(0, 3)$.
  - $\theta \in \Gamma(T^*M)$.
  - Torsion-free connection ∇ s.t. $\nabla h = \nabla\theta = 0$, so $\theta = dt$.
- Einstein eq: Galilean coordinates $x^a = (t, \mathbf{x}), h = \mathrm{diag}(0, 1, 1, 1)$.
- $\nabla \leftrightarrow F \in \Lambda^2(M)$, $\Gamma^c_{ab} = \theta_{(a}F_{b)d}h^{dc}$. Trautman condition $dF = 0$.

- Physical consequences of NC formalism



  - Free fall $\longleftrightarrow$ geodesics of ∇.
  - Time simultaneity $\longleftrightarrow$ fibration $M \rightarrow \mathbb{R}$.
  - NO absolute space. Needs a choice of Galilean coordinates.
- Newton–Cartan from GR.
  - Null Kaluza-Klein reductions (Eisenhart, Duval, Gibbons, . . .)
  - Non–relativistic limit $\lim_{c\rightarrow\infty} GR = NC$ (Dautcourt, Kunzle, Ehlers).
- One parameter family of (pseudo)–Riemannian metrics $g(\epsilon)$ such that
  - $h^{ab} \equiv \lim_{\epsilon\rightarrow 0} g^{ab}(\epsilon)$ exists and has signature $(0, 3)$.
  - $\lim_{\epsilon\rightarrow 0} \epsilon g_{ab}(\epsilon) = \theta_a\theta_b$, where $d\theta = 0$, and $h^{ab}\theta_a = 0$.

Newtonian limit of Gravitational Instantons

- Abelian monopole. $V \in C^\infty(\mathbb{R}^3), A \in \Lambda^1(\mathbb{R}^3)$, and $*dV = dA$.
- One–parameter family of Gibbons–Hawking (GH) metrics

$$g = (1 + \epsilon V)(dx^2 + dy^2 + dz^2) + \frac{1}{\epsilon(1 + \epsilon V)}(d\tau + \epsilon^{3/2} A)^2.$$

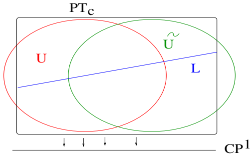
- Anti–self-dual and Ricci flat for all $\epsilon \in \mathbb{R}^+$.
- Newtonian limit

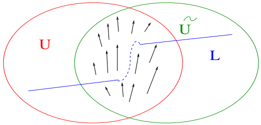
$$h^{ij} = \delta^{ij}, \quad \Gamma^i{}_{\tau\tau} = \frac{1}{2}\delta^{ij}\frac{\partial V}{\partial x^j}, \quad \theta = d\tau.$$

- Example. ASD Taub–NUT.

$$V = m/r, \quad \text{ALF} \stackrel{c\rightarrow\infty}{\longrightarrow} \text{AF}.$$

- Curvature on $(M, g) \longleftrightarrow$ Holomorphic deformations of PT_c.





- Kodaira theorems: Normal bundle $N(L_p) \equiv T(PT_c)|_{L_p}/TL_p$

$$H^1(L_p, N(L_p)) = 0, \quad H^0(L_p, N(L_p)) \cong T_p M_{\mathbb{C}}.$$

- Deforming PT_∞ MAY lead to curves jumping to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as

$$H^1(\mathbb{CP}^1, \mathsf{End}(N)) = \mathbb{C} \quad \text{if} \quad N = \mathcal{O} \oplus \mathcal{O}(2).$$

- Example. Replace $PT_\infty = \mathbb{C} \times \mathcal{O}(2)$ by a line bundle $L \to \mathcal{O}(2)$.

$$\widetilde{T} = T + \epsilon f, \quad \text{where} \quad f \in H^1(\mathbb{CP}^1, \mathcal{O}).$$

- $L \to \mathcal{O}(2)$ leads to a GH metric with $V(x, y, z) = \frac{\epsilon}{2\pi i} \oint_{\Gamma \subset L_p} \frac{\partial f}{\partial Q} d\lambda$.

Inertial Electromagnetism

- EM forces are inertial in the Newton-Cartan theory

$$\ddot{\mathbf{x}} = \mathbf{E} + 2\mathbf{B} \wedge \dot{\mathbf{x}}, \quad \Gamma^i_{00} = -E^i, \quad \Gamma^i_{0j} = \epsilon^i{}_{kj} B^k.$$

- Electric field=Gravitational force. Magnetic field=Coriolis force.
- $B = \nabla W$. Time independence$\rightarrow E = -\nabla V$.

$$\nabla^2 W = 0, \quad \nabla^2 (V + W^2) = 0.$$

- Holomorphic patching for GH twistor space

$$\widetilde{Q} = \frac{1}{\lambda^2} Q, \quad \widetilde{T} = T - \frac{Q}{c\lambda} - \frac{1}{c^3} f, \quad f = f(Q, T, \lambda) \in H^1(\mathbb{CP}^1, \mathcal{O}).$$

- Line bundle $\nu : E \rightarrow PT_\infty$, patching $F \equiv \tilde{\chi} \circ \chi^{-1} = e^f$

$$\chi : \nu^{-1}(U) \rightarrow U \times \mathbb{C}, \quad \tilde{\chi} : \nu^{-1}(\widetilde{U}) \rightarrow \widetilde{U} \times \mathbb{C}$$

- $f \in H^1(\mathcal{PT}_\infty, \mathcal{O}) \longrightarrow$ harmonic Newtonian potential V.

- Second order tangent bundle $T^{[2]}M = \cup_{p\in M}(J_p/{J_p}^3)^*$.

$$V^{[2]} \in \Gamma(T^{[2]}M), \quad V^{[2]} = V^a(x)\partial_a + V^{ab}(x)\partial_a\partial_b.$$

- Torsion free affine connection

$$\gamma : T^{[2]}M_{\mathbb{C}} \to TM_{\mathbb{C}}, \quad \gamma(V^{[2]}) = (V^a + \Gamma^a{}_{bc}V^{bc})\partial_a.$$

- Construction of γ (after S. Merkulov)
  1. $w^A = (T, Q), \tilde{w}^A = (\widetilde{T}, \tilde{Q})$.

$$F^A_B := \left[\frac{\partial \tilde{w}^A}{\partial w^B}\right]_{L_p}, \quad \text{patching matrix for} \quad N(L_p) \to L_p.$$

  2. Find 0-cochain $(\sigma, \tilde{\sigma})$ such that

$$\partial^2\tilde{w}^A/(\partial w^B \partial w^C) = -\tilde{\sigma}^A_{EF}F^E_BF^F_C + F^A_D\sigma^D_{BC}.$$

  3. Read off $\Gamma^a{}_{bc}$ from $\partial_a\partial_b w^A + \sigma^A_{BC}\partial_a w^B\partial_b w^C = \Gamma^c_{ab}\partial_c w^A$.

Twistor theory of newtonian connections

- Existence: $H^1\left(\mathbb{CP}^1, N \otimes \left(\odot^2 N^*\right)\right) = 0$.
- Uniqueness: $H^0\left(\mathbb{CP}^1, N \otimes \left(\odot^2 N^*\right)\right) = 0$.
  - $c \neq \infty$. Unique Levi–Civita connection for g.
  - $c = \infty$. Nonuniqueness.

$$(\sigma^Q{}_{TT}, \tilde{\sigma}^Q_{TT}) \sim (\sigma^Q{}_{TT} + E, \tilde{\sigma}^Q_{TT} + E), \quad \text{where} \quad E \in H^0(\mathbb{CP}^1, \mathcal{O}(2)).$$

  - $c \to \infty$ limit:

$$E = -\nabla \frac{1}{2\pi i} \oint_{\Gamma \subset L_p} \frac{\partial f}{\partial Q} d\lambda.$$

  - Serre duality: $H^0(\mathbb{CP}^1, \mathcal{O}(2)) = H^1(\mathbb{CP}^1, \mathcal{O}(-4)) = \mathbb{C}^3$.

Conclusions

- Newtonian limit = jumping twistor lines, $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O}(2)$.
- GH gravitational instantons $\to$ NC connections with no Coriolis force.
- Twistor space unstable under general Kodaira deformations.
- Newtonian connection from formal neighbourhoods.
- Coriolis force. Vector bundles non–trival on twistor lines (?). (Sparling equation)
- NC in $(2+1)$ dimensions. Integral formula

$$\psi(x, y, u) = \frac{1}{2\pi i} \oint_{\Gamma \subset \mathbb{CP}^1} e^{-\frac{1}{2}mi(x-iy)\lambda} g(x + iy + \lambda u, \lambda) d\lambda,$$

2+1 Schrodinger equation $2m\partial_u \psi = i({\partial_x}^2 + {\partial_y}^2)\psi$

Thank You.