Anti-Self-Duality and Solitons

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To Mum, Dad and my sister Prae.

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Cambridge
7 September 2009

Summary

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This thesis demonstrates how relations between anti-self-duality and integrability provide a useful framework to answer various open questions in mathematical physics. The thesis focuses on four main studies. Firstly, the classical energy quantisation of a class of moving solitons of the integrable chiral model is explained topologically using the associated linear system. The model arises as a symmetry reduction of the anti-self-dual Yang-Mills equation and hence inherits the Lax pair. Secondly, a compactified twistor picture of the Yang-Mills-Higgs system, which yields the integrable chiral model under a gauge fixing, is explored. Thirdly, another symmetry reduction of the anti-self-dual Yang-Mills equation, namely the elliptic affine sphere equation, is considered in the context of the Loftin-Yau-Zaslow construction of semi-flat Calabi-Yau metrics. Finally, an isomonodromic approach is used to construct non-diagonal anti-self-dual cohomogeneity-one metrics of Bianchi type V.

Declaration

This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and acknowledgements. No part of this dissertation submitted has been, or is concurrently being, submitted for any degree, diploma, or other qualification, nor is substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University.

The research presented in chapter 3 and chapter 5, except sections 5.4.2, 5.4.3 and 5.5, of this dissertation is based on the following papers respectively.

- Dunajski, M. and Plansangkate P. (2007) Topology and energy of time-dependent unitons, Proc. R. Soc. A **463**, 945-959.
- Dunajski, M. and Plansangkate P. (2009) Strominger-Yau-Zaslow geometry, affine spheres and Painlevé III, Commun. Math. Phys. **290**, 997-1024.

These papers are references [1] and [2] in the bibliography.

Contents

Chapter 1

Introduction

We build too many walls and not enough bridges.
Sir Isaac Newton, 1642-1727

In August 1834, Sir J. Scott Russell observed *"a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed"*, while riding alongside a canal[1]. The discovery of this solitary wave, a soliton, led to one of the most well known integrable equations, the Korteweg-de Vires (KdV) equation. This thesis explores the relations between solitons, integrable systems that admit soliton solutions and modern geometric theories of anti-self-duality.

The missing link between anti-self-duality and solitons is integrability. Integrability theory is the study of a fundamental class of nonlinear differential equations which, in principle, can be solved analytically. This means that it is possible to construct explicitly a large class of solutions to an integrable system by some general method. Apart from the ability to be solved analytically, there is no universal definition of integrability. Methods and criteria that apply to some cases may not do so to others, which may be solvable by different means. Nevertheless, one can look for common integrable properties. One common feature of many integrable systems is the existence of an overdetermined linear system for which the nonlinear equation of interest arises as the compatibility condition[2]. This is called the Lax pair formalism, which forms the basis of many methods for solving nonlinear differential equations. In some cases, the solutions of the nonlinear equations can be constructed directly from the extended solutions to the Lax pair. The equivalent zero curvature representation is also fundamental to the twistor transform for the anti-self-dual equations, by which we mean the anti-self-dual Yang-Mills equation and the anti-self-dual conditions for the conformal structures in four dimensions. Other common

[1]The passage is taken from J. Scott Russell, Report on waves, Fourteenth meeting of the British Association for the Advancement of Science, 1844.

[2]The linear system by itself, however, does not guarantee integrability.

features among integrable systems include the existence of many conserved quantities and the Painlevé property. Many integrable systems admit soliton solutions. These are smooth, localised configurations which are non-dispersive and retain their shapes after collision, such as the solitary wave that Sir Scott Russell observed. The term solitons is also used in the context of Lagrangian field theory to describe non-singular, static, finite energy solutions of the field equations. However, there also exist genuine non-dispersive time-dependent solitons (that cannot be obtained from static ones by Lorentz boosts) in classical field theories. These are usually solutions to integrable field equations.

Despite the fact that integrable systems form only a tiny fraction of all nonlinear differential equations, integrability theory has become a significant branch of mathematics. This is not least because integrable equations arise in a wide range of interesting nonlinear phenomena, from fluid dynamics, particle physics and general relativity to nonlinear optics. Theories of integrability also have deep connections with other areas in mathematics, notably complex analysis, differential geometry and algebraic geometry. However, it should be emphasised that the appeal of integrability goes beyond its wide applications and mathematical beauty, and is derived from its defining character - its tractability. Given a problem in physics or mathematics, it is always constructive to start with a simplified problem, perhaps a special case under certain assumptions, which can be solved analytically. Integrability has proved useful in solving many such problems. An important example is the four-dimensional Yang-Mills equation. The Yang-Mills equation is an Euler-Lagrange equation for a connection on a vector bundle over a spacetime manifold. It is a system of highly nonlinear partial differential equations and is consequently hard to solve. The anti-self-dual Yang-Mills equation is the condition that the curvature of a connection is anti-self-dual[3] (ASD) with respect to the Hodge star operator. The Bianchi identity implies that any ASD connection is a solution of the Yang-Mills equation. The anti-self-dual Yang-Mills equation is integrable, and all analytic solutions can be constructed using the twistor transform.

There is a deep connection between the anti-self-dual Yang-Mills (ASDYM) equation and integrability. Not only is the ASDYM equation itself an integrable equation, it also gives rise to many well known integrable systems in lower dimensions via symmetry reductions. The ASDYM equation is integrable in the sense that it is solvable by the twistor construction, which is inherited by the symmetry-reduced ASDYM equations. Twistor theory also plays an important role in the study of anti-self-dual conformal

[3]The self-duality and anti-self-duality are equivalent under orientation reversal. However, anti-self-duality occurs naturally in the context of Kähler geometry.

structures on four-manifolds. These are conformal structures with ASD Weyl tensors. Many integrable systems also arise from the ASD conditions on the conformal structures. In some cases, the relations between the ASD conditions for the Yang-Mills connections and the conformal structures can be made explicit.

Many reductions of the anti-self-dual equations are soliton equations. Although soliton solutions are not exclusive to integrable systems, the solitons of integrable equations are of particular interest. Firstly, integrability allows a large class of explicit solutions to be constructed. Secondly, the solitons usually have interesting properties, which can often be traced back to the existence of conserved quantities or other connections with geometry and topology.

1.1 Overview, Motivation and Outline

This thesis explores the relations between anti-self-duality, integrability and solitons, and their implications. The thesis is divided into 6 further chapters. In chapter 2 we give brief backgrounds of the subjects relevant to our studies, starting with the anti-self-dual Yang-Mills equation and its integrability in terms of the existence of an overdetermined linear system or the zero curvature representation. Then we introduce the Ward integrable chiral model as a symmetry reduction of the ASDYM equation in section 2.1.2. In section 2.2 the twistor correspondences for both the ASDYM equation and the Yang-Mills-Higgs system are discussed. We then go on to explain how one can classify or characterise integrable systems arising as symmetry reductions of the ASDYM equation, and introduce the Painlevé equations and their isomonodromic Lax pairs in section 2.3. Finally, in section 2.4 we discuss the anti-self-dual conditions on the conformal structures.

Chapter 3 is concerned with a symmetry reduction of the ASDYM equation, namely the Ward integrable chiral model. This particular integrable chiral model was first considered by Ward in [3], as a rare example of an integrable system in $2+1$ dimensions that admits soliton solutions and yet is close to being Lorentz invariant (the system admits $SO(1,1)$ symmetry). Ward and others [3, 4, 5, 6] constructed many static and time-dependent soliton solutions explicitly; they include both non-scattering and scattering solitons. Later, Dai and Terng [7] constructed all soliton solutions to the integrable chiral model using a Bäcklund transformation. The integrability of the model can be explained by the fact that it is equivalent, under a gauge choice, to the Yang-Mills-Higgs system in $2+1$ dimensions, which in turn is the time-translation reduction of the

ASDYM equation in $2+2$ dimensions. Our work was motivated by the observation of Ioannidou and Manton [8] that the total energy of a particular time-dependent $SU(2)$ soliton solution, called a 2-uniton solution, of the integrable chiral model is quantised. The observation is surprising because the total energy of a time-dependent soliton is the sum of the kinetic and potential terms, and hence the usual argument involving the Bogomolny bound for the potential energy is inadequate.

The aim of our work is to give a topological explanation to this classical quantisation of energy and the answer turns out to lie in the integrability of the model. As a reduction of the ASDYM equation, it inherits an overdetermined linear system. Moreover, Ward has shown [9] that a topological degree can be associated to a class of extended solutions to the Lax pair which satisfies a certain boundary condition, called the trivial scattering condition. It can be shown that the extended solutions associated to the n-uniton solutions, such as those studied by Ioannidou and Manton, indeed satisfy this condition. Using the Bäcklund transformation of Dai and Terng, we show in section 3.3 that the total energy of n-uniton solutions in the $U(N)$ integrable chiral model is proportional to the topological degree of the associated extended solutions. We then generalise the result to an $SO(1,1)$-invariant statement, involving both the energy and the conserved momentum. This work has been published in [1].

Since the integrable chiral model (or equivalently the Yang-Mills-Higgs system) is a symmetry reduction of the ASDYM equation, it can be solved by the twistor construction. The fundamental idea of twistor theory, founded by Penrose [10], is the correspondence between spacetime and its so-called twistor space, complemented by the twistor transforms relating the solutions of field equations on spacetime to geometric structures on the twistor space. Its main applications in integrability were established largely by Ward [11], who found a one-to-one correspondence between solutions of the $GL(N,\mathbb{C})$ ASDYM equation on an open set U in the complexified Minkowski spacetime $M \simeq \mathbb{C}^4$ and certain holomorphic rank N vector bundles E over the twistor space of U. From this starting point, extra conditions can be added in order to apply the correspondence to the ASDYM equation on real spacetimes with other gauge groups.

If a solution of the ASDYM equation is invariant under some symmetry transformation, the corresponding holomorphic vector bundle is also invariant under the induced transformation on the twistor space. In some cases, these invariant vector bundles can be realised as pull-backs of generic holomorphic vector bundles over a 'symmetry reduced' twistor space of lower dimensions. In the case of the Yang-Mills-Higgs system on complex

3-dimensional space, each solution corresponds to a holomorphic vector bundle over the *minitwistor space*, which is the tangent bundle $T\mathbb{P}^1$ of the Riemann sphere. The holomorphic vector bundles which correspond to solutions of the Yang-Mills-Higgs system in $\mathbb{R}^{2,1}$ will satisfy certain reality conditions. The twistor picture of the Yang-Mills-Higgs system, and hence of the integrable chiral model, provides a natural setting to study geometric and topological properties of the system. In fact, the trivial scattering boundary condition was discussed by Ward in [9] as the condition that the corresponding vector bundles can be extended to the *compactified minitwistor space*. Ward also identified the topological degree of the extended solutions with the second Chern number of the holomorphic vector bundles over the compactified minitwistor space, although the proof was not given.

As a starting point toward a proof of this identification, in chapter 4 we consider the twistor picture of the Yang-Mills-Higgs system. In sections 4.2.1 and 4.2.2 we explore and give a detailed exposition of Ward's correspondence between the compactified complexified spacetime $M \simeq \mathbb{CP}^3$ and the compactified minitwistor space, which can be obtained from a cone in another complex projective 3-space after blowing up its vertex. Then, the correspondence space of a double fibration to the compactified spacetime and the compactified minitwistor space is investigated in section 4.2.3. The trivial scattering boundary condition is realised by restricting the extended solutions to a spacelike $\mathbb{R}^2$-plane in $\mathbb{R}^{2,1}$. Therefore, in section 4.3 we define a restricted correspondence space which fibres over an $\mathbb{RP}^2$ regarded as a compactification of a spacelike surface $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$, and show that it admits a surjective map to the compactified minitwistor space.

In chapter 5 we turn to another integrable soliton equation, namely the Tzitzéica equation, which also arises as a symmetry reduction of the ASDYM equation. The study is actually motivated by an equation closely related to the Tzitzéica equation, which we call the affine sphere equation, in the context of the mirror symmetry conjecture. Mirror symmetry is a symmetry proposed in string theory, between pairs of 3-complex dimensional Calabi-Yau (CY) manifolds, i.e. 6-real dimensional Kähler manifolds with Ricci-flat metrics and holonomy group $SU(3)$. In a well known formulation of the mirror conjecture known as the Strominger-Yau-Zaslow conjecture [12], the semi-flat Calabi-Yau metrics play an important role. The affine sphere equation arises in the construction of a class of semi-flat Calabi-Yau metrics by Loftin, Yau and Zaslow [13], and the local expression of the metrics are determined by its solutions.

Inspired by this work, we set out to explore the integrability of the affine sphere equation. The fact that the affine sphere equation and the Tzitzéica equation are different real forms of the same holomorphic equation implies that the affine sphere equation can also be realised as a reduction of the ASDYM equation. In section 5.2 we characterise the holomorphic equation, which we call the holomorphic Tzitzéica equation, as a symmetry reduction of the ASDYM equation on $\mathbb{C}^4$ with gauge group $GL(3, \mathbb{C})$ via the Hitchin equations. The characterisations for the Tzitzéica equation and the affine sphere equation follow by imposing different reality conditions. As a by-product, we also obtain a characterisation for the $\mathbb{Z}_3$ 2-dimensional Toda chain. In section 5.3 we show that by imposing a radial symmetry the affine sphere equation reduces to the Painlevé III equation, except in one special case where the resulting ordinary differential equation (ODE) is solvable by elliptic functions. This gives an alternative isomonodromic Lax pair for the Painlevé III equation, complementing the standard one in [14]. Section 5.4 is concerned with the semi-flat Calabi-Yau metrics determined by the affine sphere equation. We write down an explicit expression for the metrics in terms of solutions of the affine sphere equation, discuss the isometries and finally consider a special case where the metrics are determined by elliptic functions. Finally, in section 5.5 the twistor construction for the holomorphic Tzitzéica equation is considered and we obtain a matrix transition function of the holomorphic vector bundle corresponding to a trivial (vanishing) solution of the holomorphic Tzitzéica equation. This chapter is based on publication [2].

Chapter 6 is devoted to the anti-self-duality conditions on 4-dimensional conformal structures. In particular, we consider cohomogeneity-one metrics, which are the metrics which admit isometry groups acting transitively on 3-dimensional hypersurfaces. Hence, there are two classes of such metrics, class A and class B, according to the Bianchi classification [15] of real 3-dimensional Lie algebras. A cohomogeneity-one metric only depends on one variable which parametrises the space of orbits of the isometry group. Hence, any field equation imposed on the metric reduces to a system of ODEs. Moreover, the ODEs from the anti-self-duality conditions are found to be integrable, since the anti-self-duality conditions are solvable by the twistor construction. Many results about the cohomogeneity-one metrics can be obtained by direct calculation, especially when the metric is diagonalisable in the basis of invariant one-forms (see for example [16, 17]). An interesting result is the restriction of class B metrics due to the diagonalisability assumption. For example, it was shown in [17] that there is no diagonal ASD Kähler metric with group invariant Kähler form in class B.

Non-diagonal cohomogeneity-one metrics are harder to handle as direct calculation becomes more difficult. Nevertheless, we shall use an indirect approach based on the isomonodromic construction to study a particular class of ASD non-diagonal class B metrics. In the diagonal case, Tod [18, 19] showed that many $SU(2)$-invariant ASD conformal structures are determined by the Painlevé equations. This suggests a link between ASD cohomogeneity-one metrics and isomonodromic deformations in which the Painlevé equations play the role of the deformation equation [14]. The idea of the isomonodromic approach to ASD cohomogeneity-one metrics is due to Hitchin [20], whose results inspired a number of works, including the construction of ASD cohomogeneity-one metrics from the isomonodromic Lax pair of the Painlevé equations given in [21][4].

In chapter 6 we consider metrics of a certain Bianchi type in class B, namely type V metrics. The diagonal metrics of type V are particularly restrictive. The anti-self-duality conditions imply that the metrics are conformally flat [16]. The aim of our work is two-fold. Firstly, we will give an explicit example of an ASD non-diagonal type V metric which is not conformally flat. Secondly, we aim to find whether there exists class B ASD Kähler metrics with group-invariant Kähler forms. Such a Kähler metric must be non-diagonal due to the result of [17]. We begin by introducing cohomogeneity-one metrics and complex Kähler metrics, and then explain in section 6.2.1 how one can construct an ASD cohomogeneity-one metric which is conformally related to a Kähler metric by the isomonodromic construction. Next, an explicit expression for a class of holomorphic Bianchi type V metrics which are ASD and conformal to Kähler metrics is derived. Simple examples of non-diagonal non-conformally flat ASD type V metrics are given in section 6.2.2, including an example with Euclidean signature. Finally, in section 6.3 we discuss how the results can be used to check for the existence of a real Kähler type V metric, and end with a note exploring fibrations of the semi-flat Calabi-Yau metrics determined by the affine sphere equation over 4-dimensional metrics.

The summary and outlook are given in chapter 7.

[4]A more general construction which applies to a wider class of ASD metrics, called the switch map, is also discussed in [21]. The switch map construction is based on the relation between the ASD conditions on the conformal structures and the ASDYM equation. Note that the Painlevé equations arise as symmetry reductions of the ASDYM equation [22].

Chapter 2

Integrability and Anti-Self-Dual Equations

In this chapter we introduce the concepts and ideas which are relevant to the studies in the thesis. Firstly, the anti-self-dual Yang-Mills equation and one of its symmetry reduction, the integrable chiral model, are introduced. The latter is the main subject of the study in chapter 3. Then, we discuss the twistor correspondences for the anti-self-dual Yang-Mills equation on the complexified Minkowski spacetime and the Yang-Mills-Higgs system, which gives rise to the integrable chiral model under a gauge choice. Next, we give a brief overview of how one can classify or characterise symmetry reductions of the anti-self-dual Yang-Mills equation. It should be pointed out that the criteria are strongly oriented towards our work in chapter 5. After that, we introduce the Painlevé equations and their isomonodromic Lax pairs in the context of reductions of the anti-self-dual Yang-Mills equation. Finally, anti-self-dual conformal structures on four-manifolds are discussed. This is intended to provide an introduction for the material in chapter 6.

2.1 Anti-self-dual Yang-Mills equation

The Yang-Mills equation is a system of partial differential equations whose dependent variables are components of a connection on a vector bundle over a spacetime manifold. In this thesis we only consider the Yang-Mills equation on a flat 4-dimensional real or complex space, i.e. we take the spacetime M to be $\mathbb{C}^4$ or its real slices.

Let V be a rank N vector bundle over $M \simeq \mathbb{C}^4$. Since M is contractible, V is a trivial bundle. Suppose V is equipped with a connection D which is compatible with a structure group $G \subset GL(N, \mathbb{C})$. That means the connection one-form takes values in the Lie algebra $\mathfrak{g}$ of the group G. Writing the connection as $D = d + A$, where d is the exterior derivative and A denotes the connection (potential) one-form, the curvature two-form F of D is given by

$$F = dA + A \wedge A.$$

Such a connection D is said to be a Yang-Mills connection if it is a critical point of the Yang-Mills functional

$$S = \int_M \mathrm{Tr}(F \wedge *F), \tag{2.1.1}$$

where $*$ is the Hodge star operator defined with respect to the flat metric and a volume form on M. In other words, the Yang-Mills equation is the Euler-Lagrange equation of the action (2.1.1), varying with respect to A. It is given by

$$D * F = 0, \tag{2.1.2}$$

where the covariant derivative of a $\mathfrak{g}$-valued p-form α is defined as

$$D\alpha = d\alpha + A \wedge \alpha - (-1)^p \alpha \wedge A.$$

Note that (2.1.2) is invariant under gauge transformations of the form

$$A \longrightarrow g^{-1}Ag + g^{-1}dg, \tag{2.1.3}$$

where g is a G-valued function on M.

Equation (2.1.2) is a system of highly nonlinear partial differential equations for components of A and thus is hard to solve. However, by the Jacobi identity of D, F also satisfies the Bianchi identity

$$DF = 0.$$

It follows that F which satisfies

$$*F = \pm F \tag{2.1.4}$$

is automatically a solution of (2.1.2) using the Bianchi identity. Equation (2.1.4) with $-$ sign is called the anti-self-dual Yang-Mills equation. Unlike the full Yang-Mills equation, the anti-self-dual equation is an integrable system.

The self-dual equation (with $+$ sign) and the anti-self-dual equation are equivalent under the change of orientation of M. However, we choose to work with the anti-self-dual Yang-Mills equation because anti-self-duality is a more natural choice when we come to consider conformal structures with Kähler geometry in chapter 6.

The 4-dimensional real space $\mathbb{R}^4$ can be endowed with a flat metric of either Euclidean $(++++)$, Lorentz $(-+++)$ or ultrahyperbolic $(--++)$ signature. We shall follow [23] and regard the three cases as real slices of the complexified Minkowski space $M \simeq \mathbb{C}^4$. The Yang-Mills equation on either the Euclidean space, the Minkowski space or the ultrahyperbolic space can be studied in the unified framework of the holomorphic Yang-Mills equation on M by taking the gauge group to be a real subgroup of $GL(N,\mathbb{C})$ and restricting the bundle V to the corresponding real slice in M. However, there is no real nontrivial self-dual or anti-self-dual Yang-Mills connection in a real Minkowski space. This fact can be deduced from the definition of the Hodge star operator on the complexified Minkowski space. In a general basis of tangent vector fields $\{e^a\}$ and its dual basis of one-forms $\{e_b\}$ on M, the action of the Hodge star operator $*$ on a two-form β is defined by

$$(*\beta)_{ab} = \frac{1}{2}\triangle\varepsilon_{abcd}\eta^{ce}\eta^{df}\beta_{ef}, \tag{2.1.5}$$

where η is the metric on M, $\triangle^2 = \det(\eta_{ab})$, ε_{abcd} is the 4-dimensional alternating symbol and the volume form ν is chosen such that ν_{abcd} is proportional to $\triangle\varepsilon_{abcd}$. Restricting (2.1.5) to the Minkowski real slice one sees that a two-form β satisfying $*\beta = \pm\beta$ cannot be real, because $\triangle = i$ (in a positively oriented basis). Therefore, in this thesis we only consider the anti-self-dual Yang-Mills equation on the complexified Minkowski space and its real forms on the Euclidean and the ultrahyperbolic spaces.

2.1.1 Integrability of the ASDYM equation

The integrability of the anti-self-dual Yang-Mills (ASDYM) equation on the complexified Minkowski space can be explained by the fact that the ASDYM equation arises as the compatibility condition of an overdetermined linear system of equations. The existence of such 'Lax pair' is one of the common features among integrable systems.

The Lax pair of the ASDYM equation has the following geometric interpretation. It determines covariantly constant sections of the bundle V over a totally null self-dual 2-plane in M. To see this, let us begin by defining a totally null plane. A 2-plane in $M \simeq \mathbb{C}^4$ is called totally null if each pair of its tangent vectors A, B satisfies $\eta(A,B) = 0$, where η is the flat metric. It can be shown that a tangent bivector $A \wedge B$ of a totally null plane is either self-dual (SD) or anti-self-dual (ASD). The totally null planes with SD tangent bivectors are called α-planes and the ones with ASD bivectors are called β-planes. Since an ASD 2-form is orthogonal to an SD bivector, the ASD curvature F of a Yang-Mills connection vanishes when restricted to an α-plane Z. This means that there exist N linearly independent covariantly constant sections of the restricted bundle $V|_Z$. The pair of equations for covariantly constant sections over an α-plane is the overdetermined linear system mentioned above.

In details, let us follow the conventions used in [23] and let $(w, \tilde{w}, z, \tilde{z})$ be double null coordinates on $M \simeq \mathbb{C}^4$, such that the flat holomorphic metric and the volume form are given by

$$ds^2 = 2(dz\ d\tilde{z} - dw\ d\tilde{w}), \qquad \nu = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}, \tag{2.1.6}$$

respectively. By considering a general expression for SD tangent bivectors, it can be shown that the tangent space of each α-plane in M is spanned by

$$l = \partial_w - \lambda \partial_{\tilde{z}}, \quad m = \partial_z - \lambda \partial_{\tilde{w}},$$

where ∂_w denotes $\frac{\partial}{\partial w}$, etc., and λ is a complex parameter. The parameter λ is allowed to be infinity, to include the α-planes which are parallel to the $w = 0 = z$ plane. It then follows that a covariantly constant section of V over an α-plane is given by an N-column vector solution of the linear system or the Lax pair

$$(D_z - \lambda D_{\tilde{w}})\Psi = 0, \qquad (D_w - \lambda D_{\tilde{z}})\Psi = 0, \tag{2.1.7}$$

where we have written the connection as $D = D_z dz + D_w dw + D_{\tilde{z}} d\tilde{z} + D_{\tilde{w}} d\tilde{w}$ and $D_z = \partial_z + A_z$, etc. The system (2.1.7) is overdetermined, and there exist N linearly independent solutions if and only if the compatibility condition

$$[D_z - \lambda D_{\tilde{w}}, D_w - \lambda D_{\tilde{z}}] = 0 \tag{2.1.8}$$

holds for every value of λ. Equation (2.1.8) is equivalent to the ASDYM equation

$$F_{zw} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad F_{\tilde{z}\tilde{w}} = 0,$$

where $F_{z\tilde{z}} = [D_z, D_{\tilde{z}}] = \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + [A_z, A_{\tilde{z}}]$, etc.

The existence of the Lax pair, or the zero-curvature representation, of the ASDYM equation underlies its integrability. The geometric interpretation of the Lax pair as the equations defining the covariantly constant sections over α-planes leads to the twistor approach for solving the equation. More directly, if one knows N linearly independent vector solutions to the system (2.1.7), one can form a fundamental $N \times N$ matrix solution, also denoted by Ψ, by taking the vector solutions to be its columns. The components of the connection one-form A are given in terms of Ψ by

$$(\partial_z \Psi - \lambda \partial_{\tilde{w}} \Psi)\, \Psi^{-1} = -\left(A_z - \lambda A_{\tilde{w}}\right), \quad (\partial_w \Psi - \lambda \partial_{\tilde{z}} \Psi)\, \Psi^{-1} = -\left(A_w - \lambda A_{\tilde{z}}\right).$$

2.1.2 A symmetry reduction: the integrable chiral model

In four dimensions the Hodge star operation on two-forms (2.1.5) is invariant under conformal transformations of spacetime. This means that a conformal transformation maps one ASD connection to another. Therefore it makes sense to consider ASD connections which are invariant under a subgroup of the conformal group. It is this symmetry assumption that 'reduces' the ASDYM equation.

The interest in the ASDYM equation in the context of integrable systems is centred around the concept of symmetry reduction. It was noted by Richard Ward [24] that many of known integrable systems can be realised as symmetry reductions of the ASDYM equation in four dimensions. The ASDYM equation is completely solvable by the aforementioned twistor approach, which is inherited by the reduced equations. The symmetry reduction of the ASDYM equation not only provides a natural framework for a classification of integrable systems and for a study of relations among them, but also leads to an introduction of new integrable equations which have interesting features.

In this section we consider a reduction of the ASDYM equation on the ultrahyperbolic spacetime $\mathbb{R}^{2,2}$ to $2+1$ dimension, namely the Ward integrable chiral model. The model admits soliton solutions which possess some interesting properties that can be understood using the Lax pair inherited from the ASDYM equation. The integrable chiral model is the main subject of the study in chapter 3.

The equation

The Ward integrable chiral model, also known as the Ward model, was first introduced by Ward in [3]. The equation is a rare example of an integrable system in 2+1 dimensions that admits soliton solutions and yet is close to being Lorentz invariant. Although there are a few known examples of $SO(1,1)$-invariant soliton equations in $1+1$ dimensions, no soliton system with Lorentz symmetry in more than one space dimension has been found so far. Despite not being $SO(2,1)$-invariant, the Ward model admits $SO(1,1)$ symmetry.

The equation appears as a modification of the standard chiral field equation in $\mathbb{R}^{2,1}$ and is given by

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0, \qquad (2.1.9)$$

where J is a map from $\mathbb{R}^3$ to $U(N)$ and (t,x,y) are coordinates on $\mathbb{R}^3$ such that the flat metric is given by $\eta = -dt^2 + dx^2 + dy^2$. As before, we use the notation $J_\mu := \partial_\mu J$.

The equation is not fully Lorentz invariant as the commutator term fixes a space-like direction.

We shall now derive (2.1.9) from a symmetry reduction of the $U(N)$ ASDYM equation in $\mathbb{R}^{2,2}$ by a non-null translation. For convenience let us consider the ASDYM equation on $\mathbb{R}^{2,2}$ in Cartesian coordinates $x^a, a = 0, 1, 2, 3$, such that the line element is $ds^2 = -dx^0 + dx^1 + dx^2 - dx^3$. In this coordinate basis, the ASDYM equation is given by

$$*F_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd} = -F_{ab}. \tag{2.1.10}$$

where the indices are raised and lowered by the 4-dimensional metric. Now, the Yang-Mills connection D is assumed to be invariant under the translation generated by $\frac{\partial}{\partial x^3}$. This is equivalent to saying that all components of the potential one-form A are independent of x^3. Moreover, in the invariant gauges where the matrices $g(x^a)$ in (2.1.3) are also independent of x^3, the component A_3 no longer transforms as a component of a one-form, but as a $\mathfrak{u}(N)$-valued function: $A_3 \longrightarrow g^{-1}A_3 g$. Let us rename it as $\Phi := A_3$ and call it the Higgs field. Under this reduction, the system (2.1.10) becomes

$$D_\mu \Phi = \frac{1}{2}\varepsilon_{\mu\alpha\beta}F^{\alpha\beta}, \tag{2.1.11}$$

where the indices now run from 0 to 2 and the alternating symbol in 3 dimensions is defined by $\varepsilon_{\mu\alpha\beta} = \varepsilon_{3\mu\alpha\beta}$. This is the Yang-Mills-Higgs system in $2 + 1$ dimensions.

The integrable chiral model (2.1.9) is equivalent to (2.1.11) under a gauge choice. Let $x^\mu = (x^0 = t,\ x^1 = x,\ x^2 = y)$ be coordinates on $\mathbb{R}^{2,1}$ and choose a gauge where $A_t = A_y$ and $A_x = -\Phi$. By expressing the components of A in terms of a $U(N)$-valued function J such that

$$A_t = A_y = \frac{1}{2}J^{-1}(J_t + J_y), \qquad A_x = -\Phi = \frac{1}{2}J^{-1}J_x, \tag{2.1.12}$$

(2.1.11) becomes (2.1.9).

Conserved energy functional

Writing (2.1.11) in the form of (2.1.9) involves a gauge choice which breaks the $SO(2, 1)$ symmetry. However, it has an advantage that equation (2.1.9) admits a positive-definite and conserved energy functional, whereas (2.1.11) does not. To see this, consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \mathrm{Tr}(D_\mu\Phi)(D^\mu\Phi), \tag{2.1.13}$$

which is obtained from the Lagrangian density of the Yang-Mills action (2.1.1) by substituting $F_{3\mu} = -D_\mu \Phi$. Its Euler-Lagrange equations are given by

$$D_\mu D^\mu \Phi = 0, \qquad D^\mu F_{\mu\nu} = [D_\nu \Phi, \Phi]. \tag{2.1.14}$$

Any solution of the Yang-Mills-Higgs (2.1.11) satisfies (2.1.14) in the same way as a solution of the ASDYM equation is a solution of the Yang-Mills equation. However, it turns out that the energy functional associated with (2.1.13) is not positive-definite. In fact, the energy density vanishes for all solutions of (2.1.11).

The form of (2.1.9) on the other hand allows one to associate with it the energy-momentum tensor $T_{\mu\nu}$ of the chiral model

$$T_{\mu\nu} = (-\delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta})\mathrm{Tr}(J^{-1}J_\alpha J^{-1}J_\beta). \tag{2.1.15}$$

It turns out conveniently that $\partial^\mu T_{\mu 0} = 0$ for solutions of (2.1.9), and thus (2.1.15) provides a conserved energy functional.

One can generalise the gauge choice (2.1.12) to a choice determined by a unit vector V_α, such that (2.1.11) takes a more general form

$$\eta^{\mu\nu}\partial_\mu(J^{-1}J_\nu) + V_\alpha \varepsilon^{\alpha\mu\nu}\partial_\mu(J^{-1}J_\nu) = 0.$$

This equation was considered in [25] with a time-like V_α. However, the choice does not lead to a conserved energy because the divergence of the energy-momentum tensor $\partial^\mu T_{\mu\nu}$ is proportional to V_ν, and consequently one needs the time component V_0 to vanish for a conserved energy functional. The choice which leads to (2.1.9) is $V_\mu = (0, 1, 0)$. Apart from the conserved energy, solutions of (2.1.9) also admit the conserved y-momentum.

2.2 Twistor correspondence

Twistor theory was founded by Roger Penrose [10], who proposed a way of understanding physical phenomena on spacetime by reformulating the problems in a corresponding space, namely the twistor space. The characteristic feature of the twistor transform is that solutions of field equations on spacetime are determined purely by geometric structures on the twistor space. Since its introduction, twistor theory has had significant applications in the area of integrable systems. In particular, it provides a framework to understand the integrability of the ASDYM equation and its reductions. Below, we give a brief overview of the twistor correspondences for the ASDYM equation and for the Yang-Mills-Higgs system. Our references on this subject are [23, 26, 27, 28].

2.2.1 The setting

The twistor space of complexified Minkowski space

Consider the complexified Minkowski spacetime $M \simeq \mathbb{C}^4$, with the flat metric and the volume form given by (2.1.6). The twistor space of M is defined to be the space of all α-planes in M. The definition is local, that is, one can define the twistor space of an open set $U \subset M$ to be the space of all α-planes that intersect U, assuming each intersection is connected. The twistor space $\mathcal{P}$ of M is a 3-dimensional complex manifold and is in fact biholomorphic to $\mathbb{CP}^3 - \mathbb{CP}^1$.

To see this, let us consider how α-planes are labelled. Recall that there is a $\mathbb{CP}^1$ worth of α-planes passing through each point $x \in M$. Each plane is spanned by

$$l = \partial_w - \lambda\partial_{\tilde{z}}, \qquad m = \partial_z - \lambda\partial_{\tilde{w}}, \tag{2.2.1}$$

where $\lambda \in \mathbb{CP}^1$, and $\lambda = \infty$ corresponds to the plane parallel to the $w = 0 = z$ plane. This implies the equation for an α-plane given by

$$\kappa = \lambda w + \tilde{z}, \qquad \mu = \lambda z + \tilde{w}, \tag{2.2.2}$$

where κ, μ are constant. Hence, each α-plane is labelled by 3 complex numbers κ, μ, λ. To see that $\mathcal{P}$ is biholomorphic $\mathbb{CP}^3 - \mathbb{CP}^1$, we note that $(\kappa, \mu, \lambda) \in \mathbb{C}^3$ are not global coordinates on $\mathcal{P}$ as they miss out those α-planes parallel to the $w = 0 = z$ plane. To describe these planes one needs to consider another coordinate patch on $\mathbb{CP}^1$ labelled by $\tilde{\lambda} = 1/\lambda$. In this coordinate the α-planes are spanned by

$$\tilde{l} = \tilde{\lambda}\partial_w - \partial_{\tilde{z}}, \qquad \tilde{m} = \tilde{\lambda}\partial_z - \partial_{\tilde{w}}$$

and are labelled by three complex numbers

$$\tilde{\kappa} = w + \tilde{\lambda}\tilde{z}, \qquad \tilde{\mu} = z + \tilde{\lambda}\tilde{w}, \qquad \tilde{\lambda}.$$

The planes parallel to the $w = 0 = z$ plane correspond to those with $\tilde{\lambda} = 0$.

The global picture can be seen more clearly by writing the equations for α-planes (2.2.2) in homogeneous form

$$Z_0 = Z_3 w + Z_2 \tilde{z}, \qquad Z_1 = Z_3 z + Z_2 \tilde{w},$$

where $Z_\alpha,\ \alpha = 0, 1, 2, 3$, are four complex constants and

$$\kappa = \frac{Z_0}{Z_2}, \qquad \mu = \frac{Z_1}{Z_2}, \qquad \lambda = \frac{Z_3}{Z_2}.$$

This means that $(\kappa, \mu, \lambda) \in \mathbb{C}^3$ are defined in the region of $\mathbb{C}^4 - \{0\}$ where $Z_2 \neq 0$. On the other hand, the α-planes with $\lambda = \infty$ are included in the open set where $Z_2 = 0$ and $Z_3 \neq 0$, where we have

$$\tilde{\kappa} = \frac{Z_0}{Z_3}, \qquad \tilde{\mu} = \frac{Z_1}{Z_3}, \qquad \tilde{\lambda} = \frac{Z_2}{Z_3}.$$

This shows that α-planes in $\mathbb{C}^4$ correspond to points $[Z_\alpha] \in \mathbb{CP}^3$, except those with $Z_2 = Z_3 = 0$. Hence one concludes that $\mathcal{P} \simeq \mathbb{CP}^3 - \mathbb{CP}^1$.

Equation (2.2.2) implies that a point $\hat{z} \in \mathcal{P}$ corresponds to an α-plane in M. Conversely, fixing a point $x \in M$, κ and μ are completely determined by λ. Thus, each point $x \in M$ corresponds to a $\mathbb{CP}^1 \subset \mathcal{P}$, which we denote by $\hat{x}$.

The correspondence space

A correspondence space comes up naturally when one considers the relation between the spacetime M and its twistor space $\mathcal{P}$. In the present case, the correspondence space $\mathcal{F}$ is the space of pairs (x, Z), each of which consists of a point $x \in M$ and an α-plane Z which passes through x. This leads to a double fibration from $\mathcal{F}$ to M and $\mathcal{P}$.

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ M & & \mathcal{P} \end{tikzcd}$$

Since there is a $\mathbb{CP}^1$ worth of α-planes passing through each point $x \in M$, $\mathcal{F}$ is a 5-dimensional complex manifold biholomorphic to $\mathbb{C}^4 \times \mathbb{CP}^1$, each point of which is labelled by $(w, z, \tilde{w}, \tilde{z}, \lambda)$. In these coordinates the two fibrations q, p are given by

$$q : (w, z, \tilde{w}, \tilde{z}, \lambda) \mapsto (w, z, \tilde{w}, \tilde{z}), \qquad p : (w, z, \tilde{w}, \tilde{z}, \lambda) \mapsto (\kappa = \lambda w + \tilde{z},\ \mu = \lambda z + \tilde{w},\ \lambda). \tag{2.2.3}$$

One sees that the preimage in $\mathcal{F}$ of a point $x \in M$ is a $\mathbb{CP}^1$ line, while the preimage of a point $\hat{z} \in \mathcal{P}$ is the lift of the α-plane Z to $\mathcal{F}$. These lifts of α-planes are also spanned by l, m in (2.2.1), now regarded as vector fields in $\mathcal{F}$. It follows that the twistor space $\mathcal{P}$ is the quotient space $\mathcal{F}/\{l, m\}$.

2.2.2 Ward correspondence

The main applications of twistor theory to integrable systems were established largely by Ward [11], who gave a $1:1$ correspondence between the solutions of the $GL(N,\mathbb{C})$ ASDYM equation on an open set $U \subset M \simeq \mathbb{C}^4$ and certain holomorphic rank N vector bundles E over the twistor space of U. From this starting point, extra conditions can be added in order to apply the correspondence to the ASDYM equation on real spacetimes with other gauge groups.

Theorem 2.2.1 [11] *Let U be an open set in $M \simeq \mathbb{C}^4$ such that the intersection of U with every α-plane that passes U is connected and simply connected. Then, there is a* $1:1$ *correspondence between*
(a) *a gauge equivalent class of solutions of the $GL(N,\mathbb{C})$ ASDYM equation on U; and*
(b) *holomorphic rank N vector bundle E over the twistor space of U, such that E restricted to $\hat{x}$ is trivial for all $x \in U$.*

We refer the readers to [26] for a complete proof. Let us only explain the theorem and describe the constructions which will be relevant to us later. First, consider how to go from (a) to (b). For simplicity, take the open set U to be the entire $M \simeq \mathbb{C}^4$. Let V denote the trivial rank N vector bundle over M. Given an ASDYM connection on V, it follows that its curvature restricted to any α-plane vanishes. Thus, there exists an N-dimensional space of covariantly constant sections over each α-plane. This is because the vanishing curvature allows one to parallel propagate any element of the fibre V_x over $x \in M$ along an α-plane Z on which x lies, to generate a covariantly constant section over Z, and there are N linearly independent elements on V_x. This assigns a vector space $\mathbb{C}^N$ to each point $\hat{z} \in \mathcal{P}$ corresponding to the α-plane Z, and gives a holomorphic rank N vector bundle $E \to \mathcal{P}$. To see that $E|_{\hat{x}}$ is trivial, consider an element of the fibre $E_{\hat{z}}$ over a point $\hat{z} \in \hat{x}$. Such an element is a covariantly constant section of V over the α-plane Z, and is completely determined by its value at a point on Z. Thus, we can identify the elements of two fibres $E_{\hat{z}_1}, E_{\hat{z}_2}$ over $\hat{z}_1, \hat{z}_2 \in \hat{x}$ which take the same value in V_x. Therefore, N linearly independent elements of V_x determines N holomorphic sections of $E|_{\hat{x}}$. These sections form a global frame field and hence $E|_{\hat{x}}$ is trivial.

The bundle $E \to \mathcal{P}$ can be constructed more explicitly by obtaining the transition functions of E from the ASDYM connection. One can cover $\mathcal{P}$ by two open sets W, $\widetilde{W}$, where W is the complement of $\lambda = \infty$ and $\widetilde{W}$ is the complement of $\lambda = 0$, and the transition function F on the overlap $W \cap \widetilde{W}$ determines the bundle. First, recall that an α-plane Z is spanned by l, m in (2.2.1) for a particular value of λ. A covariantly constant section v of $V|_Z$ is represented by an N-column vector solution of the linear system (2.1.7). The anti-self-duality of the connection guarantees that there exist N linearly independent solutions of (2.1.7). These column vector solutions can then be put together to form an $N \times N$ matrix fundamental solution Ψ. With a fixed λ, Ψ is regarded as a function on M, however one can also consider the Lax pair (2.1.7) with λ as a variable and regard l, m as vector fields on the correspondence space $\mathcal{F}$. It is possible to find a fundamental solution $\Psi(z, w, \tilde{z}, \tilde{w}, \lambda)$, now as a function on $\mathcal{F}$, that is holomorphic in $\lambda \in \mathbb{C}$ as well as in $z, w, \tilde{z}, \tilde{w} \in \mathbb{C}^4$. By construction the columns of Ψ form a local frame field of the pull-back bundle $p^*E|_W$ to the corresponding open set $p^*W \subset \mathcal{F}$.

In general we cannot extend Ψ to a regular function, i.e. a function which is holomorphic and whose inverse is also holomorphic, on the whole $\mathbb{CP}^1 \ni \lambda$. This is because by Liouville's theorem if Ψ is regular on $\mathbb{CP}^1$ then it is independent of λ. This would mean that the columns of Ψ were covariantly constant in all directions and thus the connection has to be flat everywhere on M. However, one can find a solution $\widetilde{\Psi}$ holomorphic in $\tilde{\lambda} = 1/\lambda$ to the Lax pair

$$(\tilde{\lambda} D_z - D_{\tilde{w}})\tilde{\Psi} = 0, \qquad (\tilde{\lambda} D_w - D_{\tilde{z}})\tilde{\Psi} = 0 \tag{2.2.4}$$

defined in the open set $p^*\widetilde{W} \subset \mathcal{F}$. Similarly, $\widetilde{\Psi}$ provides a local frame field of $p^*E|_{\widetilde{W}}$. In the overlap $p^*W \cap p^*\widetilde{W}$, one has

$$\Psi = \widetilde{\Psi} F,$$

where F is a $GL(N, \mathbb{C})$-valued holomorphic transition function of p^*E. By (2.1.7) and (2.2.4) F is annihilated by l, m in (2.2.1), which means that it is constant on α-planes and thus descends to $\mathcal{P}$. This is consistent with the fact that $F(\lambda w + \tilde{z}, \lambda z + \tilde{w}, \lambda)$ is the pull-back of the transition function $F(\kappa, \mu, \lambda)$ of $E \to \mathcal{P}$.

Now, to go from (b) to (a), suppose we are given a holomorphic vector bundle of rank N, $E \to \mathcal{P}$, which is trivial when restricted to $\hat{x} \simeq \mathbb{CP}^1$ for every $x \in M$. Since $E|_{\hat{x}}$ is trivial, the space of holomorphic sections of $E|_{\hat{x}}$ is isomorphic to $\mathbb{C}^N$ because any holomorphic function on $\mathbb{CP}^1$ is constant. This implies that one can assign a vector space $\mathbb{C}^4$ to each $x \in M$ to form a trivial vector bundle $V \to M$. We will see that V is equipped with a connection which is ASD. To define a connection is equivalent to defining a parallel transport on V. Consider two $\mathbb{CP}^1$, $\hat{x}$ and $\hat{y}$, which intersect at a point $\hat{z} \in \mathcal{P}$. Sections of $E|_{\hat{x}}$ can be identified with sections of $E|_{\hat{y}}$ by comparing their values at the point $\hat{z}$, and this gives a way of propagating a vector from x to y for any x, y lying on the same α-plane Z. Moreover, knowing how to parallel transport along null directions is enough to define a connection on V as the tangent space of M is spanned by null vectors. By construction, this connection is flat on α-planes and hence it is ASD.

Let us end this section with a discussion of how to obtain the ASDYM connection explicitly from the transition function F of E. Since $E|_{\hat{x}}$ is trivial, $F|_{\hat{x}}$ is equivalent to the identity matrix. That is,

$$F(\lambda w + \tilde{z}, \lambda z + \tilde{w}, \lambda) = \widetilde{H}^{-1}(w, z, \tilde{w}, \tilde{z}, \tilde{\lambda})\; H(w, z, \tilde{w}, \tilde{z}, \lambda),$$

where we have written $F(\kappa, \mu, \lambda) = F(\lambda w + \tilde{z}, \lambda z + \tilde{w}, \lambda)$ on restriction to $\hat{x}$, and $H, \widetilde{H}$ are $GL(N, \mathbb{C})$-valued functions holomorphic in λ and $\tilde{\lambda}$ respectively. Let us emphasise that $H, \widetilde{H}$ are defined on $\hat{x}$, where $w, z, \tilde{w}, \tilde{z}$ are constants, but not on $\mathcal{P}$. However, $H, \widetilde{H}$ can be regarded as functions of $w, z, \tilde{w}, \tilde{z}, \lambda$ on the correspondence space $\mathcal{F}$, where $F(\lambda w + \tilde{z}, \lambda z + \tilde{w}, \lambda)$ is the pull back of the transition function to $\mathcal{F}$. Now, since $F = \widetilde{H}^{-1} H$ is annihilated by l, m in (2.2.1), it follows that

$$(\partial_w H - \lambda \partial_{\tilde{z}} H) H^{-1} = (\partial_w \widetilde{H} - \lambda \partial_{\tilde{z}} \widetilde{H}) \widetilde{H}^{-1} \tag{2.2.5}$$

in the overlap $p^* W \cap p^* \tilde{W}$. The left-hand side of (2.2.5) is holomorphic in $|\lambda| \leq 1$ and the right-hand side is holomorphic in $|\lambda| \geq 1$ except a simple pole at infinity. Thus, by an extension of Liouville's theorem both sides must be of the form $-A_w + \lambda A_{\tilde{z}}$, where $A_w, A_{\tilde{z}}$ are independent of λ. Taking $A_w, A_{\tilde{z}}$ to be two components of a connection one-form A on V and obtaining $A_z, A_{\tilde{w}}$ in the same way, one has

$$(D_z - \lambda D_{\tilde{w}}) H = 0, \qquad (D_w - \lambda D_{\tilde{z}}) H = 0,$$

where $D = d + A$. Finally, the existence of a solution H implies that the above linear system is integrable and hence the connection D is ASD.

2.2.3 Minitwistor space

If an ASDYM connection is invariant under a subgroup of the conformal group, the corresponding vector bundle over the twistor space is also invariant under the induced symmetry on the twistor space. For some symmetry groups it is possible to reduce the domains of the ASDYM solutions and work with the corresponding reduced twistor spaces. Then the invariant vector bundle on $\mathcal{P}$ is the pull-back of a holomorphic vector bundle over the reduced twistor space. This is the case for the reduction by a non-null translation which yields the Yang-Mills-Higgs (YMH) system. In this section we shall introduce the *minitwistor space* for the YMH system on a complex 3-dimensional space $M \simeq \mathbb{C}^3$.

The reduction considered here is the complexified version of what is discussed in section 2.1.2. In the double null coordinates $(w, \tilde{w}, z, \tilde{z})$ of the complexified Minkowski space, the connection is assumed to be invariant under a non-null translation generated by $\partial_{\tilde{z}} - \partial_z$. Let

$$\tilde{z} = \frac{x^1 + x^3}{\sqrt{2}}, \; z = \frac{x^1 - x^3}{\sqrt{2}}, \; w = \frac{x^0 + x^2}{\sqrt{2}}, \; \tilde{w} = \frac{x^0 - x^2}{\sqrt{2}}.$$

The line element (2.1.6) and the Lax pair (2.1.7) are given in the coordinates (x^0, x^1, x^2, x^3) by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2,$$

$$\text{and} \qquad (D_1 - D_3 - \lambda(D_0 - D_2))\Psi = 0, \qquad (D_0 + D_2 - \lambda(D_1 + D_3))\Psi = 0,$$

where $D_a := D_{x^a}, \; a = 0, 1, 2, 3$. The reduced Lax pair is obtained by demanding that all functions on $\mathbb{C}^4$ are independent of x^3. As discussed in section 2.1.2, the component A_3 becomes the Higgs field Φ after the reduction. Renaming the coordinates $(x^0, x^1, x^2) \to (t, x, y)$, the reduced Lax pair is given by

$$(D_x - \Phi - \lambda(D_t - D_y))\Psi = 0, \qquad (D_t + D_y - \lambda(D_x + \Phi))\Psi = 0, \tag{2.2.6}$$

where (A_μ, Φ) are now functions on the reduced domain $M \simeq \mathbb{C}^3$ and $\Psi = \Psi(t, x, y, \lambda)$. The ASDYM equation (2.1.10) reduces to the YMH equation (2.1.11).

Following [27, 29], we shall now describe the minitwistor space for $M \simeq \mathbb{C}^3$ with the flat metric

$$ds^2 = -dt^2 + dx^2 + dy^2. \tag{2.2.7}$$

The minitwistor space is the space of null planes in M. A null plane in $\mathbb{C}^3$ is defined with respect to the flat metric (2.2.7) as a 2-plane whose normal vector is null. Hence, the equation for a null plane is given by

$$\eta_{\mu\nu} k^\mu x^\nu = -\frac{1}{2}\hat{\omega}, \tag{2.2.8}$$

where $x^\mu = (x^0 = t, \; x^1 = x, \; x^2 = y)$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, k^μ is the normal null vector field and $\hat{\omega}$ is a constant. The factor of $-\frac{1}{2}$ is introduced for convenience, which will become apparent shortly. To parametrise null vector fields, it is useful to use the spinor formalism based on the identification

$$TM = \mathcal{S} \odot \mathcal{S},$$

where TM is the holomorphic tangent bundle of M and $\mathcal{S}$ is a rank two vector bundle over M. A tangent vector k can be written as a symmetric two-spinor

$$k^{AB} = \begin{pmatrix} k^0 + k^2 & k^1 \\ k^1 & k^0 - k^2 \end{pmatrix},$$

such that $\eta_{\mu\nu}k^\mu k^\nu = -\det(k^{AB}) = -\frac{1}{2}\varepsilon_{AC}\varepsilon_{BD}k^{AB}k^{CD}$. It follows that a null vector field corresponds to a symmetric two-spinor of rank 1. That is, every null vector field is given by $k^{AB} = \pi^A\pi^B$ for $\pi^A \neq (0,0)$, where $\pi^A,\ A = 0,1$, denotes the fibre coordinates of $\mathcal{S}$. Now, writing the spacetime coordinates also as a symmetric two-spinor

$$x^{AB} = \begin{pmatrix} t+y & x \\ x & t-y \end{pmatrix},$$

the null plane equation (2.2.8) becomes

$$\hat{\omega} = x^{AB}\pi_A\pi_B. \tag{2.2.9}$$

Moreover, let us assume that $\pi_1 \neq 0$. Defining $\omega = \hat{\omega}/\pi_1^2$ and $\lambda = \frac{\pi_0}{\pi_1}$, equation (2.2.9) now reads

$$\omega = (t+y)\lambda^2 + 2x\lambda + (t-y). \tag{2.2.10}$$

The null planes with $\pi_1 = 0$ can also be captured by (2.2.10) by allowing λ to go to infinity. This implies that every null plane in $\mathbb{C}^3$ is labelled by (ω, λ), where $\omega \in \mathbb{C}$ and $\lambda \in \mathbb{CP}^1$. The minitwistor space, which is the space of null planes in $\mathbb{C}^3$, is therefore a line bundle over $\mathbb{CP}^1$. It is in fact the tangent bundle $T\mathbb{P}^1$ of $\mathbb{CP}^1$. To see this, note that under the change of coordinate $\lambda \to \tilde{\lambda} = \lambda^{-1}$ the fibre coordinate changes by $\omega \to \tilde{\omega} = \omega\lambda^{-2}$.

Similar to the 4-dimensional case, it follows from (2.2.10) that a point $p \in M$ corresponds to a $\hat{p} \simeq \mathbb{CP}^1 \subset T\mathbb{P}^1$. As before, we can define the correspondence space $\mathcal{F}$ to be the space of pairs (p, Z), of a point $p \in M$ and a null plane Z passing through p. There is a $\mathbb{CP}^1$ worth of null planes passing through each point $p \in M \simeq \mathbb{C}^3$, and thus $\mathcal{F} = \mathbb{C}^3 \times \mathbb{CP}^1$. Note that the two vector fields in the Lax pair (2.2.6)

$$l_0 = \partial_x - \lambda(\partial_t - \partial_y), \qquad l_1 = \partial_t + \partial_y - \lambda\partial_x$$

span a null plane, as they annihilate $\omega = (t+y)\lambda^2 + 2x\lambda + (t-y)$. The twistor space $T\mathbb{P}^1$ can therefore be regarded as the quotient space of $\mathbb{C}^3 \times \mathbb{CP}^1$ by the distribution $\{l_0, l_1\}$.

Since the YMH equation (2.1.11) is the compatibility condition of the Lax pair (2.2.6), there exist N linearly independent column vector solutions of (2.2.6) if (A_μ, Φ) is a solution of (2.1.11). These solutions are the covariantly constant sections of the trivial $\mathbb{C}^N$ bundle $V \to M$ restricted to null planes, with respect to (A_μ, Φ). One can construct a holomorphic rank N vector bundle over the twistor space $T\mathbb{P}^1$ by taking the fibre over each point $\hat{z} \in T\mathbb{P}^1$ to be the space of covariantly constant sections of $V|_Z$. This leads to another correspondence by Ward.

Theorem 2.2.2 **[29]** *There is a* $1:1$ *correspondence between*
(a) *a gauge equivalent class of solutions* (A_μ, Φ) *of* (2.1.11) *on* $\mathbb{C}^3$; *and*
(b) *holomorphic vector bundle* $E \to T\mathbb{P}^1$ *such that* $E|_{\hat{p}}$ *is trivial for all* $p \in M$.

The proof is analogous to the 4-dimensional case.

2.3 Symmetry reductions

We have seen an example of symmetry reductions of the ASDYM equation in the integrable chiral model in section 2.1.2. It is a reduction by a 1-dimensional group of non-null translation, which is a subgroup of the conformal group. A conformal transformation ρ of a 4-dimensional spacetime M endowed with a metric g and a volume form ν is defined by

$$\rho^* g = \Omega^2 g, \qquad \rho^* \nu = \Omega^4 \nu,$$

where ρ^* denotes the pull-back action of the map ρ on covariant tensors and Ω is a non-vanishing function on M. By definition one sees that a conformal transformation preserves the Hodge star operation on two-forms in four dimensions, since

$$(*\beta)_{ab} \ \propto \ \nu_{abcd}\, g^{ce} g^{df}\, \beta_{ef},$$

where β is a two-form on M. Therefore the ASDYM equation (2.1.4) is conformally invariant. The conformal group of the complexified Minkowski spacetime $M \simeq \mathbb{C}^4$, with the holomorphic flat metric and the volume form (2.1.6), has 15 generators. These are the conformal Killing vectors K which satisfy $\mathcal{L}_K g_{ab} \propto g_{ab}$, or equivalently

$$\partial_{(a} K_{b)} = \frac{1}{4} g_{ab} \partial_c K^c. \tag{2.3.1}$$

A general solution of (2.3.1) is given by

$$K_a = T_a + L_{ab}x^b + Rx_a + x^b x_b S_a - 2S_b x^b x_a,$$

where $T_a,\ L_{ab} = -L_{ba},\ R,\ S_a$ are constant coefficients which label translations, Lorentz transformations, dilatations (rescalings by constant factors), and special conformal transformations, respectively.

Since conformal transformations preserve the anti-self-duality of a curvature two-form of a connection, it makes sense to impose a further condition that the connection is invariant under a subgroup of the conformal group. The ASDYM equation together with such a symmetry assumption leads to a 'reduced' equation. Many well known integrable systems can be realised as symmetry reductions of the ASDYM equation in this sense. The classification of possible symmetry reductions thus provides a classification of the integrable systems arising from them.

2.3.1 Classification

Here we discuss four criteria in a classification of symmetry reductions of the ASDYM equation. We emphasise again that the list is chosen to provide a background for the research discussed in chapter 5. A full exposition of symmetry reductions of the ASDYM equation can be found in [23].

• **Subgroup of the conformal group.** In this thesis we shall only consider subgroups of the conformal group which act freely[1] on M. An action of a group H on M is said to be free if every element of H except the identity has no fixed points in M. Infinitesimally, an action of a group is replaced by an action of the Lie algebra, which is free if none of the generators has zeros in M.

When a group acts freely on a vector bundle $V \to M$ it is always possible to construct a local frame field of V which is invariant under the group action. In such an invariant gauge, the assumption that a connection on V is invariant under the group transformation is equivalent to the assumption that the components of the potential one-form are constant along the generators of the group. An example of this is the reduction by a non-null translation discussed in sections 2.1.2 and 2.2.3. There, the YMH system is obtained by assuming that the potential one-form is independent of the spacetime

[1] In one of the reductions we study, the action of the symmetry group on M is free except at one point. See section 5.3.

coordinate that parametrises the orbits of the group. The matrices of gauge transformations are also assumed to be independent of that coordinate. This is to ensure that the connection stays in an invariant gauge. If the action is not free then an invariant gauge may not exist.

• **Gauge group.** Here the term gauge group is used in the same way as the term structure group. The latter is the group of transformations of local trivialisations of the vector bundle on which the Yang-Mills connection is defined. As the potential one-form takes values in the Lie algebra of the gauge group, different gauge groups typically lead to different equations. For example, under a reduction by a 3-dimensional subgroup of the conformal group called a Painlevé group, the ASDYM equation with gauge group $SL(2,\mathbb{C})$ reduces to a Painlevé equation, whereas if the gauge group is one of all other Bianchi groups[2], the reduced equations are linear [30].

• **Normal forms of the Higgs fields and constants of integration.** Recall that the Higgs fields transform by conjugation under gauge transformations. Thus, it is possible to use gauge symmetry to put the Higgs fields in Jordan normal forms and classify the reductions according to the normal forms of the Higgs fields. Also, in some cases some of the reduced equations may be solved to yield first integrals, which are functions of one variable less than the number of independent variables. These first integrals may appear in the final form of the reduced equation and different choices lead to different equations.

• **Reality conditions.** This is important when one considers real equations from reductions of the ASDYM equation on complex spacetime. In chapter 5 we make use of two types of reality conditions. One is the choice of real slices of the complex spacetime and another is the choice of real forms of the gauge group.

An example of a successful classification is that of the nonlinear Schrödinger (NLS) equation and the Korteweg-de Vries (KdV) equation. It is shown in [31] that under a symmetry reduction by a 2-dimensional translation group such that the metric restricted to a 2-plane spanned by the two generators is of rank 1, the ASDYM equation with gauge group $SL(2,\mathbb{C})$ reduces to either the NLS equation when a Higgs field is semi-simple, or to the KdV equation when the Higgs field is nilpotent.

[2]These are the groups obtained via the exponential map from 3-dimensional Lie algebras, which are classified according to the Bianchi classification.

We emphasise that this gives a complete classification of the reductions of the $SL(2,\mathbb{C})$ ASDYM equation by the 2-dimensional translation group, denoted by H_{+0} in [23]. With gauge group $SL(2,\mathbb{C})$, the Higgs fields are represented by 2×2 matrices and one has to analyse the normal forms of 2×2 matrices. It becomes more difficult to obtain a complete classification with a larger gauge group. However, one can still characterise a particular integrable system as a reduction of the ASDYM equation by specifying the ingredients mentioned above. For example, in chapter 5 we shall give a characterisation of the Tzitzéica equation as a reduction of the $SL(3,\mathbb{R})$ ASDYM equation on $\mathbb{R}^{2,2}$.

Finally, let us note that in a classification, or a characterisation, one usually makes use of gauge symmetry of the ASDYM equation to put the Higgs fields or the components of the potential one-form in some particular forms, in which the reduction is done. Moreover, the reduced equation may possess an additional coordinate symmetry which can be used to put the reduced equation in a canonical form. These procedures will be demonstrated in chapter 5.

2.3.2 Painlevé equations and the isomonodromic Lax pairs

One of the successes of the programme studying the reductions of the ASDYM equation in relation to integrable systems is to obtain the six Painlevé equations. To see the significance of this result, let us start with a brief introduction of how the Painlevé equations came about. A nonlinear ordinary differential equation (ODE) is said to have the Painlevé property if the movable singularities of its solutions are at worst poles. Around the turn of the 20th century Painlevé [32] and others investigated second-order ODEs of the form

$$\frac{d^2y}{dt^2} = F\left(\frac{dy}{dt}, y, t\right),$$

where F is rational in dy/dt and y, and found that there are 50 canonical types of such ODEs with the Painlevé property. Out of the 50, 44 of them are solvable in terms of elementary functions, elliptic functions, or solutions to linear ODEs. The other 6 equations require an introduction of new transcendental functions. These are called Painlevé equations and their transcendental solutions are called Painlevé transcendents.

The six Painlevé equations are given by

$$\begin{aligned}
\text{PI} \quad \frac{d^2y}{dt^2} &= 6y^2 + t,\\
\text{PII} \quad \frac{d^2y}{dt^2} &= 2y^3 + ty + \alpha,\\
\text{PIII} \quad \frac{d^2y}{dt^2} &= \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},\\
\text{PIV} \quad \frac{d^2y}{dt^2} &= \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},\\
\text{PV} \quad \frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1},\\
\text{PVI} \quad \frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}\\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right),
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are constant parameters. The Painlevé equations can be regarded as nonlinear analogues of the special linear equations which define the usual special functions such as the Airy function and the Bessel function.

The Painlevé property is normally regarded as one of the common features among integrable systems. It was found by the authors of [33] that integrable equations solvable by the inverse scattering method reduce to ODEs with the Painlevé property. The converse leads to the so-called Painlevé test, where given a nonlinear partial differential equation (PDE), one obtains a symmetry reduction and check if the resulting ODE has the Painlevé property. If it does, the nonlinear PDE may be integrable.

The Painlevé equations also play an important role in the theory of isomonodromic deformation. This is the study of linear ODEs of the form

$$\frac{dY}{d\lambda} = A(\lambda; x^\mu)Y, \tag{2.3.2}$$

where Y is an $N \times N$ matrix solution and $A(\lambda; x^\mu)$ is an $N \times N$ matrix coefficient which is rational in λ and with some constant parameters x^μ. Since A is rational in λ, a solution Y will in general have singular points. Moreover, the values of Y at the initial and final points of a closed contour encircling a singular point are generally different. The final value Y_f is related to the initial value Y_i by $Y_f = Y_i M$, where M is an $N \times N$ matrix called the monodromy matrix. The monodromy matrices together with some other information about Y around singular points form a set of monodromy data (see, for example [34]). A priori the monodromy data will depend on the parameters x^μ. The isomonodromic deformation is the study of a family of linear ODEs (2.3.2) parametrised by x^μ which all have the same monodromy. Research in this area was pioneered by Schlesinger [35] and later generalised by Jimbo, Miwa, and Ueno [14, 36]. It was found that the monodromy data are independent of the parameters x^μ in A if Y also satisfies a system of linear PDEs of the form

$$\frac{\partial Y}{\partial x^\mu} = B_\mu(\lambda; x^\nu)Y, \tag{2.3.3}$$

where $B_\mu(\lambda; x^\nu)$ is an $N \times N$ matrix coefficient. Hence the isomonodromy condition is equivalent to the compatibility condition for (2.3.2) and (2.3.3). The compatibility condition, called the deformation equation, is a nonlinear differential equation for $A(\lambda; x^\mu)$ with x^μ as independent variables. It was shown in [37] that the deformation equation possesses the Painlevé property. In particular, Jimbo and Miwa [14] showed that the six Painlevé equations arise as the deformation equations for certain linear ODEs of the form (2.3.2), where A are 2×2 matrices.

It was shown in [22] that the six Painlevé equations can be obtained from the reductions of the $SL(2, \mathbb{C})$ ASDYM equation by five of the 3-dimensional Abelian subgroups of the conformal group. There are 14 such groups up to conjugations, two of which are degenerate in the sense that the orbits are 2-dimensional. Another six cases are called null groups, where the 3-dimensional orbits are null. The rest are the groups which have 3-dimensional non-null orbits. These are the translation group and the five Painlevé groups. The first and the second Painlevé equations are obtained from the reduction by the same Painlevé group.

Moreover, the isomonodromic Lax pairs (2.3.2) and (2.3.3) of the six Painlevé equations can be derived from the Lax pair of the ASDYM equation, the details of which can be found in [23]. The isomonodromic Lax pairs are given by the reduced linear systems, which we discuss next. We have seen an example of a reduced linear system in the case of the reduction by a 1-parameter non-null translation to the YMH system. The reduced Lax pair (2.2.6) is obtained from (2.1.7) by assuming not only the components of the potential one-form A, but also the fundamental solution Ψ to be independent of the ignorable coordinates. That is, in the case of (2.2.6) Ψ is assumed to depend on t, x, y and λ only. The reduced equations arise as the compatibility conditions of the reduced linear systems. A reduced Lax pair can always be derived in this way[3] when the

[3] In general we require the solutions of the Lax pair to be Lie derived along the generators of the symmetry group.

symmetry group maps an α-plane to a parallel α-plane. This means that the spectral parameter λ is constant along the lift of the generators to the correspondence space $\mathcal{F}$ (recall that λ labels the α-planes passing through the origin). This is the case for translations. In general, however, one needs to introduce an invariant spectral parameter ζ which is constant along the lift of the symmetry generators in $\mathcal{F}$. This allows us to express solutions of the Lax pair in terms of the non-ignorable coordinates and ζ.

Following [23], let us now discuss a lift of a conformal Killing vector on the complexified Minkowski spacetime M to the correspondence space $\mathcal{F}$ and the induced vector field on the twistor space $\mathcal{P}$. Consider a conformal Killing vector

$$K = a\partial_w + b\partial_z + \tilde{a}\partial_{\tilde{w}} + \tilde{b}\partial_{\tilde{z}}$$

on M, where $a, b, \tilde{a}, \tilde{b}$ are functions on M. The commutation relations with l, m (2.2.1),

$$l = \partial_w - \lambda\partial_{\tilde{z}}, \qquad m = \partial_z - \lambda\partial_{\tilde{w}},$$

are given by

$$[K, l] = Q\partial_{\tilde{z}}, \qquad [K, m] = Q\partial_{\tilde{w}}$$

modulo l, m, where

$$Q = \lambda^2 a_{\tilde{z}} + \lambda(\tilde{b}_{\tilde{z}} - a_w) - \tilde{b}_w.$$

The conformal Killing equation satisfied by K implies that Q is constant along l, m. Hence, one sees that there exists a lift K'' in $\mathcal{F}$ of K given by

$$K'' = K + Q\partial_\lambda,$$

such that

$$[K'', l] = 0, \qquad [K'', m] = 0,$$

modulo l, m. The conformal Killing vector field K on M induces a flow on $\mathcal{P}$ generated by the projection $K' = p_* K''$, where $p : \mathcal{F} \to \mathcal{P}$ is defined in (2.2.3). Since K'' preserves the distribution spanned by l, m, the projection K' is well defined on $\mathcal{P}$.

Let us end this section by looking at an example which will be relevant to the study in section 5.3.2. Consider a rotation on the z-$\tilde{z}$ plane generated by

$$K = z\partial_z - \tilde{z}\partial_{\tilde{z}}.$$

One calculates the lift of K to $\mathcal{F}$ to be

$$K'' = z\partial_z - \tilde{z}\partial_{\tilde{z}} - \lambda\partial_\lambda,$$

and sees that the spectral parameter λ is not constant along K''. To define an invariant spectral parameter we introduce new coordinates adapted to the symmetry: $z = se^{i\theta}$, $\tilde{z} = se^{-i\theta}$. Then

$$K'' = -i\partial_\theta - \lambda\partial_\lambda,$$

and an invariant spectral parameter is given by $\zeta = \lambda e^{i\theta}$.

2.4 Anti-self-dual conformal structures

So far we have only discussed the ASD condition for a Yang-Mills connection on a vector bundle over a flat 4-manifold. In the same way, one can impose the ASD condition on the Levi-Civita connection on the tangent bundle of a curved manifold. This results in the Riemann curvature being anti-self-dual, with respect to the Hodge star operator defined by the curved metric and a preferred volume form. However, since the Ricci tensor of an ASD Riemann curvature has to vanish, one often relaxes the condition and requires only the Weyl tensor to be anti-self-dual. The Weyl tensor is invariant under conformal transformations and thus one talks about the anti-self-duality of a conformal structure. Following [23, 28], we shall now give a brief overview of the anti-self-duality of conformal structures and its geometric interpretations. Similar to the setting of the ASDYM equation, we shall discuss the general framework using complex manifolds.

2.4.1 Anti-self-duality condition

Let M be a complex 4-dimensional manifold, and g_1, g_2 be two holomorphic metrics on M. The metric g_2 is in the same conformal class as g_1 if and only if $g_2 = \Omega^2 g_1$, where Ω is a non-vanishing function on M. This equivalence class together with the manifold M defines a conformal structure. A conformal structure $(M, [g])$ is said to be anti-self-dual if the Weyl tensor of any $g \in [g]$ is anti-self-dual.

Given a Levi-Civita connection ∇ and a local frame field $\{e_a\}$ of the tangent bundle, the connection one-form $\Gamma^a{}_b$ is defined by $\nabla e_b = e_a \Gamma^a{}_b$. The curvature two-form is given in terms of $\Gamma^a{}_b$ as

$$\mathcal{R}^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b, \tag{2.4.1}$$

from which the Riemann tensor $R^a{}_{bcd}$ is defined by

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d,$$

where $\{e^a\}$ is the dual basis of $\{e_b\}$ such that $e_b \lrcorner e^a = \delta^a_b$. The Riemann tensor can be decomposed into the Weyl tensor C_{abcd}, the Ricci tensor R_{ab} and the scalar curvature R as

$$R_{ab}{}^{cd} = C_{ab}{}^{cd} + 2\Phi^{[c}_{[a}\delta^{d]}_{b]} + \frac{1}{6}R\delta^{[c}_{[a}\delta^{d]}_{b]},$$

where $\Phi_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$, $R_{ab} = R^c{}_{acb}$ and $R = R^a{}_a$. Now, the symmetry of the Riemann tensor

$$R_{abcd} = R_{[ab][cd]}$$

implies that $\mathcal{R}$ belongs to the tensor product of two copies of the space of two-forms, i.e.

$$\mathcal{R} = \frac{1}{4}R_{abcd}\,(e^a \wedge e^b) \otimes (e^c \wedge e^d). \tag{2.4.2}$$

The space of two-forms is the direct sum of the space of SD forms and the space of ASD forms. Therefore, we can decompose $\mathcal{R}$ into 4 parts

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}^{++} & \mathcal{R}^{+-} \\ \mathcal{R}^{-+} & \mathcal{R}^{--} \end{pmatrix}, \tag{2.4.3}$$

where $\mathcal{R}^{++}$ belongs to the tensor product of the space of SD two-forms with itself, while $\mathcal{R}^{+-}$ belongs to the tensor product of the space of SD two-forms with the space of ASD two-forms, and similarly for $\mathcal{R}^{-+}$ and $\mathcal{R}^{--}$. When we decompose $\mathcal{R}$ into the Weyl tensor, the Ricci tensor and the Ricci scalar using the basis of SD and ASD two-forms as above, we have

$$\mathcal{R} = \begin{pmatrix} C^+ + \frac{R}{12} & \Phi \\ \Phi & C^- + \frac{R}{12} \end{pmatrix},$$

where C^+, C^- are SD and ASD parts of the Weyl tensor respectively and Φ is the trace-free part of the Ricci tensor. It should be emphasised that, unlike the Riemann curvature, the Weyl tensor decomposes completely into the SD and ASD parts

$$C_{abcd} = C^+_{abcd} + C^-_{abcd}.$$

Hence, the requirement that the Weyl tensor is ASD is equivalent to that the SD part vanishes. A convenient way to express the ASD condition on a conformal structure is summarised in the following proposition. The proposition is originally due to Penrose [38], but taken in this form from [23].

Proposition 2.4.1 *Let $V_{AA'}$ be four independent holomorphic vector fields on a 4-dimensional complex manifold M. Then $V_{AA'}$ determine an ASD conformal structure if* *and only if there exist two holomorphic functions f_A on $M \times \mathbb{CP}^1$ such that the distribution spanned by*

$$l = V_{00'} - \lambda V_{01'} + f_0 \frac{\partial}{\partial \lambda}, \quad m = V_{10'} - \lambda V_{11'} + f_1 \frac{\partial}{\partial \lambda} \tag{2.4.4}$$

is integrable, where $\lambda \in \mathbb{CP}^1$.

To see how proposition 2.4.1 comes about, let us go back and consider the Riemann curvature (2.4.1) of a complex 4-dimensional manifold. In an orthonormal basis of one-forms $\{e^a\}$ a metric is of the form

$$g = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2.$$

The form of the metric and the orientation are preserved under the change of basis $\hat{e}_a = A^a{}_b e_b$ if and only if $A^a{}_b \in SO(4, \mathbb{C})$. Considering only orthonormal bases, the structure group of the tangent bundle reduces to $SO(4, \mathbb{C})$ and the curvature $\mathcal{R}$ takes values in $\mathfrak{so}(4, \mathbb{C})$. On the other hand, one can write the metric in a basis of null tetrad. A null tetrad is a set of four linearly independent null vector fields $V_{AA'}$, $A, A' \in \{0, 1\}$, such that a metric takes the form

$$g = 2\left(V^{00'} \odot V^{11'} - V^{01'} \odot V^{10'}\right),$$

where $V^{AA'}$ are the dual one-forms of $V_{BB'}$. That is, the metric is given by $g = 2\det(V)$, where

$$V = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix}.$$

This leads to a group isomorphism

$$SO(4, \mathbb{C}) = (SL(2, \mathbb{C}) \times SL(2, \mathbb{C}))/\mathbb{Z}_2,$$

as multiplying $\Lambda, \tilde{\Lambda} \in SL(2, \mathbb{C})$ on the left and on the right of V respectively preserves g. This means that we can decompose the values of $\mathcal{R}$, which is in $\mathfrak{so}(4, \mathbb{C})$, as a sum of elements of $\mathfrak{sl}(2, \mathbb{C})$. The tangent bundle TM is factored by

$$TM = \mathcal{S} \otimes \mathcal{S}',$$

where $\mathcal{S}, \mathcal{S}'$ are the unprimed and primed spin bundles, which are the fundamental representation spaces of the two $SL(2, \mathbb{C})$. The connection one-form Γ can now be written as

$$\Gamma = \gamma + \tilde{\gamma},$$

where γ and $\tilde{\gamma}$ are $\mathfrak{sl}(2,\mathbb{C})$-valued connection one-forms on $\mathcal{S}$ and $\mathcal{S}'$, respectively. The curvature $\mathcal{R}$ is then given by the sum of the curvatures of the two connections.

To summarise, $\mathcal{R}$ is a two-form field and hence can be decomposed into SD and ASD parts, and at each point the value of $\mathcal{R}$ is the sum of two elements of $\mathfrak{sl}(2,\mathbb{C})$ coming from the connections on the unprimed and primed spin bundles. One can connect this picture with $\mathcal{R}$ in (2.4.2), which is a section of the product of two copies of two-form bundles over M, by identifying elements of $\mathfrak{so}(4,\mathbb{C})$ with two-forms on M. The decomposition of $\mathfrak{so}(4,\mathbb{C})$ into the sum of two $\mathfrak{sl}(2,\mathbb{C})$ corresponds precisely to the decomposition of the space of two-forms into SD and ASD parts.

Let us use the convention that the connection $\tilde{\gamma}$ on $\mathcal{S}'$ corresponds to the SD part and γ on $\mathcal{S}$ to the ASD part. Then the curvature of $\tilde{\gamma}$ is the sum of $\mathcal{R}^{++}$ and $\mathcal{R}^{+-}$ in (2.4.3). Therefore, we conclude that the connection $\tilde{\gamma}$ on the primed spin bundle encodes the information about the self-dual part of the Weyl tensor, the trace-free part of the Ricci tensor and the Ricci scalar.

We shall now explain proposition 2.4.1. Let $\pi_{A'}$ be coordinates on $\mathcal{S}'$ and consider a one-form $\theta = \tilde{\gamma}_{A'B'}\pi^{A'}\pi^{B'}$. After projecting the fibre of $\mathcal{S}'$ to a $\mathbb{CP}^1$ and setting $\lambda = \frac{\pi_{0'}}{\pi_{1'}}$, θ becomes a polynomial of one-forms quadratic in λ. Now consider two vector fields

$$L_0 = V_{00'} - \lambda V_{01'}, \quad \text{and} \quad M_0 = V_{10'} - \lambda V_{11'}, \tag{2.4.5}$$

where $V_{AA'}$ is a null tetrad. One can form two vector fields on $M \times \mathbb{CP}^1$ by

$$l = L_0 + (L_0 \lrcorner\, \theta)\frac{\partial}{\partial\lambda}, \quad m = M_0 + (M_0 \lrcorner\, \theta)\frac{\partial}{\partial\lambda}. \tag{2.4.6}$$

The vector fields l, m encodes the information about the connection $\tilde{\gamma}$ on the primed spin bundle. Moreover, by direct calculation one can show that the commutation $[l, m]$ is given by

$$[l, m] = (\dots)l + (\dots)m + (\dots)\frac{\partial}{\partial\lambda},$$

where the coefficient of $\frac{\partial}{\partial\lambda}$ is given in terms of the SD Weyl tensor C^+ of the metric defined by the null tetrad $V_{AA'}$, and it vanishes if and only if C^+ vanishes. Hence, given a metric with ASD Weyl tensor, the vector fields l, m in (2.4.6) span an integrable distribution. Conversely, if the vector fields (2.4.4) span an integrable distribution, the two functions f_A are forced to be $L_0 \lrcorner\, \theta$ and $M_0 \lrcorner\, \theta$ for θ associated to the metric defined by $V_{AA'}$, and hence the metric has ASD Weyl tensor. As the argument remains valid upon rescaling the tetrads, the vector fields l, m in (2.4.4) define an ASD conformal structure.

Proposition 2.4.1 has a geometric interpretation. For each λ the vector fields L_0, M_0 in (2.4.5) span a tangent α-plane passing through the origin of T_xM for each point $x \in M$. One can construct a correspondence space $\mathcal{F}$ as in the flat case, where each point of $\mathcal{F}$ is a pair (x, λ) of a point $x \in M$ and an α-plane passing through the origin of T_xM, the latter of which is labelled by λ. The vector fields L_0, M_0 have horizontal lifts in $\mathcal{F}$ given precisely by l, m in (2.4.6). In a curved manifold, instead of α-planes we have a notion of α-surfaces. These are surfaces whose tangent plane at each point is an α-plane. An α-surface can be lifted to $\mathcal{F}$ where its tangent plane at each point is spanned by l, m. By Frobenius theorem there exists a maximal (three-parameter) family of the lift of α-surfaces, which means the surfaces foliate $\mathcal{F}$, if and only if the distribution spanned by l, m is integrable. By projection it follows that there exists a three-parameter family of α-surfaces in M, and vice versa. This leads us to the original theorem by Penrose.

Theorem 2.4.2 **[38]** *Let $(M, [g])$ be a 4-dimensional conformal structure. There exists a three parameter family of α-surfaces in M if and only if $[g]$ is anti-self-dual.*

2.4.2 Twistor space of ASD conformal structure

One can define a curved twistor space of an ASD conformal structure in a similar way to that of the complexified Minkowski spacetime which is a flat conformal structure. The twistor space $\mathcal{P}$ of an ASD conformal structure $(M, [g])$ is the space of all α-surfaces that intersect[4] M. As in the flat case, each point $x \in M$ corresponds to a $\mathbb{CP}^1$ line $\hat{x} \subset \mathcal{P}$ of all α-surfaces that pass through x, and $\hat{x}$ is parametrised by a complex parameter λ which labels each tangent α-plane at x. The curved twistor space of a conformal structure with $M \simeq \mathbb{C}^4$ is therefore topologically the same as $\mathbb{CP}^3 - \mathbb{CP}^1$. However, the complex structure is different. In fact, it is the complex geometry on $\mathcal{P}$ that encodes the information of the conformal structure. We refer the readers to references [23, 26] for details of the twistor construction, and shall only state important results which will be used later in chapter 6.

There is a natural line bundle over $\mathcal{P}$. To see this, let us first consider an α-surface $\Sigma \subset M$. A natural line bundle over Σ is formed by taking the fibre at each point to be the one-dimensional space of tangent bivectors of the tangent α-plane at that point. Since Σ has a lift in the correspondence space $\mathcal{F}$ as a surface $\hat{\Sigma}$, one can define a line bundle over $\hat{\Sigma}$ to be the pull-back of the bundle over Σ. This line bundle extends to the

[4]Strictly speaking, one needs to impose a convexity assumption on M - that the intersection of each α-surface with M is connected and simply-connected - for $\mathcal{P}$ to be Hausdorff.

whole correspondence space $\mathcal{F}$, where the fibre at each point (x, λ) is the same as the fibre over x of the bundle over the α-surface Σ whose tangent α-plane at x is labelled by λ. Let us denote this line bundle by $\mathcal{O}(-2)$ as it restricts to a standard $\mathcal{O}(-2) \to \mathbb{CP}^1$ on each $\mathbb{CP}^1$ fibre of $\mathcal{F}$.

Given a representative tetrad, the Levi-Civita connection induces a connection on $\mathcal{O}(-2) \to \mathcal{F}$ and hence one can define a parallel transport on the bundle. The potential one-form restricted to $\hat{\Sigma}$ is given by $\partial_\lambda \theta$ (see, for example, [23]). Therefore a covariantly constant section of $\mathcal{O}(-2) \to \hat{\Sigma}$ is a function f on $\hat{\Sigma}$ such that

$$l(f) + (L_0 \lrcorner\, \partial_\lambda \theta) f = 0, \qquad m(f) + (M_0 \lrcorner\, \partial_\lambda \theta) f = 0. \tag{2.4.7}$$

The compatibility condition for (2.4.7) is guaranteed by the anti-self-duality of the conformal structure which implies that the curvature on $\mathcal{O}(-2) \to \hat{\Sigma}$ is flat.

Now, one can define a line bundle over $\mathcal{P}$, also denoted by $\mathcal{O}(-2)$, by taking the fibre at each point $p \in \mathcal{P}$ to be the space of covariantly constant sections of $\mathcal{O}(-2)$ over the surface $\hat{\Sigma}$ that corresponds to p. Having this line bundle, one can consider its dual bundle, which we denote by $\mathcal{O}(2)$, and their respective kth tensor products $\mathcal{O}(-2k)$ and $\mathcal{O}(2k)$. An important point is that since the lift of α-surfaces foliate $\mathcal{F}$, the pull-back of a section of $\mathcal{O}(-2) \to \mathcal{P}$ to $\mathcal{F}$ is a section of $\mathcal{O}(-2) \to \mathcal{F}$, represented by a function on $\mathcal{F}$ which satisfies (2.4.7). This result will be used in chapter 6.

2.4.3 Relation to ASDYM equation

Since ASD conformal structures admit the twistor construction, one expects the field equations for the ASD condition to be integrable. In fact, it turns out that under certain circumstance there is an explicit transformation which relates the ASD condition on a conformal structure to a reduction of the ASDYM equation. The transformation is called the switch map, which was first introduced by Mason and Woodhouse in [22] and studied further in [21]. The details of the transformation will not be discussed in this thesis and we shall refer the readers to [23] for reference on the subject. However, let us briefly mention what the map does.

Suppose we are given an ASD conformal structure $(M, [g])$ and an ASDYM connection on a vector bundle over M, both of which are invariant under a subgroup H of the conformal group, and the dimension of H is the same as that of the gauge group G. Starting from $(M, [g])$ and the ASDYM connection, one can use the switch map to construct another pair of a conformal structure $(M', [g'])$ invariant under the group G

and a connection on a vector bundle over M' with H as the gauge group. An important point is that both the new conformal structure and the connection are ASD. Therefore, the switch map can be used to construct a curved ASD conformal structure from an ASDYM connection on a flat spacetime, and the relation between the ASD conditions of conformal structures and the reductions of the ASDYM equation comes in this way.

A list of the conformal structures constructed using the switch map can be found in [23]. We are particularly interested in the case of cohomogeneity-one metrics. These are the metrics which are invariant under 3-dimensional symmetry groups. Hence, one expects them to be determined by the reductions of the ASDYM equation by 3-dimensional subgroups of the conformal group. In fact, one has the following proposition from [30].

Proposition 2.4.3 [30] *Let G be a 3-dimensional complex Lie group which acts freely on a 4-dimensional manifold M with non-null orbits. Then, using the switch map every conformally G-invariant ASD metric can be obtained from a conformally flat metric and a $\mathfrak{g}$-valued ASDYM connection which are invariant under some 3-dimensional Abelian subgroup $\widetilde{G} \subset PGL(4, \mathbb{C})$.*

The proposition is stated using the fact that the complex conformal group is isomorphic to the projective general linear group $PGL(4, \mathbb{C})$. In fact, the group $\widetilde{G}$ in proposition 2.4.3 also acts freely on the flat spacetime $\mathbb{C}^4$ with non-null orbits. Thus, $\widetilde{G}$ is either the translation group or one of the five Painlevé groups as discussed in section 2.3.2. For example, as the ASDYM equation with gauge group $SL(2, \mathbb{C})$ reduces to the Painlevé equations by the Painlevé groups, one expects $SL(2, \mathbb{C})$-invariant ASD metrics to be determined by the Painlevé equations. This is indeed what was found in [17, 19, 39].

Chapter 3

Integrable Chiral Model

We explore the energy quantisation of a class of time-dependent soliton solutions of the Ward integrable chiral model. The integrable chiral model is equivalent to the Yang-Mills-Higgs system in $2+1$ dimensions with a gauge choice, therefore one can associate a solution of the model with an extended solution of the Lax pair of the Yang-Mills-Higgs system. We show that the total energy of each soliton solution in the class considered is proportional to the third homotopy class of the extended solution, and this explains the energy quantisation topologically. This work has been published in [1].

3.1 Energy of soliton solutions

The fact that the allowed energy levels of some physical systems can take only discrete values has been well known since the the early days of quantum theory. The hydrogen atom and the harmonic oscillator are two well known examples. In these two cases the boundary conditions imposed on the wave function imply discrete spectra of the Hamiltonians. The reasons are therefore global.

The quantisation of energy can also occur at the classical level in nonlinear field theories if the energy of a smooth field configuration is finite. The reasons are again global, but one needs more subtle ideas from topology to understand what is going on. The potential energy of static soliton solutions in the Bogomolny limit of certain field theories must be proportional to integer homotopy classes of smooth maps. The details depend on the model. In pure gauge theories the energy of solitons satisfying the Bogomolny equations is given by one of the Chern numbers of the curvature. In scalar 2+1 dimensional sigma models, allowed energies of Bogomolny solitons are given by elements of $\pi_2(\Sigma)$, where the manifold Σ is the target space. In both cases the boundary conditions are used to show that the finite energy configurations extend to the compactification of space. See [40] for a detailed exposition of these constructions.

The situation is different for moving solitons. The total energy is the sum of kinetic and potential terms, and the Bogomolny bound is not saturated. One expects that the moving (non-periodic) solitons will have continuous energy. Attempts to construct theories with quantised total energy based on compactifying the time direction are physically unacceptable, as they lead to paradoxes related to the existence of closed time-like curves. A soliton moving along such curve could eventually reach its own past thus opening possibilities to sinister scenarios usually involving a death of somebody's great grandparents.

In a recent publication [8] Ioannidou and Manton made the surprising observation that the total energy of the time-dependent $SU(2)$ 2-uniton solution of Ward's $2+1$ dimensional chiral model [3, 4] is quantised in the units of 8π when the pole of the corresponding extended solution is at $\pm i$. They have shown that the 2-uniton energy density calculated at any instant of time t is the same as the energy density of a static $\mathbb{CP}^3$ multi-lump with a parameter t. The total (potential) energy of the latter model is quantised [41] which leads to the total (kinetic+potential) energy quantisation of the time-dependent unitons. The quantisation was also obtained by Lechtenfeld and Popov [42, 43] whose method was based on large time asymptotic analysis.

One expects that there are deeper topological reasons for this quantisation, and the purpose of this chapter is to show that this is indeed the case.

Recall from section 2.1.2 that the Ward chiral model is given by

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0, \tag{3.1.1}$$

where $J : \mathbb{R}^3 \longrightarrow U(N)$, $x^\mu = (t, x, y)$ are coordinates on $\mathbb{R}^3$ and the line element is $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$. Again, we use the notation $J_\mu := \partial_\mu J$. The positive-definite conserved energy functional coming from the energy-momentum tensor of the chiral model is

$$E = \int_{\mathbb{R}^2} \mathcal{E}\mathrm{d}x\mathrm{d}y, \tag{3.1.2}$$

where the energy density is given by

$$\mathcal{E} = -\frac{1}{2}\mathrm{Tr}((J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2). \tag{3.1.3}$$

The integrability of (3.1.1) allows a construction of explicit static and also time-dependent solutions by twistor or inverse-scattering methods [3, 44]. There are time-dependent solutions with non-scattering solitons [3], and also solitons that scatter [4]. A class of scattering solutions to (3.1.1) is given by the so called time-dependent unitons

$$J(x, y, t) = M_1 M_2 \dots M_n, \tag{3.1.4}$$

where the unitary matrices $M_k, k = 1, \dots, n$, are given by

$$M_k = i\Big(\mathbf{1} - \Big(1 - \frac{\mu}{\bar{\mu}}\Big)R_k\Big), \qquad R_k \equiv \frac{q_k^* \otimes q_k}{||q_k||^2}. \tag{3.1.5}$$

Here $\mu \in \mathbb{C}\backslash\mathbb{R}$ is a non-real constant and $q_k = (1, f_{k1}, ..., f_{k(N-1)}) \in \mathbb{C}^N$, with $k = 1, \dots, n$, are vectors whose components $f_{kj} = f_{kj}(x^\mu) \in \mathbb{C}$ are smooth functions which tend to a constant at spatial infinity[1], and q_k^* are their Hermitian conjugates.

If $n = 1$ then q_1 is holomorphic and rational in $\omega = x + \frac{1}{2}\mu(t+y) + \frac{1}{2}\mu^{-1}(t-y)$ [3]. Note that if $\mu = \pm i$, q_1 does not depend on t and the corresponding 1-uniton is static. If $n > 1$, q_1 is still holomorphic and rational in ω, but $q_2, q_3, ...$ are not holomorphic. The exact form of these functions is known explicitly for $n = 2, 3$ [4, 5] for the case $N = 2$. For $n > 3$ the Bäcklund transformations [6, 7] can be used to determine the fs recursively. The total energy (3.1.2) of n-uniton solutions is finite.

In general the finiteness of E is ensured by imposing the boundary condition (valid for all t)

$$J = J_0 + J_1(\varphi)r^{-1} + O(r^{-2}) \qquad \text{as} \qquad r \longrightarrow \infty, \qquad x + iy = re^{i\varphi} \tag{3.1.6}$$

and so for a fixed value of t the matrix J extends to a map from S^2 (conformal compactification of $\mathbb{R}^2$) to $U(N)$. The homotopy group $\pi_2(U(N)) = 0$, so there is no topological information in J defined on $\mathbb{R} \times S^2$ which could be related to the total energy. We shall nevertheless show that the energy of (3.1.4) is quantised and given by the third homotopy class of the extended solution to the associated Lax equations (2.2.6) with a spectral parameter λ (see section 2.2.3). The extended solution also depends on the spectral parameter and hence is defined on $\mathbb{R}^3 \times \mathbb{CP}^1$. Restricting it to a space-like plane in $\mathbb{R}^3$ and an equator in a Riemann sphere of the spectral parameter gives a map ψ, whose domain is $\mathbb{R}^2 \times S^1$. If J is an n-uniton solution (3.1.4), the corresponding extended solution satisfies a stronger boundary condition which promotes ψ to a map $S^3 \longrightarrow U(N)$. In the next section we shall introduce the extended solution, impose boundary conditions on J which are stronger than (3.1.6), and in fact provide a coordinate-free characterisation of the uniton solutions (3.1.4). In section 3.3 we establish the following result:

[1]The matrix R_k is a hermitian projection satisfying $(R_k)^2 = R_k$, and the corresponding M_k is a Grassmanian embedding of $\mathbb{CP}^{N-1}$ into $U(N)$. The results in this chapter apply to the more general class of unitons obtained from the complex Grassmanian embeddings of $Gr(K, N)$ into the unitary group. For μ pure imaginary, a complex K-plane $V \subset \mathbb{C}^N$ corresponds to a unitary transformation $i(\pi_V - \pi_{V^\perp})$, where π_V denotes the hermitian orthogonal projection onto V. The formula (3.1.5) with $\mu = i$ corresponds to $K = 1$ where $Gr(1, N) = \mathbb{CP}^{N-1}$.

Theorem 3.1.1 *The total energy of the n-uniton solution* (3.1.4) *with complex number* $\mu = me^{i\phi}$ *is quantised and equal to*

$$E_{(n)} = 4\pi\Big(\frac{1+m^2}{m}\Big)|\sin(\phi)|\;[\psi], \tag{3.1.7}$$

where for any fixed value of t the map $\psi : S^3 \longrightarrow U(N)$ is given by

$$\psi = \prod_{k=n}^{1}\Big(\mathbf{1} + \frac{\overline{\mu}-\mu}{\mu + \cot\left(\frac{\theta}{2}\right)}R_k\Big), \qquad \theta \in [0, 2\pi], \tag{3.1.8}$$

and

$$[\psi] = \frac{1}{24\pi^2}\int_{S^3} Tr\,((\psi^{-1}\mathrm{d}\psi)^3) \tag{3.1.9}$$

is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$.

The model (3.1.1) is $SO(1,1)$-invariant, and in section 3.3 it is shown that the Lorentz boosts correspond to rescaling μ by a real number. The rest frame corresponds to $|\mu| = 1$ where the y-component of the momentum vanishes. The $SO(1,1)$-invariant generalisation of (3.1.7) is given by theorem 3.3.3. Energies of soliton solutions more general than (3.1.4) are briefly discussed in section 3.4.

3.2 Extended solution and its homotopy

3.2.1 Lax pair and trivial scattering

The proof of theorem 3.1.1 relies on integrability of (3.1.1) and its Lax formulation, which we introduced in section 2.2.3. Recall from section 2.1.2 that the integrable chiral model is equivalent to the Yang-Mills-Higgs (YMH) system in $2+1$ dimensions with a gauge choice. That is,

$$D_\mu\Phi = \frac{1}{2}\varepsilon_{\mu\alpha\beta}F^{\alpha\beta} \tag{3.2.1}$$

becomes (3.1.1) in a gauge where $A_t = A_y$ and $A_x = -\Phi$ and

$$A_t = A_y = \frac{1}{2}J^{-1}(J_t + J_y), \qquad A_x = -\Phi = \frac{1}{2}J^{-1}J_x,$$

where J is a map from $\mathbb{R}^3$ to $U(N)$ and we use the same notation as in section 2.1.2. The system (3.2.1) is the integrability condition $[L_0, L_1] = 0$ for an overdetermined system of linear equations

$$L_0\Psi := (D_y + D_t - \lambda(D_x + \Phi))\Psi = 0, \qquad L_1\Psi := (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0, \tag{3.2.2}$$

where Ψ is a $GL(N,\mathbb{C})$-valued function of x^μ and a complex parameter $\lambda \in \mathbb{CP}^1$. Moreover, since we now work with gauge group $U(N)$, Ψ is required to satisfy the unitary reality condition

$$\Psi(x^\mu, \overline{\lambda})^* \Psi(x^\mu, \lambda) = \mathbf{1}.$$

The matrix fundamental solution Ψ is subject to gauge transformation $\Psi \longrightarrow b^{-1}\Psi$ corresponding to the gauge transformations of (A, Φ)

$$A \longrightarrow b^{-1}Ab + b^{-1}\mathrm{d}b, \qquad \Phi \longrightarrow b^{-1}\Phi b, \qquad b = b(x^\mu) \in U(N).$$

Given a solution Ψ to the linear system (3.2.2) one can construct a solution to (3.1.1) by

$$J(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0) \tag{3.2.3}$$

and all solutions to (3.1.1) arise from some Ψ's. The detailed exposition of this is presented, for example, in [27].

Let us restrict Ψ from $\mathbb{R}^{2,1} \times \mathbb{CP}^1$ to the space-like plane $t = 0$. We shall also restrict the spectral parameter λ to lie in the real equator $S^1 \subset \mathbb{CP}^1$ parametrised by θ:

$$\Psi(t, x, y, \lambda) \longrightarrow \psi(x, y, \theta) := \Psi\left(x, y, 0, -\cot\left(\frac{\theta}{2}\right)\right), \tag{3.2.4}$$

where now ψ is a map from $\mathbb{R}^2 \times S^1$ to $U(N)$ and we have made change of variable for real $\lambda = -\cot\left(\frac{\theta}{2}\right)$. Note that ψ automatically satisfies

$$(u^\mu D_\mu - \Phi)\psi = 0, \tag{3.2.5}$$

where the operator annihilating ψ is the spatial part of the Lax pair (3.2.2), given by

$$\frac{\lambda L_0 + L_1}{1+\lambda^2} = u^\mu D_\mu - \Phi, \qquad \text{where} \qquad \mathbf{u} = \left(0, \frac{1-\lambda^2}{1+\lambda^2}, \frac{2\lambda}{1+\lambda^2}\right) = (0, -\cos\theta, -\sin\theta).$$

We impose the 'trivial scattering' boundary condition [9, 45]

$$\psi(x, y, \theta) \longrightarrow \psi_0(\theta) \qquad \text{as} \qquad r \longrightarrow \infty, \tag{3.2.6}$$

where $\psi_0(\theta)$ is a $U(N)$-valued function on S^1. We shall now demonstrate that this enables us to extend ψ to a map from S^3 to $U(N)$.

First note that (3.2.6) implies the existence of the limit of ψ at spatial infinity for all values of θ, while the finite energy boundary condition (3.1.6) only implies the limit at $\theta = \pi$. Thus the condition (3.2.6) extends the domain of ψ to $S^2 \times S^1$. However, it turns out that (3.2.6) is also a sufficient condition for ψ to extend to the suspension

$SS^2 = S^3$ of S^2. This can be seen as follows. The domain $S^2 \times S^1$ can be considered as $S^2 \times [0,1]$ with $\{0\}$ and $\{1\}$ identified. Recall that a suspension SX of a manifold X is the quotient space [46]

$$SX = ([0,1] \times X)/((\{0\} \times X) \cup (\{1\} \times X)),$$

This definition is compatible with spheres in the sense that $SS^d = S^{d+1}$.

Now the only condition ψ needs to fulfil for the suspension is an equivalence relation between all the points in $S^2 \times \{0\}$, since such relation for $S^2 \times \{1\}$ will follow from the identification of $\{0\}$ and $\{1\}$. This equivalence can be achieved by choosing a gauge

$$\psi(x, y, 0) = \mathbf{1}. \tag{3.2.7}$$

Therefore ψ extends to a map from $SS^2 = S^3$ to $U(N)$ if it satisfies the trivial scattering boundary condition.

In addition, after fixing the gauge (3.2.7) there is still some residual freedom in ψ given by

$$\psi \longrightarrow \psi K, \tag{3.2.8}$$

where $K = K(\theta, x, y) \in U(N)$ is annihilated by $u^\mu \partial_\mu$. Setting $K = (\psi_0(\theta))^{-1}$ results in

$$\psi(\{\infty\}, \theta) = \mathbf{1}. \tag{3.2.9}$$

The choice (3.2.9) picks a base point $\{x_0 = \infty\} \in S^2$, and this implies that the trivial scattering condition is also sufficient for ψ to extend to the reduced suspension of S^2 given by

$$S_{red}S^2 = ([0,1] \times S^2)/((\{0\} \times S^2) \cup (\{1\} \times S^2) \cup ([0,1] \times \{x_0\})).$$

This is also homeomorphic to S^3. The idea of (reduced) suspension is illustrated in (Fig. 3.1).

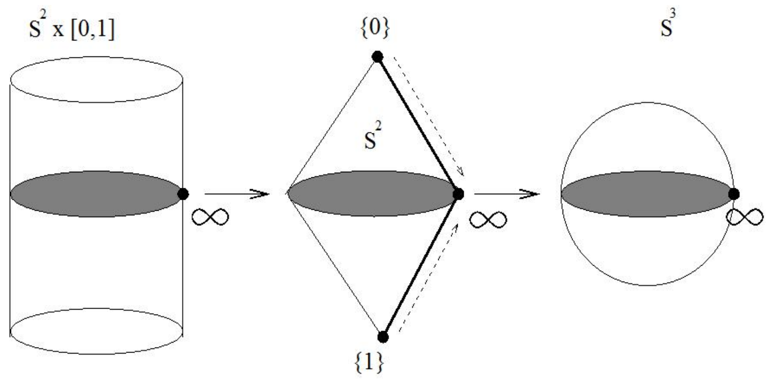
Now let us justify the term ‘trivial scattering’ in (3.2.6). Consider equation (3.2.5) and restrict it to a line $(x, y) = (x_0 - \sigma \cos\theta, y_0 - \sigma \sin\theta), \sigma \in \mathbb{R}$. Equation (3.2.5) becomes an ODE describing the propagation of

$$\psi = \psi(x_0 - \sigma\cos\theta, y_0 - \sigma\sin\theta, \theta)$$

along an oriented line through (x_0, y_0) in $\mathbb{R}^2$. We can choose a gauge such that

$$\lim_{\sigma \to -\infty} \psi = \mathbf{1},$$

Figure 3.1: Suspension and Reduced Suspension



and define the scattering matrix $S : TS^1 \to U(N)$ on the space of oriented lines in $\mathbb{R}^2$ as

$$S = \lim_{\sigma\to\infty} \psi. \tag{3.2.10}$$

The trivial scattering condition (3.2.6) then implies this matrix is trivial,

$$S = \mathbf{1}. \tag{3.2.11}$$

The boundary conditions (3.1.6) and (3.2.6) imply that for each value of θ the function ψ extends to a one-point compactification S^2 of $\mathbb{R}^2$. The straight lines on the plane are then replaced by the great circles, and in this context the trivial scattering condition implies that the differential operator $u^\mu D_\mu - \Phi$ has trivial monodromy along the compactification $S^1 = \mathbb{R} \cup \{\infty\}$ of a straight line parametrised by σ.

3.2.2 Topology of extended solution

In the last section we have seen that ψ can be regarded as a map from S^3 to $U(N)$. All such maps are characterised by their homotopy type [46]

$$[\psi] = \frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}((\psi^{-1}\mathrm{d}\psi)^3). \tag{3.2.12}$$

The element $[\psi]$ is an integer taking values in $\pi_3(U(N)) = \mathbb{Z}$ and is invariant under continuous deformations of ψ.

In the next section we will need the following result. Let g_1 and g_2 be maps from S^3 to $U(N)$ and let $g_1g_2 : S^3 \longrightarrow U(N)$ be given by

$$g_1g_2(x) := g_1(x)g_2(x), \qquad x \in S^3,$$

where the product on the RHS is the point-wise group multiplication. Then

$$[g_1g_2] = [g_1] + [g_2]. \tag{3.2.13}$$

This is because

$$\mathrm{Tr}[((g_1g_2)^{-1}\mathrm{d}(g_1g_2))^3] = \mathrm{Tr}[(g_1{}^{-1}\mathrm{d}g_1)^3 + (g_2^{-1}\mathrm{d}g_2)^3] + \mathrm{d}\beta,$$

where β is a two-form and so $\mathrm{d}\beta$ integrates to zero by Stokes' theorem. This was explicitly demonstrated by Skyrme [47] in the case of $SU(2)$.

Rather than exhibiting the exact form of β we shall use the following general argument. The higher homotopy groups $\pi_d(G)$ of a Lie group G are abelian, and the group multiplication in G induces the addition in the homotopy groups: if g_1 and g_2 are maps from S^d to G then the homotopy class of the map $g_1g_2 : S^d \longrightarrow G$ defined by the group multiplication is the sum of homotopy classes of g_1 and g_2. The proof of this is presented for example in [46] and essentially follows the proof that the fundamental group of a topological group is abelian. Now $\pi_3(G) = \mathbb{Z}$ for any compact simple Lie group. If $G = SU(2)$ this result just reproduces the calculation done by Skyrme as two continuous maps from S^3 to itself are homotopic if and only if they have the same topological degree. Theorem 3.1.1 holds for unitons with value in $G = U(N)$, where $[\psi]$ in (3.1.7) is the sum of homotopy classes which arise from integrals of elements of $H^3(G)$. To find out a homotopy class of a map ψ we can use the formula (3.2.12), where the integrand is a left-invariant three-form on the group manifold pulled back to S^3. This is because $\pi_3(G)$ is isomorphic to the integral homology group $H_3(G, \mathbb{Z})$, and the RHS of (3.2.12) coincides with the homology class of the cycle $\psi(S^3) \subset G$.

3.3 Time-dependent unitons and energy quantisation

A class of extended solutions which satisfy the trivial scattering condition (3.2.6) give rise to the n-uniton solutions defined in (3.1.4). These extended solutions factorise into the so called n-uniton factors [4]

$$\Psi = G_nG_{n-1}\ \ldots\ G_1, \quad \text{where} \quad G_k = \Big(\mathbf{1} - \frac{\overline{\mu} - \mu}{\lambda - \mu}R_k\Big) \in GL(N, \mathbb{C}), \quad R_k = \frac{q_k^* \otimes q_k}{||q_k||^2}, \tag{3.3.1}$$

where $q_k = q_k(x, y, t) \in \mathbb{C}^N$, $k = 1, \ldots, n$, and μ is a non-real constant. The terminology here is rather confusing, as the maxima of the energy density of the corresponding soliton solutions of (3.1.1) do physically scatter. The exact form of q_ks is determined from (3.2.2) by demanding that the expressions

$$(\partial_x\Psi - \lambda(\partial_t - \partial_y)\Psi)\Psi^{-1}, \qquad \text{and} \qquad ((\partial_t + \partial_y)\Psi - \lambda\partial_x\Psi)\Psi^{-1} \tag{3.3.2}$$

are independent of λ. In practice one determines the q_ks by a limiting procedure from solutions of a Riemann problem with simple poles [3].

The restricted map ψ (3.2.4) corresponding to (3.3.1) is given by

$$\psi = g_n g_{n-1} \ \ldots \ g_1, \qquad \text{where} \qquad g_k = \mathbf{1} + \frac{\bar{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_k \in U(N), \tag{3.3.3}$$

where $\lambda = -\cot\left(\frac{\theta}{2}\right) \in S^1 \subset \mathbb{CP}^1$ as before and all the maps are restricted to the $t = 0$ plane. Each element g_k has the limit at spatial infinity for all values of θ.

$$g_k(x, y, \theta) \longrightarrow g_{0k}(\theta) = \mathbf{1} + \frac{\overline{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_{0k} \qquad \text{as} \qquad x^2 + y^2 \longrightarrow \infty.$$

The existence of the limit of R_k at spatial infinity $R_{0k} = \lim_{r\to\infty} R_k(x, y) = const$ is guaranteed by the finite energy condition (3.1.6). Hence ψ (3.3.3) satisfies the trivial scattering condition (3.2.6) and extends to a map from S^3 to $U(N)$. The scattering matrix (3.2.10) is $S = \mathbf{1}$.

Note, however, that the g_ks and ψ in (3.3.3) only extend to the ordinary suspension of S^2. One needs to perform the transformation (3.2.8) with $K = \prod_{k=1}^n g_{0k}^{-1}$ for ψ to extend to the reduced suspension of S^2. We shall use ψ as in (3.3.3) because (3.2.13) implies that the transformation (3.2.8) does not contribute to the degree and $[K(\theta)\psi] = [\psi]$.

Proposition 3.3.1 *The third homotopy class of ψ in* (3.3.3) *is given by*

$$[\psi] = \pm\frac{i}{2\pi}\int_{\mathbb{R}^2}\sum_{k=1}^n Tr(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\mathrm{d}y \qquad \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi, \end{cases} \tag{3.3.4}$$

where $\mu = me^{i\phi}$.

Proof. The recursive application of (3.2.13) implies that

$$[\psi] = \sum_{k=1}^n [g_k].$$

Using (3.2.12), with $z = x + iy$

$$\begin{aligned} [g_k] &= \frac{1}{8\pi^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}(g_k^{-1}\partial_\theta g_k\, [g_k^{-1}\partial_z g_k, g_k^{-1}\partial_{\overline{z}} g_k])\; \mathrm{d}\theta\wedge \mathrm{d}z\wedge \mathrm{d}\overline{z} \\ &= \frac{1}{16\pi^2}\Theta(\mu)\int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_z R_k, \partial_{\overline{z}} R_k])\; \mathrm{d}z\wedge \mathrm{d}\overline{z}, \end{aligned}$$

where

$$\begin{aligned} \Theta(\mu) &= \int_0^{2\pi} \frac{(\bar{\mu}-\mu)^3\sin^2\left(\frac{\theta}{2}\right)}{\left(|\mu|^2+(1-|\mu|^2)\cos^2\left(\frac{\theta}{2}\right)+(\mu+\bar{\mu})\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2}\mathrm{d}\theta \\ &= \pm 8\pi i \quad \begin{cases} 0<\phi<\pi \\ \pi<\phi<2\pi. \end{cases} \end{aligned}$$

Hence, changing back to the (x, y) coordinates we obtain

$$[g_k] = \pm\frac{i}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\mathrm{d}y \qquad \begin{cases} 0<\phi<\pi \\ \pi<\phi<2\pi. \end{cases} \tag{3.3.5}$$

Therefore, the third homotopy class of ψ is given by (3.3.4).

□

The proof of theorem (3.1.1) makes use of the above proposition and a recursive procedure of adding unitons to a given solution which we shall now explain. Let Ψ be an extended solution of the Lax pair (3.2.2) corresponding to a solution J satisfying (3.1.1). Set

$$\hat{\Psi} = G\Psi = \Big(\mathbf{1} - \frac{\overline{\mu}-\mu}{\lambda-\mu}R\Big)\Psi, \qquad \hat{J} = \hat{\Psi}^{-1}|_{\lambda=0} = JM, \tag{3.3.6}$$

where M is of the form (3.1.5), up to a constant phase factor which is irrelevant. The matrix $\hat{\Psi}$ will be an extended solution if the expressions in (3.3.2) with Ψ replaced by $\hat{\Psi}$ are independent of λ. This leads to the Bäcklund relations [6, 7]. These are first order PDEs for M, which can be regarded as a generalisation of Uhlenbeck's method of adding unitons for harmonic maps [48]. In terms of the hermitian projection R, these PDEs are

$$R(R_t - J^{-1}J_t(\mathbf{1}-R)) = B \qquad \text{and} \qquad RR_t = C, \tag{3.3.7}$$

where

$$\begin{aligned} B &= (\mu R_x - R_y + RJ^{-1}J_y)(\mathbf{1}-R) \\ C &= \frac{1}{\mu}((\mu R_y + R_x - RJ^{-1}J_x)(\mathbf{1}-R)). \end{aligned}$$

Proof of Theorem 3.1.1 We first consider a solution of the form $\hat{J} = JM$, where J is an arbitrary solution of (3.1.1). Noting that M is unitary and writing it in terms of R leads to the difference between the energy densities (3.1.3) of $\hat{J}$ and J given by

$$\Delta\mathcal{E} \equiv \hat{\mathcal{E}} - \mathcal{E} = \sum_a \mathrm{Tr}\,(\kappa\bar{\kappa}R_aR_aR + \kappa(\mathbf{1} - \bar{\kappa}R)J^{-1}J_aR_a), \tag{3.3.8}$$

where a stands for (t,x,y), $\hat{\mathcal{E}}$ and $\mathcal{E}$ are the energy densities of $\hat{J}$ and J respectively and $\kappa = \left(1 - \frac{\mu}{\bar{\mu}}\right)$. Multiplying the relations (3.3.7) and their Hermitian conjugates with each other yields the following identities

$$\begin{aligned}
\mathrm{Tr}(R_tR_tR) &= \mathrm{Tr}(CC^*) \\
\mathrm{Tr}(J^{-1}J_tR_t) &= \mathrm{Tr}(CB^* - BC^*) \\
\mathrm{Tr}(RJ^{-1}J_tR_t) &= \mathrm{Tr}((C-B)C^*).
\end{aligned} \tag{3.3.9}$$

The terms in (3.3.8) which involve the time derivatives are of the form R_tR_tR, $J^{-1}J_tR_t$ and $RJ^{-1}J_tR_t$, which, by (3.3.9), can be written in terms of the spatial derivatives of J and R only. Thus by direct substitution and some rearrangements (3.3.8) becomes

$$\Delta\mathcal{E} = -\frac{\kappa}{\mu}\mathrm{Tr}\Big((1+|\mu|^2)R[R_x,R_y] + \mathcal{T}\Big),$$

where $\mathcal{T} = \partial_x(RJ^{-1}J_y) - \partial_y(RJ^{-1}J_x)$ gives no contribution to the difference in the energy functionals of $\hat{J}$ and J. This is because

$$\begin{aligned}
&\mathrm{Tr}\int_{\mathbb{R}^2}\mathcal{T}\mathrm{d}x\wedge\mathrm{d}y = \lim_{r\to\infty}\int_{D_r}\mathrm{d}(\mathrm{Tr}(RJ^{-1}\mathrm{d}J)) \\
=\;& \lim_{r\to\infty}\oint_{C_r}\mathrm{Tr}(RJ^{-1}\mathrm{d}J) = \mathrm{Tr}\Big(\lim_{r\to\infty}\oint_{C_r}(JR)^*\mathrm{d}J\Big) \\
\leq\;& \lim_{r\to\infty}\bigg(\mathrm{Tr}\Big(\frac{((JR)_0)^*}{r}(J_1(\varphi=2\pi) - J_1(\varphi=0))\Big) + 2\pi r\Big\{\frac{|c_2|}{r^2} + \frac{|c_3|}{r^3} + \ldots\Big\}\bigg) = 0,
\end{aligned}$$

by Stokes's theorem, where C_r denotes the circle enclosing the disc D_r of radius r, φ is a coordinate on C_r, and $|c_i|$ is the bound of $\mathrm{Tr}((JR)_i^*\partial_\varphi J)$, $i = 1,2,\ldots$. We have used the boundary condition

$$\lim_{r\to\infty} JR = (JR)_0 + (JR)_1(\varphi)r^{-1} + O(r^{-2}), \tag{3.3.10}$$

which follows from (3.1.6) for $\hat{J} = JM$, and the fact that integrands are continuous on the circle and hence bounded. Since $(JR)_0$ is a constant matrix, the first term in the series is a total derivative.

So far we have only used the assumption that J is a solution of (3.1.1), but not that it is a uniton solution defined by (3.1.4). Therefore, we have a more general result for the total energy of a solution of the form $\hat{J} = JM$, where J is an arbitrary solution to the Ward model. Let $\hat{E}$ and E be the total energies of $\hat{J}$ and J respectively, then

$$\hat{E} = E + \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \int_{\mathbb{R}^2} \mathrm{Tr}(R[R_x, R_y])\mathrm{d}x\mathrm{d}y. \tag{3.3.11}$$

From this, the explicit expression for the total energy of an n-uniton solution (3.1.4) follows. First, consider a 1-uniton solution $J_{(1)} = M_1$. It can be written as $J_{(1)} = J_{(0)}M_1$, where the constant matrix $J_{(0)}$, which satisfies (3.1.1) trivially, is chosen to be the identity matrix. Then it follows from (3.3.11) that the total energy of a 1-uniton solution is given by

$$E_{(1)} = \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \int_{\mathbb{R}^2} \mathrm{Tr}(R_1[\partial_x R_1, \partial_y R_1])\mathrm{d}x\mathrm{d}y. \tag{3.3.12}$$

Therefore, using (3.3.11) we show by induction that the total energy of an n-uniton solution (3.1.4) is given by

$$\begin{aligned} E_{(n)} &= \frac{(\mu - \bar{\mu})(1 + |\mu|^2)}{|\mu|^2} \sum_{k=1}^{n} \int_{\mathbb{R}^2} \mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\mathrm{d}y \qquad (3.3.13) \\ &= \pm 4\pi \left(\frac{1 + m^2}{m}\right) \sin(\phi)\ [\psi] \quad \begin{cases} 0 < \phi < \pi \\ \pi < \phi < 2\pi, \end{cases} \end{aligned}$$

where $\mu = me^{i\phi}$ and we have used (3.3.4).

□

We remark that the formula (3.3.5) reveals another topological interpretation of the energy quantisation which is useful in practical calculations. Consider the group element (3.3.3) with the index k dropped. The Grassmanian projector R corresponds to a smooth map from the compactified space to the projective space $q : S^2 \longrightarrow \mathbb{CP}^{N-1}$. The homotopy group $\pi_2(\mathbb{CP}^{N-1}) = \mathbb{Z}$ is non–trivial and the degree of q is obtained by evaluating the homology class on a standard generator for $H_2(\mathbb{CP}^{N-1})$ represented in a map $q = (1, f_1, ..., f_{N-1})$ by the Kähler form

$$\Omega = -4i\partial\bar{\partial} \ln(1 + \sum_{j=1}^{N-1} |f_j|^2).$$

This evaluation is just the integration and thus

$$[q] = \frac{i}{8\pi} \int_{\mathbb{R}^2} q^*(\Omega).$$

Evaluating the integrand we verify that

$$i\mathrm{Tr}(R[R_x, R_y]) = \frac{1}{4}q^*(\Omega).$$

We conclude that the energy is proportional to the sum of the topological degrees of Grassmanian projectors involved in the definition of unitons.

In the remaining part of this section we shall prove a Lorentz invariant generalisation of theorem 3.1.1. Let us begin by looking at the quantisation of the momentum. Following [3], we have chosen the conserved energy functional for a solution of (3.1.1) to be that obtained from the energy-momentum tensor of the associated standard chiral model. However, only the energy and the y-component of momentum are conserved, whereas the x-component of momentum is not. The conserved y-momentum is given by

$$P = \int_{\mathbb{R}^2} \mathcal{P}\mathrm{d}x\mathrm{d}y, \tag{3.3.14}$$

where the momentum density is

$$\mathcal{P} = -\mathrm{Tr}(J^{-1}J_tJ^{-1}J_y). \tag{3.3.15}$$

It turns out that this is also quantised and proportional to the third homotopy class of the restricted extended solution.

Proposition 3.3.2 *The y-momentum of the n-uniton solution* (3.1.4) *is given by*

$$P_{(n)} = -4\pi\Big(\frac{1-m^2}{m}\Big)|\sin(\phi)|\,[\psi]. \tag{3.3.16}$$

Thus, unless $[\psi] = 0$, $P = 0$ if and only if $m = 1$.

Proof. We first consider $\hat{J} = JM$ as in the the proof of theorem 3.1.1. The difference between the y-momentum densities (3.3.15) of $\hat{J}$ and J is given by

$$\Delta\mathcal{P} \equiv \hat{\mathcal{P}} - \mathcal{P} = \kappa\mathrm{Tr}\Big((\mathbf{1} - \bar{\kappa}R)(J^{-1}J_tR_y + J^{-1}J_yR_t) + \bar{\kappa}(R_yR_t)\Big), \tag{3.3.17}$$

where $\kappa = \Big(1 - \frac{\mu}{\bar{\mu}}\Big)$. Then the substitution

$$\begin{aligned} \mathrm{Tr}(J^{-1}J_tR_yR) &= \mathrm{Tr}\Big((C-B)R_y\Big) \\ \mathrm{Tr}(J^{-1}J_tRR_y) &= \mathrm{Tr}\Big((B^*-C^*)R_y\Big) \end{aligned}$$

from the Bäcklund relations (3.3.7) gives

$$\Delta\mathcal{P} = \frac{\kappa}{\mu}\mathrm{Tr}\Big((1-|\mu|^2)R[R_x, R_y] + \mathcal{T}\Big).$$

The term $\mathcal{T} = \partial_x(RJ^{-1}J_y) - \partial_y(RJ^{-1}J_x)$ gives no contribution to the difference in the y-momenta of $\hat{J}$ and J, in the same way as to the difference in the energies. Thus, we have an expression for the y-momentum of a solution $\hat{J}$ of the form $\hat{J} = JM$ where J is an arbitrary solution to the Ward model as

$$\hat{P} = P - \frac{(\mu-\bar{\mu})(1-|\mu|^2)}{|\mu|^2}\int_{\mathbb{R}^2}\mathrm{Tr}(R[R_x, R_y])\mathrm{d}x\mathrm{d}y, \tag{3.3.18}$$

where $\hat{P}$ and P are y-momenta of $\hat{J}$ and J respectively.

We then proceed by induction to obtain the expression for the y-momentum of an n-uniton solution (3.1.4), in the same way as for the total energy. This gives

$$\begin{aligned} P_{(n)} &= -\frac{(\mu-\bar{\mu})(1-|\mu|^2)}{|\mu|^2}\sum_{k=1}^{n}\int_{\mathbb{R}^2}\mathrm{Tr}(R_k[\partial_x R_k, \partial_y R_k])\mathrm{d}x\mathrm{d}y \qquad (3.3.19)\\ &= \mp 4\pi\Big(\frac{1-m^2}{m}\Big)\sin(\phi)\ [\psi] \quad \begin{cases} 0<\phi<\pi \\ \pi<\phi<2\pi. \end{cases} \end{aligned}$$

$\square$

Let us now exploit the $SO(1,1)$ symmetry of (3.1.1) and combine theorem 3.1.1 and proposition 3.3.2 into a Lorentz invariant form.

Theorem 3.3.3 *For an n-uniton solution the $SO(1,1)$ invariant relation*

$$E_{(n)}^2 - P_{(n)}^2 = 64\pi^2\sin^2(\phi)[\psi]^2 \tag{3.3.20}$$

holds.

Proof. Since equation (3.1.1) is invariant under $SO(1,1)$, we can generate new solutions from a given one by boosts in the $y-t$ plane. In the coordinates

$$x, \quad u = \frac{1}{2}(t+y), \quad v = \frac{1}{2}(t-y)$$

the boosts are given by $x \to x, \quad u \to su, \quad v \to s^{-1}v, \quad s \in \mathbb{R}^*$. We shall show that a boost of an n-uniton solution with a pole at $\lambda = \mu$ in the extended solution gives rise to another n-uniton solution with the pole $\mu' = s\mu$.

Consider the Bäcklund relations (3.3.7) expressed in the (x, u, v) coordinates,

$$\begin{aligned} (\mu R_x - R_u + RJ^{-1}J_u)(\mathbf{1} - R) &= 0 \\ (\mu R_v - R_x + RJ^{-1}J_x)(\mathbf{1} - R) &= 0. \end{aligned} \tag{3.3.21}$$

Let J be an arbitrary solution of (3.1.1) and $R(x, u, v)$ be a Hermitian projector satisfying (3.3.21). Under the boost to another solution $J \longrightarrow J'$ we have $R \rightarrow R' = R(x, su, s^{-1}v)$. Changing the coordinates, one sees that R' will satisfy (3.3.21) with μ and J replaced by μ' and J' if $\mu' = s\mu$. That is, each restricted uniton factor transforms as

$$g_k = \mathbf{1} + \frac{\bar{\mu} - \mu}{\mu + \cot\left(\frac{\theta}{2}\right)} R_k(x, u, v, \mu) \longrightarrow g'_k = \mathbf{1} + \frac{s\bar{\mu} - s\mu}{s\mu + \cot\left(\frac{\theta}{2}\right)} R_k(x, su, s^{-1}v, \mu).$$

Since boost is a continuous transformation it does not change the homotopy types and we have

$$[\psi(x, u, v)] = [\psi(x, su, s^{-1}v)].$$

Hence, under the transformation $E_{(n)}$ and $P_{(n)}$ only change by the explicit factors of μ in (3.1.7) and (3.3.16) respectively. The boosts rescale μ by $m \longrightarrow sm$, keeping the phase ϕ fixed. This leads to the $SO(1,1)$ invariance of $E^2_{(n)} - P^2_{(n)}$. The formula (3.3.20) follows directly from (3.1.7) and (3.3.16).

□

Examples. Consider the $SU(2)$ case where the third homotopy class is equal to the topological degree and set $\mu = i$. The uniton factors are of the form

$$M_k = \frac{i}{1 + |f_k|^2} \begin{pmatrix} |f_k|^2 - 1 & -2f_k \\ -2\overline{f_k} & 1 - |f_k|^2 \end{pmatrix}.$$

$\mathbf{n} = \mathbf{1}$. In the 1-uniton case $\partial_t M_1 = 0$, and M_1 is given by (3.1.5) with $f_1 = f_1(z)$ a rational function of some fixed degree Q. The energy density is

$$\mathcal{E}_1 = \frac{8|f'_1|^2}{(1 + |f_1|^2)^2} = -i\mathrm{Tr}\left(M_1[\partial_z M_1, \partial_{\bar{z}} M_1]\right)$$

and $E = 8\pi \deg(g_1)$ in agreement with (3.1.7). In this case g_1 is a suspension of a rational map $f_1 : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ and $\deg(g_1) = \deg(f_1)$ is a simple illustration of the Freundenthal theorem which says that a suspension of maps of d-spheres induces an isomorphism of the homotopy groups.

$\mathbf{n=2}$. In the 2-uniton case M_1 and M_2 are given by (3.1.5) with $\mu = i$ and

$$q_1 = (1, f), \qquad q_2 = (1 + |f|^2)(1, f) - 2i(tf' + h)(\overline{f}, -1),$$

where f and h are rational functions of z [4]. Define $k = 2(tf' + h)$. The total energy density is

$$\mathcal{E} = \frac{8|(1 + |f|^2)k' - 2k\overline{f}f'|^2 + 16|kf'|^2 + 16(1 + |f|^2)^2|f'|^2}{(|k|^2 + (1 + |f|^2)^2)^2} \tag{3.3.22}$$

and

$$E = \int_{\mathbb{R}_2} \mathcal{E}\, \mathrm{d}x\mathrm{d}y = 8\pi(\deg(g_1) + \deg(g_2))$$

for all t. The quantisation of energy in this case has first been observed in [8], where it was shown that $E = 8\pi Q$ where generically $Q = 2\deg f + \deg h$. However, if both f and h are polynomials, $Q = \max\,(2\deg f, \deg h)$. Our formula (3.1.7) is valid for all pairs (f, h).

3.4 Discussion

We have established the relation between the total energy of time-dependent solitons (3.1.4) and homotopy classes of associated extended solutions. To the best of our knowledge this is the first example of a topological mechanism ensuring the classical energy quantisation of moving solitons.

The n-uniton solutions (3.1.4) form a subclass of all finite energy solitons which satisfy the trivial scattering boundary condition (3.2.6). Dai and Terng [7] have demonstrated that the extended solution corresponding to a general trivial scattering soliton has poles at non-real points $\mu_1, ..., \mu_r$ with multiplicities $n_1, ..., n_r$, and is a product of simple elements $G_{k,\alpha}$ $\alpha = 1, ..., r$ of the form (3.3.1). Our case (3.3.1) corresponds to $r = 1$, but the method used in the proof of theorem 3.1.1 applies to the general case as one can choose a different μ at each iteration of the Bäcklund transformations (3.3.7). Formulae (3.3.5) and (3.3.11) lead to a general form of the total energy of trivial scattering solitons

$$E = 4\pi \sum_{\alpha=1}^{r} \sum_{k=1}^{n_r} \frac{1 + m_\alpha^2}{m_\alpha} |\sin\phi_\alpha| [g_{k,\alpha}], \qquad \mu_\alpha = m_\alpha e^{i\phi_\alpha}, \tag{3.4.1}$$

where

$$g_{k,\alpha} = \mathbf{1} + \frac{\bar{\mu}_\alpha - \mu_\alpha}{\mu_\alpha + \cot\left(\frac{\theta}{2}\right)} R_{k,\alpha} \in U(N)$$

and $R_{k,\alpha}$ are hermitian projections whose form is determined by the Bäcklund relations. This agrees with the result of Lechtenfeld and Popov [42]. The formula (3.4.1) is not directly linked to the homotopy type of the extended solution and the $SO(1,1)$ invariance can not be easily incorporated. This is why we have focused on the special case (3.1.4).

In [49] the $SU(2)$ integrable chiral model (3.1.1) has been analysed in the moduli space approximation, when the time dependent slowly moving solitons correspond to curves in the moduli space of static solitons which are geodesic with respect to the natural metric

$$h(\dot{\gamma},\dot{\gamma}) = \frac{1}{2}\dot{\gamma}^p\dot{\gamma}^q\int_{\mathbb{R}^2}\frac{|\partial_p f\overline{\partial_q f}|}{(1+|f|^2)^2}\mathrm{d}x\mathrm{d}y$$

on the space of rational maps. Here $f = f(z,\gamma)$ is a rational meromorphic function of $z = x + iy$ which depends on real parameters (positions of zeroes and poles) γ^p and $\partial_p f = \partial f/\partial\gamma^p$.

The kinetic energy of these approximate solitons is small, and their total energy is close (in the units of 8π) to the degree of the associated rational map. Theorem (3.1.1) gives a class of exact solutions with quantised total energy, and one may expect that the approximate solitons of [49] arise from the time-dependent unitons by some limiting procedure.

Chapter 4

Compactified Twistor Fibration for Yang-Mills-Higgs System

A compactification of the minitwistor space was introduced by Ward in [44], in order to discuss a particular class of solutions to the Yang-Mills-Higgs system in $2+1$ dimensions. In this chapter we give a detailed exposition of the correspondence, which was described briefly in [9], between the compactified spacetime and the compactified twistor space, starting from the identification between $T\mathbb{P}^1$ and a cone in a $\mathbb{CP}^3$ minus the vertex. Then we go on to discuss the correspondence space of an associated double fibration. A double fibration picture was discussed in [50], where one of the target spaces is the cone and the correspondence space was taken to be a singular variety in the direct product of the compactified spacetime and the $\mathbb{CP}^3$ where the cone lives. Here we explore a fibration from a blow up of the variety, taken as the correspondence space, to the compactified twistor space $\overline{T\mathbb{P}^1}$. Moreover, we define a restricted correspondence space which fibres over an $\mathbb{RP}^2$ regarded as a compactification of a spacelike surface $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$ and show that it admits a surjective map to $\overline{T\mathbb{P}^1}$.

4.1 Compactification: spacetime and twistor space

The minitwistor space discussed in section 2.2.3 arises in a number of contexts. The description in section 2.2.3 is taken from [27, 29], where it appears in the context of the Yang-Mills-Higgs system on $\mathbb{R}^{2,1}$. Another construction of the minitwistor space is given by Hitchin [51] in the Euclidean setting. Here we continue from chapter 3 and consider the Yang-Mills-Higgs (YMH) system in $2+1$ dimensions, which gives the integrable chiral model under a gauge choice[1]. In [9] Ward discussed a particular class of the YMH

[1]The twistor correspondence for the Ward model follows readily from the correspondence for the Yang-Mills-Higgs system. However, since the field J is obtained from (A, Φ) by integration, it contains more information than the YMH fields. This additional data is a holomorphic framing of the bundle E over $T\mathbb{P}^1$ along the fibres of $T\mathbb{P}^1$ over two points in the spectral parameter [44].

fields (and the associated chiral fields J) whose corresponding vector bundles extend to the compactified twistor space $\overline{T\mathbb{P}^1}$, which was defined in [44] to be the fibrewise compactification of $T\mathbb{P}^1$ where each $\mathbb{C}$-fibre becomes a copy of $\mathbb{CP}^1$. One can also think of $\overline{T\mathbb{P}^1}$ as a cone in $\mathbb{CP}^3$ with blown-up vertex. The finite-dimensional class of the YMH fields are those which satisfy the trivial scattering condition[2] [9]. A significant feature of this class of solutions is that they admit a well defined topological degree associated with their extended solutions, as we discuss in chapter 3. On the other hand, the holomorphic vector bundles, now over the compactified twistor space, have Chern numbers as topological invariants. The fact that the bundle is trivial on the $\mathbb{CP}^1$ lines corresponding to spacetime points implies that the first Chern number vanishes. The next non-trivial invariant is the second Chern number. The identification between the third homotopy class of the extended solutions of Ward solitons and the second Chern number of the corresponding vector bundles was stated in [9] and described in a particular case of gauge group $SU(2)$, although the proof is not given. This identification is supported by the work of Anand in [50], which gives the relation between the second Chern number and the energy of static $U(N)$-unitons, together with theorem 3.1.1 in this thesis.

In the next section we shall give a detailed explanation, in the holomorphic setting, of the correspondence between the compactified spacetime $\overline{M} \simeq \mathbb{CP}^3$ and $\overline{T\mathbb{P}^1}$, which was described briefly by Ward in [9]. Starting from the identification between the minitwistor space $T\mathbb{P}^1$ and a cone minus the vertex in a complex projective 3-space, we will show how the natural compactification of $T\mathbb{P}^1$, obtained by adding a $\mathbb{CP}^1$ where the fibre coordinate goes to infinity, is equivalent to the cone with blown-up vertex. Then, we will define a correspondence space, from which we can define a natural restricted correspondence space which fibres over an $\mathbb{RP}^2$-compactification of a spacelike $\mathbb{R}^2$-plane in $\mathbb{R}^{2,1}$. We are interested in a spacelike $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$ as it is the domain of the restricted extended solutions ψ featured in chapter 3. The study of the compactified twistor fibrations should be regarded as a starting point towards a proof of the identification between the third homotopy class of the extended solutions of the YMH fields and the second Chern number of the corresponding vector bundles.

Our main result is that the defined restricted correspondence space admits a surjective map to $\overline{T\mathbb{P}^1}$. This is partly due to the following proposition, which we shall prove

[2] The compactified twistor space was already used in [44] to describe static 1-uniton solutions and the time-dependent non-scattering solitons of the integrable chiral model, both of which satisfy the trivial scattering condition.

in section 4.3.

Proposition 4.1.1 *Let $(P^0, P^1, P^2, P^3) \in \mathbb{C}^4 - \{0\}$ be homogeneous coordinates of a compactified complexified spacetime $\overline{M} \simeq \mathbb{CP}^3$, and let $\tau_{\mathbb{R}}$ denote an $\mathbb{RP}^2 \subset \overline{M}$ defined by $(P^0, P^1, P^2, P^3) \in \mathbb{R}^4 - \{0\}$ and setting $P^3 = 0$. Define a null plane in $\overline{M}$ to be a $\mathbb{CP}^2$ given by*

$$Z_0 P^0 + Z_1 P^1 + Z_2 P^2 - Z_3 P^3 = 0,$$

where $(Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 - \{0\}$ satisfy

$$(Z_1)^2 + (Z_2)^2 - (Z_3)^2 = 0.$$

Then, every null plane in $\overline{M}$ intersects $\tau_{\mathbb{R}}$.

4.2 Correspondence in the compact case

4.2.1 Minitwistor space as $T\mathbb{P}^1$ and a cone in $\mathbb{CP}^3$

In [9, 44] Ward gave us another picture of the (non-compact) minitwistor space as a cone in a $\mathbb{CP}^3$ minus its vertex, complementing the standard one of the tangent bundle $T\mathbb{P}^1$ of the Riemann sphere. The cone picture proves to be convenient in the study of the compactified double fibrations. Therefore we shall start with the non-compact case and explain how the tangent bundle $T\mathbb{P}^1$ of a $\mathbb{CP}^1$ can be identified with a cone in a complex projective 3-space, without the vertex.

Let us first consider the cone. Let $Z_\alpha = (Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 - \{0\}$ be homogeneous coordinates of a complex projective 3-space, which we will denote by $\mathbb{CP}^{3*}$. Then a cone $\mathcal{C}$ in $\mathbb{CP}^{3*}$ is given by

$$(Z_1)^2 + (Z_2)^2 - (Z_3)^2 = 0. \tag{4.2.1}$$

Note our convention of one minus sign. The vertex is the point $\mathbf{z_0} = [Z_0, 0, 0, 0],\ Z_0 \neq 0$. For any point on the cone except the vertex, we can parametrise $Z_i := (Z_1, Z_2, Z_3) \neq (0, 0, 0)$ in the same way as a null vector in $\mathbb{C}^3$ as discussed in section 2.2.3. Let

$$Z^{AB} = \begin{pmatrix} \frac{Z_3+Z_2}{2} & \frac{Z_1}{2} \\ \frac{Z_1}{2} & \frac{Z_3-Z_2}{2} \end{pmatrix}.$$

Then equation (4.2.1), which is the same as $-4\,\det(Z^{AB}) = 0$, implies that rank $(Z^{AB}) = 1$. Hence

$$Z^{AB} = \pi^A \pi^B = \begin{pmatrix} (\pi^0)^2 & \pi^0\pi^1 \\ \pi^0\pi^1 & (\pi^1)^2 \end{pmatrix},$$

where $\pi^A \in \mathbb{C}^2 - \{0\}$. In other words, we can parametrise solutions of (4.2.1) in $\mathbb{C}^4 - \{0\}$ with $Z_i \neq (0,0,0)$ by

$$Z_\alpha = (\hat{\omega},\ -2\pi_0\pi_1,\ {\pi_1}^2 - {\pi_0}^2,\ {\pi_0}^2 + {\pi_1}^2), \tag{4.2.2}$$

where $\hat{\omega} \in \mathbb{C}$ is arbitrary, $\pi_A = \pi^B \varepsilon_{BA}$ and ε_{BA} is the alternating symbol. Dividing through by π_1^2 and π_0^2 in the patches where $\pi_1 \neq 0$ and $\pi_0 \neq 0$ respectively, we have that a point $[Z_\alpha] \in \mathcal{C} - \mathbf{z_0}$ is given by

$$[\omega, -2\lambda, 1-\lambda^2, \lambda^2+1] \qquad \text{and} \qquad [\tilde{\omega}, -2\tilde{\lambda}, \tilde{\lambda}^2 - 1, 1+\tilde{\lambda}^2] \tag{4.2.3}$$

in the two patches, where $\omega := \frac{\hat{\omega}}{\pi_1^2}$, $\lambda := \frac{\pi_0}{\pi_1}$ and $\tilde{\omega} := \frac{\hat{\omega}}{\pi_0^2}$, $\tilde{\lambda} := \frac{\pi_1}{\pi_0}$. In the overlap, the inhomogeneous coordinates are related by $\tilde{\lambda} = \frac{1}{\lambda}$ and $\tilde{\omega} = \frac{\omega}{\lambda^2}$ and this identifies $\mathcal{C} - \{\mathbf{z_0}\}$ with $T\mathbb{P}^1$. Formally, one sees that the local maps

$$(\omega, \lambda) \mapsto [\omega, -2\lambda, 1-\lambda^2, 1+\lambda^2] \quad \text{and} \quad (\tilde{\omega}, \tilde{\lambda}) \mapsto [\tilde{\omega}, -2\tilde{\lambda}, \tilde{\lambda}^2 - 1, 1+\tilde{\lambda}^2] \tag{4.2.4}$$

from $T\mathbb{P}^1$ to $\mathcal{C} - \{\mathbf{z_0}\}$ are $1:1$ and onto, and well defined in the overlap. In fact, (4.2.4) gives a biholomorphism between $T\mathbb{P}^1$ and $\mathcal{C} - \{\mathbf{z_0}\} \subset \mathbb{CP}^{3*}$.

Let us now describe the correspondence between the complexified spacetime $M \simeq \mathbb{C}^3$ and the minitwistor space in the cone picture. It is convenient to embed M in a $\mathbb{CP}^3$. Let $P^\alpha = (P^0, P^1, P^2, P^3) \in \mathbb{C}^4 - \{0\}$ be homogeneous coordinates on $\mathbb{CP}^3$ and take the open set $P^0 \neq 0$ to be our spacetime $M \simeq \mathbb{C}^3$. A plane in $\mathbb{CP}^3$ is defined to be the projection of a 3-dimensional subspace of the associated $\mathbb{C}^4$, given by

$$Z_0P^0 + Z_1P^1 + Z_2P^2 - Z_3P^3 = 0. \tag{4.2.5}$$

Note again our convention of one minus sign. Each plane is thus labelled by $Z_\alpha \in \mathbb{C}^4 - \{0\}$ up to a constant multiplication. That is, the space of planes in $\mathbb{CP}^3$ is another complex projective 3-space, which we called $\mathbb{CP}^{3*}$ earlier. Then in this setting, 2-planes in $M \simeq \mathbb{C}^3$ are the $\mathbb{C}^2$-intersections of planes in $\mathbb{CP}^3$ with M.

The picture suggests a natural compactification of the spacetime M to $\overline{M} \simeq \mathbb{CP}^3$. One can think of $\overline{M}$ as $M + \mathbb{CP}^2$, where $\mathbb{CP}^2$ is the complement region $P^0 = 0$. Let

$$x = \frac{P_1}{P_0}, \quad y = \frac{P_2}{P_0}, \quad t = \frac{P_3}{P_0} \tag{4.2.6}$$

be coordinates on M. Then, one can interpret the complement $\mathbb{CP}^2$ as the infinity boundary, which we shall denote by $\mathbb{CP}^2_\infty$. To make contact with a real setting, since $\mathbb{CP}^2 \simeq S^5/S^1$ the $\mathbb{CP}^2_\infty$ can be thought of as the S^5 infinity boundary of $\mathbb{C}^3 = \mathbb{R}^6$ with the points on S^1 orbits identified.

Definition 4.2.1 *A plane* (4.2.5) *in* $\overline{M} \simeq \mathbb{CP}^3$ *is called a null plane if* $[Z_\alpha] \in \mathbb{CP}^{3*}$ *lies in the cone* $\mathcal{C}$ (4.2.1).

Let us now show that null planes in $\overline{M}$ indeed give rise to null planes in M as defined in section 2.2.3. Since $P^0 \neq 0$ in M, we can divide (4.2.5) by P^0 and use the coordinates (4.2.6). By substituting in the parametrisation (4.2.3) for Z_α, equation (4.2.5) becomes

$$\omega = 2x\lambda + y(\lambda^2 - 1) + t(1 + \lambda^2), \tag{4.2.7}$$

which is indeed the equation for null planes in $\mathbb{C}^3$. Note that the parametrisation (4.2.3) is only valid for the points on $\mathcal{C} - \mathbf{z_0}$. From the plane equation (4.2.5) one sees that the vertex $\mathbf{z_0} = [Z_0, 0, 0, 0]$ corresponds to the infinity boundary $\mathbb{CP}^2_\infty$, which we shall regard as a null plane by definition. Hence, the natural extension from $\mathbb{C}^3$ to $\mathbb{CP}^3$ makes the inclusion of the vertex $\mathbf{z_0}$. The correspondence between the compactified spacetime $\overline{M} \simeq \mathbb{CP}^3$ and the cone $\mathcal{C} \subset \mathbb{CP}^{3*}$ is summarised in lemma 4.2.3 below. First, let us define what we mean by a conic section of $\mathcal{C}$.

Definition 4.2.2 *A conic section of a cone* $\mathcal{C} \subset \mathbb{CP}^{3*}$ *is given by the intersection of a plane in* $\mathbb{CP}^{3*}$ *with* $\mathcal{C}$.

Lemma 4.2.3 *There is a* $1:1$ *correspondence between points on the cone minus the vertex* $\mathcal{C} - \{\mathbf{z_0}\} \subset \mathbb{CP}^{3*}$ *and null planes in* $M \simeq \mathbb{C}^3 \subset \overline{M} \simeq \mathbb{CP}^3$. *The vertex* $\mathbf{z_0}$ *corresponds to the infinity boundary* $\mathbb{CP}^2_\infty \subset \overline{M}$.

On the other hand, there is a $1:1$ *correspondence between points on* $\overline{M}$ *and conic sections of* $\mathcal{C}$, *where*

1. *points in* M *correspond to the conic sections that do not intersect* $\mathbf{z_0}$,
2. *points in* $\mathbb{CP}^2_\infty$ *correspond to the conic sections, each of which consists of two* $\mathbb{C}$*-lines, counting multiplicity, which meet at* $\mathbf{z_0}$.

Proof. We have already established the first part of the lemma. The second part can be proved by considering equation (4.2.5). By fixing $[P^\alpha]$ and varying $[Z_\alpha]$ one sees that (4.2.5) is also the equation for planes in $\mathbb{CP}^{3*}$. That is, a point $[P^\alpha] \in \mathbb{CP}^3$ labels a plane in $\mathbb{CP}^{3*}$. Moreover, for a given $[P^\alpha]$ it is always possible to find common solutions $[Z_\alpha]$ to (4.2.1) and (4.2.5), which means that any plane in $\mathbb{CP}^{3*}$ intersects $\mathcal{C}$.

Now consider points in M. Since $P^0 \neq 0$ for a point in M, no plane labelled by $[P^\alpha] \in M$ passes through $\mathbf{z_0}$. Hence we have that each point in M corresponds to a conic section on $\mathcal{C} - \mathbf{z_0}$. On the other hand, for a point on $\mathbb{CP}^2_\infty$, with $P^0 = 0$ the corresponding plane in $\mathbb{CP}^{3*}$ is given by

$$P^1 Z_1 + P^2 Z_2 - P^3 Z_3 = 0. \tag{4.2.8}$$

Equation (4.2.8) admits the vertex $[Z_0, 0, 0, 0]$ as a solution. Thus, the plane passes through the vertex $\mathbf{z_0}$. Thinking analogously of a cone in $\mathbb{R}^3$, one would expect the conic section to consist of two lines coming together at the vertex. This is indeed the case. For $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$ we can use the parametrisation (4.2.2) to label Z_i. For concreteness, let us consider the patch where $\pi_1 \neq 0$ and use the first parametrisation in (4.2.3). Equation (4.2.8) becomes

$$(P^2 + P^3)\lambda^2 + 2P^1\lambda + (P^3 - P^2) = 0. \tag{4.2.9}$$

This is a quadratic equation for λ. Since ω (corresponding to Z_0) is arbitrary, it implies that a conic section corresponding to a point on $\mathbb{CP}^2_\infty$ consists of two $\mathbb{C}$-lines of constant λ, whose values are given by the two roots of (4.2.9) counting multiplicity. In the limit where ω approaches infinity the two lines meet at $\mathbf{z_0}$.

□

4.2.2 Compactified twistor space and the blow-up of the cone

The compactified minitwistor space was defined in [44] to be the fibrewise compactification $\overline{T\mathbb{P}^1}$ of $T\mathbb{P}^1$, where each fibre is extended from $\mathbb{C}$ to $\mathbb{CP}^1$. This can be regarded as adding a $\mathbb{CP}^1$ at $\omega = \infty$. We shall denote the additional $\mathbb{CP}^1$ by L_∞. In the cone picture, the compactified twistor space is the cone $\mathcal{C}$ with the vertex blown up to a $\mathbb{CP}^1$.

Proposition 4.2.4 *The fibrewise compactification $\overline{T\mathbb{P}^1}$ of $T\mathbb{P}^1$, which is constructed by adding $L_\infty \simeq \mathbb{CP}^1$ to extend each $\mathbb{C}$-fibre of $T\mathbb{P}^1$ to a $\mathbb{CP}^1$, is isomorphic to the blow-up $\widetilde{\mathcal{C}}$ of the cone $\mathcal{C} \subset \mathbb{CP}^{3*}$ at the vertex $\mathbf{z_0}$, where the blow-up of $\mathbf{z_0}$ is identified with L_∞.*

Before giving a proof to the proposition, let us start by describing the blow-up $\widetilde{\mathcal{C}}$ in details. A reference on the blow-up can be found, for example, in [52, 53].

First, consider a blow-up of an open set $U \simeq \mathbb{C}^3 \subset \mathbb{CP}^{3*}$ with $Z_0 \neq 0$. Let

$$z_1 = \frac{Z_1}{Z_0}, \quad z_2 = \frac{Z_2}{Z_0}, \quad z_3 = \frac{Z_3}{Z_0} \tag{4.2.10}$$

be coordinates of U, so that the vertex $\mathbf{z_0}$ coincides with the origin. By definition, the blow-up $\widetilde{U}$ of U at the origin is given by

$$\{(z,l) \in U \times \mathbb{CP}^2 : z_i l_j = z_j l_i,\ i \neq j\}, \tag{4.2.11}$$

where $\{l_i\}$ are homogeneous coordinates of the $\mathbb{CP}^2$, $i = 1,2,3$. In other words, $\widetilde{U}$ is a 3-dimensional subspace of $U \times \mathbb{CP}^2$ defined by the relation in (4.2.11). Geometrically, z lies on a line labelled by $l \in \mathbb{CP}^2$ passing through the origin in $\mathbb{C}^3$. One can consider $\widetilde{U}$ in three coordinate neighbourhoods: $\widetilde{U}^k$, where $l_k \neq 0$. A point in $\widetilde{U}^k$ is labelled by $(z_k, \frac{l_j}{l_k})$ where $j \neq k$.

There exists a surjective map from $\widetilde{U}$ to U which is given locally in a coordinate patch $\widetilde{U}^k$ by

$$\pi : \left(z_k, \frac{l_j}{l_k}\right) \mapsto \left(z_k,\ z_j = z_k \frac{l_j}{l_k}\right). \tag{4.2.12}$$

One sees that for a point (z_k, z_j) with $z_k \neq 0$ there is a unique preimage $\left(z_k,\ \frac{l_j}{l_k} = \frac{z_j}{z_k}\right)$ in $\widetilde{U}$. However, if $z_k = 0$ all points with coordinates $\left(0, \frac{l_j}{l_i}\right)$ are mapped to the origin. Hence, the preimage $\mathtt{E}$ of the origin is isomorphic to $\mathbb{CP}^2$. $\mathtt{E}$ is called the exceptional divisor. It is important to note that the map $\pi : \widetilde{U} - \mathtt{E} \longrightarrow U - \{\mathbf{z_0}\}$ is $1:1$.

Now, let us look at the blow-up $\widetilde{\mathcal{C}}$ at the vertex of the cone $\mathcal{C} \subset \mathbb{CP}^{3*}$. We are only interested in the region around the vertex, as the projection from $\widetilde{\mathcal{C}}$ to $\mathcal{C}$ is $1:1$ elsewhere. The blow-up $\widetilde{\mathcal{C}}_U = \widetilde{\mathcal{C}} \cap \widetilde{U}$ is obtained from on $\widetilde{U}$ by imposing the cone equation (4.2.1) on $\widetilde{U}$. In a coordinate patch, say $\widetilde{U}^1$ where $l_1 \neq 0$, (4.2.1) becomes

$$z_1^2 \left(1 + \left(\frac{l_2}{l_1}\right)^2 - \left(\frac{l_3}{l_1}\right)^2\right) = 0.$$

The continuity implies that $\widetilde{\mathcal{C}}_U \cap \mathtt{E}$ are given locally in $\widetilde{U}^1$ by the points with

$$z_1 = 0 \quad \text{and} \quad 1 + \left(\frac{l_2}{l_1}\right)^2 - \left(\frac{l_3}{l_1}\right)^2 = 0.$$

Since $l_1 \neq 0$ in $\widetilde{U}^1$, the second condition can be written as

$$l_1^2 + l_2^2 - l_3^2 = 0. \tag{4.2.13}$$

One obtains similar description in patches $\widetilde{U}^2$ and $\widetilde{U}^3$. Then it follows from (4.2.13) that $\widetilde{\mathcal{C}}_U \cap \mathtt{E} = \widetilde{\mathcal{C}} \cap \mathtt{E}$ is $\{\mathbf{z_0}\} \times \mathbb{CP}^1$, where the $\mathbb{CP}^1$ is embedded in the $\mathbb{CP}^2 \ni [l_i]$ by

$$[l_1, l_2, l_3] = [-2\pi_0\pi_1,\ {\pi_1}^2 - {\pi_0}^2,\ {\pi_0}^2 + {\pi_1}^2] \tag{4.2.14}$$

with $\pi_A \in \mathbb{C}^2 - \{0\}$, where we have used the same parametrisation as for null vectors. In exactly the same way as in section 4.2.1, the $\mathbb{CP}^1$ in (4.2.14) can be parametrised by a single variable as

$$[-2\lambda, 1-\lambda^2, 1+\lambda^2] \qquad \text{and} \qquad [2\tilde{\lambda}, \tilde{\lambda}^2 - 1, 1+\tilde{\lambda}^2], \tag{4.2.15}$$

in the patches with $\pi_1 \neq 0$ and $\pi_0 \neq 0$ respectively, where $\tilde{\lambda} = \frac{1}{\lambda}$ in the overlap.

Note that we deliberately denote the inhomogeneous coordinate of the $\mathbb{CP}^1$ by λ, to be the same as the base coordinate of $T\mathbb{P}^1$. We shall now show that the $\widetilde{\mathcal{C}} \cap \mathtt{E}$ indeed corresponds to the additional $\mathbb{CP}^1$ at $\omega = \infty$ of $\overline{T\mathbb{P}^1}$.

Proof of Proposition 4.2.4. A bijection from $T\mathbb{P}^1$ to $\mathcal{C} - z_0$ is already given by (4.2.4). Here, we shall extend the map (4.2.4) to a bijection from $\overline{T\mathbb{P}^1}$ to $\widetilde{\mathcal{C}}$. Although the fibre of $\overline{T\mathbb{P}^1}$ is a $\mathbb{CP}^1$, we shall avoid using two fibre-coordinate patches, but rather we will define a map by taking the limit $\omega \to \infty$. Since the projection $\pi : \widetilde{\mathcal{C}} - (\widetilde{\mathcal{C}} \cap \mathtt{E}) \longrightarrow \mathcal{C} - \mathbf{z_0}$ is $1:1$, we only need to consider the map locally in a neighbourhood of $\mathbf{z_0}$. Assuming $\omega \neq 0$, then (4.2.4) can be written as

$$(\omega, \lambda) \mapsto [1, \frac{-2\lambda}{\omega}, \frac{1-\lambda^2}{\omega}, \frac{1+\lambda^2}{\omega}], \quad \text{and} \quad (\tilde{\omega}, \tilde{\lambda}) \mapsto [1, \frac{-2\tilde{\lambda}}{\tilde{\omega}}, \frac{\tilde{\lambda}^2-1}{\tilde{\omega}}, \frac{1+\tilde{\lambda}^2}{\tilde{\omega}}]. \tag{4.2.16}$$

To extend the domain of (4.2.16) to $\overline{T\mathbb{P}^1}$ minus the $\omega = 0$ section, we shall take the limit $\omega \to \infty$. For concreteness, let us consider the first local map of (4.2.16). In the inhomogeneous coordinates z_1, z_2, z_3 in (4.2.10) of $\mathcal{C}_U - \{\mathbf{z_0}\}$, the first map of (4.2.16) is given by

$$(\omega \neq 0, \lambda) \longmapsto (z_1, z_2, z_3) = \left(\frac{-2\lambda}{\omega}, \frac{1-\lambda^2}{\omega}, \frac{1+\lambda^2}{\omega}\right). \tag{4.2.17}$$

We can now define another map from the image of (4.2.17), which is $\mathcal{C}_U - \{\mathbf{z_0}\}$, to the blow-up $\widetilde{\mathcal{C}}_U$ in terms of three local maps from the regions: $U^1 = \{\lambda \neq 0\}$, $U^2 = \{\lambda \neq \pm 1\}$ and $U^3 = \{\lambda \neq \pm i\}$ to the blow-up neighbourhood $\widetilde{U}^1 = \{l_1 \neq 0\}$, $\widetilde{U}^2 = \{l_2 \neq 0\}$ and $\widetilde{U}^3 = \{l_3 \neq 0\}$ in $\widetilde{U}$ respectively.

In U^1 for example, the local map is defined by

$$(z_1, z_2, z_3) \longmapsto \left(z_1,\ \frac{l_2}{l_1} = \frac{z_2}{z_1},\ \frac{l_3}{l_1} = \frac{z_3}{z_1}\right).$$

Composing it with the map (4.2.17), we have

$$(\omega \neq 0, \lambda) \longmapsto \left(z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}\right) = \left(\frac{-2\lambda}{\omega}, \frac{1-\lambda^2}{-2\lambda}, \frac{1+\lambda^2}{-2\lambda}\right). \tag{4.2.18}$$

One sees that this is consistent with the parametrisation of $[l_i]$ in (4.2.15). Since at the moment ω is still finite and $\lambda \neq 0$ in U^1, $z_1 \neq 0$. Therefore (4.2.18) is a $1:1$ map from U^1 to $\widetilde{U}^1$, whose image is $(\widetilde{\mathcal{C}} \cap \widetilde{U}^1) - (\widetilde{U}^1 \cap \mathtt{E})$. The local maps $U^2 \to \widetilde{U}^2$ and $U^3 \to \widetilde{U}^3$ are defined similarly from the inverse of the projection (4.2.12).

Now, consider the limit $\omega \to \infty$ of the map (4.2.18)

$$(\omega \neq 0, \lambda) \longmapsto \left(z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}\right) = \left(0, \frac{1-\lambda^2}{-2\lambda}, \frac{1+\lambda^2}{-2\lambda}\right) \quad \text{as } \omega \to \infty. \tag{4.2.19}$$

We define a bijection from $\overline{T\mathbb{P}^1} \cap U^1$ to $\widetilde{\mathcal{C}} \cap \widetilde{U}^1$ to be the extension of the map (4.2.18) by the limit (4.2.19). Comparing this with the local expression of $\widetilde{\mathcal{C}} \cap \mathtt{E} \cap \widetilde{U}^1$ obtained from (4.2.15), and similarly for the other two neighbourhoods, one deduces that L_∞ is mapped onto the restricted exceptional divisor $\widetilde{\mathcal{C}} \cap \mathtt{E}$.

□

For convenience, we will now use L_∞ to denote both $\omega = \infty$ section in $\overline{T\mathbb{P}^1}$ and the restricted exceptional divisor $\widetilde{\mathcal{C}} \cap \mathtt{E}$.

Remarks.

1) Since every map is given in holomorphic coordinates, one deduces that $\overline{T\mathbb{P}^1}$ is biholomorphic to $\widetilde{\mathcal{C}}$.

2) Note that as $\mathbf{z_0}$ is blown up to L_∞, a conic section which corresponds to a spacetime point on $\mathbb{CP}^2_\infty$ now consists of two $\mathbb{CP}^1$ fibres of $\overline{T\mathbb{P}^1}$, which includes two points counting multiplicity on L_∞. The two points are the two roots of (4.2.9), where here λ is the inhomogeneous coordinate of L_∞.

4.2.3 The correspondence space

We are interested in a correspondence space, which admits a double fibration onto the spacetime and the twistor space, because it is the domain of the extended solution of the Lax pair (2.2.6) (see section 2.2.3). A compactified double fibration was considered in [50] where a singular variety in $\mathbb{CP}^3 \times \mathbb{CP}^{3*}$ plays the role of the correspondence space. Here we shall discuss the use of its blow-up as the domain of a double fibration.

Recall that the correspondence space in the non-compact case, denoted here as $\mathtt{F}$, is the space of pairs of a spacetime point in $\mathbb{C}^3$ and a null plane on which the point lies, and the null plane corresponds to a point on the minitwistor space $T\mathbb{P}^1$. Hence $\mathtt{F}$ is a subset of $\mathbb{C}^3 \times T\mathbb{P}^1$ defined by

$$\mathtt{F} := \{(p, z) \in \mathbb{C}^3 \times T\mathbb{P}^1 \ : \omega = 2x\lambda + y(\lambda^2 - 1) + t(1 + \lambda^2))\},$$

where (x, y, t) are coordinates of a point $p \in \mathbb{C}^3$ and (ω, λ) are those of a point $z \in T\mathbb{P}^1$. Given $(x, y, t) \in \mathbb{C}^3$ and $\lambda \in \mathbb{CP}^1$, ω is determined uniquely by the incidence relation, and hence $\mathtt{F}$ is isomorphic to $\mathbb{C}^3 \times \mathbb{CP}^1$.

For the compactified case, one can start by considering a singular algebraic variety in $\mathbb{CP}^3 \times \mathbb{CP}^{3*}$ given by

$$\hat{f} := \{(p, z) \in \mathbb{CP}^3 \times \mathbb{CP}^{3*} \ : Z_1^2 + Z_2^2 - Z_3^2 = 0,\ P^0 Z_0 + P^1 Z_1 + P^2 Z_2 - P^3 Z_3 = 0\}. \quad (4.2.20)$$

This is effectively a subset of $\mathbb{CP}^3 \times \mathcal{C}$ which consists of pairs of a point $p \in \mathbb{CP}^3$ and a point $z \in \mathcal{C}$ corresponding to a null plane passing through p. Equivalently, it is the space of pairs of a point $z \in \mathcal{C}$ and a point $p \in \mathbb{CP}^3$ which corresponds to a plane in $\mathbb{CP}^{3*}$ that passes through z. Hence, $\hat{f}$ has a natural double fibration

$$\hat{r} : \hat{f} \longrightarrow \overline{M} \simeq \mathbb{CP}^3 \qquad \text{and} \qquad \hat{q} : \hat{f} \longrightarrow \mathcal{C} \subset \mathbb{CP}^{3*}, \quad (4.2.21)$$

where $\hat{q} \circ \hat{r}^{-1}(p)$ is the corresponding conic section $l_p \subset \mathcal{C}$ and $\hat{r} \circ \hat{q}^{-1}(z)$ is the null plane in $\overline{M}$ which is a $\mathbb{CP}^2$, and $\mathbf{z_0}$ corresponds to $\mathbb{CP}^2_\infty$. This is the double fibration discussed in [50].

In the double fibration (4.2.21), every point $z \in \mathcal{C}$ is on an equal footing: each point corresponds to a $\mathbb{CP}^2$ plane, including $\mathbf{z_0}$. This is not the case for points on $\overline{T\mathbb{P}^1}$. Since $\mathbf{z_0}$ is blown up to $L_\infty \simeq \mathbb{CP}^1$, one can now consider each point on L_∞. In fact, since the finite points on M are holomorphic sections (4.2.7) in $T\mathbb{P}^1 \subset \overline{T\mathbb{P}^1}$ which do not intersect L_∞, we know that a point on L_∞ has to correspond to a subset of $\mathbb{CP}^2_\infty$.

Lemma 4.2.5 *A point on $L_\infty \subset \overline{T\mathbb{P}^1} \simeq \widetilde{\mathcal{C}}$ corresponds to a $\mathbb{CP}^1 \subset \mathbb{CP}^2_\infty \subset \overline{M}$.*

Proof. Recall that (4.2.9) is the equation for the intersection of a conic section corresponding to a point in $\mathbb{CP}^2_\infty$ with L_∞. Conversely, fixing λ the equation determines what $\lambda \in L_\infty$ corresponds to in $\mathbb{CP}^2_\infty$. To see it, we look for the solutions $[P^1, P^2, P^3] \in \mathbb{CP}^2_\infty$ for a given λ. Given a value of λ, (4.2.9) is one linear equation for 3 unknowns. Since it is not possible for all the coefficients to vanish at the same time, one can always determine one variable in terms of the other two. Hence, there are two degrees of freedom in the homogeneous coordinates in $\mathbb{C}^2 - \{0\}$, and we conclude that each point in L_∞

corresponds to a $\mathbb{CP}^1$ line in $\mathbb{CP}^2_\infty$. This is unlike the points on $T\mathbb{P}^1 \subset \overline{T\mathbb{P}^1}$, each of which gives a $\mathbb{CP}^2$-null plane in $\overline{M} \simeq \mathbb{CP}^3$.

□

We shall now present a fibration onto the compactified twistor space, where each point of $\overline{T\mathbb{P}^1}$ has an equal footing. That is, a point on L_∞ is also a $\mathbb{CP}^2$. This is achieved simply by defining the correspondence space to be the blow-up of $\hat{f}$ along its singularity. The singularity of $\hat{f}$ comes from the conic singularity $\mathbf{z_0} \in \mathcal{C}$, which now corresponds to the points $(p, \mathbf{z_0}) \in \hat{f}$ for all p such that $P^0 Z_0 = 0$ while $Z_0 \neq 0$. That is, $P^0 = 0$. Thus the singularity is $\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}$.

The correspondence space

We define a correspondence space $\hat{\mathcal{F}}$ of a double fibration to the compactified spacetime $\overline{M} \simeq \mathbb{CP}^3$ and the compactified twistor space $\widetilde{\mathcal{C}}$ to be

$$\hat{\mathcal{F}} = \textit{the blow-up of the algebraic variety } \hat{f} \text{ (4.2.20) } \textit{along } \mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}.$$

$\hat{\mathcal{F}}$ has the following properties.

1. The blow-up of $\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}$ is $\mathbb{CP}^2_\infty \times L_\infty$.

This fact can be derived from the direct construction of the blow-up as follows. Let us first consider the blow-up of $\mathbb{CP}^3 \times \mathbb{CP}^{3*}$ along $\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}$ locally in each coordinate patch. Recall that $\mathbb{CP}^2_\infty = \{[P^\alpha] \ : P^0 = 0\}$ and $\mathbf{z_0} = [1, 0, 0, 0]$. Since we know that away from the singularity, the projection $\rho : \hat{\mathcal{F}} \longrightarrow \hat{f}$ is a $1:1$ and onto, we need to consider only three coordinate patches of $\mathbb{CP}^3 \times \mathbb{CP}^{3*}$ that include the singularity, namely $U_i = \{Z_0 \neq 0, P^i \neq 0,\}$ $i = 1, 2, 3$. Then, the blow-up of $\hat{f} \subset \mathbb{CP}^3 \times \mathbb{CP}^{3*}$ is obtained by imposing the incidence relations in (4.2.20). Note the lower indices of U_i to be distinguished from U^i in the previous sections.

First, consider the patch $U_1 = \{Z_0 \neq 0, P^1 \neq 0\} \simeq \mathbb{C}^3 \times \mathbb{C}^3 = \mathbb{C}^6$ with coordinates

$$y_i = (y_0 = p^0,\ y_1 = z_1,\ y_2 = z_2,\ y_3 = z_3,\ y_4 = p^2,\ y_5 = p^3),$$

where $z_j = \frac{Z_j}{Z_0}$ and $p^j = \frac{P^j}{P^1}$. The intersection $(\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}) \cap U_1$ is then given by $\mathbb{C}^2_{\infty(1)} := \{(0, 0, 0, 0, p^2, p^3)\} \simeq \mathbb{C}^2$. The blow-up of U_1 along $\mathbb{C}^2_{\infty(1)}$ is by definition (see for example [53]) given by

$$\widetilde{U}_1 := \{(y, l) \in \mathbb{C}^6 \times \mathbb{CP}^3 \ : y_i l_j = y_j l_i,\ i \neq j \in \{0, 1, 2, 3\}\}, \tag{4.2.22}$$

where $\{l_i\}$ are homogeneous coordinates of the $\mathbb{CP}^3$. The projection $\rho : (y, l) \mapsto y$ is bijective to the region away from $\mathbb{C}^2_{\infty(1)}$. If $y \in \mathbb{C}^2_{\infty(1)}$, then l is arbitrary, and hence the preimage of $\mathbb{C}^2_{\infty(1)}$ is $\hat{\mathtt{E}}_1 := \rho^{-1}(\mathbb{C}^2_{\infty(1)}) = \mathbb{C}^2_{\infty(1)} \times \mathbb{CP}^3$.

The blow-up $\hat{\mathcal{F}} \cap \widetilde{U}_1$ is obtained from $\widetilde{U}_1$ by imposing the incidence relations in (4.2.20)

$$z_1^2 + z_2^2 - z_3^2 = 0, \;\; \text{and} \;\; p^0 + z_1 + p^2 z_2 - p^3 z_3 = 0. \tag{4.2.23}$$

Lifting the relations in (4.2.23) to $\widetilde{U}_1$, they can be written locally in the four coordinate neighbourhoods of $\widetilde{U}_1$. First, in the patch $l_0 \neq 0$, with the coordinates $(p^0, \frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}, p^2, p^3)$ equation (4.2.23) becomes

$$(p^0)^2 \left(\left(\frac{l_1}{l_0}\right)^2 + \left(\frac{l_2}{l_0}\right)^2 - \left(\frac{l_3}{l_0}\right)^2 \right) = 0 \quad \text{and} \quad p^0 \left(1 + \frac{l_1}{l_0} + \frac{l_2}{l_0}p^2 - \frac{l_3}{l_0}p^3\right) = 0. \tag{4.2.24}$$

Recall that the exceptional divisor of $\widetilde{U}_1$ is given by

$$\hat{\mathtt{E}}_1 = \{(y, l) \in \widetilde{U}_1 \; : y_0 = y_1 = y_2 = y_3 = 0\}.$$

This is the preimage of $\mathbb{C}^2_{\infty(1)}$ where $p^0 = 0$ and $(z_1, z_2, z_3) = 0$. The continuity of (4.2.24) implies that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}_1$ is given locally in the patch $l_0 \neq 0$ by the points with $p^0 = 0$ and

$$\left(\frac{l_1}{l_0}\right)^2 + \left(\frac{l_2}{l_0}\right)^2 - \left(\frac{l_3}{l_0}\right)^2 = 0, \qquad 1 + \frac{l_1}{l_0} + \frac{l_2}{l_0}p^2 - \frac{l_3}{l_0}p^3 = 0.$$

Since $l_0 \neq 0$, the last two equations can be written as

$$l_1^2 + l_2^2 - l_3^3 = 0 \tag{4.2.25}$$

$$l_0 + l_1 + l_2 p^2 - l_3 p^3 = 0. \tag{4.2.26}$$

Similarly, in the patch $l_1 \neq 0$, with coordinates $(\frac{l_0}{l_1}, z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}, p^2, p^3)$ equation (4.2.23) becomes

$$(z_1)^2 \left(1 + \left(\frac{l_2}{l_1}\right)^2 - \left(\frac{l_3}{l_1}\right)^2\right) = 0, \qquad z_1 \left(\frac{l_0}{l_1} + 1 + \frac{l_2}{l_1}p^2 - \frac{l_3}{l_1}p^3\right) = 0,$$

and $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}_1$ is locally given in this neighbourhood by the points with $z_1 = 0$ and

$$1 + \left(\frac{l_2}{l_1}\right)^2 - \left(\frac{l_3}{l_1}\right)^2 = 0 \quad \text{and} \quad \frac{l_0}{l_1} + 1 + \frac{l_2}{l_1}p^2 - \frac{l_3}{l_1}p^3 = 0.$$

Now, since $l_1 \neq 0$, these can also be written in the homogeneous coordinates $\{l_i\}$ as (4.2.25, 4.2.26). The equations in the other two patches $l_2 \neq 0$ and $l_3 \neq 0$ are similar to those in the patch $l_1 \neq 0$.

This shows that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}_1$ is the subset of

$$\hat{\mathtt{E}}_1 = \{(0,0,0,0,p^2,p^3,l_0,l_1,l_2,l_3)\} = (\mathbb{C}^2_{\infty(1)} \times \mathbb{CP}^3) \subset (\mathbb{C}^6 \times \mathbb{CP}^3)$$

given by (4.2.25, 4.2.26). Given (p^2,p^3), l_0 is uniquely determined from (l_1,l_2,l_3) by (4.2.26). This, together with (4.2.25), defines a $\mathbb{CP}^1 \subset \mathbb{CP}^3$ given by

$$[l_0,\ -2\alpha_0\alpha_1,\ \alpha_1^2-\alpha_0^2,\ \alpha_0^2+\alpha_1^2],$$

where $l_0 = 2\alpha_0\alpha_1 + (\alpha_0^2-\alpha_1^2)p^2 + (\alpha_0^2+\alpha_1^2)p^3$ and $\alpha_A \in \mathbb{C}^2 - \{0\}$. Hence, we conclude that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}_1 = \mathbb{C}^2_{\infty(1)} \times \mathbb{CP}^1$. Note that the local equations for $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}_1$ are smooth and in fact holomorphic in each of the four patches of $\widetilde{U}_1$.

In the other two patches $U_2 = \{Z_0 \neq 0, P^2 \neq 0\}$ and $U_3 = \{Z_0 \neq 0, P^3 \neq 0\}$, the blow-up follows similarly. Let $\hat{\mathtt{E}} = \hat{\mathtt{E}}_1 \cup \hat{\mathtt{E}}_2 \cup \hat{\mathtt{E}}_3$ denotes the union of the exceptional divisors of $\widetilde{U}_1, \widetilde{U}_2$ and $\widetilde{U}_3$. The blow-up is defined such that the parts in coordinate patches glue naturally, therefore we conclude that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}} = \mathbb{CP}^2_\infty \times \mathbb{CP}^1$. Note that since $\{U_i\}$, $i = 1,2,3$, are the only patches that include the singularity $\mathbb{CP}^2_\infty \times \mathbf{z_0}$, $\hat{\mathtt{E}}$ is also the exceptional divisor of the blow-up of $\mathbb{CP}^3 \times \mathbb{CP}^{3*}$ along $\mathbb{CP}^2_\infty \times \mathbf{z_0}$.

The $\mathbb{CP}^1$ in $\hat{\mathcal{F}} \cap \hat{\mathtt{E}}$ is precisely L_∞ of $\widetilde{\mathcal{C}}$. To see this, consider the incidence relation (4.2.22) in $\widetilde{U}_1$. Equation $y_i l_j = y_j l_i$ implies $z_i l_j = z_j l_i$, $i = 1,2,3$, and the same in $\widetilde{U}_2$ and $\widetilde{U}_3$. This is the same expression for the blow up of $\mathcal{C}$ along $\mathbf{z_0}$.

2. $\hat{\mathcal{F}}$ is a $\mathbb{CP}^2$ bundle over $\widetilde{\mathcal{C}}$.

This feature gives a direct way to show that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}} = \mathbb{CP}^2_\infty \times L_\infty$. Let us start with the fact that the algebraic variety $\hat{f} \subset \mathbb{CP}^3 \times \mathcal{C}$ given by (4.2.20) is a $\mathbb{CP}^2$ bundle over $\mathcal{C}$. To see this, consider the following. Let us denote the neighbourhood $\{Z_0 \neq 0\} \subset \mathbb{CP}^3 \times \mathbb{CP}^{3*}$ by U, the same as the neighbourhood $\{Z_0 \neq 0\} \subset \mathbb{CP}^{3*}$. Locally in U, with coordinates $z_i = \frac{Z_i}{Z_0}$ the incidence relations in (4.2.20) become

$$z_1^2 + z_2^2 - z_3^2 = 0 \tag{4.2.27}$$
$$P^0 + P^1 z_1 + P^2 z_2 - P^3 z_3 = 0. \tag{4.2.28}$$

Equation (4.2.28) is homogeneous, i.e. given a point (z_1,z_2,z_3) on $\mathcal{C}_U$ satisfying (4.2.27), a solution $[P^\alpha] \in \mathbb{CP}^3$ is given by

$$[-P^1 z_1 - P^2 z_2 + P^3 z_3, P^1, P^2, P^3],$$

which is determined by (P^1,P^2,P^3) up to a non-zero multiplication. This implies that $\hat{f}_U \simeq \mathbb{CP}^2 \times \mathcal{C}_U$, where $\hat{f}_U := \hat{f} \cap U$, and $\mathbb{CP}^2_\infty$ is the $\mathbb{CP}^2$ corresponding to $\mathbf{z_0}$. In the other neighbourhoods of $\mathcal{C}$, for example $W = \{z \in \mathcal{C} : Z_1 \neq 0\}$, the subset $\hat{f}_W := \hat{f} \cap W$ is also isomorphic to $\mathbb{CP}^2 \times \mathcal{C}_W$, where in this case the $\mathbb{CP}^2$ is given by $[P^0, \frac{(-P^0 Z_0 - P^2 Z_2 + P^3 Z_3)}{Z_1}, P^2, P^3]$. This shows that $\hat{f}$ is a $\mathbb{CP}^2$ bundle over $\mathcal{C}$.

From the blow-up of $\hat{f}$ along $\mathbb{CP}^2_\infty \times \mathbf{z_0}$, it follows that $\hat{\mathcal{F}}$ is also a $\mathbb{CP}^2$ bundle over $\widetilde{\mathcal{C}}$. Locally the blow-up $\hat{\mathcal{F}}_U$ is isomorphic to $\mathbb{CP}^2 \times \widetilde{\mathcal{C}}_U$, where $\widetilde{\mathcal{C}}_U$ is the blow-up of $\mathcal{C}_U$ along $\mathbf{z_0}$. To see this, consider the blow-up locally in the three regions of $\hat{f}_U$, namely $U_1 = \{Z_0 \neq 0, P^1 \neq 0\}$, $U_2 = \{Z_0 \neq 0, P^2 \neq 0\}$ and $U_3 = \{Z_0 \neq 0, P^3 \neq 0\}$ as discussed previously. Note that $\hat{f}_U$ is completely covered by these three open sets. The points in U which are omitted by U_1, U_2, U_3 are the ones with $(P^1, P^2, P^3) = (0,0,0)$ and are not solutions of (4.2.28).

We have already done the blow-up $\hat{\mathcal{F}}_{U_1}$ of $\hat{f}_{U_1}$ explicitly, where we describe it in coordinate patches, $\{l_0 \neq 0\}$, $\{l_1 \neq 0\}$, $\{l_2 \neq 0\}$, and $\{l_3 \neq 0\}$. However, we note here that $\hat{\mathcal{F}}_{U_1}$ can in fact be described completely in the patches $\{l_i \neq 0\}$, $i = 1,2,3$, because the point $(l_0 \neq 0, 0, 0, 0)$ is not a solution of (4.2.26). In the patch $l_1 \neq 0$, with coordinates $(\frac{l_0}{l_1}, z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}, p^2, p^3)$ we can label a point in $\hat{\mathcal{F}}_{U_1,\, l_1 \neq 0}$ by

$$(-1 - \frac{l_2}{l_1} p^2 + \frac{l_3}{l_1} p^3,\ z_1,\ \frac{l_2}{l_1},\ \frac{l_3}{l_1},\ p^2,\ p^3),$$

as a consequence of (4.2.26). The set $(z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1})$ can be identified with a point in $\widetilde{\mathcal{C}}_U$. Hence, given a point $z \in \widetilde{\mathcal{C}}_U$ we only have freedom in (p^2, p^3). Let $\mathbb{C}^2_{(1)}$ be the $\mathbb{C}^2$ defined by (p^2, p^3). Then, we have that $\hat{\mathcal{F}}_{U_1\ l_1 \neq 0} \simeq \mathbb{C}^2_{(1)} \times \widetilde{\mathcal{C}}_U$. One can deduce the same result for the patches $l_2 \neq 0$, $l_3 \neq 0$, and therefore $\hat{\mathcal{F}}_{U_1} \simeq \mathbb{CP}^2 \times \widetilde{\mathcal{C}}_{U_1}$. This, together with similar results from the neighbourhood U_2 and U_3, imply that

$$\hat{\mathcal{F}}_U \simeq \mathbb{CP}^2 \times \widetilde{\mathcal{C}}_U,$$

where it follows that $\hat{\mathcal{F}} \cap \hat{\mathtt{E}} = \mathbb{CP}^2_\infty \times L_\infty$. Moreover, since $\hat{\mathcal{F}} - \hat{\mathcal{F}}_U$ is isomorphic to $\hat{f} - \hat{f}_U$, we conclude that $\hat{\mathcal{F}}$ is a $\mathbb{CP}^2$ bundle over $\widetilde{\mathcal{C}}$.

3. The double fibration (4.2.21) extends to $\hat{\mathcal{F}}$.

In particular, the fibration $\hat{q} : \hat{f} \longrightarrow \mathcal{C}$ in (4.2.21) extends to a map $q : \hat{\mathcal{F}} \longrightarrow \widetilde{\mathcal{C}}$. Denote $e := \mathbb{CP}^2_\infty \times L_\infty$, then

$$q|_e : \mathbb{CP}^2_\infty \times L_\infty \longrightarrow L_\infty. \tag{4.2.29}$$

Locally, q is just the right projection, for example

$$q : \hat{\mathcal{F}}_U \simeq \mathbb{CP}^2 \times \widetilde{\mathcal{C}}_U \longrightarrow \widetilde{\mathcal{C}}_U.$$

In local coordinates, say in $U_{1,\, l_1 \neq 0}$, q is given by

$$q : (1 + \frac{l_2}{l_1} p^2 - \frac{l_3}{l_1} p^3, z_1,\ \frac{l_2}{l_1}, \frac{l_3}{l_1}, p^2, p^3) \longmapsto (z_1, \frac{l_2}{l_1}, \frac{l_3}{l_1}).$$

4.3 Restricted correspondence space

Recall that the topological degree of a YMH field or a chiral field satisfying the trivial scattering condition comes from the third homotopy class of the restricted extended solution $\psi(x, y, \theta)$. With this in mind we aim to define a 'restricted' correspondence space such that it gives rise to the domain of ψ in the non-compact case.

We shall consider what we call the 'constant time' slice τ, which is a $\mathbb{CP}^2 \subset \overline{M}$ obtained by setting $P^3 = 0$. It is clear that the intersection of τ with the non-compact spacetime $M \simeq \mathbb{C}^3$, where $P^0 \neq 0$, is the $t = 0$ $\mathbb{C}^2$-plane in the (x, y, t) coordinates (4.2.6). We will also consider the 'real slice' $\mathbb{RP}^3 \subset \mathbb{CP}^3$, which consists of the points $[P^\alpha]$ whose homogeneous representatives can be chosen to be in $\mathbb{R}^4 - \{0\}$. Since we write the line element on M as $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$, the finite part of this $\mathbb{RP}^3$ is an $\mathbb{R}^{2,1}$. Then, the intersection $\tau_{\mathbb{R}} \simeq \mathbb{RP}^2$ of τ with the $\mathbb{RP}^3$ can be thought of as the extension of the $t = 0$ $\mathbb{R}^2$-plane to the compactified space.

We define the *restricted correspondence space* $\mathcal{F}$ to be the restriction of $\hat{\mathcal{F}}$ to $\tau_{\mathbb{R}}$,

$$\mathcal{F} := \hat{\mathcal{F}}|_{\tau_{\mathbb{R}}}.$$

This means away from the singularity $\mathcal{F}$ is isomorphic to the algebraic variety

$$f := \{(p, z) \in \mathbb{RP}^3 \times \mathbb{CP}^{3*} : Z_1^2 + Z_2^2 - Z_3^2 = 0, P^0 Z_0 + P^1 Z_1 + P^2 Z_2 - P^3 Z_3 = 0, P^3 = 0\} \tag{4.3.1}$$

minus its singularity. Similar to the complex case, the singularity of f consists of the points $\{[P^\alpha] \times \mathbf{z_0}\}$ for all $[P^\alpha]$ such that $P^0 = 0$. Since $P^3 = 0$, this is an $\mathbb{RP}^1 \subset \mathbb{CP}^2_\infty$, which we shall denote by $\mathbb{RP}^1_\infty$. The preimage in $\mathcal{F}$ of the singularity $\mathbb{RP}^1_\infty \times \mathbf{z_0}$ under the usual projection map from the blow-up is $e_{\tau_{\mathbb{R}}} := \mathbb{RP}^1_\infty \times L_\infty$.

Proposition 4.3.1 *The restriction of the map* $q : \hat{\mathcal{F}} \longrightarrow \widetilde{\mathcal{C}}$ *to* $\mathcal{F}$

$$q|_{\mathcal{F}} : \mathcal{F} \longrightarrow \widetilde{\mathcal{C}} \tag{4.3.2}$$

is surjective.

Proof. First, it follows readily from (4.2.29) that

$$q|_{e_{\tau_{\mathbb{R}}}} : \mathbb{RP}^1_\infty \times L_\infty \longrightarrow L_\infty$$

is onto, as a right projection. However, it is not obvious that

$$q|_{\mathcal{F}-e_{\tau_{\mathbb{R}}}} : \mathcal{F} - e_{\tau_{\mathbb{R}}} \longrightarrow \widetilde{\mathcal{C}} - L_\infty \tag{4.3.3}$$

is also surjective. The map (4.3.3) is equivalent to the restriction of the map $\hat{q}$ in (4.2.21) to $f - \mathbb{RP}^1_\infty \times \{\mathbf{z_0}\}$,

$$\hat{q} : f - \mathbb{RP}^1_\infty \times \{\mathbf{z_0}\} \longrightarrow \mathcal{C} - \{\mathbf{z_0}\}. \tag{4.3.4}$$

Therefore, the question whether $q|_{\mathcal{F}-e_{\tau_{\mathbb{R}}}}$ is onto comes down to whether, given a point $[Z_\alpha] \in \mathcal{C} - \mathbf{z_0}$, one can find a point in $\tau_{\mathbb{R}} \simeq \mathbb{RP}^2$ which lies on the corresponding null plane. In other words, one needs to ask whether all null planes in $\overline{M} \simeq \mathbb{CP}^3$ intersect $\tau_{\mathbb{R}}$. If this is the case, then we can always find a (non-empty) preimage of (4.3.4) for every point in $\mathcal{C} - \mathbf{z_0}$, which implies that (4.3.4) is onto, and hence so are (4.3.3) and (4.3.2). Therefore, proposition 4.3.1 is a corollary of proposition 4.1.1.

□

Before giving a proof to proposition 4.1.1, let us note that locally, say in the open set $\{Z_0 \neq 0\} \subset \mathbb{RP}^3 \times \mathbb{CP}^{3*}$, which we shall also denote by U to be the same as the open set $\{Z_0 \neq 0\} \subset \mathbb{CP}^{3*}$, the algebraic variety f_U is not $\mathbb{RP}^1 \times \mathcal{C}_U$. This can be seen from the second incidence relation in the definition of f (4.3.1)

$$P^0 Z_0 + P^1 Z_1 + P^2 Z_2 = 0, \tag{4.3.5}$$

which is given locally in U by

$$P^0 + P^1 z_1 + P^2 z_2 = 0, \tag{4.3.6}$$

where $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$. Given a point $[Z_\alpha] \in \mathcal{C}$, it is not always the case that $(-(P^1 z_1 + P^2 z_2), P^1, P^2)$ belongs to $\tau_{\mathbb{R}}$ for any (P^1, P^2). Nevertheless, by direct calculation we find that given a point $[Z_\alpha] \in \mathcal{C}$ there always exists a solution $[P^0, P^1, P^2] \in \mathbb{RP}^2$ to (4.3.5). This means that each null plane in $\overline{M} \simeq \mathbb{CP}^3$ intersects $\tau_{\mathbb{R}}$ and it proves proposition 4.1.1.

Proof of Proposition 4.1.1. Let us first consider a class of null planes which we will call real null planes. Let $\mathbb{RP}^{3*}$ be the subset of $\mathbb{CP}^{3*}$ that consists of points $[Z_\alpha]$

whose representatives can be chosen to be in $\mathbb{R}^4 - \{0\}$, and let $\mathcal{C}_\mathbb{R}$ be the intersection $\mathcal{C} \cap \mathbb{RP}^{3*}$. We shall call the planes corresponding to $z \in \mathcal{C}_\mathbb{R}$ real null planes. To see the intersection of real null planes with $\mathcal{T}_\mathbb{R}$, we first look at the real null planes with $Z_0 \neq 0$, i.e. $z \in \mathcal{C}_\mathbb{R} \cap U$. Such a null plane is given by (4.3.6) with (z_1, z_2, z_3) all real. Since everything is real, given (z_1, z_2, z_3), P^0 is determined in terms of P^1, P^2 by (4.3.6). Thus, the intersection of a null plane in $\mathcal{C}_\mathbb{R} \cap U$ with $\mathcal{T}_\mathbb{R}$ is an $\mathbb{RP}^1$. For a real null plane with $Z_0 = 0$, either Z_1 or Z_2 must be non-zero. Similar calculation for these planes shows that their intersections with $\mathcal{T}_\mathbb{R}$ are also $\mathbb{RP}^1$. Therefore, one concludes that each real null plane intersections $\mathcal{T}_\mathbb{R}$ in an $\mathbb{RP}^1$. Note that $\mathbb{CP}^2_\infty$ is a real null plane, whose intersection with $\mathcal{T}_\mathbb{R}$ is $\mathbb{RP}^1_\infty$.

Let us call the real null planes with $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$ finite real null planes. For finite real null planes we have the parametrisation (4.2.2), and the coordinates $[\hat{\omega}, \pi_A]$ can be chosen such that $\hat{\omega} \in \mathbb{R}$, $\pi_A \in \mathbb{R}^2 - \{0\}$. We will now show that these planes intersect $\mathcal{T}_\mathbb{R}$ in oriented lines which are the extension of straight lines in $t = 0$ $\mathbb{R}^2$-plane. First, we note that such a real null plane with $(Z_1, Z_2, Z_3) \neq (0, 0, 0)$ corresponds to a null plane in $\mathbb{R}^{2,1}$ given by

$$\hat{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) + t(\pi_0^2 + \pi_1^2),$$

where we use (x, y, t) in (4.2.6) as coordinates on $\mathbb{R}^{2,1}$. Using the diffeomorphism between $\mathbb{RP}^1$ and S^1

$$\pi_0 = \cos\left(-\frac{\theta}{2}\right), \quad \pi_1 = \sin\left(-\frac{\theta}{2}\right),$$

the null plane equation becomes

$$\hat{\omega} = t - x\sin\theta + y\cos\theta. \tag{4.3.7}$$

Now, the intersection with $\mathcal{T}_\mathbb{R}$ is obtained by restricting (4.3.7) to the $t = 0$ $\mathbb{R}^2$-plane, which results in

$$\hat{\omega} = -\sin\theta\, x + \cos\theta\, y. \tag{4.3.8}$$

This is the equation for oriented lines in $\mathbb{R}^2$. Hence, we have that the space of finite real null planes (in spacetime $\mathbb{RP}^3$ or $\mathbb{CP}^3$) is the space of oriented lines in $\mathbb{R}^2$, which is $S^1 \times \mathbb{R}$. From (4.3.8) we note that $(\hat{\omega}, \theta)$ and $(-\hat{\omega},\ \theta + \pi)$ give the same unoriented line. This means that the two orientations of a line correspond to a pair of null planes labelled by $(\hat{\omega}, \pi_0, \pi_1)$ and $(-\hat{\omega}, -\pi_1, \pi_0)$, or $[Z_0, Z_1, Z_2, Z_3]$ and $[Z_0, Z_1, Z_2, -Z_3]$.

Now, let us consider non-real null planes, given by the points $[Z_\alpha] \in \mathcal{C} - \mathcal{C}_\mathbb{R}$, and again first look at $[Z_\alpha]$ with $Z_0 \neq 0$. Such a null plane must have either z_1 or z_2 non real, or both. Writing $z_1 = l + im$ and $z_2 = k + in$, equation (4.3.5) becomes

$$P^0 = -(P^1 l + P^2 k) \qquad \text{and} \qquad P^1 m + P^2 n = 0.$$

There are two cases. If $m \neq 0$, then

$$P^1 = -P^2 \frac{n}{m} \qquad \text{and} \qquad P^0 = P^2 \left(\frac{n}{m} l - k\right),$$

and if $n \neq 0$,

$$P^2 = -P^1 \frac{m}{n}, \qquad \text{and} \qquad P^0 = P^1 \left(\frac{m}{n} k - l\right).$$

In other words, each non-real null plane with $Z_0 \neq 0$ intersects $\tau_{\mathbb{R}}$ in a single point given by

$$[\frac{n}{m} l - k,\ -\frac{n}{m},\ 1] \qquad \text{and} \qquad [\frac{m}{n} k - l,\ 1,\ -\frac{m}{n}]$$

for $m \neq 0$ and $n \neq 0$ respectively. Note that if $m \neq 0$ it follows that $P^2 \neq 0$ and if $n \neq 0$ then $P^1 \neq 0$.

Now consider non-real null planes with $Z_0 = 0$. First, note that since $Z_0 = 0$, both Z_1, Z_2 must be non-zero, otherwise, for example if $Z_2 = 0$ the cone equation $(Z_1)^2 - (Z_3)^2 = 0$ implies that the plane labelled by $[0, Z_1, 0, Z_3]$ is a real null plane. Now, let us write $Z_1 = L + iM,\ Z_2 = K + iN$. Then (4.3.5) implies that

$$LP^1 + KP^2 = 0, \qquad MP^1 + NP^2 = 0.$$

For a non-real null plane, at least one of M or N must be non-zero. Suppose $M \neq 0$ then $P^1 = -\frac{N}{M} P^2$. There are 2 cases.

i) $L \neq 0$. Then $P^1 = -\frac{K}{L} P^2$. Since either N or K must be non-zero, in a generic case where $\frac{N}{M} \neq \frac{K}{L}$ we have $P^1 = 0 = P^2$. Therefore the plane intersects $\tau_{\mathbb{R}}$ at a single point $[P^0, P^1, P^2] = [1, 0, 0]$. It can be shown using the cone equation (4.2.1) that, if $\frac{N}{M} = \frac{K}{L}$, then the null planes are real null planes.

ii) $L = 0$. Then $KP^2 = 0$. If $K \neq 0$ we again have $P^1 = 0 = P^2$. On the other hand $K = 0$ means Z_1, Z_2 are pure imaginary, which implies that Z_3 is also pure imaginary. Thus the plane is a real null plane.

If we suppose $L \neq 0$ at the beginning, interchanging the roles of (L, M) and (K, N) yields the same result. Hence we conclude that each non-real null plane intersects $\tau_{\mathbb{R}}$ at a single point.

Therefore, every null plane in $\mathbb{CP}^3$ intersects $\tau_{\mathbb{R}}$.

$\square$

Remark 1. The above consideration also gives us a local expression of the algebraic variety f (4.3.1). In the patch U where $Z_0 \neq 0$, f_U can be described in two disjoint region $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C} - \mathcal{C}_{\mathbb{R}}$. The points on $f_U|_{\mathcal{C}_{\mathbb{R}}}$, as a subset of $\mathbb{RP}^3 \times \mathcal{C}$, are given by

$$(-(P^1 z_1 + P^2 z_2), P^1, P^2, 0, z_1, z_2, z_3)$$

and is isomorphic to $\mathbb{RP}^1 \times (\mathcal{C}_{\mathbb{R}} \cap U)$. Over $\mathcal{C} - \mathcal{C}_{\mathbb{R}}$, f_U is an embedding of $\mathcal{C} - \mathcal{C}_{\mathbb{R}}$ in $\mathbb{RP}^2 \times (\mathcal{C} - \mathcal{C}_{\mathbb{R}})$ given by

$$(l + im, k + in, z_3) \longmapsto \Big([\frac{n}{m} l - k, -\frac{n}{m}, 1], l + im, k + in, z_3\Big) \tag{4.3.9}$$

and

$$(l + im, k + in, z_3) \longmapsto \Big([\frac{m}{n} k - l, 1, -\frac{m}{n}], l + im, k + in, z_3\Big), \tag{4.3.10}$$

for $m \neq 0$ and $n \neq 0$ respectively.

Now, it can be shown that the two descriptions are well defined where $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C} - \mathcal{C}_{\mathbb{R}}$ meet if the limit $\lim\limits_{n,m \to 0} \frac{n}{m}$ exists. First, suppose $P^2 \neq 0$, the points on $f_U|_{\mathcal{C}_{\mathbb{R}}}$ are given locally by

$$\Big([-(\frac{P^1}{P^2} z_1 + z_2),\ \frac{P^1}{P^2},\ 1],\ z_1,\ z_2,\ z_3\Big).$$

On the other hand, in the limit $m \to 0$ and $n \to 0$, (4.3.9) becomes

$$(l, k, z_3) \longmapsto ([\gamma l - k,\ -\gamma, 1], l, k, z_3),$$

where $\gamma = \lim\limits_{n,m \to 0} \frac{n}{m}$ is real and hence can be identified with $-\frac{P^1}{P^2}$. Similarly, if $P^1 \neq 0$ we can identify $\lim\limits_{n,m \to 0} \frac{m}{n}$ with $-\frac{P^2}{P^1}$ by taking the limit $m, n \to 0$ of (4.3.10).

Remark 2. The surjectivity of the restricted map (4.3.2) is due to the fact that under the map (4.2.29) each point on L_∞ corresponds to $\mathbb{CP}^2_\infty$. We have (4.2.29) essentially because we take the correspondence space $\hat{\mathcal{F}}$ of the fibration to be the blow-up of the variety $\hat{f}$ along $\mathbb{CP}^2_\infty \times \mathbf{z_0}$.

Recall lemma 4.2.5 which states that each point on L_∞ corresponds to a $\mathbb{CP}^1$ line in $\mathbb{CP}^2_\infty$ under (4.2.9). We note here that not every point in L_∞ gives a $\mathbb{CP}^1$ that intersects $\tau_{\mathbb{R}}$. Consider equation (4.2.9) with $P^3 = 0$ for the intersection of such a $\mathbb{CP}^1$ with the constant time slice τ

$$-2\lambda P^1 + (1 - \lambda^2) P^2 = 0. \tag{4.3.11}$$

Since the coefficients in front of P^1 and P^2 cannot be zero at the same time, we have one degree of freedom in (P^1, P^2). Hence, the $\mathbb{CP}^1$ intersects τ in a single point. For example if $\lambda \neq 0$, the intersection point is given by

$$[P^1, P^2] = [\frac{(1-\lambda^2)}{2\lambda}, 1]. \tag{4.3.12}$$

Note that the map is $2:1$ as λ and $-\frac{1}{\lambda}$ give the same point in τ.

Now assume that $[P^1, P^2] \in \tau_{\mathbb{R}}$. From (4.3.12), we need $\frac{(1-\lambda^2)}{2\lambda}$ to be real. Writing $\lambda = l+im$, the imaginary part of $\frac{(1-\lambda^2)}{2\lambda}$ is $\frac{-m(1+l^2+m^2)}{l^2+m^2}$. This vanishes if and only if $m=0$. Therefore, we conclude that a point λ in L_∞ gives rise to a $\mathbb{CP}^1$ in $\mathbb{CP}^2_\infty$ which intersects $\tau_{\mathbb{R}}$ if and only if $\lambda \in \mathbb{RP}^1 \subset L_\infty$. Then the intersection is a single point determined by (4.3.11).

4.4 Involution maps

The constant time slice τ and the real slice $\mathbb{RP}^3$ considered in section 4.3 can be regarded as the sets of fixed points of a holomorphic involution and an anti-holomorphic involution, respectively, on the complexified spacetime. The maps induce the corresponding involutions on the twistor space.

4.4.1 Time reversal

Let us start by considering the non-compact case. In the complexified spacetime $M \simeq \mathbb{C}^3$ with coordinates (x, y, t), we define the time reversal map as usual as

$$\sigma : (x, y, t) \longmapsto (x, y, -t). \tag{4.4.1}$$

The fixed points of (4.4.1) are of course those with $t = 0$. The map (4.4.1) induces a holomorphic involution on $T\mathbb{P}^1$ via the null plane equation

$$\hat{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) + t(\pi_0^2 + \pi_1^2). \tag{4.4.2}$$

Under (4.4.1), equation (4.4.2) becomes

$$\hat{\omega} = 2x\pi_0\pi_1 + y(\pi_0^2 - \pi_1^2) - t(\pi_0^2 + \pi_1^2). \tag{4.4.3}$$

We now want to define a map σ (keeping the same name) acting on a point $(\hat{\omega}, \pi_A) \in T\mathbb{P}^1$ such that the image $(\hat{\omega}', \pi'_A)$ corresponds to the null plane defined by (4.4.3). Multiplying (4.4.3) by -1 on both side does not change the plane and we have

$$\sigma : (\hat{\omega}, \pi_0, \pi_1) \longmapsto (\hat{\omega}' = -\hat{\omega},\ \pi'_0 = -\pi_1,\ \pi'_1 = \pi_0).$$

We could equally well define the map σ with $\pi'_A = (\pi_1, -\pi_0)$, but since π_A are homogeneous coordinates of $\mathbb{CP}^1$, the choice does not matter. In the inhomogeneous coordinates $(\omega = \frac{\hat{\omega}}{\pi_1^2},\ \lambda = \frac{\pi_0}{\pi_1})$ we have

$$\sigma : (\omega, \lambda) \longmapsto (-\tilde{\omega}, -\tilde{\lambda}),$$

where we recall that $\tilde{\omega} = \frac{\omega}{\lambda^2}$ and $\tilde{\lambda} = \frac{1}{\lambda}$ in the overlap. It is immediate that a holomorphic section (4.4.2) labelled by (x, y, t) is preserved by σ if and only if $t = 0$.

For the compact case, we want to extend the map (4.4.1) to $\overline{M} \simeq \mathbb{CP}^3$ and define the corresponding involutions on the cone $\mathcal{C} \subset \mathbb{CP}^{3*}$ and its blow-up $\widetilde{\mathcal{C}} \simeq \overline{T\mathbb{P}^1}$. Recall our convention that the extension of $t = 0$ $\mathbb{C}^2$-plane to $\mathbb{CP}^3$ is $\tau \simeq \mathbb{CP}^2$ defined by $P^3 = 0$, and (x, y, t) is given by (4.2.6). Then the extension of (4.4.1) to $\overline{M}$ is

$$\sigma : [P^0, P^1, P^2, P^3] \longmapsto [P^0, P^1, P^2, -P^3].$$

This induces a map σ on $\mathbb{CP}^{3*}$ via (4.2.5), given by

$$\sigma : [Z_0, Z_1, Z_2, Z_3] \longmapsto [Z_0, Z_1, Z_2, -Z_3], \tag{4.4.4}$$

and hence a map on the cone $\mathcal{C}$.

By generalising the discussion in section 4.3, we can show that each null plane in $\overline{M}$ corresponding to each point on $\mathcal{C}$ intersects τ in an $\mathbb{CP}^1$. Moreover, two null planes labelled by

$$[Z_0,\ Z_1,\ Z_2,\ +\sqrt{Z_1^2 + Z_2^2}] \qquad \text{and} \qquad [Z_0,\ Z_1,\ Z_2,\ -\sqrt{Z_1^2 + Z_2^2}]$$

have the same intersection line. Hence, geometrically σ interchanges the two members of such pair. The fixed points of the map are the planes which do not form a pair. A special case is the vertex of the cone $\mathbf{z_0}$ with $\mathbb{CP}^1_\infty$ intersection. The rest are those with $Z_2 = \pm i Z_1$. These are the points $[\frac{Z_0}{Z_1}, 1, \pm i, 0] \in \mathcal{C}$, and there are two $\mathbb{C}$-worth sets of these points. Choosing new representatives as $[\pm 2i\frac{Z_0}{Z_1}, \pm 2i, 2, 0]$, we see that these are the fibres $\lambda = \pm i$ of $T\mathbb{P}^1$.

We can extend the map (4.4.4) to the compactified twistor space $\overline{T\mathbb{P}^1}$, which is biholomorphic to $\widetilde{\mathcal{C}}$, by demanding that it gives back (4.4.4) under the projection $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$. Locally, say in the blow-up $\widetilde{\mathcal{C}}_U \subset U \times \mathbb{CP}^2$ of the patch U with $Z_0 \neq 0$, the map is defined by its action on $U \times \mathbb{CP}^2$ as

$$\sigma : (z_1, z_2, z_3) \times [l_1, l_2, l_3] \longmapsto (z_1, z_2, -z_3) \times [l_1, l_2, -l_3],$$

where $z_i = \frac{Z_i}{Z_0}$ are local coordinates on U. Note that the blow-up vertex $L_\infty \subset \overline{T\mathbb{P}^1}$ is fixed by σ, although it is not the set of fixed points.

4.4.2 Reality condition

Define a map $\varphi : M \to M$ by complex conjugation

$$\varphi : (x, y, t) \longmapsto (\bar{x}, \bar{y}, \bar{t}). \tag{4.4.5}$$

The set of fixed points of (4.4.5) is the real slice $\mathbb{R}^{2,1} \subset M \simeq \mathbb{C}^3$. Now, considering the null plane equation (4.4.2) one sees that the map φ induces an anti-holomorphic involution on $T\mathbb{P}^1$ which maps each point to its complex conjugate

$$\varphi : (\hat{\omega}, \pi_A) \longmapsto (\bar{\hat{\omega}}, \bar{\pi}_A). \tag{4.4.6}$$

The fixed points of (4.4.6) corresponds to real null planes discussed in section 4.3. Hence the set of fixed point is $TS^1 \subset T\mathbb{P}^1$.

There are unique extensions of (4.4.6) to $\overline{M} \simeq \mathbb{CP}^3$ and $\mathbb{CP}^{3*}$, sending

$$[P^0, P^1, P^2, P^3] \longmapsto [\bar{P}^0, \bar{P}^1, \bar{P}^2, \bar{P}^3] \quad \text{and} \quad [Z_0, Z_1, Z_2, Z_3] \longmapsto [\bar{Z}_0, \bar{Z}_1, \bar{Z}_2, \bar{Z}_3],$$

respectively. The cone $\mathcal{C} \subset \mathbb{CP}^{3*}$ is preserved by the map, with the vertex $\mathbf{z_0}$ being another fixed point in addition to the set TS^1.

The extension to the blow-up $\widetilde{U}$ of the neighbourhood U around $\mathbf{z_0}$ is obtained similarly to the previous section, where the map is given locally by the complex conjugation of the coordinates of $\widetilde{U}$. The involution φ maps the blow-up vertex

$$L_\infty = \{[l_i] \in \mathbb{CP}^2 : l_1^2 + l_2^2 - l_3^2 = 0\}$$

to itself, and the fixed points are those with $[l_i] \in \mathbb{RP}^2 \subset \mathbb{CP}^2$. In the coordinate $\lambda \in L_\infty$, these are the points with real λ.

Finally, φ preserves the sections in $\overline{T\mathbb{P}^1}$ corresponding to $[P^\alpha] \in \mathbb{RP}^3$. For finite-point sections it follows readily from (4.2.7). For the sections corresponding to the points at infinity, the pairs of lines of constant λ are determined by (4.2.9). We see that for $\{P^\alpha\}$ real, the two roots can either be both real or complex conjugates, and thus the pairs of lines are preserved by φ.

4.5 Discussion

Our motivation for studying the compactified double fibration is to provide a proof of the identification between the third homotopy class of the restricted extended solution ψ of the YMH field and the second Chern number $c_2(E)$ of the corresponding vector bundle E over $\overline{T\mathbb{P}^1}$. We are interested in a restricted correspondence space which gives rise to the non-compact domain $\mathbb{R}^2 \times S^1$ of the restricted extended solution $\psi(x, y, \theta)$ and admits a surjective map onto $\overline{T\mathbb{P}^1}$. This leads us to the restricted correspondence space $\mathcal{F}$ and the map $q|_{\mathcal{F}}$ as defined in section 4.3. The hope is to use $q|_{\mathcal{F}}$ to pull back the vector bundle E to $\mathcal{F}$ and calculate $c_2(q|_{\mathcal{F}}^* E)$, which should be related to $c_2(E)$. Moreover, $c_2(q|_{\mathcal{F}}^* E)$ is hoped to be expressible in terms of $\psi(x, y, \theta)$ which serves as a local trivialisation of $q|_{\mathcal{F}}^* E$. This is in analogy with the instanton number calculation in the Yang-Mills theory on S^4. The study towards the proof is unfinished.

Chapter 5

Tzitzéica Equation and Calabi-Yau Metrics

In this chapter we look at a symmetry reduction of the $SL(3,\mathbb{C})$ ASDYM equation. Two real forms of the reduction give the Tzitzéica equation, which is a well known integrable system, and the affine sphere equation. The affine sphere equation is currently of interest in the context of mirror symmetry as its solutions determine a class of semi-flat Calabi-Yau metrics locally. Here we give characterisations of the Tzitzéica equation and the affine sphere equation as reductions of the ASDYM equation. We show that the affine sphere equation reduces to the Painlevé III equation under a radial symmetry assumption and derive a 3×3 isomonodromic Lax pair. Finally, we use the Lax pair formulation of the affine sphere equation to write down an explicit expression of the semi-flat Calabi-Yau metrics. The research presented in this chapter, except sections 5.4.2, 5.4.3 and 5.5, is based on the article [2].

5.1 Semi-flat Calabi-Yau metrics and affine sphere equation

Let us begin with a background on mirror symmetry. Mirror symmetry is a symmetry conjectured in string theory between pairs of Calabi-Yau (CY) three-folds. These are 3-complex dimensional Kähler manifolds with covariantly constant holomorphic three-form Ω. Any such manifold admits a Ricci-flat Kähler metric with holonomy contained in $SU(3)$. A well known formulation of mirror symmetry is the conjecture by Strominger, Yau and Zaslow [12], known as the SYZ conjecture. It involves a class of CY manifolds which are fibred over a real 3-dimensional manifold, and the fibres are special Lagrangian tori T^3. Let X denote such a CY manifold and B a real 3-manifold, this means that there exists a projection

$$\pi : X \longrightarrow B$$

such that the restrictions of the Kähler form ω and the real part of the holomorphic three-form $\mathrm{Re}(\Omega)$ vanish on any fibre $\pi^{-1}(p) \simeq T^3$ over a point $p \in B$. We refer the readers to [54] for a reference on Calabi-Yau manifolds in mirror symmetry.

The main idea of the SYZ conjecture is that if X, Y are mirror Calabi-Yau manifolds, then there exists a compact real three-manifold B such that

- $\pi : X \longrightarrow B, \quad \rho : Y \longrightarrow B$ are special Lagrangian fibrations by tori (the fibres can be singular at some points of B).
- The fibres of π and ρ are dual tori.

It turns out that one can define a duality between tori equipped with Riemannian metrics only if the metrics are flat. Therefore the conjecture holds in the *large complex structure limit*, where the volume of the fibres is small in comparison to the volume of the base space and the metric on the fibres is approximately flat (see [54]). Such a CY metric which is flat along the fibres is called semi-flat. The work of Cheng and Yau [55] shows that semi-flat CY metrics on compact complex three-folds are flat, so in what follows we allow CY manifolds to be non-compact, and some fibres of π to be singular.

A natural class of semi-flat CY manifolds are the T^3-invariant manifolds, where the Kähler potential ϕ can be chosen not to depend on the coordinates of the fibres of π. Following [56], let us briefly discuss how one can construct a T^3-invariant semi-flat CY manifold as the tangent bundle TB (with compactified fibres) of a 3-dimensional real affine manifold B with a Hessian metric of the form[1]

$$g_B = \frac{\partial^2 \phi}{\partial x^j \partial x^k} dx^j \otimes dx^k, \tag{5.1.1}$$

where $\phi(x^i)$ satisfies the real Monge-Ampère equation

$$\det\Big(\frac{\partial^2 \phi}{\partial x^j \partial x^k}\Big) = 1. \tag{5.1.2}$$

First, consider a Calabi-Yau three-fold X with holomorphic coordinates $z^j = x^j + iy^j$, and let $\phi(z^j, \bar{z}^j)$ be the Kähler potential, multiplied by 2, such that the Kähler form $\omega = i\partial\overline{\partial}\phi/2$. The Ricci-flat condition for the corresponding Riemannian metric is

$$\det\left(\frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k}\right) = 1. \tag{5.1.3}$$

[1]It follows from the work of Hitchin [57] that the natural Weil-Petersson metric on the space of special Lagrangian submanifolds has this form. More precisely, it is shown in [57] that the Kähler potentials of X and its mirror Y both satisfy the Monge-Ampère equation (5.1.2) and are related by a Legendre transform on the base. The fibres of the special Lagrangian fibration of Y are dual (by a Fourier transform) tori to the fibres of $\pi : X \longrightarrow B$.

Now, assume that the potential ϕ is invariant under translations in the imaginary directions y^j. The complex Monge-Ampère equation (5.1.3) reduces to the real Monge-Ampère equation (5.1.2) for $\phi = \phi(x^1, x^2, x^3)$, and the Riemannian metric and the Kähler form are given by

$$g = \phi_{jk}(dx^j \otimes dx^k + dy^j \otimes dy^k), \qquad \omega = \frac{i}{2}\phi_{jk} dz^j \wedge d\bar{z}^k, \tag{5.1.4}$$

where $\phi_{jk} := \frac{\partial^2 \phi}{\partial x^j \partial x^k}$. The metric (5.1.4) is a semi-flat CY metric. One can regard the x^j as local coordinates in an open set $B \subset \mathbb{R}^3$. The freedom in choosing the coordinates x^j without changing the equation (5.1.2) is given by affine transformations[2] $\mathbf{x} \mapsto M\mathbf{x} + \mathbf{b}$, where $M \in SL(3, \mathbb{R})$ and $\mathbf{b}$ is a constant vector in $\mathbb{R}^3$.

Conversely, given a 3-real dimensional affine manifold B with a metric of Hessian type (5.1.1) where ϕ satisfies the Monge-Ampère equation (5.1.2), one can construct the Calabi-Yau metric on $X = TB$ by (5.1.4). We then compactify the fibres by quotienting them by a lattice. This leads to a T^3-invariant Calabi-Yau structure on the total space of a toric fibration $\pi : X \longrightarrow B$.

One consequence of mirror conjecture is that the base metric g_B (5.1.1) should have singularities in codimension two. The research presented in this chapter is motivated by the work of Loftin, Yau and Zaslow (LYZ) [13], who were interested in a local metric model near the trivalent vertex of a Y-shaped singularity. LYZ constructed a candidate for such metrics as a cone over the definite elliptic affine sphere metric with three singular points. This is where the affine sphere equation comes in. The relation between affine spheres and a Hessian metric of the form (5.1.1) is discussed in section 5.1.1.

The LYZ construction of the metric comes down to looking for solutions of the affine sphere equation [58]

$$\psi_{z\bar{z}} + \frac{1}{2}e^{\psi} + |U|^2 e^{-2\psi} = 0, \qquad U_{\bar{z}} = 0, \tag{5.1.5}$$

where ψ and U are real and complex functions respectively on an open set in $\mathbb{C}$. LYZ set $U = z^{-2}$ to account for the singularity of the metric they considered. They then proved the existence of a radially symmetric solution ψ of (5.1.5) with a prescribed behaviour near the singularity $z = 0$, and established the existence of the global solution to the coordinate-independent version of (5.1.5) on S^2 minus three points. The monodromy

[2] The affine transformations induce a change in the potential $\phi \rightarrow (\det M)^2\phi$, thus ϕ should be regarded as a section of the second power of the real determinant line bundle over B.

of the resulting affine structure has not been calculated, so it is not yet clear that the metric coincides with the one predicted by Gross-Siebert [59, 60] and Haase-Zharkov [61].

As one of our results, we show below that the radially symmetric solutions $\psi(|z|)$ of (5.1.5) with $U = z^{-2}$ are the Painlevé III transcendents.

Proposition 5.1.1 *Solutions to* (5.1.5) *with* $U = z^{-2}$ *invariant under a group of rotations* $z \longrightarrow e^{ic}z,\ c \in \mathbb{R}$ *are of the form*

$$\psi(z, \bar{z}) = \log H(\rho) - 3\log(\rho), \qquad \rho = |z|^{1/2},$$

where H *satisfies*

$$H_{\rho\rho} = \frac{(H_\rho)^2}{H} - \frac{H_\rho}{\rho} - \frac{8H^2}{\rho} - \frac{16}{H}$$

which is the Painlevé III equation with parameters $(-8, 0, 0, -16)$.

Proof. Set $U = z^{-2}$ and look for solutions of (5.1.5) of the form $\psi = \psi(s)$, where $s = |z|$. Making a substitution $\psi(s) = \log(s^{-3/2}H(s))$ and introducing a new independent variable by $s = \rho^2$ yields the the following ODE for $H = H(\rho)$

$$H_{\rho\rho} = \frac{(H_\rho)^2}{H} - \frac{H_\rho}{\rho} - \frac{8H^2}{\rho} - \frac{16}{H}. \tag{5.1.6}$$

This is the celebrated Painlevé III equation [62]

$$H_{\rho\rho} = \frac{(H_\rho)^2}{H} - \frac{H_\rho}{\rho} + \frac{\alpha H^2 + \beta}{\rho} + \gamma H^3 + \frac{\delta}{H}$$

with special values of parameters

$$(\alpha, \beta, \gamma, \delta) = (-8, 0, 0, -16).$$

In the classification of Okamoto [63] it falls into type $D7$.

$\square$

This proves that under the radial symmetry assumption $\psi = \psi(|z|)$ the affine sphere equation (5.1.5) with $U = z^{-2}$ reduces to a Painlevé III equation. The following argument shows that real solutions which determine semi-flat CY metrics are the Painlevé III transcendents. In general, a Painlevé III equation may have two types of special (i.e. non-transcendental) solutions: the finite number of rational solutions and a one parameter family of Riccati type solutions expressible by special functions [62]. For the values of parameters in (5.1.6) the Riccati solutions do not exist, and there exists a unique algebraic solution

$$H = -(2\rho)^{1/3}.$$

This corresponds to

$$\psi = \frac{1}{3}\log(2) - \frac{4}{3}\log(|z|) + \log(-1)$$

which is not real. There are Bäcklund transformations leading to new solutions, but they change the value of the parameters. This shows that a real radial solution to the affine sphere equation (5.1.5) is transcendental.

The fact that equation (5.1.5) reduces to a Painlevé equation is perhaps not surprising. As we shall prove in section 5.2, the affine sphere equation is a symmetry reduction of the ASDYM equation on $\mathbb{R}^4$ with gauge group $SU(2,1)$. Therefore solutions of (5.1.5) can be described by holomorphic twistor data and any ODE arising as a reduction of (5.1.5) by another symmetry must be of Painlevé type in agreement with an integrable dogma [23, 33].

The main result of this chapter is a novel gauge-invariant characterisation of the affine sphere equation (5.1.5) as a special case of the $SU(2,1)$ Hitchin equations [64], which in turn arise from a reduction of the ASDYM equation by a 2-dimensional group of translation. This work is motivated by the fact that (5.1.5) is closely related to a well known integrable equation, namely the Tzitzéica equation

$$u_{xy} = e^u - e^{-2u}, \tag{5.1.7}$$

which was shown via an explicit ansatz to be a reduction of the ASDYM equation [65]. Here the variables x, y and the function $u(x,y)$ are real. We note that the Tzitzéica equation and the affine sphere equation are two different real forms of the same holomorphic equation, which we shall call 'holomorphic Tzitzéica equation'. In section 5.2.1 we characterise the holomorphic Tzitzéica equation as a reduction of the ASDYM equation on $\mathbb{C}^4$ with gauge group $SL(3,\mathbb{C})$, and impose reality conditions to obtain the corresponding results for equations (5.1.5) and (5.1.7). The gauge invariant characterisation of the affine sphere equation (5.1.5) via Hitchin equations is given in theorem 5.1.2 below.

Let A be an $\mathfrak{su}(2,1)$-valued connection on a rank 3 complex vector bundle $V \to \mathbb{C}$ with the curvature $F_A = dA + A \wedge A$, and let Φ be a one-form with values in $\mathrm{adj}(V)$.

Choose a local trivialisation of V and set

$$A = A_z dz + (A_z)^* d\bar{z}, \quad \Phi = Q d\bar{z}, \quad D = d + A,$$

where $m^* := -\eta^{-1}\bar{m}^t\eta$ with $\eta = \text{diag}(1, 1, -1)$, so that $\Phi^* = Q^* dz$.

Theorem 5.1.2 *The Hitchin equations*

$$F_A - \Phi \wedge \Phi^* - \Phi^* \wedge \Phi = 0, \qquad D\Phi = 0 \tag{5.1.8}$$

hold with

$$A_z = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2}\psi_z & -Ue^{-\psi} \\ 0 & 0 & \frac{1}{2}\psi_z \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.1.9}$$

if the functions (ψ, U) *satisfy the affine sphere equation* (5.1.5).

Conversely, any solution to the $SU(2,1)$ *Hitchin equations such that*

1. Q *has minimal polynomial* t^2 *and* $Tr(QQ^*) \neq 0$,
2. $Tr\left((D_z Q^*)^2\right) = 0, \quad Tr\left((D_z Q^*)^2 (D_{\bar{z}} Q)^2\right) \neq 0,$
3. $Tr[(QQ^*)^4 - (Q^*Q)^2(D_z Q^*)(D_{\bar{z}}Q) + Q^*Q(D_z Q^*)QQ^*(D_{\bar{z}}Q)] = 0$

is equivalent to (5.1.9) *by gauge and coordinate transformations.*

As a consequence of theorem 5.1.2 and proposition 5.1.1, the reduced Lax pair of the ASDYM equation (or the Hitchin equations) associated with the affine sphere equation gives rise to an alternative isomonodromic Lax pair for the Painlevé III equation in terms of 3×3 matrices. We derive this Lax pair in section 5.3.

Finally, we come to consider the semi-flat Calabi-Yau metrics constructed by Loftin, Yau and Zaslow. The connection between solutions of the affine sphere equation (5.1.5) and the Calabi-Yau metric (5.1.4) has not been made explicit in [13]. In section 5.4 we derive a local expression for a semi-flat Calabi-Yau metric (5.1.4) explicitly in terms of a solution of the affine sphere equation. This result is summarised in the following proposition.

Proposition 5.1.3 *Given a semi-flat Calabi-Yau metric* (5.1.4), *where* $\phi(x)$ *satisfies the Monge-Ampère equation* (5.1.2), *and* $\phi(cx) = c^2\phi(x)$ *where c is a non-zero constant,* *there exist complex coordinates $\{z, w, \xi\}$ such that the metric g and the Kähler form ω can be written as*

$$\begin{aligned} g &= e_1\bar{e}_1 + e_2\bar{e}_2 + e_3\bar{e}_3, \\ \omega &= \frac{i}{2}\left(e_1 \wedge \bar{e}_1 + e_2 \wedge \bar{e}_2 + e_3 \wedge \bar{e}_3\right), \end{aligned} \tag{5.1.10}$$

where

$$\begin{aligned} e_1 &= dw - \frac{i}{2}e^{\psi}(\bar{\xi}dz + \xi d\bar{z}), \\ e_2 &= \frac{e^{\psi/2}}{\sqrt{2}}\left(\,(w + i\xi\psi_z)dz + i(d\xi + e^{-\psi}\bar{U}\bar{\xi}d\bar{z})\,\right) \\ e_3 &= \frac{e^{\psi/2}}{\sqrt{2}}\left(\,i(d\bar{\xi} + e^{-\psi}U\xi dz) + (w + i\bar{\xi}\psi_{\bar{z}})d\bar{z}\,\right), \end{aligned} \tag{5.1.11}$$

and $\psi(z, \bar{z})$, $U(z)$ are real and complex functions respectively defined on an open set in $\mathbb{C}$ which satisfy the affine sphere equation (5.1.5).

5.1.1 Affine geometry and Hessian metrics

In this section we introduce affine spheres and summarise their relations with a Hessian metric (5.1.1). Interested readers are referred to [58, 66, 67, 68] for details. Affine differential geometry is the study of properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under volume-preserving affine transformations

$$\mathbf{x} \longmapsto A\mathbf{x} + \mathbf{b},$$

where $\mathbf{x} \in \mathbb{R}^{n+1}$, $A \in SL(n+1, \mathbb{R})$ and $\mathbf{b}$ is a constant vector in $\mathbb{R}^{n+1}$. In affine differential geometry, one only assumes the existence of a flat connection D and a parallel volume element on $\mathbb{R}^{n+1}$, but not an ambient metric. Let us compare this with the Euclidean geometry of a hypersurface Σ in $\mathbb{R}^{n+1}$. The Euclidean metric induces the first fundamental form h_I on Σ and determines the unit normal vector $\mathbf{n}$. One has the Gauss formula

$$D_X Y = \nabla_X Y + h_{II}(X, Y)\,\mathbf{n},$$

where ∇ is the Levi-Civita connection of h_I, h_{II} is the second fundamental form and X, Y are tangent vector fields to Σ. Without an ambient metric, one has a choice of transversal vector fields to the hypersurface Σ. For an arbitrary transversal vector field ξ, the Gauss formula defines a symmetric bilinear form h called the affine second fundamental form, instead of h_{II}, and an induced connection ∇ on Σ. One can choose

ξ uniquely up to sign to satisfy the following properties. Let θ be an induced volume element on Σ defined by $\theta(X_1, \ldots, X_n) = \mathrm{Det}[X_1, \ldots, X_n, \xi]$, where the determinant function Det is regarded as a parallel volume form on $\mathbb{R}^{n+1}$ and X_i, $i = 1, \ldots, n$ are tangent vector fields to Σ. Then,

1. θ is parallel with respect to the induced connection ∇, i.e. $\nabla\theta = 0$,

2. θ coincides with the volume element of the affine second fundamental form h. That is, $\theta(X_1, \ldots, X_n) = |\det h(X_i, X_j)|^{\frac{1}{2}}$.

Such a transversal vector field ξ is called the affine normal field and is affine invariant. Other affine invariants on Σ can be defined using ξ. In particular one has the structure equations which are the analogues of the Gauss and Weingarten formulas:

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \tag{5.1.12}$$

$$D_X \xi = -S(X). \tag{5.1.13}$$

The affine-invariant symmetric bilinear form h is called the Blaschke metric or the affine metric and is conformal to the second fundamental form h_{II}. Again X, Y are tangent vector fields to Σ. The operator $S : T\Sigma \longrightarrow T\Sigma$ in (5.1.13) is called the affine shape operator and $H = \frac{1}{n}\mathrm{Tr}(S)$ is the affine mean curvature.

We are now in a position to define affine spheres. An affine sphere is a hypersurface Σ whose affine shape operator S is a multiple of the identity. That is, $S = HI$, I being the identity matrix. If $H = 0$, Σ is called an improper affine sphere, and if $H \neq 0$, Σ is called a proper affine sphere. Equivalently, an improper affine sphere is a hypersurface whose affine normal fields are all parallel, whereas a proper affine sphere is a hypersurface whose affine normal lines meet at a single point, called the centre. In this chapter we are mainly interested in convex affine spheres, in which the affine normal fields ξ can be chosen to point to the convex side of the hypersurfaces and the Blaschke metrics are positive definite metrics (a discussion of this can be found in [68]). In such case an improper affine sphere is called a parabolic affine sphere, and there are two types of proper affine spheres: elliptic type where the affine mean curvature $H > 0$ and ξ points towards the centre, and parabolic type where $H < 0$ and ξ points away from the centre.

Let us now discuss the relation between affine spheres and a Hessian metric of the form (5.1.1) and how this is useful in solving the Monge-Ampère equation (5.1.2), which determines semi-flat CY metrics. Equation (5.1.2) in 3 dimensions is known not to be integrable, at least in the sense of the hydrodynamic reductions [69]. However, using its interpretation in affine differential geometry, its homogeneous solutions are characterised by a PDE in 2 dimensions; the affine sphere equation (5.1.5), which we show in this chapter to be integrable as a reduction of the ASDYM equation.

It is known [66] that a Hessian metric (5.1.1) which satisfies the Monge-Ampère equation (5.1.2) in $n+1$ dimensions is the Blaschke metric of an improper affine sphere in $\mathbb{R}^{n+1}$, given by a graph $(x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n, \phi(x^i))$. Moreover, it was shown by Baues and Cortés [70] that a metric cone $\frac{1}{H}dr^2 + r^2 h$ over a proper affine sphere metric h is the Blaschke metric of an improper affine sphere. In particular, for convex hypersurfaces we have that a cone over an elliptic affine sphere is a parabolic affine sphere, with positive definite affine metric. This means that a class of solutions of the 3-dimensional Monge-Ampère equation (5.1.2) are determined by the equation which governs elliptic affine 2-spheres, which is (5.1.5).

Below we present an alternative proof of the above result by showing that if a solution $\phi(x^i)$ of (5.1.2) is homogeneous of degree 2, then an improper affine hypersphere metric g_B (5.1.1) is the metric cone over a proper affine hypersphere. We shall carry over the homogeneity analysis for a general Hessian metric in $n+1$ dimensions, and then restrict our attention to $n = 2$ where there is a direct connection with the semi-flat CY manifolds on one side and integrability on the other. The following proposition follows from combining the results of Calabi [66] and Baues-Cortés [70]. However, here we give a direct elementary proof not based on affine differential geometry. It has certain advantages as it exhibits explicit coordinate transformations between solutions to various forms of homogeneous Hessian equations.

Proposition 5.1.4 *Let $\phi = \phi(x^i)$ be a solution to the Monge-Ampère equation* (5.1.2) *on an open ball $B \subset \mathbb{R}^{n+1}$ such that $\phi(cx) = c^2\phi(x)$ for any non-zero constant c. Then there exists a local coordinate system $(p_1, \ldots, p_n, r)$ on B such that the metric* (5.1.1) *is*

$$g_B = dr^2 + r^2 \frac{1}{w}\Big(\frac{\partial^2 w}{\partial p_\alpha \partial p_\beta}\Big) dp_\alpha dp_\beta, \qquad \alpha, \beta = 1, \ldots, n, \tag{5.1.14}$$

where $w = w(p_\alpha)$ satisfies

$$\det\Big(\frac{\partial^2 w}{\partial p_\alpha \partial p_\beta}\Big) = \frac{1}{w^{n+2}}. \tag{5.1.15}$$

Proof. Consider the Hessian metric (5.1.1) with ϕ homogeneous of degree 2. Therefore $V = x^i\partial/\partial x^i$ is a homothety with $\mathcal{L}_V g_B = 2g_B$. Locally there exists a function $r : B \longrightarrow \mathbb{R}$ such that $V = r\partial/\partial r$ and

$$g_B = \gamma(dr + r\alpha)^2 + r^2 h$$

where h, α, γ are a metric, a one-form and a function respectively on the space of orbits of V. The relation $\partial_i(x^j\phi_j) = 2\phi_i$ gives

$$g_B(V, ...) = x^i\phi_{ij}dx^j = d\phi.$$

Thus $d(\gamma(dr + r\alpha)) = 0$ and we can redefine r to set $\alpha = 0$ and $\gamma = 1$. We also note that $|V|^2 = x^i x^j \phi_{ij} = 2\phi$, and recognise g_B as a cone over h

$$g_B = dr^2 + r^2 h, \quad \phi = \frac{r^2}{2}. \tag{5.1.16}$$

Now let us consider the surface $r = 1$ given by a graph in $\mathbb{R}^{n+1}$

$$(\tilde{x}^1, \ldots, \tilde{x}^n) \longmapsto (\tilde{x}^1, \ldots, \tilde{x}^n, v(\tilde{x}^\alpha)),$$

where $\tilde{x}^\alpha, \alpha = 1, \ldots, n$, parametrise the surface. We shall show that its induced metric h is given by

$$h = \frac{\partial_\alpha\partial_\beta v}{\tilde{x}^\gamma\partial_\gamma v - v} d\tilde{x}^\alpha d\tilde{x}^\beta, \tag{5.1.17}$$

where $\partial_\alpha := \partial/\partial\tilde{x}^\alpha$. To prove it, restrict the function ϕ to the surface $r = 1$. This gives an identity $\phi(\tilde{x}^\alpha, v(\tilde{x}^\alpha)) = 1/2$. We differentiate this identity implicitly with respect to $\tilde{x}^\alpha$ and express the first and second derivatives of ϕ in terms of the derivatives of v

$$\begin{aligned}
0 &= \partial_\alpha\phi + \partial_{n+1}\phi\,\partial_\alpha v, \\
0 &= \partial_\alpha\partial_\beta\phi + \partial_\alpha\partial_{n+1}\phi\,\partial_\beta v + \partial_\beta\partial_{n+1}\phi\,\partial_\alpha v + \partial_{n+1}^2\phi\,\partial_\alpha v\partial_\beta v + \partial_{n+1}\phi\,\partial_\alpha\partial_\beta v, \\
2\phi &= \tilde{x}^\alpha\partial_\alpha\phi + v\partial_{n+1}\phi = 1,
\end{aligned}$$

where the last relation is just the homogeneity condition restricted to the hypersurface $\phi = 1/2$. Substituting all that to g_B gives (5.1.17).

Now if the function ϕ in the Hessian metric g_B satisfies the Monge-Ampère equation (5.1.2) then v satisfies

$$\det \frac{\partial^2 v}{\partial\tilde{x}^\alpha\partial\tilde{x}^\beta} = (\tilde{x}^\alpha\partial_\alpha v - v)^{n+2}. \tag{5.1.18}$$

To see it, let us write the coordinates x^i on $\mathbb{R}^{n+1}$ as $(x^1, ..., x^n, x^{n+1}) = (r\tilde{x}^1, ..., r\tilde{x}^n, rv(\tilde{x}^\alpha))$, i.e. regard $\mathbb{R}^{n+1}$ as the cone over the $r = 1$ surface. Now consider the invariant volume element

$$\sqrt{|g_B|}\, dx^1 \wedge \ldots \wedge dx^n \wedge dx^{n+1} = \sqrt{|\tilde{g}_B|}\, d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n \wedge dr \tag{5.1.19}$$

where $|g_B|$ is the absolute value of the determinant of the Hessian metric (5.1.1) written in the coordinates x^i and $\tilde{g}_B$ is the same metric expressed in the basis $\{d\tilde{x}^\alpha, dr\}$. We contract both sides of (5.1.19) with V. On the LHS of (5.1.19) we use the form $V = x^i\partial/\partial x^i$ and on the RHS use $V = r\partial/\partial r$. We now set $r = 1$ and impose the Monge-Ampère equation (5.1.2), $\det g_B = \det \phi_{ij} = 1$. This yields

$$v - \tilde{x}^\alpha \partial_\alpha v = \sqrt{|\tilde{g}_B|}.$$

On the surface $r = 1$, one has $\det \tilde{g}_B = \det h$ where h is given by (5.1.17). Substituting this in the above formula and taking squares of both sides yields (5.1.18). Note[3] that we have taken $\det h > 0$ from the assumption that $\det g_B = \det \phi_{jk} = 1$.

To obtain the statement in the proposition, perform a Legendre transform

$$p_\alpha = \frac{\partial v}{\partial \tilde{x}^\alpha}, \quad w(p_\alpha) = \tilde{x}^\alpha \frac{\partial v}{\partial \tilde{x}^\alpha} - v, \quad \tilde{x}^\alpha = \frac{\partial w}{\partial p_\alpha}.$$

Using $dp_\alpha = \partial_\alpha \partial_\beta v \, d\tilde{x}^\beta$ yields

$$h = \frac{1}{w} \frac{\partial^2 w}{\partial p_\alpha \partial p_\beta} dp_\alpha dp_\beta \tag{5.1.20}$$

and

$$\frac{\partial^2 w}{\partial p_\alpha \partial p_\beta} = \Big(\frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} \Big)^{-1},$$

which implies (5.1.14) and (5.1.15).

□

The metric (5.1.20) which satisfies (5.1.15) is indeed the Blaschke metric of a proper affine sphere (see for example [68]).

Before we go on to discuss the affine sphere equation (5.1.5), let us digress briefly to explain how the above calculation makes contact with the affine spheres as first introduced by Tzitzéica. Without making any reference to affine differential geometry, Tzitzéica [71, 72] has studied surfaces Σ in $\mathbb{R}^3$ for which the ratio of the Gaussian curvature $\mathcal{K}$ to the fourth power of a distance from a tangent plane to some fixed point is a constant. If $\mathcal{K} \neq 0$, one can always rescale the coordinates to set this constant to $+1$ or -1 depending on the sign of the Gaussian curvature. We call this the Tzitzéica condition. The generalisation of the Tzitzéica condition to hypersurfaces in $\mathbb{R}^{n+1}$ is given by

$$\mathcal{K} = \pm \mathcal{D}^{n+2}, \tag{5.1.21}$$

[3] If we started with $\det \phi_{ij} = -1$, which implies $\det h < 0$, the analogous argument would lead to $\det \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} = -(\tilde{x}^\alpha \partial_\alpha v - v)^{n+2}$ in (5.1.18)

where $\mathcal{D} = \mathbf{r} \cdot \mathbf{n}$ is the same as the distance up to sign. It is now well known in affine differential geometry that an immersed hypersurface Σ in $\mathbb{R}^{n+1}$ is a proper affine hypersphere with the origin as its centre if and only if the Tzitzéica condition (5.1.21) holds [67].

In more details, let us consider a hypersurface Σ immersed in $\mathbb{R}^{n+1}$ with the flat metric $\delta_{jk}\, dx^j dx^k$, given by a graph

$$\mathbf{r} = (\tilde{x}^1, \dots, \tilde{x}^n, v(\tilde{x}^1, \dots, \tilde{x}^n)). \tag{5.1.22}$$

The first and the second fundamental forms on Σ are given by

$$\begin{aligned} h_I &= d\mathbf{r} \cdot d\mathbf{r} = (\delta_{\alpha\beta} + \partial_\alpha v \partial_\beta v) d\tilde{x}^\alpha d\tilde{x}^\beta, \\ h_{II} &= -d\mathbf{r} \cdot d\mathbf{n} = \frac{1}{\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2}} \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} d\tilde{x}^\alpha d\tilde{x}^\beta, \end{aligned}$$

where $\mathbf{n}$ is the unit normal to Σ. In these coordinates, $\mathcal{D}$ and the Gaussian curvature $\mathcal{K}$ are given by

$$\begin{aligned} \mathcal{D} &= \frac{v - \tilde{x}^\alpha \partial_\alpha v}{\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2}}, \\ \mathcal{K} &= \frac{1}{(\sqrt{1 + (\partial_1 v)^2 + \cdots + (\partial_n v)^2})^{n+2}} \det\left(\frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}\right). \end{aligned}$$

It follows that the Tzitzéica condition (5.1.21) holds if and only if v satisfies

$$\det \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} = \pm (v - \tilde{x}^\alpha \partial_\alpha v)^{n+2}, \tag{5.1.23}$$

where plus and minus signs correspond to positive and negative Gaussian curvature respectively.

The Blaschke metric, which is conformally related to the second fundamental form, can be defined in the context of Euclidean geometry. Let $\mathbf{N}$ denote the transversal vector field of the surface Σ such that the unit normal $\mathbf{n}$ is given by $\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$, i.e. $\mathbf{N} = \nabla(\tilde{x}^{n+1} - v(\tilde{x}^1, ..., \tilde{x}^n))$. Consider a bilinear form

$$\hat{h} = -d\mathbf{r} \cdot d\mathbf{N} = |\mathbf{N}|\; h_{II}.$$

The Blaschke metric is then given by

$$h := |\det \hat{h}|^{-\frac{1}{n+2}}\; \hat{h}. \tag{5.1.24}$$

Therefore, for the surface Σ given by the graph (5.1.22), we have

$$h = \left| \det \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} \right|^{-\frac{1}{n+2}} \frac{\partial^2 v}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta} d\tilde{x}^\alpha d\tilde{x}^\beta,$$

which coincides with the metric (5.1.17) if equation (5.1.18) holds.

We shall now end this section by coming back to dimension $n = 2$ and consider the metric h (5.1.17). For $n = 2$, $\det h > 0$ implies that h is a definite metric. In the context of the Calabi-Yau manifolds, the metric g_B is Riemannian, hence one is interested in positive definite h. The result of Baues and Cortés [70] then implies that h is the Blaschke metric of an elliptic affine sphere, with affine mean curvature $H = 1$. Since h is positive definite we can adopt isothermal coordinates for the affine metric (which are asymptotic coordinates for the second fundamental form h_{II}) and write it as

$$h = e^{\psi} dz d\bar{z}, \tag{5.1.25}$$

for some real valued function $\psi = \psi(z, \bar{z})$. In this form, Simon and Wang [58] proved that the structure equations (5.1.12, 5.1.13) of affine spheres imply that ψ necessarily satisfies the equation (5.1.5)

$$\psi_{z\bar{z}} + \frac{1}{2}e^{\psi} + |U|^2 e^{-2\psi} = 0, \qquad U_{\bar{z}} = 0.$$

The holomorphic cubic differential $U(z)dz^3$ is related to another affine invariant quantity: a totally symmetric tensor called the cubic form $\hat{C}$. It is defined by

$$\hat{C}(X, Y, Z) = h(C(X, Y), Z),$$

where C is the difference tensor $C = \hat{\nabla} - \nabla$ and $\hat{\nabla}$ is the Levi-Civita connection of the Blaschke metric h. Consider h as in (5.1.25) and let $C^i_{jk},\ i, j, k \in \{1, \bar{1}\}$ be the components of C in the basis $e^1 = dz, e^{\bar{1}} = d\bar{z}$. Then it can be shown that the only nonvanishing components of C are $C^{\bar{1}}_{11}$ and $C^{1}_{\bar{1}\bar{1}} = \overline{C^{\bar{1}}_{11}}$, and the function U in (5.1.5) is defined by $U = C^{\bar{1}}_{11} e^{\psi}$. It follows that the cubic form is $\hat{C} = U dz^3 + \bar{U} d\bar{z}^3$ (see e.g. [13]).

Conversely, given a solution of (5.1.5) one can construct an affine sphere with (5.1.25) as its Blaschke metric. We note here that if the holomorphic cubic differential $U(z)dz^3$ is non-zero, one can choose the isothermal coordinates such that $U = 1$. For example, defining $\zeta = \zeta(z)$ by $d\zeta = 2^{-1/3} U^{1/3} dz$ transforms (5.1.5) into

$$\hat{\psi}_{\zeta\bar{\zeta}} + e^{\hat{\psi}} + e^{-2\hat{\psi}} = 0, \tag{5.1.26}$$

where

$$\hat{\psi} = \psi - \frac{1}{3}\log U - \frac{1}{3}\log \bar{U} - \frac{1}{3}\log 2.$$

We will make use of such coordinate transformation[4] in section 5.2.

Loftin, Yau and Zaslow [13] constructed of a class of semi-flat Calabi-Yau metrics (5.1.4) using the base metric g_B as the metric cone over an elliptic affine sphere

$$g_B = \phi_{ij} dx^i dx^j = dr^2 + r^2 e^{\psi} dz d\overline{z}. \tag{5.1.27}$$

The construction given in [13] is implicit. However, as one of our main results, we shall show in section 5.4 that the semi-flat CY metric (5.1.4) can be expressed explicitly in terms of solutions of (5.1.5).

To go further one needs to solve the affine sphere equation (5.1.5). This motivates the study of the integrability of (5.1.5), which we shall present in section 5.2. As mentioned previously, (5.1.5) is closely related to the Tzitzéica equation (5.1.7), which is a well known integrable system[5]. The difference between the equations (5.1.5) and (5.1.7) lies in the relative sign of the two exponential terms on the RHS. For the Tzitzéica equation, $u = 0$ is a solution and other solutions may be constructed using Darboux and Bäcklund transformations, see for example [74]. The affine sphere equation does not seem to have such obvious solutions. However, Calabi [66] has shown that an elliptic affine hypersphere with complete Blaschke metric is an ellipsoid. This is in agreement with the fact that (5.1.5) admits solutions in term of elliptic functions, which can be found by making an ansatz $\psi(\zeta, \overline{\zeta}) = f(\zeta + \overline{\zeta})$ in (5.1.26).

[4] We note that the analytic continuation

$$\hat{\psi}_{\zeta\overline{\zeta}} + e^{\hat{\psi}} - e^{-2\hat{\psi}} = 0$$

of equation (5.1.26) was used by McIntosh [73] to describe minimal Lagrangian immersions in $\mathbb{CP}^2$ and special Lagrangian cones in $\mathbb{C}^3$.

[5] In the context of affine spheres, the Tzitzéica equation arises if the metric h (5.1.17) has $\det h < 0$. By writing the indefinite metric in isothermal coordinates as $h = 2e^u\ dxdy$, Simon and Wang [58] showed that h is the Blaschke metric of an indefinite affine sphere (with negative affine mean curvature) if and only if u satisfies $u_{xy} = e^u - r(x)b(y)e^{-2u}$, where $r(x), b(y)$ are arbitrary non-vanishing functions of one variable which can be normalised by rescaling the isothermal coordinates. Thus, we obtain

$$u_{xy} = e^u - \epsilon e^{-2u}, \tag{5.1.28}$$

where $\epsilon = \pm 1$. The equation with $\epsilon = 1$, (5.1.7), was first derived in [71, 72] for the Tzitzéica surface in $\mathbb{R}^3$ with negative Gaussian curvature $\mathcal{K} = -\mathcal{D}^4$, where the indefinite second fundamental form is written in asymptotic coordinates as $h_{II} = 2e^u\mathcal{D}\ dxdy$.

5.2 Characterisations

In this section we prove theorem 5.1.2 and establish the integrability of the affine sphere equation as a symmetry reduction of the ASDYM equation on $\mathbb{R}^4$ with gauge group $SU(2,1)$.

5.2.1 Tzitzéica equation and affine sphere equation

It was shown in [65] that the Tzitzéica equation (5.1.7) can be obtained from a special ansatz to the ASDYM equation in $\mathbb{R}^{2,2}$ with gauge group $SL(3,\mathbb{R})$. We shall now characterise the Tzitzéica equation and the affine sphere equation as different real forms of a reduction of the ASDYM equation on $\mathbb{C}^4$ with gauge group $SL(3,\mathbb{C})$, via the holomorphic Hitchin equations on $\mathbb{C}^2$. We first look at the complex form of the two equations.

Holomorphic Tzitzéica equation

Consider a holomorphic metric and volume element on $\mathbb{C}^4$

$$ds^2 = 2(dz\ d\tilde{z} - dw\ d\tilde{w}), \qquad \nu = dw \wedge d\tilde{w} \wedge dz \wedge d\tilde{z}.$$

Let $A = A_z dz + A_w dw + A_{\tilde{z}} d\tilde{z} + A_{\tilde{w}} d\tilde{w}$ be a Lie algebra valued connection on a vector bundle $V \to \mathbb{C}^4$. The ASDYM equation is given by

$$F_{zw} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0, \quad F_{\tilde{z}\tilde{w}} = 0.$$

Recall that these equations arise from a Lax pair

$$[D_z - \lambda D_{\tilde{w}}, D_w - \lambda D_{\tilde{z}}] = 0, \tag{5.2.1}$$

where $D_z = \partial_z + A_z$, etc, are covariant derivatives, $F_{z\tilde{z}} = [D_z, D_{\tilde{z}}]$, and (5.2.1) is required to hold for any value of the spectral parameter λ.

Choose the gauge group to be $SL(3,\mathbb{C})$ and assume that A is invariant under the action of two dimensional group of translations such that the metric restricted to the planes spanned by the generators of the group is non-degenerate. Let X_1, X_2 be the generators of the group, then the Higgs fields

$$P = X_1 \lrcorner A, \quad Q = X_2 \lrcorner A$$

belong to the adjoint representation. We can always choose the coordinates so that the group is generated by the two null vectors $X_1 = \partial/\partial\tilde{w}$ and $X_2 = \partial/\partial w$. The ASDYM equation reduces to the holomorphic form of the Hitchin equations [64]

$$D_z Q = 0, \tag{5.2.2a}$$

$$D_{\tilde{z}} P = 0, \tag{5.2.2b}$$

$$F_{z\tilde{z}} + [P, Q] = 0, \tag{5.2.2c}$$

where

$$F_{z\tilde{z}} = \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + [A_z, A_{\tilde{z}}]$$

is a curvature of a holomorphic connection $A = A_z dz + A_{\tilde{z}} d\tilde{z}$ on $\mathbb{C}^2$. The Hitchin equations are invariant under the gauge transformations

$$A \to g^{-1}Ag + g^{-1}\,dg, \qquad P \to g^{-1}Pg, \qquad Q \to g^{-1}Qg \tag{5.2.3}$$

and later we shall also make use of the following coordinate freedom

$$z \longrightarrow \hat{z}(z), \quad \tilde{z} \longrightarrow \hat{\tilde{z}}(\tilde{z}). \tag{5.2.4}$$

The Lax pair (5.2.1) for the ASDYM equation reduces to the following Lax pair for the holomorphic Hitchin equations

$$[D_z - \lambda P, Q - \lambda D_{\tilde{z}}] = 0. \tag{5.2.5}$$

There are several gauge inequivalent ways to embed the Tzitzéica equation (5.1.7) as a special case of the Hitchin equations. The gauge used in [65] is

$$A_{\tilde{w}} = P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_w = Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^u & 0 & 0 \end{pmatrix}, \tag{5.2.6}$$

$$A_z = \begin{pmatrix} u_z & 0 & 0 \\ 1 & -u_z & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.2.7}$$

where $u(z, \tilde{z})$ is a complex-valued function holomorphic in $(z, \tilde{z})$. With this ansatz the Hitchin equations yield the holomorphic Tzitzéica equation

$$u_{z\tilde{z}} = e^u - e^{-2u}. \tag{5.2.8}$$

Choosing the real form $SL(3, \mathbb{R})$ of $SL(3, \mathbb{C})$ and regarding $u = u(x, y)$ as a real function of real coordinates $z = x, \tilde{z} = y$ reduces (5.2.8) to (5.1.7).

On the other hand, performing the coordinate transformation

$$d\hat{z} = \left(\frac{U(z)}{2}\right)^{-\frac{1}{3}} dz, \quad d\hat{\tilde{z}} = \left(\frac{\tilde{U}(\tilde{z})}{2}\right)^{-\frac{1}{3}} d\tilde{z}$$

and setting

$$u = \psi(z,\tilde{z}) - \frac{1}{3}\log\left(\frac{U}{2}\right) - \frac{1}{3}\log\left(\frac{\tilde{U}}{2}\right) + \log\left(-\frac{1}{2}\right)$$

for any branch of $\log\left(-\frac{1}{2}\right)$ puts (5.2.8) in the form

$$\psi_{z\tilde{z}} + \frac{1}{2}e^{\psi} + U(z)\tilde{U}(\tilde{z})e^{-2\psi} = 0, \tag{5.2.9}$$

where we have dropped hats of the new variables. Equation (5.2.9) then reduces to the affine sphere equation (5.1.5) under the Euclidean reality condition $\tilde{z} = \bar{z}$ and restricting the gauge group to $SU(2,1)$, which implies the constraint $\tilde{U} = \bar{U}$.

Now we shall establish a gauge invariant characterisation of the ansatz (5.2.6), (5.2.7) in terms of the gauge and Higgs fields of the Hitchin equations. We will make use of the following lemma.

Lemma 5.2.1 *Consider* 3 *by* 3 *complex matrices* P, Q *such that*

$$P^2 = Q^2 = 0, \qquad Tr(PQ) = \omega \neq 0. \tag{5.2.10}$$

There exists a gauge transformation such that P, Q *are in the form* (5.2.6) *for some* u.

Proof. The conditions (5.2.10) are invariant under the gauge transformations

$$P \longrightarrow g^{-1}Pg, \qquad Q \longrightarrow g^{-1}Qg.$$

These conditions imply that the nullities (dimensions of the kernels of the associated linear maps) satisfy $n(QP) < 3$ and $n(P) = 2$. Thus

$$\mathrm{Ker}(QP) = \mathrm{Ker}(P).$$

Also $\mathrm{rank}(QP) = 1$ and $\mathrm{Im}(QP)$ is contained in the one-dimensional image of Q, therefore

$$\mathrm{Im}(QP) = \mathrm{Im}(Q). \tag{5.2.11}$$

Choose a Jordan basis $(\mathbf{v}, \mathbf{u}, \mathbf{w})$ of $\mathbb{C}^3$ such that

$$P(\mathbf{w}) = \mathbf{v}, \quad P(\mathbf{v}) = 0, \quad P(\mathbf{u}) = 0. \tag{5.2.12}$$

From (5.2.11) $\mathrm{Im}(Q) = \mathrm{span}\,(Q(\mathbf{v}))$, thus $Q(\mathbf{u}) = aQ(\mathbf{v})$, $Q(\mathbf{w}) = bQ(\mathbf{v})$ for some a, b so that $\mathrm{Ker}(Q) = \mathrm{span}\,(\mathbf{u} - a\mathbf{v}, \mathbf{w} - b\mathbf{v})$. Use the freedom in the basis (5.2.12) to set

$$\mathbf{w}' = \mathbf{w} - b\mathbf{v}, \quad \mathbf{u}' = \mathbf{u} - a\mathbf{v}, \quad \mathbf{v}' = \mathbf{v}.$$

Now

$$\begin{aligned} P(\mathbf{w}') &= \mathbf{v}', \quad P(\mathbf{v}') = 0, \quad P(\mathbf{u}') = 0, \\ Q(\mathbf{w}') &= 0, \quad Q(\mathbf{u}') = 0, \quad Q(\mathbf{v}') = c\mathbf{u}' + \omega\mathbf{w}', \end{aligned}$$

where $\omega \neq 0$ as $Tr(PQ) = \omega \neq 0$. There is still freedom in (5.2.12):

$$\mathbf{v}'' = \mathbf{v}', \quad \mathbf{u}'' = \mathbf{u}', \quad \mathbf{w}'' = \mathbf{w}' + (c/\omega)\mathbf{u}'$$

so that, dropping primes,

$$P(\mathbf{w}) = \mathbf{v}, \quad P(\mathbf{v}) = 0, \quad P(\mathbf{u}) = 0,$$

$$Q(\mathbf{w}) = 0, \quad Q(\mathbf{u}) = 0, \quad Q(\mathbf{v}) = \omega\mathbf{w}.$$

Ordering the basis $(\mathbf{v}, \mathbf{u}, \mathbf{w})$ yields the matrices in the desired form, i.e. $P_{13} = 1, Q_{31} = \omega$, and all other components vanish. The residual gauge freedom is

$$\mathbf{w} \to \alpha\mathbf{w}, \quad \mathbf{v} \to \alpha\mathbf{v}, \quad \mathbf{u} \to \beta\mathbf{u}$$

and the change of basis matrix gives the residual $GL(3, \mathbb{C})$ gauge transformation. In the $SL(3, \mathbb{C})$ case we set $\beta = \alpha^{-2}$. The statement of the lemma now follows by setting $\omega = e^u$.

□

We shall now give a set of necessary and sufficient conditions allowing solutions of the Hitchin equations (5.2.2a, b, c) to be transformed into (5.2.6), (5.2.7) by gauge and coordinate symmetries.

Proposition 5.2.2 *Let* $(Q, P, A = A_z dz + A_{\tilde{z}} d\tilde{z})$ *be a solution of the holomorphic Hitchin equations* (5.2.2a, b, c)*, with gauge group* $SL(3, \mathbb{C})$*. Then,* $(Q, P, A_z, A_{\tilde{z}})$ *can be transformed into* (5.2.6),(5.2.7) *by gauge symmetry and coordinate symmetry* (5.2.4) *if and only if the following conditions hold:*

(i) P *and* Q *have minimal polynomial* t^2*, with* $Tr(PQ) \neq 0$.

(ii) $Tr\left((D_z P)^2\right) = 0 = Tr\left((D_{\tilde{z}} Q)^2\right)$ *and* $Tr\left((D_z P)^2 (D_{\tilde{z}} Q)^2\right) \neq 0$.

(iii) $Tr M = 0$*, where*

$M = (PQ)^4 + (PQ)^2 (D_z P)(D_{\tilde{z}} Q) - PQ(D_z P) Q P (D_{\tilde{z}} Q)$.

Proof. The proof of the necessary conditions is straightforward. It can be shown by direct calculation that (5.2.6),(5.2.7) satisfy conditions (**i**), (**ii**), (**iii**). The three conditions are gauge invariant by the cyclic property of the trace. Under the coordinate transformation (5.2.4), the connection $(A_z, A_{\tilde{z}})$ and the Higgs fields (P, Q) transform as

$$\begin{aligned} \hat{A}_{\hat{z}} &= \left(\frac{d\hat{z}}{dz}\right)^{-1} A_z, \quad \hat{A}_{\hat{\tilde{z}}} = \left(\frac{d\hat{\tilde{z}}}{d\tilde{z}}\right)^{-1} A_{\tilde{z}}, \\ \hat{Q} &= \left(\frac{d\hat{\tilde{z}}}{d\tilde{z}}\right)^{-1} Q, \quad \hat{P} = \left(\frac{d\hat{z}}{dz}\right)^{-1} P. \end{aligned}$$

Thus, using condition (**i**), the square of the covariant derivative is given by

$$(\hat{D}_{\hat{z}}\hat{P})^2 = \left(\frac{d\hat{z}}{dz}\right)^{-4} (D_z P)^2$$

and similarly for $(D_{\tilde{z}}Q)^2$. Therefore, conditions (**i**) and (**ii**) are invariant under the coordinate transformation. A similar calculation shows that (**iii**) is also invariant under (5.2.4).

Conversely, we now show that any solution to (5.2.2a, b, c) such that all the conditions in proposition 5.2.2 hold can be gauge and coordinate transformed into the form (5.2.6),(5.2.7).

Firstly, by lemma 5.2.1, condition (**i**) implies that we can use gauge symmetry to put the Higgs fields (Q, P) in the form (5.2.6). The equations (5.2.2a) and (5.2.2b) imply that $A_z, A_{\tilde{z}}$ are of the form

$$A_z = \begin{pmatrix} n & 0 & 0 \\ r & u_z - 2n & 0 \\ m & t & n - u_z \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} p & s & h \\ 0 & -2p & k \\ 0 & 0 & p \end{pmatrix}, \tag{5.2.13}$$

where n, r, m, t, p, s, h, k are some functions of $z, \tilde{z}$. Note that we have also used the assumption that the fields are $\mathfrak{sl}(3, \mathbb{C})$ valued, hence traceless. Next, to set the diagonal elements of $(A_z, A_{\tilde{z}})$ to be as in (5.2.7), we consider the residual gauge freedom. Lemma 5.2.1 implies that the gauges preserving (Q, P) are given by

$$g(z, \tilde{z}) = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix} \tag{5.2.14}$$

for an arbitrary function $a(z, \tilde{z}) \neq 0$. Thus, using (5.2.3) we have

$$A_z \quad \longrightarrow \quad \begin{pmatrix} n + \frac{a_z}{a} & 0 & 0 \\ ra^3 & u_z - 2n - 2\frac{a_z}{a} & 0 \\ m & \frac{t}{a^3} & n - u_z + \frac{a_z}{a} \end{pmatrix},$$

$$A_{\tilde{z}} \quad \longrightarrow \quad \begin{pmatrix} p + \frac{a_{\tilde{z}}}{a} & \frac{s}{a^3} & h \\ 0 & -2p - 2\frac{a_{\tilde{z}}}{a} & ka^3 \\ 0 & 0 & p + \frac{a_{\tilde{z}}}{a} \end{pmatrix}.$$

We choose $a(z, \tilde{z})$ such that

$$(\ln a)_z = u_z - n, \quad \text{and} \quad (\ln a)_{\tilde{z}} = -p.$$

This is allowed because the compatibility condition

$$\partial_z p + \partial_z \partial_{\tilde{z}} u - \partial_{\tilde{z}} n = 0 \tag{5.2.15}$$

holds automatically as a consequence of condition (**iii**). To see it, note that equation (5.2.2c) implies

$$\partial_z p + \partial_z \partial_{\tilde{z}} u - \partial_{\tilde{z}} n + mh + tk = e^u.$$

Hence, condition (5.2.15) is equivalent to

$$mh + tk = e^u,$$

which holds by (**iii**).

Note that at this point elements of $(A_z, A_{\tilde{z}})$ will be transformed, however, for convenience we will label them with the same letters as in (5.2.13). Thus we have set $n = u_z$ and $p = 0$. We now proceed to deal with r, m, t, s, h, k. $\mathrm{Tr}\,((D_z P)^2 (D_{\tilde{z}} Q)^2) \neq 0$ in condition (**ii**) implies that $r, t, s, k \neq 0$, and

$$\mathrm{Tr}\left((D_z P)^2\right) = 0 = \mathrm{Tr}\left((D_{\tilde{z}} Q)^2\right)$$

gives

$$m = 0 = h.$$

Hence (5.2.2c) becomes

$$\begin{aligned} u_{z\tilde{z}} + rs &= e^u \\ s_z + 2su_z &= 0 \\ r_{\tilde{z}} &= 0 \\ k_z - ku_z &= 0 \\ t_{\tilde{z}} &= 0 \\ tk &= e^u. \end{aligned}$$

Since $r, t, s, k \neq 0$, we can solve the above equations. The last three equations imply that t is a constant, and thus can be set to 1 by a constant gauge transformation of the form (5.2.14) with $a = t^{-1/3}$, and s is determined to be of the form $b(\tilde{z})e^{-2u}$. This results in

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^u & 0 & 0 \end{pmatrix},$$

$$A_z = \begin{pmatrix} u_z & 0 & 0 \\ r(z) & -u_z & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = \begin{pmatrix} 0 & b(\tilde{z})e^{-2u} & 0 \\ 0 & 0 & e^u \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.16}$$

Note that the gauge is now fixed. To get to ansatz (5.2.6),(5.2.7), we will now use the coordinate symmetry. Define $\hat{z}, \hat{\tilde{z}}$ such that

$$d\hat{z} = e^{j(z)}dz, \quad d\hat{\tilde{z}} = e^{l(\tilde{z})}d\tilde{z},$$

and set

$$\hat{u} := u - j(z) - l(\tilde{z}).$$

By choosing $j(z), l(\tilde{z})$ such that $e^{3j(z)} = r(z)$ and $e^{3l(\tilde{z})} = b(\tilde{z})$, (5.2.16) becomes gauge equivalent to (5.2.6),(5.2.7) in the new variables $(\hat{z}, \hat{\tilde{z}}, \hat{u})$. The gauge transformation we need in the final step is given by (5.2.3) with

$$g(\hat{z}, \hat{\tilde{z}}) = \begin{pmatrix} e^{-j(z(\hat{z}))} & 0 & 0 \\ 0 & e^{j(z(\hat{z}))} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

We note that substituting (5.2.16) to the Hitchin equations yields

$$u_{z\tilde{z}} = e^u - r(z)b(\tilde{z})e^{-2u}. \tag{5.2.17}$$

Therefore, the change of coordinates can, roughly speaking, be regarded as setting $r(z)$ and $b(\tilde{z})$ to constants such that $r(z)b(\tilde{z}) = 1$.

Affine sphere equation

We shall now choose the Euclidean reality condition and select the real form $SU(2,1)$ of $SL(3,\mathbb{C})$ to deduce theorem 5.1.2 from the last proposition.

Proof of Theorem 5.1.2. Consider the ansatz (5.2.16) and equation (5.2.17). By changing the dependent variable from u to

$$\psi = u - \log\left(-\frac{1}{2}\right)$$

for any branch of $\log\left(-\frac{1}{2}\right)$, equation (5.2.17) becomes

$$\psi_{z\tilde{z}} + \frac{1}{2}e^{\psi} + U(z)\tilde{U}(\tilde{z})e^{-2\psi} = 0, \tag{5.2.18}$$

where $U(z) = 2r(z)$, $\tilde{U}(\tilde{z}) = 2b(\tilde{z})$. Then, after an $SL(3,\mathbb{C})$ gauge transformation with

$$g(z,\tilde{z}) = \begin{pmatrix} 0 & 0 & -\sqrt{2}e^{-\frac{\psi}{2}} \\ 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the ansatz (5.2.16) becomes

$$\begin{aligned} A_w &= Q = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_{\tilde{w}} &= P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 & 0 \end{pmatrix}, \\ A_z &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & 0 \\ 0 & -\frac{1}{2}\psi_z & -U(z)e^{-\psi} \\ 0 & 0 & \frac{1}{2}\psi_z \end{pmatrix}, \\ A_{\tilde{z}} &= \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}e^{\frac{\psi}{2}} & \frac{1}{2}\psi_{\tilde{z}} & 0 \\ 0 & -\tilde{U}(\tilde{z})e^{-\psi} & -\frac{1}{2}\psi_{\tilde{z}} \end{pmatrix}. \end{aligned} \tag{5.2.19}$$

Imposing the Euclidean reality conditions $\tilde{z} = \bar{z}$, $\tilde{w} = -\bar{w}$ results in a positive-definite metric on $\mathbb{R}^4$. The ASDYM equation with these reality conditions is given by

$$F_{zw} = 0, \tag{5.2.20}$$

$$F_{z\bar{z}} + F_{w\bar{w}} = 0. \tag{5.2.21}$$

Now, take the gauge group to be $SU(2,1)$. A matrix $\mathcal{M}$ is in the Lie algebra $\mathfrak{su}(2,1)$ if it is trace-free and satisfies

$$\bar{\mathcal{M}}^t = -\eta\, \mathcal{M}\, \eta^{-1}, \tag{5.2.22}$$

where

$$\eta = \eta^{-1} = \mathrm{diag}(1, 1, -1).$$

Let $z = p + iq$, $w = r + is$, so (p, q, r, s) are standard flat coordinates on $\mathbb{R}^4$. The gauge fields A_p, A_q, A_r, A_s are $\mathfrak{su}(2,1)$-valued. The relations

$$A_z = \frac{1}{2}(A_p - iA_q), \qquad A_{\bar{z}} = \frac{1}{2}(A_p + iA_q),$$

together with (5.2.22), imply that

$$\bar{A_z}^t = -\eta A_{\bar{z}} \eta^{-1},$$

with a similar relation between A_w and $A_{\bar{w}}$. Concretely, this means that

$$A_{\bar{z}} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}, \quad A_z = \begin{pmatrix} -\bar{a} & -\bar{d} & \bar{g} \\ -\bar{b} & -\bar{e} & \bar{h} \\ \bar{c} & \bar{f} & -\bar{k} \end{pmatrix},$$

where $a + e + k = 0$ (and of course A_w and $A_{\bar{w}}$ are related in the same way).

Choosing a real form $SU(2,1)$ of $SL(3,\mathbb{C})$ on restriction to the Euclidean slice imposes a constraint $\tilde{U} = \bar{U}$ and yields the affine sphere equation (5.1.5).

To sum up, one can achieve the characterisation of the ansatz (5.2.19), with $\tilde{z} = \bar{z}$ and $\tilde{U} = \bar{U}$, analogously to proposition 5.2.2. Let us again choose the double null coordinates such that the generators of the symmetry group of the ASDYM equation are given by $\partial_{\tilde{w}},\ \partial_w$. With the chosen reality condition the ASDYM equation reduces to the $SU(2,1)$ Hitchin equations

$$D_z A_w \;=\; 0 \tag{5.2.23}$$

$$F_{z\bar{z}} + [A_w, A_{\bar{w}}] \;=\; 0, \tag{5.2.24}$$

where

$$A_{\bar{z}} = -\eta^{-1} \bar{A_z}^t \eta \quad \text{and} \quad A_{\bar{w}} = -\eta^{-1} \bar{A_w}^t \eta. \tag{5.2.25}$$

We now consider the reduction of the system (5.2.23),(5.2.24). Theorem 5.1.2 arises as a corollary of proposition 5.2.2.

□

Tzitzéica equation

The Tzitzéica equation (5.1.7) is a different real form of (5.2.8). It arises from the ASDYM equation with the gauge group $SL(3,\mathbb{R})$ on restriction to the ultrahyperbolic real slice $\mathbb{R}^{2,2}$ in $\mathbb{C}^4$ with real

$$(w,\ \tilde{w},\ x = z,\ y = \tilde{z}).$$

The Higgs fields are given by $P = A_{\tilde{w}}, Q = A_w$ and the metric on the space of orbits of $X_1 = \partial_{\tilde{w}}$ and $X_2 = \partial_w$ has signature $(1,1)$.

The real version of the ansatz (5.2.6),(5.2.7) can be characterised analogously to the holomorphic case treated in proposition 5.2.2. However, one needs to take care of the fact that $e^{u(x,y)} > 0$ for real valued function $u(x,y)$. There are two places where this needs to be considered. First is where we use condition (**i**) in proposition 5.2.2 to put (Q,P) in the form (5.2.6),(5.2.7). To write $\mathrm{Tr}(PQ) = e^{u(x,y)}$, we require that $\mathrm{Tr}(PQ) > 0$. Assume that this can be done at a point (x_0, y_0) (if not then change coordinates $y \to -y$) and restrict the domain of u to a neighbourhood of this point where the positivity still holds.

The second place where the problem of the sign arises is when we use the coordinate symmetry to transform

$$u_{xy} = e^u - r(x)b(y)e^{-2u}$$

to the Tzitzéica equation (5.1.7). This can only be done for $r(x)b(y) > 0$. The sign of $r(x)b(y)$ is governed by the quantity $\mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right)$ in condition (**ii**). To see it, note that in the notation of (5.2.16),

$$\mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right) = (sktr)e^{2u}.$$

After we set $t = 1$, the condition (**iii**) implies that $k = e^u > 0$. Hence, the sign of sr and thus the sign $r(x)b(y)$ are the same as the sign of $\mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right)$. However, this cannot be changed by real coordinate transformation $x \to \hat{x}(x),\ y \to \hat{y}(y)$, because

$$\mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right) \longrightarrow \left(\frac{d\hat{x}}{dx}\right)^{-4}\left(\frac{d\hat{y}}{dy}\right)^{-4}\mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right),$$

where we have used $Q^2 = 0 = P^2$. Therefore condition (**ii**) in proposition 5.2.2 needs to be replaced by $\mathrm{Tr}\left((D_zP)^2\right) = 0 = \mathrm{Tr}\left((D_{\tilde{z}}Q)^2\right)$ and

$$\mathrm{Tr}\left((D_zP)^2(D_{\tilde{z}}Q)^2\right) > 0$$

in the domain of u.

We remark that $\mathrm{Tr}\,((D_xP)^2(D_yQ)^2) < 0$ corresponds to the equation

$$u_{xy} = e^u + e^{-2u},$$

whereas $\mathrm{Tr}\,((D_xP)^2(D_yQ)^2) = 0$ yields the Liouville's equation

$$u_{xy} = e^u.$$

Therefore, the sign of $\mathrm{Tr}\,((D_xP)^2(D_yQ)^2)$ corresponds to the sign of ϵ in (5.1.28).

5.2.2 $\mathbb{Z}_3$ two-dimensional Toda chain

As a by-product of the proof of proposition 5.2.2, we digress briefly to consider the case where condition (**iii**) in the proposition is omitted. We find that the Hitchin equations can be reduced to a coupled system which includes the $\mathbb{Z}_3$ two-dimensional Toda chain [75] as a special case. Recall that a two-dimensional Toda chain is given by

$$(u_\alpha)_{xy} - e^{(u_{\alpha+1}-u_\alpha)} + e^{(u_\alpha - u_{\alpha-1})} = 0, \tag{5.2.26}$$

where $\alpha \in \mathbb{Z}$. Equation (5.2.26) is called the $\mathbb{Z}_3$ two-dimensional Toda chain when
i) $\alpha \in \mathbb{Z}/\mathbb{Z}_3$ and
ii) $u_1 + u_2 + u_3 = 0$.
We summarise the result in the following proposition.

Proposition 5.2.3 *Let u_1, u_2 be functions of (x, y). The coupled system of equations*

$$\begin{aligned} (u_1)_{xy} - \epsilon_1 e^{(u_2-u_1)} + e^{2u_1+u_2} &= 0 \\ (u_2)_{xy} + \epsilon_1 e^{(u_2-u_1)} - \epsilon_2 e^{-2u_2-u_1} &= 0, \end{aligned} \tag{5.2.27}$$

where $\epsilon_1, \epsilon_2 = \pm 1$, is gauge equivalent to the $SL(3,\mathbb{R})$ Hitchin equations (5.2.2a, b, c) *with $z = x, \tilde{z} = y$ real and*
(**i**) *the Higgs fields P and Q have minimal polynomial t^2, with $Tr(PQ) \neq 0$,*
(**ii**) *$Tr((D_xP)^2) = 0 = Tr((D_yQ)^2)$ and $Tr((D_xP)^2(D_yQ)^2) \neq 0$.*

Proof. These conditions are the first two conditions in proposition 5.2.2. Following the proof and assuming condition (**i**) gives (5.2.13). However, now it is not possible to use gauge symmetry to set the diagonal elements of both A_x and A_y (corresponding to A_z and $A_{\tilde{z}}$ in (5.2.13)) to be the same as in (5.2.7) without the compatibility condition. Instead, let us use only the gauge transformation (5.2.14) to eliminate the diagonal elements of A_y by choosing $(\ln a)_y = -p$.

As before, condition (**ii**) implies that $m = h = 0$ and $sktr \neq 0$. The Hitchin equations (5.2.2a, b, c) imply that t is a function of x only. Hence, we can use the residual gauge freedom (5.2.14) with $a = a(x)$ to set $t = 1$. Equation (5.2.2c) then gives

$$\begin{aligned} n_y + r(x)s &= e^u & (5.2.28)\\ 2n_y - u_{xy} + r(x)s - k &= 0 & (5.2.29)\\ s_x + 3ns - su_x &= 0 & (5.2.30)\\ k_x + 2ku_x - 3kn &= 0. & (5.2.31) \end{aligned}$$

Equations (5.2.30) and (5.2.31) imply that $sk = c(y)e^{-u}$, where $c(y)$ is some arbitrary function which arises from the integration. Now, since $s \neq 0$ let us write

$$k = \frac{c(y)}{s}e^{-u} \quad \text{and} \quad n = \alpha_x, \ s = \pm e^{\beta},$$

for some functions $\alpha(x, y)$ and $\beta(x, y)$. Then, (5.2.30) becomes

$$e^{\beta}(\beta_x + 3\alpha_x - u_x) = 0,$$

which can be integrated to give

$$s = b(y)e^{u-3\alpha} \quad \text{and} \quad n = \alpha_x$$

for some $b = b(y) \neq 0$. Finally, (5.2.28) and (5.2.29) give a coupled system

$$\begin{aligned} \alpha_{xy} + r(x)b(y)e^{u-3\alpha} - e^u &= 0\\ 2\alpha_{xy} - u_{xy} + r(x)b(y)e^{u-3\alpha} - c(y)b^{-1}(y)e^{-2u+3\alpha} &= 0. \qquad (5.2.32) \end{aligned}$$

Set $u_1 = \alpha,\ u_2 = -2\alpha + u$, and change the coordinate $y \to -y$. The system (5.2.32) becomes

$$\begin{aligned} (u_1)_{xy} - r(x)b(y)e^{u_2-u_1} + e^{2u_1+u_2} &= 0\\ (u_2)_{xy} + r(x)b(y)e^{u_2-u_1} - c(y)b^{-1}(y)e^{-2u_2-u_1} &= 0, \end{aligned}$$

which can be transformed into (5.2.27) by the change of dependent variables and coordinates. There are four distinct cases depending on the signs of ϵ_1, ϵ_2. Since the coordinates are real, the signs of ϵ_1, ϵ_2 are the same as those of $r(x)b(y)$ and $c(y)b^{-1}(y)$, respectively. Similar to the real version of proposition 5.2.2 for the Tzitzéica equation, $r(x)b(y)$ and $c(y)b^{-1}(y)$ can be related to some gauge invariant quantities. It can be shown that at a given point (x_0, y_0) the signs of $r(x)b(y)$ and $c(y)b^{-1}(y)$ are determined by the signs of

$$\begin{aligned} (\mathbf{a}) &:= \mathrm{Tr}\left((D_xP)^2(D_yQ)^2\right),\\ (\mathbf{b}) &:= \mathrm{Tr}\left((PQ)^2(D_xP)(D_yQ) - PQ(D_xP)QP(D_yQ)\right). \end{aligned}$$

We shall analyse these signs and then restrict the domains of (u_1, u_2) to a neighbourhood of (x_0, y_0) where the signs remain constant. If $(\mathbf{a}) > 0$, setting $t = 1$ gives $skr > 0$, which gives $r(x)c(y) > 0$. This implies that $r(x)b(y)$ and $c(y)b^{-1}(y)$ have the same signs. Now if $(\mathbf{b}) > 0$, $k > 0$ means $c(y)b^{-1}(y) > 0$ and hence $r(x)b(y) > 0$. Similarly if $(\mathbf{b}) < 0$ then $c(y)b^{-1}(y)$ and $r(x)b(y) < 0$. On the other hand, $(\mathbf{a}) < 0$ implies that $r(x)b(y)$ and $c(y)b^{-1}(y)$ have opposite signs. Then, the sign of $(\mathbf{b})$ determines the sign of $c(y)b^{-1}(y)$. The important point is that the signs of $(\mathbf{a})$ and $(\mathbf{b})$ cannot be changed by real coordinate transformations. This completes the proof.

□

5.2.3 Other gauges

As a further remark, let us note that there are several gauge inequivalent ways to reduce the ASDYM equation to the Tzitzéica equation or to the affine sphere equation. The reductions are relatively easy to obtain, but their gauge invariant characterisation requires much more work. Here we shall mention one other possibility which is not gauge equivalent to (5.2.6),(5.2.7).

It can be shown that the holomorphic Tzitzéica equation (5.2.8) also arises from the Hitchin equations with

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^{u} \\ e^{u} & 0 & 0 \end{pmatrix},$$
$$A_z = \begin{pmatrix} u_z & 0 & 0 \\ 0 & -u_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\tilde{z}} = 0. \tag{5.2.33}$$

The real version of this ansatz was implicitly used by E. Wang [76].

Let us comment on how this formulation is related to (5.2.6),(5.2.7). First note that the Lax pairs (5.2.5) with (5.2.6),(5.2.7) and (5.2.33) are equal for $\lambda = 1$. Now consider the ansatz (5.2.6),(5.2.7) and set $\lambda = 1$ in the Lax pair (5.2.5). Introduce the new spectral parameter by exploiting the Lorentz symmetry and rescaling the coordinates

$$(z, \tilde{z}) \longrightarrow (\hat{\lambda} z, \hat{\lambda}^{-1} \tilde{z})$$

and read off new $A_z, A_{\tilde{z}}, P, Q$ from (5.2.5) with λ replaced by $\hat{\lambda}$. This yields the ansatz (5.2.33).

Choosing the Euclidean reality condition and reducing the gauge group to $SU(2,1)$ we find another reduction of the ASDYM equation to the affine sphere equation. Take the following ansatz, in which the gauge fields are independent of w and $\bar{w}$, $\psi = \psi(z, \bar{z})$ is a real function and $U(z, \bar{z})$ is a complex function:

$$\begin{aligned} A_w &= \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}e^{\psi/2} \\ \bar{U}e^{-\psi} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{\psi/2} & 0 \end{pmatrix}, \\ A_{\bar{w}} &= \begin{pmatrix} 0 & -Ue^{-\psi} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}e^{\psi/2} \\ \frac{1}{\sqrt{2}}e^{\psi/2} & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} A_z &= \begin{pmatrix} -\frac{1}{2}\psi_z & 0 & 0 \\ 0 & \frac{1}{2}\psi_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_{\bar{z}} &= \begin{pmatrix} \frac{1}{2}\psi_{\bar{z}} & 0 & 0 \\ 0 & -\frac{1}{2}\psi_{\bar{z}} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.2.34}$$

Recall that $A_w = Q$ and $A_{\bar{w}} = -P$. The equation $F_{zw} = 0$ is satisfied provided that

$$U_{\bar{z}} = 0,$$

i.e. U must be holomorphic. The equation $F_{z\bar{z}} + F_{w\bar{w}} = 0$ is satisfied if and only if (5.1.5) holds.

5.3 Painlevé III

We have seen in section 5.1 that under a radial symmetry assumption the affine sphere equation (5.1.5) with $U = z^{-2}$ reduces to a Painlevé III equation with special values of parameters. In this section we show that this is also the case for $U = z^{-n}$, where n is an integer, except $n = 3$. Moreover, using the fact that the affine sphere equation is a symmetry reduction of the ASDYM equation, we shall derive an alternative isomonodromic Lax pair for the Painlevé III equation from the reduced Lax pair of the ASDYM equation.

5.3.1 Symmetry reduction

If $U = z^{-n}$, $n \in \mathbb{Z}$, equation (5.1.5) admits a rotational symmetry

$$z \to e^{ic}z, \quad c \in \mathbb{R}. \tag{5.3.1}$$

Therefore one can consider invariant solutions ψ and look for the ODE characterising such reduction. The reduction is done by substitution as in the proof of proposition 5.1.1.

$\mathbf{n \neq 3}$. Changing the independent variable in (5.1.5) to

$$s = (z\bar{z})^{\frac{3-n}{4}}$$

and using the ansatz

$$\psi = \log\left(s^{-\left(\frac{1+n}{3-n}\right)} H(s)^k\right)$$

with $k = \pm 1$ reduces (5.1.5) to the Painlevé III equation with parameters

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{-8}{(3-n)^2}, 0, 0, \frac{-16}{(3-n)^2}\right) \quad \text{and} \quad (\alpha, \beta, \gamma, \delta) = \left(0, \frac{8}{(3-n)^2}, \frac{16}{(3-n)^2}, 0\right)$$

for $k = 1$ and $k = -1$, respectively. In both cases, the Painlevé III equations are of type $D7$ in Okamoto's classification. We note that it has been shown in [77, 78] that the radial solutions of the Tzitzéica equation (5.1.7) also satisfies the Painlevé III equation of type $D7$.

$\mathbf{n = 3}$. Setting $\psi = \psi(s)$ where $s = (z\bar{z})^{\frac{1}{2}}$ in equation (5.1.5) yields

$$\psi_{ss} + \frac{\psi_s}{s} + \frac{4e^{-2\psi}}{s^6} + 2e^{\psi} = 0, \tag{5.3.2}$$

which under multiplication by $\left(\frac{\psi_s}{2} + \frac{1}{s}\right)$, gives a first-order ODE

$$\frac{{\psi_s}^2}{4} + \frac{\psi_s}{s} + e^{\psi} - \frac{e^{-2\psi}}{s^6} + \frac{1+c^2}{s^2} = 0, \tag{5.3.3}$$

where c is a constant of integration. Hence any solution to (5.3.3) such that $s\psi_s \neq -2$ gives rise to a solution to (5.3.2), and conversely all solutions to (5.3.2) arise from (5.3.3). Equation (5.3.3) is integrable by quadratures in terms of the elliptic functions.

5.3.2 Lax pair for Painlevé III

Recall from section 2.3.2 that the standard isomonodromic approach to the Painlevé III equation identifies the equation with a rank 2 isomonodromic problem with two double poles. Using the formalism of the reduced Lax pair of the ASDYM equation, we shall give an alternative isomonodromic Lax pair for the Painlevé III equation given in terms of 3×3 matrices, as opposed to the standard Lax pair with 2×2 matrices [14]. (See also [23] where the $SL(2, \mathbb{C})$ ASDYM equation has been reduced to the Painlevé III equation.) The 3×3 isomonodromic Lax pair can also be derived from the relation between the Painlevé III equation and affine differential geometry, with its underlying isospectral Lax pair, which will be discussed in the next section.

Let us now return to the holomorphic setting and consider the Lax pair for the ASDYM equation in $\mathbb{C}^4$ with gauge group $SL(3, \mathbb{C})$

$$(D_w - \lambda D_{\tilde{z}})\Psi = 0, \quad (D_z - \lambda D_{\tilde{w}})\Psi = 0,$$

where Ψ is a matrix-valued function of $w, \tilde{w}, z, \tilde{z}$ and λ. We require that the connection is invariant under the 3-dimensional subgroup of the conformal group $PGL(4, \mathbb{C})$ generated by

$$\{\partial_w,\ \partial_{\tilde{w}},\ z\partial_z - \tilde{z}\partial_{\tilde{z}}\}, \tag{5.3.4}$$

and introduce coordinates $(s, \theta) \in \mathbb{C}^2$ such that $z = se^{i\theta}$, $\tilde{z} = se^{-i\theta}$ and $z\partial_z - \tilde{z}\partial_{\tilde{z}} = -i\frac{\partial}{\partial\theta}$. Then the ASDYM Lax pair becomes

$$\begin{aligned}
\left(-\zeta\partial_s + s^{-1}\zeta^2\partial_\zeta + 2(A_w - \zeta e^{-i\theta}A_{\tilde{z}})\right)\Psi &= 0,\\
\left(\partial_s + s^{-1}\zeta\partial_\zeta + 2(e^{i\theta}A_z - \zeta A_{\tilde{w}})\right)\Psi &= 0,
\end{aligned}$$

where the gauge fields are in an invariant gauge, i.e. $(A_w, A_{\tilde{w}}, e^{i\theta}A_z, e^{-i\theta}A_{\tilde{z}})$ are functions of s only and $\zeta = \lambda e^{i\theta}$ is an invariant spectral parameter, as discussed in section 2.3.2. Taking linear combinations of these two linear PDEs gives a Lax pair of the form

$$\frac{\partial\Psi}{\partial\zeta} = \hat{L}\ \Psi, \quad \frac{\partial\Psi}{\partial s} = \hat{M}\ \Psi, \tag{5.3.5}$$

where

$$\begin{aligned}
\hat{L} &= s\zeta^{-2}\left(\zeta^2 A_{\tilde{w}} - A_w + \zeta(e^{-i\theta}A_{\tilde{z}} - e^{i\theta}A_z)\right)\\
\hat{M} &= \zeta^{-1}\left(A_w + \zeta^2 A_{\tilde{w}} - \zeta(e^{i\theta}A_z + e^{-i\theta}A_{\tilde{z}})\right).
\end{aligned}$$

The calculation leading to the Painlevé III equation (5.1.6) implies that if we gauge transform ansatz (5.2.19) with $U(z) = z^{-2}$, $\tilde{U}(\tilde{z}) = \tilde{z}^{-2}$ into an invariant gauge and substitute it into (5.3.5), then in the new coordinate $\rho = s^{1/2}$ the system (5.3.5) becomes Lax pair of the Painlevé III equation with special values of parameters (5.1.6). We shall now present this calculation.

An invariant gauge of (5.2.19) can be obtained using the gauge transformation with

$$g = \begin{pmatrix} e^{i\theta/3} & 0 & 0 \\ 0 & e^{-i2\theta/3} & 0 \\ 0 & 0 & e^{i\theta/3} \end{pmatrix},$$

which does not change A_w and $A_{\tilde{w}}$, but gives

$$\begin{aligned} e^{i\theta} A_z &= \begin{pmatrix} \frac{1}{6s} & \frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} & 0 \\ 0 & -\left(\frac{1}{4}\psi_s + \frac{1}{3s}\right) & -\frac{1}{s^2} e^{-\psi} \\ 0 & 0 & \frac{1}{4}\psi_s + \frac{1}{6s} \end{pmatrix}, \\ e^{-i\theta} A_{\tilde{z}} &= \begin{pmatrix} -\frac{1}{6s} & 0 & 0 \\ -\frac{1}{\sqrt{2}} e^{\frac{\psi}{2}} & \frac{1}{4}\psi_s + \frac{1}{3s} & 0 \\ 0 & -\frac{1}{s^2} e^{-\psi} & -\left(\frac{1}{4}\psi_s + \frac{1}{6s}\right) \end{pmatrix}. \end{aligned}$$

Then, in terms of $\rho = s^{1/2}$ and $H(\rho) = \rho^3 e^{\psi}$, the system (5.3.5) gives a Lax pair for the Painlevé III equation (5.1.6) as

$$\frac{\partial \Psi}{\partial \zeta} = L\ \Psi, \quad \frac{\partial \Psi}{\partial \rho} = M\ \Psi, \tag{5.3.6}$$

where

$$\begin{aligned} L &= -\frac{1}{\zeta^2} \begin{pmatrix} \frac{\zeta}{3} & \frac{1}{\sqrt{2}}\zeta(\rho H)^{1/2} & \frac{1}{\sqrt{2}}(\rho H)^{1/2} \\ \frac{1}{\sqrt{2}}\zeta(\rho H)^{1/2} & \zeta\left(\frac{1}{12} - \frac{\rho H_\rho}{4H}\right) & -\zeta\frac{\rho}{H} \\ \frac{1}{\sqrt{2}}\zeta^2(\rho H)^{1/2} & \zeta\frac{\rho}{H} & \zeta\left(\frac{\rho H_\rho}{4H} - \frac{5}{12}\right) \end{pmatrix}, \\ M &= \sqrt{2}\left(\frac{H}{\rho}\right)^{1/2} \begin{pmatrix} 0 & -1 & \frac{1}{\zeta} \\ 1 & 0 & \sqrt{2}\left(\frac{\rho}{H^3}\right)^{1/2} \\ -\zeta & \sqrt{2}\left(\frac{\rho}{H^3}\right)^{1/2} & 0 \end{pmatrix}. \end{aligned}$$

The matrix L has two double poles as expected for the Painlevé III equation [14], at $\zeta = 0$ and $\zeta = \infty$.

We note here that a different (i.e. gauge inequivalent) 3×3 isomonodromic Lax pair for the Painlevé III equation of type $D7$ was used by Kitaev in [77]. The Lax pair can also be derived from the ASDYM Lax pair, using a solution to the Hitchin equations which is gauge equivalent to (5.2.33).

5.4 Semi-flat Calabi-Yau metrics

5.4.1 Explicit expression

We come now to discuss the semi-flat Calabi-Yau metric which are determined by the affine sphere equation (5.1.5). In this section we show that the semi-flat Calabi-Yau metric constructed by Loftin, Yau and Zaslow can be locally expressed explicitly in terms of solution of the affine sphere equation.

Let us first summarise the construction in [13] which is based on the Simon-Wang approach to affine spheres [58]. Consider the parametrisation of an elliptic affine sphere

$$(z, \bar{z}) \mapsto f = (f^1(z, \bar{z}), f^2(z, \bar{z}), f^3(z, \bar{z})) \in \mathbb{R}^3.$$

The structure equations (5.1.12, 5.1.13) defining the affine sphere can be written as a linear first order system of PDEs in f, f_z and $f_{\bar{z}}$

$$\frac{\partial}{\partial z}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ -\frac{1}{2}e^{\psi} & 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix}, \tag{5.4.1}$$
$$\frac{\partial}{\partial \bar{z}}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2}e^{\psi} & 0 & 0 \\ 0 & \bar{U}e^{-\psi} & \psi_{\bar{z}} \end{pmatrix}\begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix},$$

where we have set the affine mean curvature[6] to 1. The compatibility condition for this overdetermined system is the affine sphere equation (5.1.5). Therefore, given a solution ψ, one can find f and hence the cone over the sphere

$$(z, \bar{z}, r) \longmapsto (x^1 = rf^1(z, \bar{z}),\ x^2 = rf^2(z, \bar{z}),\ x^3 = rf^3(z, \bar{z})). \tag{5.4.2}$$

This expression can be inverted locally to give $r = r(x)$. The metric cone over an elliptic affine sphere is given by (5.1.27) with $\phi(x) = r^2/2$ and the corresponding semi-flat metric

[6]For an elliptic affine sphere with affine mean curvature set to 1, the shape operator is $S = I$. Now, with the affine metric (5.1.25), the affine normal chosen to point inward from the surface is given by minus the position vector $-f$, and the structure equations (5.1.12) and (5.1.13) become

$$\begin{aligned} D_X Y &= \nabla_X Y + h(X, Y)(-f) \\ D_X(-f) &= -X. \end{aligned}$$

is (5.1.4). As it stands, to obtain an explicit expression of the CY metric one needs to solve the linear system (5.4.1) for f and invert (5.4.2) for $r(x)$.

However, by using the linear system (5.4.1) we obtain the local expression (5.1.10, 5.1.11) in proposition 5.1.3 without having to solve (5.4.1).

Proof of Proposition 5.1.3. The matrix ϕ_{jk} in (5.1.4) can be obtained by contracting the metric (5.1.27) with $\partial/\partial x^j, \partial/\partial x^k$. Given a solution of the affine sphere equation ψ, we know g_B in the basis $(dr, dz, d\bar{z})$, therefore we want to express $\partial/\partial x^j$ in terms of $\partial/\partial r, \partial/\partial z, \partial/\partial \bar{z}$. Now, from (5.4.2), we have that

$$\begin{pmatrix} \partial/\partial x^1 \\ \partial/\partial x^2 \\ \partial/\partial x^3 \end{pmatrix} = N^{-1} \begin{pmatrix} \partial/\partial r \\ r^{-1}\partial/\partial z \\ r^{-1}\partial/\partial \bar{z} \end{pmatrix}, \quad \text{where} \quad N = \begin{pmatrix} f^1 & f^2 & f^3 \\ f^1_z & f^2_z & f^3_z \\ f^1_{\bar{z}} & f^2_{\bar{z}} & f^3_{\bar{z}} \end{pmatrix}.$$

Moreover, N is the matrix solution of the linear system (5.4.1), whose existence and the existence of its inverse N^{-1} are guaranteed by the affine sphere equation. Writing

$$N^{-1} = \begin{pmatrix} p_1 & q_1 & \bar{q}_1 \\ p_2 & q_2 & \bar{q}_2 \\ p_3 & q_3 & \bar{q}_3 \end{pmatrix}, \tag{5.4.3}$$

one calculates ϕ_{jk} and thus the metric on the fibre to be

$$\phi_{jk} dy^j dy^k = (p_j p_k + e^{\psi} q_j \bar{q}_k) dy^j dy^k. \tag{5.4.4}$$

Now, let us introduce new coordinates

$$\tau := p_i y^i, \quad \xi := q_i y^i, \quad \bar{\xi} := \bar{q}_i y^i$$

and write $p_i dy^i = d\tau - y^i dp_i$ etc. Denote the two matrices of coefficients in the linear system (5.4.1) by $-A^{(z)}$ and $-A^{(\bar{z})}$ respectively, so that (5.4.1) is

$$\partial_z N + A^{(z)} N = 0, \qquad \partial_{\bar{z}} N + A^{(\bar{z})} N = 0.$$

Then, by considering the corresponding equation for N^{-1}, the one-forms $y^i dp_i, y^i dq_i, y^i d\bar{q}_i$ can be written in terms of the coordinates $\tau, \xi, \bar{\xi}$ and the components of $A^{(z)}$ and $A^{(\bar{z})}$ which are known in terms of ψ. Finally, we can write the metric (5.1.4) as

$$g = dr^2 + r^2 e^{\psi} |dz|^2 + |d\tau + \alpha|^2 + e^{\psi} |d\xi + \beta|^2, \tag{5.4.5}$$

where

$$\alpha = -\frac{1}{2}e^{\psi}(\bar{\xi}dz + \xi d\bar{z}), \quad \beta = (\tau + \xi\psi_z)dz + e^{-\psi}\bar{U}\bar{\xi}d\bar{z}.$$

This is the promised local expression of the semi-flat CY metric. By similar calculation the Kähler form can be written as

$$\omega = dr \wedge (d\tau + \alpha) + \frac{r}{2}e^{\psi}(d\bar{z} \wedge (d\xi + \beta) + dz \wedge (d\bar{\xi} + \bar{\beta})).$$

Using the relation between the metric, the Kähler form and the complex structure, we find holomorphic basis $\{e_1, e_2, e_3\}$ (5.1.11) and write g and ω as in proposition 5.1.3, where we have introduced a complex coordinate $w = r + i\tau$.

□

Remark 1. The Ricci-flat condition for the metric (5.1.10) is the affine sphere equation (5.1.5) for $\psi(z, \bar{z})$ and $U(z)$. Equation (5.1.5) is invariant under the transformations $\partial/\partial z \to \partial/\partial\hat{z}$, $\psi \to \hat{\psi}$, $U \to \hat{U}$, where

$$\partial/\partial_{\hat{z}} = e^{-j(z)}\partial/\partial_z, \quad \hat{\psi} = \psi - j(z) - \overline{j(z)}, \quad \text{and} \quad \hat{U} = e^{-3j(z)}U.$$

This can be understood from the fact that $e^{\psi}dzd\bar{z}$ and Udz^3 are the affine metric and the cubic differential of the affine sphere, respectively. The metric (5.1.10) is invariant under the above transformations, together with $\xi \to \hat{\xi} = e^{j(z)}\xi$.

Remark 2. One expects the linear system associated with the structure equations of affine spheres (5.4.1) to be equivalent to the Hitchin Lax pair (5.2.5) giving rise to the affine sphere equation. The matrices $A^{(z)}$ and $A^{(\bar{z})}$ in (5.4.1) are unique up to gauge transformations

$$A^{(z)} \longrightarrow g^{-1}A^{(z)}g + g^{-1}\partial_z g, \quad A^{(\bar{z})} \longrightarrow g^{-1}A^{(\bar{z})}g + g^{-1}\partial_{\bar{z}}g.$$

If we write

$$A^{(z)} = (A_z + \lambda P), \quad A^{(\bar{z})} = (A_{\bar{z}} + \lambda^{-1}Q) \tag{5.4.6}$$

for some value of λ, then it follows that $(A_z, A_{\bar{z}}, Q, P)$ will satisfy the Hitchin equations (5.2.2a, b, c), with reality condition $\tilde{z} = \bar{z}$. Conversely, given a solution $(A_z, A_{\bar{z}}, Q, P)$ to the Hitchin equations, we should be able to find a value of the spectral parameter λ such that $(A_z + \lambda P)$ and $(A_{\bar{z}} + \lambda^{-1}Q)$ can be gauge transformed to $A^{(z)}$ and $A^{(\bar{z})}$ respectively.

For example, we can obtain $A^{(z)}$ and $A^{(\bar{z})}$ in (5.4.1) from the ansatz (5.2.19), with $\tilde{z} = \bar{z}$ and $\tilde{U} = \bar{U}$, by gauge transformation with

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{2}e^{-\psi/2} & 0 \\ 0 & 0 & -\sqrt{2}e^{-\psi/2} \end{pmatrix},$$

and choosing the value of spectral parameter in (5.4.6) to be $\lambda = 1$. Note that we need $\det g \neq 1$, since $A^{(z)}$ and $A^{(\bar{z})}$ are not traceless.

5.4.2 Symmetry of the metrics

For convenience, let us write the semi-flat Calabi-Yau metric of Loftin-Yau-Zaslow as

$$g = dr^2 + r^2 e^{\psi(z,\bar{z})} dz d\bar{z} + \phi_{ij}(z, \bar{z}) dy^i dy^j, \tag{5.4.7}$$

where $\psi(z, \bar{z})$ satisfies the affine sphere equation (5.1.5) and ϕ_{ij} is determined from ψ by (5.4.4), which implies that $\det \phi_{ij} = 1$. By the semi-flat assumption the metric (5.4.7) already possesses three Killing symmetries: three translations along the y^i directions. Moreover, the generator $r\partial/\partial r$ of the conformal scaling symmetry of the base metric g_B can also be lifted to a conformal Killing vector of (5.4.7) given by

$$X = r\frac{\partial}{\partial r} + y^i \frac{\partial}{\partial y^i} \quad \text{and} \quad \mathcal{L}_X\, g = 2\, g.$$

Now, in the special case when ψ is a radially symmetric solution of the affine sphere equation, i.e. $\psi = \psi(s)$ where $z = se^{i\theta}$, the based metric

$$g_B = dr^2 + r^2 e^{\psi(s)} (ds^2 + s^2 d\theta^2)$$

has an additional Killing symmetry generated by $\partial/\partial\theta$. However, as we shall show below, this symmetry does not in general lift to a Killing symmetry of (5.4.7) by a natural lift given by (5.4.8) in proposition 5.4.1 below. The only exception is when $U(z) = z^{-3}$ in the affine sphere equation (5.1.5). We note that this is the special case when (5.1.5) does not reduce to a Painlevé III equation, but is solvable by elliptic functions, see section 5.3.

Proposition 5.4.1 *Let $z = se^{i\theta}$ and consider a metric g given by* (5.4.7) *for which ψ is a radially symmetric solution $\psi(s)$ of* (5.1.5) *with $U(z) = z^{-n}$ and ϕ_{ij} is determined from ψ by* (5.4.4)*. Then, g admits a Killing vector of the form*

$$K = \frac{\partial}{\partial\theta} + m^i(s, \theta, y^j)\frac{\partial}{\partial y^i} \tag{5.4.8}$$

if and only if $n = 3$. *In the generic case where* g *does not admit a Killing vector of the form* $f^k\partial/\partial y^k$ *other than those with constant* f^k, *the functions* $m^i(s,\theta,y^j)$ *depend only on* y^j *and are given by* $m^i = a^i{}_j y^j$, *where* $a^i{}_j$ *are constant.*

Proof. Consider the expression (5.4.5) of g in the coordinates

$$(r' = r,\ s' = s,\ \theta' = \theta,\ \tau = p_i(s,\theta)y^i,\ \xi = q_i(s,\theta)y^i,\ \bar{\xi} = \bar{q}_i(s,\theta)y^i)$$

with $U = z^{-n}$. In these coordinates the vector field (5.4.8) takes the form

$$K = \frac{\partial}{\partial\theta'} + f_\tau\frac{\partial}{\partial\tau} + f_\xi\frac{\partial}{\partial\xi} + f_{\bar{\xi}}\frac{\partial}{\partial\bar{\xi}}$$

for some functions $f_\tau, f_\xi, f_{\bar{\xi}}$. If K is a Killing vector of g, then g is invariant under a transformation which involves $\theta' \to \theta' + c$, where c is a constant, and some changes in $\tau, \xi, \bar{\xi}$. A direct calculation shows that this is possible only in the case $U = z^{-3}$. Conversely, if $U = z^{-3}$ then g admits a Killing vector field

$$K = \frac{\partial}{\partial\theta'} + i\xi\frac{\partial}{\partial\xi} - i\bar{\xi}\frac{\partial}{\partial\bar{\xi}}$$

which is of the form (5.4.8) after change of coordinates. Finally, the form of m^i can be deduced from the commutators $[K, \partial/\partial y^k]$ and $[K, r\partial/\partial r + y^k\partial/\partial y^k]$ using the fact that a commutation of (conformal) Killing vectors is a (conformal) Killing vector.

□

5.4.3 Solutions to case $U = z^{-3}$

Recall from section 5.3 that under the radial symmetry reduction the affine sphere equation (5.1.5) with $U = z^{-n}$ reduces to the Painlevé III equation of type $D7$, except $n = 3$. With $U = z^{-3}$ the reduced ODE can be integrated and $\psi(s)$ is given in terms of the elliptic functions. In this section, we shall show that the Lax pair (5.4.1) with $U = z^{-3}$ and a radially symmetric $\psi(s)$ is completely solvable. First, consider the three equations encoded in (5.4.1)

$$\begin{aligned}
f_{zz} &= \psi_z f_z + \frac{1}{s^3 e^{i3\theta}} e^{-\psi} f_{\bar{z}}, \\
f_{\bar{z}\bar{z}} &= \psi_{\bar{z}} f_{\bar{z}} + \frac{1}{s^3 e^{-i3\theta}} e^{-\psi} f_z, \\
f_{z\bar{z}} &= -\frac{1}{2} e^{\psi} f.
\end{aligned}$$

In term of the (s, θ) coordinates, $z = se^{i\theta}$, the above equations become

$$f_{ss} = \left(\frac{\psi_s}{2} + \frac{e^{\psi}}{s^3}\right) f_s - e^{\psi} f, \tag{5.4.9}$$

$$\frac{1}{s^2} f_{\theta\theta} = -e^{\psi} f - \left(\frac{\psi_s}{2} + \frac{e^{-\psi}}{s^3} + \frac{1}{s}\right) f_s, \tag{5.4.10}$$

$$f_{s\theta} = \left(\frac{\psi_s}{2} - \frac{e^{-\psi}}{s^3} + \frac{1}{s}\right) f_\theta. \tag{5.4.11}$$

Equation (5.4.11) can be integrated to give

$$f_s = H(s) f + h(s), \tag{5.4.12}$$

for some function $h(s)$, where

$$H(s) := \frac{\psi_s}{2} - \frac{e^{-\psi}}{s^3} + \frac{1}{s}.$$

Equation (5.4.12) can be integrated once more and one has

$$f(s, \theta) = e^{\int H ds} \left(\int h e^{-\int H ds} ds + g(\theta) \right) \tag{5.4.13}$$

for some function $g(\theta)$. For f in (5.4.13) to be a solution of (5.4.9), ψ is required to satisfy the radial reduction of the affine sphere equation (5.3.2), which is already assumed. In addition, $h(s)$ is determined by (5.4.9) to be

$$h(s) \propto e^{\int \left(\frac{2e^{-\psi}}{s^3} - \frac{1}{s}\right) ds}.$$

On the other hand the compatibility condition with (5.4.10) gives equation (5.3.3) for ψ. We remind the readers that this is not inconsistent, as every solution of (5.3.2) satisfies (5.3.3).

To sum up, given a solution ψ of (5.3.2), one can solve for three linearly independent solutions (f^1, f^2, f^3) of the system (5.4.9, 5.4.10, 5.4.11) explicitly. There are two cases: $c \neq 0$ and $c = 0$, where c is the constant parameter in (5.3.3). For $c \neq 0$, (f^1, f^2, f^3) is given by

$$f^1 = Ae^{\int H(s)ds + c\theta}, \qquad f^2 = Be^{\int H(s)ds - c\theta}, \qquad f^3 = Ce^{-\int \frac{e^{\psi(s)}}{K(s)} ds}, \tag{5.4.14}$$

and for $c = 0$

$$f^1 = c_1 e^{\int H(s)ds}\theta, \; f^2 = e^{\int H(s)ds} \left(\int h(s) e^{-\int H(s)ds} ds + c_2\theta^2 \right), \; f^3 = c_0 e^{\int H(s)ds}, \tag{5.4.15}$$

where

$$K(s) := \frac{\psi_s}{2} + \frac{e^{-\psi}}{s^3} + \frac{1}{s},$$

and A, B, C, c_0, c_1, c_2 are constants of integration. Note that for every value of c, there exists a solution which is a function of s only. In other words, the solution which depends only on s is insensitive to c in (5.3.3). This is because when assuming the θ-independence, equation (5.4.11) is satisfied trivially and hence the system (5.4.9, 5.4.10, 5.4.11) requires only one compatibility condition - the affine sphere equation (5.3.2).

Explicit expression of the semi-flat metric

Solutions (f_1, f_2, f_3) of the system (5.4.9, 5.4.10, 5.4.11) and their derivatives form a fundamental solution N of the Lax pair (5.4.1). Recall that one can obtain ϕ_{ij} from N^{-1} using (5.4.4).

For the case $c \neq 0$ with N constructed from (5.4.14), N^{-1} as (5.4.3) is given by

$$N^{-1} = \begin{pmatrix} \tilde{p}_1(s)e^{-c\theta} & \tilde{q}_1(s)e^{(i-c)\theta} & \overline{\tilde{q}_1(s)}e^{(-i-c)\theta} \\ \tilde{p}_2(s)e^{c\theta} & \tilde{q}_2(s)e^{(i+c)\theta} & \overline{\tilde{q}_2(s)}e^{(-i+c)\theta} \\ \tilde{p}_3(s) & \tilde{q}_3(s)e^{i\theta} & \overline{\tilde{q}_3(s)}e^{-i\theta} \end{pmatrix},$$

where
$\tilde{p}_1(s) = -A^{-1}\frac{s^2}{2c^2}e^{\psi - \int H(s)ds}$, $\tilde{p}_2(s) = -B^{-1}\frac{s^2}{2c^2}e^{\psi - \int H(s)ds}$, $\tilde{p}_3(s) = C^{-1}\left(1 + \frac{s^2 e^{\psi}}{c^2}\right)e^{\int \frac{e^{\psi}}{K(s)}ds}$
$\tilde{q}_1(s) = -A^{-1}\frac{s^2}{2c^2}\left(K(s) - i\frac{c}{s}\right)e^{-\int H(s)ds}$, $\tilde{q}_2(s) = -B^{-1}\frac{s^2}{2c^2}\left(K(s) + i\frac{c}{s}\right)e^{-\int H(s)ds}$, and
$\tilde{q}_3(s) = C^{-1}\frac{s^2}{c^2}K(s)e^{\int \frac{e^{\psi}}{K(s)}ds}$.

For the case $c = 0$ with N determined by (5.4.15)

$$N^{-1} = \begin{pmatrix} \tilde{p}_1(s)\theta & e^{i\theta}\left(\tilde{q}_1(s)\theta + i\hat{q}_1(s)\right) & e^{-i\theta}\left(\tilde{q}_1(s)\theta - i\hat{q}_1(s)\right) \\ \tilde{p}_2(s) & \tilde{q}_2(s)e^{i\theta} & \tilde{q}_2(s)e^{-i\theta} \\ \tilde{p}_3(s)\theta^2 + \hat{p}_3(s) & e^{i\theta}\left(\tilde{q}_3(s)\theta^2 + i\hat{q}_3(s)\theta + \check{q}_3(s)\right) & e^{-i\theta}\left(\tilde{q}_3(s)\theta^2 - i\hat{q}_3(s)\theta + \check{q}_3(s)\right) \end{pmatrix},$$

where
$\tilde{p}_1(s) = \frac{2c_2}{c1}\frac{H(s)}{h(s)}$, $\tilde{p}_2(s) = -\frac{H(s)}{h(s)}$, $\tilde{p}_3(s) = -\frac{c_2}{c_0}\frac{H(s)}{h(s)}$,
$\hat{p}_3(s) = \frac{1}{c_0}\left(\frac{\int h(s)e^{-\int H(s)ds}ds}{h(s)} + e^{-\int H(s)ds}\right)$, $\tilde{q}_1(s) = -\frac{2c_2}{c_1 h(s)}$, $\hat{q}_1(s) = \frac{s}{c1}e^{-\int H(s)ds}$,
$\tilde{q}_2(s) = \frac{1}{h(s)}$, $\tilde{q}_3(s) = \frac{c_2}{c_0 h(s)}$, $\hat{q}_3(s) = -\frac{s}{c_0}e^{-\int H(s)ds}$, and $\check{q}_3(s) = -\frac{\int h(s)e^{-\int H(s)ds}ds}{c_0 h(s)}$.

This shows that in the case $U = z^{-3}$ we can write down the semi-flat metric (5.1.4) explicitly in the original coordinates (x^i, y^i), as ϕ_{ij} are completely determined.

Remark. The explicitly known θ-dependence of (5.4.14, 5.4.15) enables us to conclude the following. In the case $U = z^{-3}$, a coordinate system can be chosen such that the metric (5.4.7)

$$g = dr^2 + r^2 e^{\psi(s)}(ds^2 + s^2 d\theta^2) + \phi_{ij}(s,\theta)dy^i dy^j$$

admits a Killing vector field of the form

$$K = \frac{\partial}{\partial\theta} + a^i{}_j y^j \frac{\partial}{\partial y^i},$$

where

i) $a^1{}_1 = c,\ a^2{}_2 = -c$ and $a^i{}_j = 0$ otherwise, if ψ satisfies (5.3.3) for $c \neq 0$, or

ii) $a^1{}_3 = \frac{c_1}{c_0},\ a^2{}_1 = 2\frac{c_2}{c_1}$ and $a^i{}_j = 0$ otherwise, if ψ satisfies (5.3.3) for $c = 0$ and c_0, c_1, c_2 are constants in the metric coming from (5.4.15).

5.5 Patching matrices

As a reduction of the ASDYM equation, the affine sphere equation (5.1.5) is in principle solvable by the twistor approach. The problem comes down to finding a transition function of the holomorphic vector bundle over the twistor space which corresponds to a general solution of (5.1.5). As a starting point towards a solution of this problem, in this final section we investigate the twistor correspondence for the holomorphic Tzitzéica equation (5.2.8), which gives (5.1.5) under a reality condition (see section 5.2.1). The holomorphic Tzitzéica equation admits the trivial (vanishing) solution $u = 0$ which gives rise to a non-flat ASDYM connection via the ansatz (5.2.6, 5.2.7). By solving the ASDYM Lax pair, we find the transition function corresponding to the solution $u = 0$, as presented below.

Let us first recall from section 2.2.2 how the transition function of a holomorphic vector bundle over the twistor space can be constructed from two matrix solutions Ψ and $\widetilde{\Psi}$ of the ASDYM Lax pair

$$(D_w - \lambda D_{\tilde{z}})\Psi = 0, \qquad (D_z - \lambda D_{\tilde{w}})\Psi = 0, \tag{5.5.1}$$

where Ψ is holomorphic in λ and $\widetilde{\Psi}$ in $\tilde{\lambda} = \lambda^{-1}$. Let $U, \widetilde{U}$ be two open sets covering the twistor space $\mathcal{P} \simeq \mathbb{CP}^3 - \mathbb{CP}^1$, where U is the complement of $\lambda = 0$ and $\widetilde{U}$ is the complement of $\lambda = \infty$. The transition function F is defined in the overlap $U \cap \widetilde{U}$ by

$$\Psi = \widetilde{\Psi} F. \tag{5.5.2}$$

Now, a solution of the holomorphic Tzitzéica equation determines an ASDYM connection which is invariant under the 2-dimensional translation group generated by $\partial_w, \partial_{\tilde{w}}$. Since the group acts freely on $M \simeq \mathbb{C}^4$, we can choose to express the potential one-form in an invariant gauge, where $A_w, A_{\tilde{w}}, A_z, A_{\tilde{z}}$ are independent of $w, \tilde{w}$. Moreover, since a translation maps an α-plane to a parallel α-plane, the spectral parameter λ is constant along the lift of $\partial_w, \partial_{\tilde{w}}$ to the correspondence space $\mathcal{F}$. Hence the reduced Lax pair, for which the Hitchin equations are the compatibility conditions, can be obtained by assuming that Ψ is independent of $w, \tilde{w}$. This yields the Lax pair (5.2.5). However, an invariant extended solution Ψ which only depends on $(z, \tilde{z}, \lambda)$ must be local and cannot be defined for all $\lambda \in \mathbb{C}$. To see this, consider the Lax pair (5.5.1). Suppose we have two solutions $\Psi, \widetilde{\Psi}$ which are holomorphic in λ and $\tilde{\lambda}$, respectively. Then

$$\begin{aligned} A_w &= -(\partial_w \Psi)\Psi^{-1}|_{\lambda=0}, \qquad & A_z &= -(\partial_z \Psi)\Psi^{-1}|_{\lambda=0}, \\ A_{\tilde{w}} &= -(\partial_{\tilde{w}} \widetilde{\Psi})\widetilde{\Psi}^{-1}|_{\lambda=\infty}, \qquad & A_{\tilde{z}} &= -(\partial_{\tilde{z}} \widetilde{\Psi})\widetilde{\Psi}^{-1}|_{\lambda=\infty}, \end{aligned}$$

which means that if such $\Psi, \widetilde{\Psi}$ are independent of $w, \tilde{w}$, then $A_w = 0 = A_{\tilde{w}}$.

Geometrically, recall that a column vector solution v of the Lax pair can be regarded as a local section of $E|_{\hat{x}}$, where E is the corresponding holomorphic vector bundle over the twistor space $\mathcal{P}$ and $\hat{x} \subset \mathcal{P}$ is the $\mathbb{CP}^1$ corresponding to a point $x \in M$. An invariant section v is then a solution of

$$\mathcal{L}_{X_1'} v = 0, \qquad \mathcal{L}_{X_2'} v = 0,$$

where X_1', X_2' are the induced vector fields on $\mathcal{P}$, obtained from the generators $X_1 = \partial_{\tilde{w}}$, $X_2 = \partial_w$ by lifting them to the correspondence space $\mathcal{F}$ and projecting down to $\mathcal{P}$, as discussed in section 2.3.2. In the twistor coordinates of the two open sets $U, \widetilde{U}$: $(\kappa = \lambda w + \tilde{z},\ \mu = \lambda z + \tilde{w},\ \lambda)$ and $(\tilde{\kappa} = w + \tilde{\lambda}\tilde{z},\ \tilde{\mu} = z + \tilde{\lambda}\tilde{w},\ \tilde{\lambda})$, the vector fields X_1', X_2' are given by

$$X_1' = \partial_\mu = \tilde{\lambda}\partial_{\tilde{\mu}}, \qquad X_2' = \lambda\partial_\kappa = \partial_{\tilde{\kappa}}.$$

This shows that X_1' has a zero at $\lambda = \infty$ and X_2' has a zero at $\lambda = 0$. Hence, it is only possible to find a solution which is holomorphic in λ and depends on $w, z, \tilde{z}, \lambda$, and another solution holomorphic in $\tilde{\lambda}$ and depends on $\tilde{w}, z, \tilde{z}, \tilde{\lambda}$. Therefore, when we solve for $\Psi, \widetilde{\Psi}$ to write down the transition function F as in (5.5.2), it is convenient to use the full Lax pair (5.5.1) of the ASDYM equation.

5.5.1 Trivial solution $u = 0$

To prove proposition 5.5.2 below, which gives the transition function for the $u = 0$ solution, we shall make use of the following lemma.

Lemma 5.5.1 *Let k_i, $i = 1, 2, 3$, be the three cube roots of λ. Then*

$$\sum_{i=1}^{3} k_i^b \, e^{k_i^2 w - k_i z}, \quad b = -2, -1, 0, 1, 2$$

are holomorphic in λ.

Proof. Let k be a chosen cube root of λ, then the other two roots can be written in terms of the cube roots of unity: $1, a, a^2$, where $a = e^{2\pi i/3}$, as

$$k_1 = k, \qquad k_2 = ka, \qquad k_3 = ka^2,$$

keeping in mind that $a^3 = 1$. Now, consider

$$\sum_{i=1}^{3} k_i^b \, e^{k_i^2 w - k_i z} = k^b e^{k^2 w - kz} + k^b a^b e^{k^2 a^2 w - kaz} + k^b a^{2b} e^{k^2 aw - ka^2 z}, \tag{5.5.3}$$

where we have used $a^4 = a$. Using

$$e^{k^2 w - kz} = \sum_{n=0}^{\infty} \frac{1}{n!} (k^2 w - kz)^n$$

and

$$(k^2 w - kz)^n = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} (-kz)^{n-m} (k^2 w)^m,$$

we have that the $\begin{pmatrix} n \\ m \end{pmatrix}$ term of (5.5.3) is proportional to

$$z^{n-m} w^m k^{n+m+b} (1 + a^{n+m+b} + a^{2n-m+2b}). \tag{5.5.4}$$

Now, denote $j = n + m + b$. The term (5.5.4) is given by a power of λ if and only if $j = 3l$, $l \in \mathbb{Z}$. In general j can take values either $3l$, $3l + 1$ or $3l + 2$. However, it turns out that the terms (5.5.4) with $j = 3l + 1$, $3l + 2$ all vanish. This is because we have

$$1 + a^{n+m+b} + a^{2n-m+2b} = 1 + a^j + a^{2j-3m} = 1 + a^j + a^{2j},$$

which, for $j = 3l+1,\ 3l+2$, gives $1+a+a^2 = 0$, while it is equal to 3 if $j = 3l$. To have only non-negative powers of λ, the minimum value j can take is -2. Since $m+n \geq 0$, it follows that the minimum value of b is -2. This shows that for $b \geq -2$ each term in the expansion of (5.5.3) is holomorphic in λ. This proves the lemma.

$\square$

Proposition 5.5.2 *Let* $(\kappa = \lambda w + \tilde{z},\ \mu = \lambda z + \tilde{w},\ \lambda)$ *be holomorphic coordinates on the twistor space* $\mathcal{P} \simeq \mathbb{CP}^3 - \mathbb{CP}^1$ *of the complexified Minkowski spacetime. Then, with respect to the ansatz* (5.2.6, 5.2.7), *the holomorphic vector bundle over* $\mathcal{P}$ *corresponding to the trivial solution* $u = 0$ *of the holomorphic Tzitzéica equation* (5.2.8) *is non-trivial, with the transition function given by*

$$F = \frac{1}{3}\begin{pmatrix} \sum_{i=1}^{3} k_i^2 e^{h_i} & \sum_{i=1}^{3} k_i e^{h_i} & \sum_{i=1}^{3} e^{h_i} \\ -\sum_{i=1}^{3} k_i e^{h_i} & -\sum_{i=1}^{3} e^{h_i} & -\sum_{i=1}^{3} k_i^{-1} e^{h_i} \\ \sum_{i=1}^{3} e^{h_i} & \sum_{i=1}^{3} k_i^{-1} e^{h_i} & \sum_{i=1}^{3} k_i^{-2} e^{h_i} \end{pmatrix}, \tag{5.5.5}$$

where k_i *are the three cube roots of* λ *and* $h_i = k_i^{-1}\kappa - k_i^{-2}\mu$.

Proof. Let us now derive the transition function (5.5.5) from the extended solutions $H, \widetilde{H}$ of the ASDYM Lax pair (5.5.1), where the solutions are holomorphic in λ and $\tilde{\lambda}$, respectively.

For convenience, we shall introduce new coordinates $(\kappa, \mu, \tau, \sigma, \lambda')$ on $\mathbb{C}^4 \times \mathbb{CP}^1$, such that

$$\kappa = \lambda w + \tilde{z}, \quad \mu = \lambda z + \tilde{w}, \quad \tau = z, \quad \sigma = w, \quad \lambda' = \lambda.$$

Then

$$\partial_w - \lambda\partial_{\tilde{z}} = \partial_\sigma, \qquad \partial_z - \lambda\partial_{\tilde{w}} = \partial_\tau$$

and the Lax pair (5.5.1) becomes

$$(\partial_\tau + A_z - \lambda A_{\tilde{w}})\Psi = 0, \qquad (\partial_\sigma + A_w - \lambda A_{\tilde{z}})\Psi = 0, \tag{5.5.6}$$

where we have dropped prime from λ. Now, substituting the ansatz (5.2.6, 5.2.7) for $A_w, A_{\tilde{w}}, A_z, A_{\tilde{z}}$ in (5.5.6) with $u = 0$ and solving (5.5.6) for a 3-column vector solution, one finds three independent solutions Ψ_i of the form

$$\Psi_i = \begin{pmatrix} k_i^2 \\ -k_i \\ 1 \end{pmatrix} \gamma_i(\kappa, \mu) e^{k_i^2 \sigma - k_i \tau},$$

where $\gamma_i(\kappa, \mu)$ is an arbitrary twistor function and $k_i,\ i = 1, 2, 3$ are the three cube roots of λ. For simplicity, set $\gamma_i(\kappa, \mu) = 1$. A fundamental matrix solution Ψ is formed by taking each column to be Ψ_i. Now, let $l_i = k_i^2 \sigma - k_i \tau = k_i^2 w - k_i z$. We have

$$\Psi = \begin{pmatrix} k_1^2 e^{l_1} & k_2^2 e^{l_2} & k_3^2 e^{l_3} \\ -k_1 e^{l_1} & -k_2 e^{l_2} & -k_3 e^{l_3} \\ e^{l_1} & e^{l_2} & e^{l_3} \end{pmatrix}. \tag{5.5.7}$$

Similarly, when the spectral parameter λ takes values in $\mathbb{CP}^1 - \{0\}$ we look for a 3-column vector solution of the Lax pair

$$(\lambda^{-1} D_w - D_{\tilde{z}})\widetilde{\Psi} = 0, \quad (\lambda^{-1} D_z - D_{\tilde{w}})\widetilde{\Psi} = 0. \tag{5.5.8}$$

With $\tilde{l}_i = k_i^{-2} \tilde{w} - k_i^{-1} \tilde{z}$, a fundamental solution $\widetilde{\Psi}$ is given by

$$\widetilde{\Psi} = \begin{pmatrix} e^{\tilde{l}_1} & e^{\tilde{l}_2} & e^{\tilde{l}_3} \\ -k_1^{-1} e^{\tilde{l}_1} & -k_2^{-1} e^{\tilde{l}_2} & -k_3^{-1} e^{\tilde{l}_3} \\ k_1^{-2} e^{\tilde{l}_1} & k_2^{-2} e^{\tilde{l}_2} & k_3^{-2} e^{\tilde{l}_3} \end{pmatrix}. \tag{5.5.9}$$

Solutions (5.5.7) and (5.5.9) depend on the cube roots of λ and $\tilde{\lambda} = \lambda^{-1}$, and hence are not holomorphic in $\lambda, \tilde{\lambda}$. However, since a twistor function $f(\kappa, \mu, \lambda)$ is annihilated by $\partial_w - \lambda \partial_{\tilde{z}}, \partial_z - \lambda \partial_{\tilde{w}}$, it follows that if Ψ is a solution of (5.5.1), $H = \Psi K(\kappa, \mu, \lambda)$ will also satisfy (5.5.1). Therefore, to obtain the holomorphic transition function one needs to find two matrix-valued twistor functions K and $\widetilde{K}$ such that the solutions of (5.5.1) and (5.5.8), $H = \Psi K$ and $\widetilde{H} = \widetilde{\Psi}\widetilde{K}$, are holomorphic and invertible in λ and $\tilde{\lambda}$ respectively. Then, the patching matrix is given by

$$F = \widetilde{H}^{-1} H = \widetilde{K}^{-1} \widetilde{\Psi}^{-1} \Psi K. \tag{5.5.10}$$

Exploiting lemma 5.5.1, we find that the following K and $\widetilde{K}$ work,

$$K = \begin{pmatrix} 1 & k_1^{-1} & k_1^{-2} \\ 1 & k_2^{-1} & k_2^{-2} \\ 1 & k_3^{-1} & k_3^{-2} \end{pmatrix}, \qquad \widetilde{K} = \begin{pmatrix} 1 & -k_1 & k_1^2 \\ 1 & -k_2 & k_2^2 \\ 1 & -k_3^{-1} & k_3^2 \end{pmatrix}.$$

This results in

$$H = \begin{pmatrix} \sum_{i=1}^{3} k_i^2 e^{l_i} & \sum_{i=1}^{3} k_i e^{l_i} & \sum_{i=1}^{3} e^{l_i} \\ -\sum_{i=1}^{3} k_i e^{l_i} & -\sum_{i=1}^{3} e^{l_i} & -\sum_{i=1}^{3} k_i^{-1} e^{l_i} \\ \sum_{i=1}^{3} e^{l_i} & \sum_{i=1}^{3} k_i^{-1} e^{l_i} & \sum_{i=1}^{3} k_i^{-2} e^{l_i} \end{pmatrix} \text{for } l_i = k_i^2 w - k_i z \tag{5.5.11}$$

and

$$\widetilde{H} = \begin{pmatrix} \sum_{i=1}^{3} e^{\tilde{l}_i} & -\sum_{i=1}^{3} k_i e^{\tilde{l}_i} & \sum_{i=1}^{3} k_i^2 e^{\tilde{l}_i} \\ -\sum_{i=1}^{3} k_i^{-1} e^{\tilde{l}_i} & \sum_{i=1}^{3} e^{\tilde{l}_i} & -\sum_{i=1}^{3} k_i e^{\tilde{l}_i} \\ \sum_{i=1}^{3} k_i^{-2} e^{\tilde{l}_i} & -\sum_{i=1}^{3} k_i^{-1} e^{\tilde{l}_i} & \sum_{i=1}^{3} e^{\tilde{l}_i} \end{pmatrix} \text{for } \tilde{l}_i = k_i^{-2}\tilde{w} - k_i^{-1}\tilde{z}. \tag{5.5.12}$$

H is holomorphic in λ, according to lemma 5.5.1. Similar to the proof of lemma 5.5.1, it can be shown that

$$\sum_{i=1}^{3} k_i^b e^{k_i^{-2}\tilde{w} - k_i^{-1}\tilde{z}}, \quad b = -2, -1, 0, 1, 2$$

is holomorphic in λ^{-1}. This is done essentially by replacing w, z, k_i in lemma 5.5.1 with $\tilde{w}, \tilde{z}, k_i^{-1}$, respectively. The only difference is that, now to have only the negative power of λ, one needs $b \leq 2$. This shows that $\widetilde{H}$ is holomorphic in λ^{-1}. Moreover, it can be verified that H and $\widetilde{H}$ in (5.5.11, 5.5.12) are regular in the regions where they are holomorphic. In particular, each element of the inverse matrix H^{-1} is of the form

$$\sum_{i=1}^{3} k_i^b e^{-(k_i^2 w - k_i z)}, \quad b = -2, -1, 0, 1, 2.$$

This is holomorphic in λ by lemma 5.5.1, with $z \to -z, w \to -w$. Similarly, $\widetilde{H}^{-1}$ is also holomorphic in $\tilde{\lambda}$.

Finally, F in (5.5.5) is obtained using (5.5.10). Again by lemma 5.5.1, the expression (5.5.5) is regular in the twistor coordinates (κ, μ, λ) in the overlap of $U \cap \widetilde{U}$, as required.

$\square$

Remark. By direct calculation we verify that the conditions in the characterisation of the holomorphic Tzitzéica equation given in proposition 5.2.2 exclude the transition functions which are extensions of the Atiyah-Ward ansatz [79, 80] of the forms

$$\begin{pmatrix} \lambda & \rho_1 & \rho_2 \\ 0 & 1 & \rho_3 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \qquad \begin{pmatrix} \lambda & \rho_1 & \rho_2 \\ 0 & \lambda & \rho_3 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

where ρ_1, ρ_2, ρ_3 are twistor functions.

Our ultimate aim is to formulate the twistor correspondence for the affine sphere equation (5.1.5). Having the transition function (5.5.5), one can investigate whether a dressing transformation [81] can be used to generate the transition function corresponds to a general solution of the holomorphic Tzitzéica equation (5.2.8). It is then hoped that the twistor correspondence for the affine sphere equation can be deduced from the holomorphic theory by imposing a reality condition. This is left for future work.

Chapter 6

Anti-Self-Dual Cohomogeneity-One Bianchi V Metrics

In this chapter we give an affirmative answer to a question raised by Tod [16] and show that there exists an anti-self-dual non-diagonal cohomogeneity-one metric of Bianchi type V which is not conformally flat, and construct some explicit examples. Each of the conformal classes we consider contains a complex Kähler metric.

6.1 Cohomogeneity-one metrics

A cohomogeneity-one metric is a metric which admits an isometry group acting transitively on codimension-one surfaces. In this chapter we study cohomogeneity-one metrics in four dimensions. They are also known as spatially-homogeneous metrics in the context of general relativity. The 4-dimensional cohomogeneity-one metrics can be classified according to the Bianchi classification of 3-dimensional real Lie algebras [15] of the isometry groups. The Lie algebras are divided into two classes: class A and class B. In class A, there are 6 types: type I, II, VI_0, VII_0, VIII and IX, and there are 5 types in class B: type III, IV, V, VI_h and VII_h. For example, type IX is referred to the Lie algebra of $SU(2)$, type VIII is $SU(1,1)$ and type II is often called the Lie algebra of the Heisenberg group. Note that some of the real Lie algebras are different real forms of the same complex Lie algebra, e g. type VIII and type IX are real forms of $SL(2,\mathbb{C})$ algebra.

A reason for studying cohomogeneity-one metrics is that they are tractable. Since the metrics only depend on one spacetime coordinate, any field equation imposed on the metrics will reduce to a system of ODEs, which in general are easier to solve than a system of PDEs. Moreover, if the metrics are anti-self-dual (ASD), i.e. having anti-self-dual Weyl tensors, one expects the ODEs to be integrable. This is because the ASD conditions on the conformal structures can in principle be solved by twistor construction.

Since the field equations are ODEs, many results about cohomogeneity-one metrics can be obtained by direct calculation, as presented in [16, 17] for example. This is the case especially when the metric takes a simple form, such as being diagonal in a basis of left-invariant one-forms of the isometry group. One of the interesting results about the diagonal Bianchi metrics is the restriction of class B metrics. For example, every diagonal ASD type V metric is conformally flat, all diagonal ASD Einstein metrics in class B are conformally flat, and there exists no genuine (Euclidean) diagonal ASD Kähler metrics with group-invariant Kähler forms in class B [16, 17].

Nevertheless, one expects non-diagonal metrics to be less restrictive. In the non-diagonal case direct calculations become more difficult. However, we shall use an indirect approach based on the isomonodromic construction to study a particular class of ASD non-diagonal class B metrics. For many diagonal cases, it has been shown by Tod [18, 19] that the ODEs which determine $SU(2)$-invariant ASD conformal structures are the Painlevé equations. The result suggests a link between ASD cohomogeneity-one metrics and isomonodromic deformations in which the Painlevé equations are the deformation equations [14]. The idea of the isomonodromic approach to ASD cohomogeneity-one metrics is due to Hitchin [20], whose results inspired the number of works including the construction of ASD cohomogeneity-one metrics from the isomonodromic Lax pair of the Painlevé equations given in [21][1].

In particular, we shall construct a class of holomorphic ASD conformal structures, each of which admits a holomorphic cohomogeneity-one metric of type V and a complex Kähler metric, and look at some explicit examples. We consider type V metrics because they belong to Bianchi class B, where the assumption of diagonalisability severely restricts properties of the metrics: every diagonal ASD type V metric is conformally flat [16].

Let G be the Lie group generated from the Lie algebra of type V by the exponential

[1]The switch map discussed in section 2.4.3 can be used to study a general cohomogeneity-one metric. Recall that the switch map [21, 22] relates the ASD conditions on the conformal structures to the reductions of the ASDYM equation. It follows from the reductions of the ASDYM equation with gauge group $SL(2,\mathbb{C})$ that the ASD Bianchi IX metrics are determined by the Painlevé equations. All other complex Bianchi types have been studied in [30], where solutions to the corresponding ASDYM equations are given explicitly. It has been shown that for complex Bianchi types other than type IX, the solutions to the reduced ASDYM equations are given in term of elementary functions and special functions. Particularly in the case of cohomogeneity-one metrics, the relation between the ASD conditions on conformal classes and the reduced ASDYM equations can also be realised via a direct correspondence between the Lax pair of vector fields (2.4.4) and the isomonodromic Lax pair for the Painlevé equations. The explicit correspondence in the case of Bianchi IX metrics and the Painlevé VI was given in [21].

map. Then we can choose a basis $\{\hat{X}, \hat{Y}, \hat{Z}\}$ of the tangent space T_eG at the identity $e \in G$ to obey

$$[\hat{X}, \hat{Y}] = \hat{Y}, \quad [\hat{X}, \hat{Z}] = \hat{Z}, \quad [\hat{Y}, \hat{Z}] = 0.$$

Associated with the generators $\hat{X}, \hat{Y}, \hat{Z}$ are left-invariant vector fields L_x, L_y, L_z on G, which satisfy the same Lie algebra

$$[L_x, L_y] = L_y, \quad [L_x, L_z] = L_z, \quad [L_y, L_z] = 0, \tag{6.1.1}$$

and the dual basis $\{\lambda^x, \lambda^y, \lambda^z\}$ of left-invariant one-forms satisfying the Maurer-Cartan's structure equation

$$d\lambda^x = 0, \qquad d\lambda^y = \lambda^y \wedge \lambda^x, \qquad d\lambda^z = \lambda^z \wedge \lambda^x. \tag{6.1.2}$$

We shall regard a holomorphic type V metric as a product metric on $\mathbb{C} \times G$ which is invariant under the left translations of the group G on itself. Any such metric can be written as

$$g = \frac{1}{2}\left(dt^2 - t^2\sigma_R^2 - \sigma_P\sigma_Q\right), \tag{6.1.3}$$

where t is the coordinate on $\mathbb{C}$ and $\sigma_P, \sigma_Q, \sigma_R$ are three linearly independent one-forms on G, which depend on t and are invariant under the left translations. Moreover, any cohomogeneity-one ASD conformal structure comes equipped with a quartic

$$Q(\lambda) = a_0 + a_1\lambda + \cdots + a_4\lambda^4$$

defined in terms of the twistor distribution l, m (2.4.4) and the lifts $\tilde{X}, \tilde{Y}, \tilde{Z}$ of the generators of G to the correspondence space $\mathcal{F}$ (see section 2.4.2) as

$$Q(\lambda) = (d\lambda \wedge \nu)(l, m, \tilde{X}, \tilde{Y}, \tilde{Z}),$$

where $\lambda \in \mathbb{CP}^1$ is the spectral parameter in the Lax pair (2.4.4) and ν is the volume form on $M \simeq \mathbb{C} \times G$. We shall prove the following proposition.

Proposition 6.1.1 *Any holomorphic ASD Bianchi type V conformal structure such that the corresponding quartic $Q(\lambda)$ has two distinct zeros of order two, can be represented by a metric of the form* (6.1.3)*, where the one-forms $\sigma_P, \sigma_Q, \sigma_R$ are given in the basis of the left invariant one-forms $\lambda^x, \lambda^y, \lambda^z$ by*

$$\begin{aligned}
\sigma_P &= \frac{1}{T^2}\left((\tilde{q}r - \tilde{r}q)\,\lambda^x + t\tilde{q}_t\,\lambda^y - tq_t\,\lambda^z\right), \\
\sigma_Q &= \frac{1}{T^2}\left((\tilde{r}p - \tilde{p}r)\,\lambda^x + t\tilde{p}_t\,\lambda^y - tp_t\,\lambda^z\right), \\
\sigma_R &= \frac{1}{T^2}\left((\tilde{p}q - \tilde{q}p)\,\lambda^x + \frac{\tilde{r}_t}{2t}\,\lambda^y - \frac{r_t}{2t}\,\lambda^z\right),
\end{aligned} \tag{6.1.4}$$

and

$$\mathbf{v} = (p(t),\ q(t),\ r(t)), \qquad \tilde{\mathbf{v}} = (\tilde{p}(t),\ \tilde{q}(t),\ \tilde{r}(t)) \tag{6.1.5}$$

belong to the solution space of a linear system of ODEs

$$\frac{d}{dt}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} r_0/t & 0 & -p_0/t \\ 0 & -r_0/t & q_0/t \\ -2tq_0 & 2tp_0 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}, \tag{6.1.6}$$

where $\mathbf{v_0} = (p_0, q_0, r_0) \neq (0,0,0)$ *is a constant vector in* $\mathbb{C}^3$, *and* $\mathbf{v}, \tilde{\mathbf{v}}$ *are chosen such that* $(\mathbf{v_0}, \mathbf{v}, \tilde{\mathbf{v}})$ *are linearly independent, with*

$$T = \det[\mathbf{v_0}, \tilde{\mathbf{v}}, \mathbf{v}].$$

The system (6.1.6) turns out to be equivalent to the Bessel equation, as we shall explain in section 6.2.1. We will also consider a Euclidean reality condition and obtain an explicit example of a Riemannian type V metric in terms of Bessel functions of the first kind $J_0(t), J_1(t)$ and the second kind $Y_0(t), Y_1(t)$ of order $0, 1$. The metric is given by equation (6.2.26)

$$g = \frac{dt^2}{4} + \frac{(\lambda^x)^2}{4} + t^2(Y_0^2 + Y_1^2)\frac{(\lambda^y)^2}{T^2} + t^2(J_0^2 + J_1^2)\frac{(\lambda^z)^2}{T^2} - 2t^2(J_0Y_0 + J_1Y_1)\frac{\lambda^y\lambda^z}{T^2},$$

where $T = 2t(Y_1J_0 - Y_0J_1)$.

Every conformal structure in proposition 6.1.1 admits a complex Kähler metric, as we shall now explain.

6.1.1 ASD Kähler metrics

To describe what we mean by a complex Kähler metric, let us begin by defining a (real) Kähler metric. Consider a 4-real dimensional manifold M with a complex structure I, and let g be a Riemannian metric on the complex manifold (M, I) which is Hermitian. That is, for any two tangent vector fields $X, Y \in TM$,

$$g(X, Y) = g(IX, IY).$$

Associated with each Hermitian metric g is the Kähler form ω, which is a real 2-form defined by

$$\omega(X, Y) = g(X, IY). \tag{6.1.7}$$

Then, the triple (M, g, ω) is a Kähler manifold if

$$d\omega = 0.$$

The metric g is called the Kähler metric of M.

Now, suppose M is a complex 4-dimensional manifold and is equipped with an endomorphism $I : TM \to TM$ of the holomorphic tangent bundle TM, such that

$$I^2 = -\text{Id},$$

where Id is the identity map on TM. A holomorphic metric g on M is a complex Kähler metric if the holomorphic two-form ω defined by (6.1.7) is closed and I is integrable in the sense that the torsion N_I of I defined by

$$N_I(X,Y) = [IX, IY] - [X,Y] - [IX,Y] - [X,IY], \quad \text{for} \quad X, Y \in TM$$

vanishes.

We shall now state a proposition which is the key to the construction of an ASD conformal structure which admits a (complex) Kähler metric. The proposition is originally due to [82], but is taken in this form from [23].

Proposition 6.1.2 *A holomorphic ASD conformal structure $(M, [g])$ admits a complex Kähler metric if and only if there exists a section s of the bundle $\mathcal{O}(2) = K^{-1/2}$ over the twistor space $\mathcal{P}$ of $(M, [g])$, with exactly two distinct zeros on each $\hat{x} \simeq \mathbb{CP}^1$ corresponding to a point $x \in M$. Here $K^{-1/2}$ is a distinguished square root of the canonical bundle K of $\mathcal{P}$.*

We shall refer the readers to [82] for a proof, and recall the notations for the curved twistor space of an ASD conformal structure from section 2.4.2. Let us remark that instead of a section of $\mathcal{O}(2) \to \mathcal{P}$, in practice we will consider its pull-back to the bundle $\mathcal{O}(2)$ over the correspondence space $\mathcal{F}$. Recall from section 2.4.2 that the pull-back is a covariantly constant section on the lift of α-planes to $\mathcal{F}$, with respect to the connection on $\mathcal{O}(2) \to \mathcal{F}$ induced by the Levi-Civita connection on M. Therefore it is represented by a quadratic function in λ, $s : \mathcal{F} \to \mathbb{C}$, such that

$$l(s) - (L_0 \lrcorner\, \partial_\lambda \theta)s = 0, \qquad m(s) - (M_0 \lrcorner\, \partial_\lambda \theta)s = 0,$$

where

$$l = V_{00'} - \lambda V_{01'} + f_0 \frac{\partial}{\partial \lambda}, \quad m = V_{10'} - \lambda V_{11'} + f_1 \frac{\partial}{\partial \lambda}, \tag{6.1.8}$$

$$L_0 = V_{00'} - \lambda V_{01'}, \qquad M_0 = V_{10'} - \lambda V_{11'},$$

and θ is a one-form determined by the connection on the prime spin bundle, see section 2.4.1. Also recall that $V_{AA'}$ is a null tetrad, such that a holomorphic metric g is given in the basis of the dual one-forms $V^{AA'}$ by

$$g = 2\left(V^{00'} \odot V^{11'} - V^{01'} \odot V^{10'}\right), \tag{6.1.9}$$

and the orientation is chosen via the volume form

$$\nu = V^{01'} \wedge V^{10'} \wedge V^{11'} \wedge V^{00'}.$$

We denote the kth tensor power of $\mathcal{O}(2)$ bundle by $\mathcal{O}(2k)$. The dual bundle $\mathcal{O}(-4)$ of $\mathcal{O}(4)$ coincides with the canonical line bundle K of $\mathcal{P}$ (see for example [23]), and hence $\mathcal{O}(2) = K^{-1/2}$ in proposition 6.1.2. For any ASD conformal class which admits a cohomogeneity-one metric, there is a natural section of $\mathcal{O}(4) \to \mathcal{F}$ determined by the three generators of the isometry group.

To see it, first note that any ASD conformal structure admits a three-form on $\mathcal{F}$ with values in $\mathcal{O}(4)$ given by

$$\xi = (d\lambda \wedge \nu)(l, m, .\,, .\,, .\,)\,.$$

Moreover, it can be shown that ξ satisfies

$$\mathcal{L}_l \xi = 2(L_0 \lrcorner\, \partial_\lambda \theta)\xi, \qquad \mathcal{L}_m \xi = 2(M_0 \lrcorner\, \partial_\lambda \theta)\xi.$$

Let us now consider an ASD conformal class which admits a cohomogeneity-one metric invariant under a 3-dimensional Lie group G, and pick a representative tetrad which is invariant under G. Let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be the lift of the three generators of G to $\mathcal{F}$ such that $[l, \tilde{X}] = 0$, $[m, \tilde{X}] = 0$ modulo l, m, and similarly for $\tilde{Y}$ and $\tilde{Z}$. Such a lift exists because G is a symmetry group. Contracting ξ with $\tilde{X}, \tilde{Y}, \tilde{Z}$ gives us a section of $\mathcal{O}(4) \to \mathcal{F}$

$$s^2 = \xi(\tilde{X}, \tilde{Y}, \tilde{Z}) = (d\lambda \wedge \nu)(l, m, \tilde{X}, \tilde{Y}, \tilde{Z}). \tag{6.1.10}$$

A section of $\mathcal{O}(4) \to \mathcal{F}$ is represented by a $\mathbb{C}$-valued function on $\mathcal{F}$ which is a polynomial of degree 4 in λ. The readers is reminded that the section (6.1.10) is the quartic $Q(\lambda)$ in proposition 6.1.1.

Now, since $\mathcal{L}_l \tilde{X} = 0$, $\mathcal{L}_m \tilde{X} = 0$ modulo l, m, etc., s^2 satisfies

$$l(s^2) = 2(L_0 \lrcorner\, \partial_\lambda \theta)s^2, \quad m(s^2) = 2(M_0 \lrcorner\, \partial_\lambda \theta)s^2,$$

because $\xi(l, \ . \ , \ . \ ) = 0 = \xi(m, \ . \ , \ . \ )$. This means that s^2 is a pull-back of a section of $\mathcal{O}(4) \to \mathcal{P}$.

In the following, we shall assume that s^2 admits a global square root s, which is a section of $\mathcal{O}(2) \to \mathcal{F}$ and can be regarded as the pull-back of a section of $\mathcal{O}(2) \to \mathcal{P}$. Moreover, s^2 as a polynomial of degree 4 in λ is assumed to have two distinct zeros of order 2, which implies that s has exactly two distinct zeros. This guarantees that the conformal class admits a complex Kähler metric by proposition 6.1.2.

6.2 ASD type V conformal to Kähler

We shall now describe a construction of an ASD conformal structure which admits a cohomogeneity-one metric together with a complex Kähler metric. The following construction is independent of the Bianchi type.

6.2.1 The construction

Let G be the 3-complex dimensional isometry group of a cohomogeneity-one metric g, which is regarded as a product metric on $\mathbb{C} \times G$ invariant under the left translations of G on itself. The metric g is of the form

$$g = dt^2 + g_{ij}(t)\sigma^i \odot \sigma^j,$$

where t is the coordinate on $\mathbb{C}$ and σ^i, $i \in \{1, 2, 3\}$, form a basis of left-invariant one-forms on G. Recall that in terms of a null tetrad g is given by (6.1.9). One can write $V_{AA'}$ in terms of the vector field ∂_t and three linearly independent vector fields X, Y, Z tangent to G which are t-dependent and invariant under the left translations, as

$$V_{00'} = \partial_t + Y, \quad V_{11'} = \partial_t - Y, \quad V_{01'} = X, \quad V_{10'} = Z. \tag{6.2.1}$$

The associated Lax pair (6.1.8) becomes

$$l = \partial_t + Y - \lambda X + f_0\partial_\lambda \quad \text{and} \quad m = Z - \lambda(\partial_t - Y) + f_1\partial_\lambda. \tag{6.2.2}$$

Now, let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be the right-invariant vector fields on G corresponding to three generators of the left translations. Since $\tilde{X}, \tilde{Y}, \tilde{Z}$ are independent of t, one has

$$[l, \tilde{X}] = -\tilde{X}(f_0)\partial_\lambda, \quad [m, \tilde{X}] = -\tilde{X}(f_1)\partial_\lambda.$$

A direct calculation shows that there is no lift of $\tilde{X}$ of the form $\tilde{X} + Q\partial_\lambda$ for some function Q such that $[l, \tilde{X} + Q\partial_\lambda] = 0$, $[m, \tilde{X} + Q\partial_\lambda] = 0$ modulo l, m, and similarly for

$\tilde{Y}, \tilde{Z}$. Hence, we conclude that $[l, \tilde{X}], [m, \tilde{X}]$, etc. are identically zero. This implies that f_0 and f_1 are constant on G, and hence they are functions of λ and t only.

We shall now show that the section s^2 of $\mathcal{O}(4) \to \mathcal{F}$ as defined in (6.1.10) is proportional to $\lambda f_0 + f_1$, with the proportionality factor given by a function $h(t, x, y, z)$ on M. First consider

$$\begin{aligned} \xi = (d\lambda \wedge \nu)(l, m, .\,, .\,, .\,) &= (d\lambda \wedge \nu)(L_0 + f_0 \partial_\lambda, M_0 + f_1 \partial_\lambda, .\,, .\,, .\,), \\ &= d\lambda \wedge \nu\, (L_0, M_0, .\,, .\,, .\,) + f_0\, M_0 \lrcorner\, \nu(.\,, .\,, .\,) - f_1\, L_0 \lrcorner\, \nu(.\,, .\,, .\,). \end{aligned}$$

Contracting this with $(\tilde{X}, \tilde{Y}, \tilde{Z})$, the first term vanishes. One calculates $L_0 \lrcorner\, \nu$ and $M_0 \lrcorner\, \nu$ to be

$$\begin{aligned} L_0 \lrcorner\, \nu &= V^{10'} \wedge V^{01'} \wedge V^{11'} + \lambda\, V^{11'} \wedge V^{10'} \wedge V^{00'} \\ M_0 \lrcorner\, \nu &= V^{11'} \wedge V^{01'} \wedge V^{00'} + \lambda\, V^{10'} \wedge V^{01'} \wedge V^{00'}. \end{aligned}$$

Recall that the tetrad is chosen so that $V^{01'}$, $V^{00'} - V^{11'}$, $V^{10'}$ are linearly independent and invariant under the left translations of G. The right-invariant vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ can be written in the basis of $\{X = V_{01'},\ Y = \frac{1}{2}(V_{00'} - V_{11'}),\ Z = V_{10'}\}$. Thus the terms in $L_0 \lrcorner\, \nu$ and $M_0 \lrcorner\, \nu$ with $V^{00'} \wedge V^{11'}$ vanish under the contraction, and we are left with

$$s^2 = (d\lambda \wedge \nu)(l, m, \tilde{X}, \tilde{Y}, \tilde{Z}) \propto (\det H)\ (\lambda f_0 + f_1),$$

where H is the matrix of coefficients of $\tilde{X}, \tilde{Y}, \tilde{Z}$ written in the basis of (X, Y, Z).

What is important to us is that, since $\det H$ does not depend on λ, the section s^2 has two distinct zeros of order two if and only if $\lambda f_0 + f_1$ has two distinct zeros of order two, as a quartic in λ. Assuming this is the case, then it is possible to use Möbius transformation to put the two zeros at 0 and ∞. The Möbius transformation in λ corresponds to a change of null tetrad by a right rotation $V_{AA'} \to V_{AA'} r^{A'}{}_{B'}$, where r is an $SL(2, \mathbb{C})$-valued function. Since the coefficients of the quartic $\lambda f_0 + f_1$ are functions of t only, the required $r^{A'}{}_{B'}$ will only depend on t. Thus, the new tetrad is still G-invariant.

With the zeros at 0 and ∞, $\lambda f_0 + f_1$ is of the form $a(t)\lambda^2$. Moreover, one still has a Möbius degree of freedom that preserves $(0, \infty)$, that is, the multiplication of λ by a function of t. Let us use this freedom to set $a(t) = 2/t$. The current tetrad is some right rotation of the original one. It is possible to use another freedom: a left rotation $V_{AA'} \to l^A{}_B V_{AA'}$, $l \in SL(2, \mathbb{C})$ to keep $V_{00'} - V_{11'}, V_{01'}, V_{10'}$ tangent to G. Hence, we still have l, m of the form (6.2.2). It can be shown that the right rotation does not change the quartic $\lambda f_0 + f_1$, and we now have f_0, f_1 of the form

$$f_0 = b(t)\lambda^2 + c(t)\lambda + d(t), \quad f_1 = -b(t)\lambda^3 + \left(\frac{2}{t} - c(t)\right)\lambda^2 - d(t)\lambda \qquad (6.2.3)$$

for some functions $b(t), c(t), d(t)$.

Now, consider a pair of linear combinations of l and m

$$L = \frac{\lambda l + m}{\lambda f_0 + f_1} = \frac{\partial}{\partial \lambda} + \frac{2\lambda Y - \lambda^2 X + Z}{\lambda f_0 + f_1} \tag{6.2.4}$$

$$M = \frac{f_1 l - f_0 m}{\lambda f_0 + f_1} = \frac{\partial}{\partial t} + \frac{(f_1 - \lambda f_0)Y - \lambda f_1 X - f_0 Z}{\lambda f_0 + f_1}. \tag{6.2.5}$$

Since the conformal class is ASD, proposition 2.4.1 means $[l, m] = (\dots)l + (\dots)m$. This in turn implies that $[L, M] = 0$, modulo L and M. However, one sees that $[L, M]$ does not contain ∂_λ or ∂_t, thus $[L, M]$ must be identically zero. It turns out that $[L, M] = 0$ implies that $b(t) = 0 = d(t)$. Renaming the vector fields to

$$P = \frac{X}{2}, \quad R = tY, \quad Q = \frac{Z}{2},$$

the compatibility conditions $[L, M] = 0$ are given by

$$\begin{aligned} tP_t - [R, P] + (tc(t) - 1)P &= 0, \\ R_t - 2t[P, Q] &= 0, \\ tQ_t + [R, Q] - (tc(t) - 1)Q &= 0. \end{aligned} \tag{6.2.6}$$

This shows that a cohomogeneity-one metric (6.1.3), in the tetrad

$$V_{00'} = \partial_t + \frac{R}{t}, \quad V_{11'} = \partial_t - \frac{R}{t}, \quad V_{01'} = 2P, \quad V_{10'} = 2Q, \tag{6.2.7}$$

is ASD if the vector fields P, Q, R satisfy the system (6.2.6), where $c(t)$ is defined in (6.2.3). Now, let

$$\hat{R} = R, \quad \hat{P} = h(t)P, \quad \hat{Q} = h^{-1}(t)Q, \quad \text{where} \quad h(t) = e^{\int \left(c(t) - \frac{1}{t}\right)dt}. \tag{6.2.8}$$

The system (6.2.6) implies that the new vector fields $\hat{P}, \hat{Q}, \hat{R}$ satisfy

$$\begin{aligned} tP_t - [R, P] &= 0, \\ R_t - 2t[P, Q] &= 0, \\ tQ_t + [R, Q] &= 0, \end{aligned} \tag{6.2.9}$$

where we have dropped the hat from the new vector fields. Moreover, the tetrad (6.2.7) constructed from a solution (P, Q, R) of (6.2.9) gives the same metric (6.1.3) as the tetrad determined from $(h^{-1}(t)P,\ h(t)Q,\ R)$ which satisfy (6.2.6) with $c(t) = \frac{h_t}{h} + \frac{1}{t}$. Under (6.2.8), the Lax pair (6.2.4, 6.2.5) becomes

$$L = \frac{\partial}{\partial\lambda} + \frac{thQ}{\lambda^2} + \frac{R}{\lambda} - \frac{tP}{h}, \quad M = \frac{\partial}{\partial t} - (th_t + h)\frac{Q}{\lambda} - \frac{h_t}{h}R + \left(\frac{th_t}{h} - 1\right)\lambda\frac{P}{h}, \tag{6.2.10}$$

where we again dropped the hat from the new vector fields.

Our derivation so far has not depended on the choice of the isometry Lie algebra. The resulting system (6.2.9) describes the general isomonodromic deformation equations with two double poles[2].

We conclude that a cohomogeneity-one metric (6.1.3), which belong to an ASD conformal structure admitting a section s^2 of $\mathcal{O}(4) \to \mathcal{F}$ defined in (6.1.10) with two distinct zeros of order two, can be written in terms of a null tetrad (6.2.7), where the vector fields P, Q, R satisfy the system (6.2.9).

The above construction leads to proposition 6.1.1. Before presenting a proof to the proposition, let us give the following remark. For generic values of (p_0, q_0, r_0), where none of the constants is zero, the solutions $\mathbf{v}, \tilde{\mathbf{v}}$ in (6.1.5) are determined by two linearly independent solutions of the Bessel equation. To see it, let $f(t) = r_t$. Differentiating the last row of (6.1.6) twice and substituting in the first two rows shows that $f(t)$ satisfies the Bessel equation

$$f_{tt} - \frac{1}{t}f_t - \left(4p_0q_0 + \frac{(r_0^2 - 1)}{t^2}\right) f = 0. \tag{6.2.12}$$

[2]The system (6.2.9) also arises from the Lax pair

$$L = \frac{\partial}{\partial\lambda} + \frac{(tQ + \lambda R - \lambda^2 tP)}{\lambda^2}, \qquad M = \frac{\partial}{\partial t} - \frac{(\lambda Q + \lambda^3 P)}{\lambda^2}, \tag{6.2.11}$$

which is equivalent to (6.2.10) if $h(t) = 1$. Since $h(t)$ does not appear in the metric, any L, M of the form (6.2.11) gives rise to the metric (6.1.3). The Lax pair (6.2.11) with P, Q, R given by 2×2 matrices was shown by Jimbo and Miwa [14] to give rise to the Painlevé III equation.

The same Lax pair also arises as the reduced Lax pair of the ASDYM equation, by the Painlevé III group. It is shown to be the isomonodromic Lax pair for the Painlevé III equation when the gauge group of the ASDYM connection is $SL(2,\mathbb{C})$ [22]. This is also the case in a special case of gauge group $SL(3,\mathbb{C})$ as discussed in chapter 5. Comparing with (5.3.5), the Lax pair (6.2.11) is obtained by choosing a gauge where the connection along for ∂_ρ is zero. That is, $\rho^{-1}(z\partial_z + \tilde{z}\partial_{\tilde{z}}) \lrcorner A = e^{i\theta}A_z + e^{-i\theta}A_{\tilde{z}} = 0$, and thus relabelling the Higgs fields $Q := A_w,\ P := A_{\tilde{w}},\ R := \rho(e^{i\theta}A_z - e^{-i\theta}A_{\tilde{z}})$ and the coordinates (ρ, ζ) to (t, λ) one obtains (6.2.11).

Given a solution $f(t)$ of (6.2.12), one obtains $r(t)$ by integration, and $p(t), q(t)$ satisfy

$$\begin{aligned} q_0 p - p_0 q &= -\frac{f}{2t}, \\ \frac{d}{dt}(q_0 p + p_0 q) &= -r_0 \frac{f}{2t^2}. \end{aligned}$$

A general solution of (6.2.12) is given by a linear combination of Bessel functions of the first kind J_{r_0} and the second kind Y_{r_0} of order r_0

$$f(t) = c_1 t J_{r_0}(2\sqrt{-p_0 q_0}\ t) + c_2 t Y_{r_0}(2\sqrt{-p_0 q_0}\ t), \tag{6.2.13}$$

where c_1, c_2 are constants of integration. We note here that the relation with the Bessel equation is already expected from the result of [30].

Note also that if $p_0, q_0 \neq 0$, one can set both of them to ± 1 (the sign does not matter in the complex case). This rescales the metric (6.1.3) by $\pm p_0 q_0$.

Proof of Proposition 6.1.1. The metric (6.1.3) is written in terms of the dual basis

$$V^{00'} = \frac{1}{2}\left(dt + t\sigma_R\right), \quad V^{11'} = \frac{1}{2}\left(dt - t\sigma_R\right), \quad V^{01'} = \frac{1}{2}\sigma_P, \quad V^{10'} = \frac{1}{2}\sigma_Q,$$

of a null tetrad of the form (6.2.7), where $\sigma_P, \sigma_Q, \sigma_R$ are the dual one-forms of the vector fields P, Q, R, which are assumed to be tangent to the group G and are invariant under the left translations.

The ASD condition means that the Lax pair of vector fields (6.1.8) span an integrable distribution. This, together with the assumption that $\lambda f_0 + f_1$ has two distinct zeros of order two, implies that the vector fields P, Q, R satisfy the system (6.2.9), as discussed above. The three vector fields can be written in the basis of left-invariant vector fields L_x, L_y, L_z satisfying (6.1.1) as

$$\begin{aligned} P &= p_0(t)\ L_x + p(t)\ L_y + \tilde{p}(t)\ L_z, \\ Q &= q_0(t)\ L_x + q(t)\ L_y + \tilde{q}(t)\ L_z, \\ R &= r_0(t)\ L_x + r(t)\ L_y + \tilde{r}(t)\ L_z, \end{aligned} \tag{6.2.14}$$

for some functions $p_0(t), p(t), \tilde{p}(t)$, etc. Now, since the system (6.2.9) does not depend on the representation of the Lie algebra, one can represent the Lie algebra (6.1.1) by 3×3 matrices as

$$L_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$P = \begin{pmatrix} p_0(t) & p(t) & \tilde{p}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} q_0(t) & q(t) & \tilde{q}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} r_0(t) & r(t) & \tilde{r}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The system (6.2.9) implies p_0, q_0, r_0 are constant and

$$tp_t = r_0 p - p_0 r, \qquad tq_t = -r_0 q + q_0 r, \qquad r_t = 2t(p_0 q - q_0 p), \tag{6.2.15}$$

and $\tilde{p}, \tilde{q}, \tilde{r}$ satisfy the same equations as (6.2.15).

Equations (6.2.15) form the linear system (6.1.6), and one obtains the expression (6.1.4) by finding the dual one-forms $\sigma_P, \sigma_Q, \sigma_R$ to the vector fields (6.2.14) and uses the system (6.2.15) to eliminate p_0, q_0, r_0.

□

6.2.2 Examples

In the following, we shall look at some special cases and give explicit examples of non-diagonal cohomogeneity-one metrics of type V which are ASD and not conformally flat.

First, one notes that if $r_0 = 0$, equations (6.2.15) are simplified. In particular, $r(t)$ now satisfies the Bessel equation

$$r_{tt} - \frac{1}{t} r_t - 4p_0 q_0 r = 0 \tag{6.2.16}$$

and $p(t), q(t)$ are determined from $r(t)$ simply by

$$tp_t = -p_0 r, \quad tq_t = q_0 r. \tag{6.2.17}$$

The same set of equations are satisfied by $\tilde{r}(t), \tilde{p}(t), \tilde{q}(t)$. With $r_0 = 0$ there are two cases.

Case 1: $\mathbf{r_0 = 0, p_0 = 0, q_0 \neq 0}$.

Setting $r_0 = 0$ constrains either p_0 or q_0 to be non-zero, otherwise the metric is degenerate. Suppose $q_0 \neq 0$ and consider the case where $p_0 = 0$. We can rescale the tetrad to set $q_0 = 1$. Solving the system (6.2.15) results in

$$r(t) = -pt^2 + k, \quad q(t) = -\frac{p}{2} t^2 + k \ln t + l,$$

where p is now a constant and k, l are constants of integration. As usual, we obtain the tilde variables in the same way. The function $T(t)$ is now also a constant

$$T = \tilde{r}p - \tilde{p}r = p\tilde{k} - \tilde{p}k.$$

The only constraint on the constants of integration is that $T \neq 0$ for the metric to be non-degenerate. The metric (6.1.3) is then given by

$$2g = dt^2 - (t^2 A(t)^2 + TB(t))\,(\lambda^0)^2 - t^2(\lambda^+)^2 + t^2(2A(t) + T)\,\lambda^0\lambda^+ - T\lambda^0\lambda^-,$$

where

$$\lambda^0 = \frac{\lambda^x}{T}, \quad \lambda^+ = \frac{\tilde{p}\lambda^y - p\lambda^z}{T}, \quad \lambda^- = \frac{\tilde{k}\lambda^y - k\lambda^z}{T}, \quad \text{and}$$

$$\begin{aligned} A(t) &= -T\ln t + \tilde{p}l - \tilde{l}p \\ B(t) &= t^2((1 - 2\ln t)T + 2(\tilde{p}l - \tilde{l}p)) + (\tilde{l}k - \tilde{k}l). \end{aligned}$$

We note a particularly simple case where the constants of integration $l = 0 = \tilde{l}$. Then $A(t)$ and $B(t)$ are simplified and we have

$$2g = dt^2 - t^2T^2\left((\ln t)^2 + \frac{1}{2}(1 - 2\ln t)\right)(\lambda^0)^2 - t^2(\lambda^+)^2 + t^2T(1 - 2\ln t)\lambda^0\lambda^+ - T\lambda^0\lambda^-.$$

A direct calculation using MAPLE verifies that the metric is indeed ASD and not conformally flat.

Remark. For the Lie algebra of type V, the left-invariant one-forms $(\lambda^x, \lambda^y, \lambda^z)$ have simple expression in a coordinate basis of the group G. Let (x, y, z) be a set of coordinates on G such that a general element of G is given by

$$g = \begin{pmatrix} (1+x) & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any group H, the left-invariant one-forms $\{\lambda^a\}$ corresponding to a basis $\{e_a\}$ of a Lie algebra are given by

$$h^{-1}dh = \lambda^a e_a, \quad \text{where} \quad h \in H.$$

It then follows that the left-invariant one-forms $(\lambda^x, \lambda^y, \lambda^z)$ satisfying (6.1.2) are given in the coordinates (x, y, z) simply by $\lambda^x = dx/(1+x)$, $\lambda^y = dy/(1+x)$, $\lambda^z = dz/(1+x)$. For convenience let us change the coordinate $x \to x' = 1 + x$ so that

$$\lambda^x = \frac{dx}{x}, \quad \lambda^y = \frac{dy}{x}, \quad \lambda^z = \frac{dz}{x},$$

where we have dropped the prime. In the above example, we can also introduce coordinates associated to $\lambda^{\pm}$:

$$m = \frac{\tilde{p}y - pz}{T} \quad \text{and} \quad n = \frac{\tilde{k}y - kz}{T},$$

where

$$\lambda^{+} = \frac{dm}{x} \quad \text{and} \quad \lambda^{-} = \frac{dn}{x}.$$

Case 2: $\mathbf{r_0 = 0, p_0 \neq 0, q_0 \neq 0}$.

In this case, one can set $p_0 q_0 = \pm 1$ by rescaling the metric by $\pm p_0 q_0$ and reparametrise the time coordinate $t \to t' = \sqrt{\pm p_0 q_0}\, t$. We shall only consider complex metrics for the moment, so the sign does not matter, and set $p_0 = 1$ and $q_0 = -1$. Then the system (6.2.15) becomes

$$r_t = 2t(p+q), \qquad tp_t = -r, \qquad tq_t = -r. \tag{6.2.18}$$

These conditions imply that $r(t)$ satisfies

$$r_{tt} - \frac{1}{t} r_t + 4r = 0, \tag{6.2.19}$$

with a general solution

$$r(t) = c_1 t J_1(2t) + c_2 t Y_1(2t),$$

where J_1, Y_1 are Bessel functions of the first kind and the second kind, respectively, of order 1, and c_1, c_2 are constants of integration. The last two equations of system (6.2.18) imply that

$$p(t) = -\int \frac{r(t)}{t} dt \quad \text{and} \quad q(t) = p(t) + k, \tag{6.2.20}$$

where k is the constant of integration. Now let $t' = 2t$ A recurrence relation for Bessel functions

$$\int z^{-n} J_{n+1}(z) dz = -z^{-n} J_n(z) \quad \text{and} \quad \int z^{-n} Y_{n+1}(z) dz = -z^{-n} Y_n(z) \tag{6.2.21}$$

implies that

$$p(t) = \frac{c_1}{2} J_0(t') + \frac{c_2}{2} Y_0(t'),$$

where J_0, Y_0 are Bessel functions of the first kind and the second kind, respectively, of order zero. Using another recurrence relation

$$\frac{d}{dz}\left(z^n J_n\right) = z^n J_{n-1} \qquad \text{and} \qquad \frac{d}{dz}\left(z^n Y_n\right) = z^n Y_{n-1},$$

one sees that the constant k in (6.2.20) has to vanish, to satisfy the first equation of (6.2.18). Thus we have $q(t') = p(t')$. We shall now drop the prime from t coordinate

A simple example is

$$r(t) = tJ_1(t), \qquad \tilde{r}(t) = tY_1(t).$$

Then

$$p(t) = q(t) = J_0(t), \qquad \tilde{p}(t) = \tilde{q}(t) = Y_0(t),$$

and the Bianchi V metric is given by

$$2g = \frac{dt^2}{4} + \frac{(\lambda^x)^2}{4} - t^2(Y_0^2 + Y_1^2)\frac{(\lambda^y)^2}{T^2} - t^2(J_0^2 + J_1^2)\frac{(\lambda^z)^2}{T^2} + 2t^2(Y_0J_0 + Y_1J_1)\frac{\lambda^y\lambda^z}{T^2}, \quad (6.2.22)$$

where $T(t) = 2t(Y_0J_1 - Y_1J_0)$. A direct calculation again verifies that the metric is ASD and not conformally flat.

6.2.3 Reality condition

In order to find a genuine real Kähler metrics in a conformal class, one needs to look at Riemannian metrics. By construction, if the coordinate t and one-forms σ_P, σ_Q, σ_R are real then the metric (6.1.3) has ultrahyperbolic signature $(++--)$. To select a Euclidean reality condition, let us write $\sigma_P = e_0 + e_1$ and $\sigma_Q = e_0 - e_1$ for some one-forms e_0, e_1. Then

$$2g = dt^2 - t^2\sigma_R^2 - e_0^2 + e_1^2.$$

This implies that one can choose t and e_1 to be real and σ_R and e_0 to be pure imaginary for a Euclidean signature $(++++)$. That is

$$t \in \mathbb{R}, \quad \sigma_Q = -\overline{\sigma}_P, \quad \sigma_R = -i\hat{\sigma}_R, \quad \text{where} \quad \hat{\sigma}_R \text{ is real.} \qquad (6.2.23)$$

Therefore, a Riemannian metric can be obtained using the construction described in section 6.2.1 by replacing

$$R \to iR \quad \text{and} \quad Q \to -\bar{P}.$$

We now have

$$\begin{aligned} P &= p_0\, L_x + p\, L_y + \tilde{p}\, L_z, \\ Q &= -\left(\bar{p}_0\, L_x + \bar{p}\, L_y + \bar{\tilde{p}}\, L_z\right), \\ R &= i\left(r_0\, L_x + r\, L_y + \tilde{r}\, L_z\right), \end{aligned}$$

where $(r_0, r, \tilde{r})$ and (L_x, L_y, L_z) are real. The system (6.2.15) becomes

$$tp_t = i(r_0 p - p_0 r), \qquad r_t = 2t(\bar{p}_0 p - p_0 \bar{p}), \tag{6.2.24}$$

where the equation for $q(t)$ in (6.2.15) becomes the complex conjugate of the first equation of (6.2.24). Again a real function $f = r_t$ satisfies

$$f_{tt} - \frac{1}{t} f_t + \left(4|p_0|^2 + \frac{(r_0^2 + 1)}{t^2} \right) f = 0. \tag{6.2.25}$$

Possible cases

Below we discuss all distinct cases of the cohomogeneity-one Riemannian metrics constructed using the reality condition (6.2.23). Due to the constraint $q_0 = -\bar{p}_0$, there are only three possible cases. Note that the metrics in case 1 in section 6.2.2 do not satisfy the reality condition (6.2.23).

• $\mathbf{r_0 = 0, p_0 \neq 0}$. This is case 2 in section 6.2.2. The reality condition implies that $p_0 q_0 = -|p_0|^2 < 0$. Since we want to keep t real, we can only rescale the metric by $-p_0 q_0 = |p_0|^2$, which sets $p_0 = 1$. With $r_0 = 0$, equation (6.2.25) reduces to (6.2.19) and $r(t)$ is given by the Bessel functions of order one. Since $r(t)$ is real, the first equation of (6.2.24) implies that $p(t)$ is pure imaginary and therefore the reality condition $q(t) = -\bar{p}(t)$ just becomes $q(t) = p(t)$ as before. The Riemannian form of the metric (6.2.22) is given in term of the real Bessel functions $J_0(t), J_1(t), Y_0(t), Y_1(t)$ by

$$g = \frac{dt^2}{4} + \frac{(\lambda^x)^2}{4} + t^2(Y_0^2 + Y_1^2)\frac{(\lambda^y)^2}{T^2} + t^2(J_0^2 + J_1^2)\frac{(\lambda^z)^2}{T^2} - 2t^2(J_0 Y_0 + J_1 Y_1)\frac{\lambda^y \lambda^z}{T^2}, \tag{6.2.26}$$

where $T = 2t(Y_1 J_0 - Y_0 J_1)$ and we have changed $t \to 2t$ as before.

• $\mathbf{r_0 \neq 0, p_0 = 0}$. The system (6.2.24) becomes

$$tp_t = ir_0 p \qquad r_t = 0,$$

and the same for the tilde variables. Thus, $r(t)$ and $\tilde{r}(t)$ are arbitrary constants and

$$p(t) = c_1(\cos(r_0 \ln t) + i \sin(r_0 \ln t)), \qquad \tilde{p}(t) = c_2(\cos(r_0 \ln t) + i \sin(r_0 \ln t)),$$

where c_1 and c_2 are complex numbers such that $(c_1 \bar{c}_2 - \bar{c}_1 c_2) \neq 0$. The metrics are conformally flat.

• $\mathbf{r_0 \neq 0, p_0 \neq 0}$. In this case, a general solution of the Bessel equation (6.2.25) is given in terms of Bessel functions of pure imaginary order ir_0

$$f(t) = c_1 t \; J_{ir_0}(2p_0 t) + c_2 t \; Y_{ir_0}(2p_0 t).$$

Since the order is pure imaginary, as opposed to a general complex order, the complex coefficients c_1, c_2 can be chosen so that f is real, see for example [83].

To summarise, together with the reality condition (6.2.23), proposition 6.1.1 gives an explicit expression for a Riemannian ASD Bianchi type V metric, and our main result is the non-conformally flat example (6.2.26) in terms of the Bessel functions. We leave the problem of finding a Kähler metric in the same conformal class for future work.

6.3 Further issues

• Ultrahyperbolic signature.

Non-conformally flat Riemannian metrics obtained by imposing the Euclidean reality condition (6.2.23) are given in terms of Bessel functions of either order zero and one, or pure imaginary order. However, for the ultrahyperbolic signature $(++--)$, where every variable is real, the $r_0 \neq 0$ case may admit non-flat examples in terms of elementary functions. For example, all Bessel functions of half-integer order can be expressed in terms of trigonometric functions.

• Kähler metrics.

It is an interesting question whether there exists a real ASD Kähler metric of Bianchi type V. We do not know a priori that the complex Kähler metrics of the conformal structures in proposition 6.1.1 will be group-invariant. Given a Bianchi V metric g in each conformal class, there exists a complex Kähler metric $\hat{g}$ given by $\hat{g} = \Omega^2 g$ for some non-vanishing function Ω. Therefore, the problem comes down to specify the conformal factor Ω. If Ω is a function of t only, then one obtain a type V Kähler metric, whereas if Ω also depends on other coordinates the Kähler metric may not respect the symmetry. It is also interesting to check if the conformal classes admit Ricci-flat metrics, as any Ricci-flat ASD metrics are hyperKähler [84].

As mentioned earlier, it was shown in [17] that there exists no real diagonal scalar-flat Kähler metrics for the entire Bianchi class B. The construction via the isomonodromic Lax pair, as well as the switch map, can also be applied to other Bianchi types in class

B to obtain examples of Kähler metrics in the non-diagonal cases.

• Fibration from Calabi-Yau metrics.

Recall from section 5.4 that the semi-flat Calabi-Yau (CY) metrics of Loftin-Yau-Zaslow are determined by the Painlevé III equation in the case where the solutions to the affine sphere equation have radial symmetry. The relation to the Painlevé III group brings about the question whether there is any relation between the Calabi-Yau metrics and some 4-dimensional ASD Kähler metrics, possibly under some fibrations.

Also recall that even when the CY metric is determined by the Painlevé III transcendent, the radial Killing vector field of the 3-dimensional based metric g_B cannot be lifted to the 6-dimensional space at least in a natural way, except in the special case where the function $U(z)$ in the affine sphere equation (5.1.5) is equal to z^{-3} (see section 5.4.2). Therefore, in general the 4-dimensional metrics obtained from the CY metrics by the Kaluza-Klein reduction could have at most two conformal symmetries. Thus they will not be cohomogeneity-one metrics. This is perhaps not surprising. Since they are determined by the Painlevé III transcendents, if they were of cohomogeneity-one and ASD, one would expect them to be of Bianchi type IX.

We explore three possible Kaluza-Klein reductions of the metric (5.4.7) to four dimensions. The three cases are the reductions by 1) $\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}$, 2) $\{\frac{\partial}{\partial y^3}, r\frac{\partial}{\partial r} + y^i\frac{\partial}{\partial y^i}\}$ and 3) $\{\frac{\partial}{\partial y^3}, \frac{\partial}{\partial \theta} + m^i\frac{\partial}{\partial y^i}\}$. Reduction 3) is only valid in the case $U = z^{-3}$ where the CY metric admits the Killing vector field $\frac{\partial}{\partial \theta} + m^i\frac{\partial}{\partial y^i}$. In the cases 1) and 2) we perform the Kaluza-Klein reductions without assuming that the solution $\psi(z, \bar{z})$ to the affine sphere equation is radially symmetric. In case 1) we find that the 4-dimensional metric inherits two Killing symmetry $\frac{\partial}{\partial y^3}$ and $r\frac{\partial}{\partial r} + y^3\frac{\partial}{\partial y^3}$. However, the ASD condition forces the metric to be conformally flat. We note that if $U = z^{-3}$ and ψ is radially symmetric, the 4-dimensional metric is of diagonal Bianchi type III. A general 4-dimensional metric of case 2) inherits no symmetry. It only admits the radial symmetry in the case $U = z^{-3}$ and $\psi = \psi(s)$. In case 3) the metric inherits a symmetry from $r\frac{\partial}{\partial r} + y^i\frac{\partial}{\partial y^i}$.

In most cases the expressions of the resulting 4-dimensional metrics turn out to be complicated and not readily illuminating. The work of Tod [19] shows that there exist some ASD Kähler metrics with one Killing symmetry, which are determined by the Painlevé III equation, but not having a full $SU(2)$ of isometries. The work is based on the relation between an ASD Kähler metric with a Killing vector preserving the Kähler form and the $SU(\infty)$-Toda field equation [85]. However, it turns out that in our case the reduced metrics that have a Killing symmetry are those special cases determined by elliptic functions, and the reduced metrics that are determined by the Painlevé III transcendents inherit no symmetry. Nevertheless, one could explore if the relevant elliptic equation (5.3.3) can come up as a special case of the $SU(\infty)$-Toda field equation in a similar way to which the Painlevé III equation does in [19].

Recall from chapter 5 the Calabi-Yau metric (5.4.7)

$$g = dr^2 + r^2 e^{\psi(z,\bar{z})} dz d\bar{z} + \phi_{ij}(z,\bar{z}) dy^i dy^j, \tag{6.3.1}$$

where ψ is a solution of the affine sphere equation (5.1.5) and ϕ_{ij} is determined from ψ by (5.4.4), which implies that $\det \phi_{ij} = 1$. The metric (6.3.1) has three Killing vector fields $\partial/\partial y^i$ and one conformal Killing vector field $r\partial/\partial r + y^i \partial/\partial y^i$. In a special case where ψ is given by an elliptic function, the metric also admits a radial symmetry (see section 5.4.2). The details of the three Kaluza-Klein reductions of the metric (6.3.1) by two conformal Killing vector fields to four dimensions are:

Reduction 1: $\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}$

In this case, the metric (6.3.1) can be written as

$$g = g_4 + \phi_{11}(dy^1 + A)^2 + \phi_{22}(dy^2 + B)^2 + 2\phi_{12}(dy^1 + A)(dy^2 + B),$$

where g_4, A, B are a metric and one-forms respectively on the space of orbits of $\partial/\partial y^1$, $\partial/\partial y^2$. Denote $y := y^3$ and $\Phi(z,\bar{z}) := 1/(\phi_{11}\phi_{22} - \phi_{12}^2)$. Then

$$g_4 = dr^2 + r^2 e^{\psi} dz d\bar{z} + \Phi(z,\bar{z}) dy^2,$$

$$\text{and} \quad A = \Phi(z,\bar{z})(\phi_{13}\phi_{22} - \phi_{12}\phi_{23})dy, \qquad B = \Phi(z,\bar{z})(\phi_{11}\phi_{23} - \phi_{13}\phi_{12})dy.$$

The 4-dimensional metric g_4 inherits the Killing vector field $\partial/\partial y$ and the conformal Killing vector field $r\partial/\partial r + y\partial/\partial y$ from g.

Recall that if the solution ψ of the affine sphere equation is radially symmetric, i.e. ψ depends on s only, where $z = se^{i\theta}$, then the affine sphere equation reduces to the Painlevé III equation, except when $U = z^{-3}$. Therefore, in general with $\psi = \psi(s)$, g_4 is determined by the Painlevé III transcendents. However, a direct calculation shows that for g_4 to be ASD, $\Phi(z,\tilde{z})$ needs to be constant and this forces g_4 to be flat.

Note that in the exceptional case where $U = z^{-3}$ and $\psi(s)$ is an elliptic function satisfying (5.3.3) with $c \neq 0$,

$$g_4 = dr^2 + r^2 e^{\psi}(ds^2 + s^2 d\theta^2) + \Phi(s) dy^2,$$

where Φ is now a function of s only. Hence, g_4 also admits $\partial/\partial\theta$ as a Killing vector. Let $r = e^R$, one can write $g_4 = e^{2R}\tilde{g}_4$, where

$$\tilde{g}_4 = dR^2 + e^{\psi}(ds^2 + s^2 d\theta^2) + \Phi(s)(e^{-R}dy)^2.$$

This is a diagonal Bianchi type III metric. We have shown that a non-flat metric is not ASD. Moreover, it cannot be Kähler Einstein, since Kähler Einstein diagonal class B metrics must belong to type V [17].

Reduction 2: $\{\frac{\partial}{\partial y^3},\ r\frac{\partial}{\partial r} + y^i\frac{\partial}{\partial y^i}\}$

First, introduce new coordinates $(R, \hat{y}^a, y),\ a = 1, 2$, adapted to the symmetries, where

$$R = \ln(r), \quad \hat{y}^a = r^{-1}y^a, \quad y = y^3.$$

Then, the metric (6.3.1) can be written as

$$\begin{aligned} g &= e^{2R}\left(dR^2 + e^{\psi}dzd\bar{z} + \phi_{ab}(y^a dR + dy^a)(y^b dR + dy^b)\right) \\ &\quad + e^{2R}\left(2e^{-R}\phi_{c3}(y^c dR + dy^c)dy + e^{-R}\phi_{33}dy^2\right), \end{aligned}$$

where we have dropped the hat from the coordinates $\hat{y}^a$. The two chosen conformal Killing vectors are now

$$X := \frac{\partial}{\partial R} + y\frac{\partial}{\partial y} \quad \text{and} \quad Y := \frac{\partial}{\partial y}.$$

Let

$$e^X = dR \quad \text{and} \quad e^Y = e^{-R}dy.$$

One can write

$$g = e^{2R}\left(g_4 + \Phi(e^X + A)^2 + \phi_{33}(e^Y + B)^2 + 2\chi(e^X + A)(e^Y + B)\right),$$

where g_4, A, B, Φ, χ are a metric, two one-forms and two functions respectively on the space of orbits of the group generated by $\{X, Y\}$. The 4-dimensional metric is given by

$$g_4 = e^{\psi}dzd\bar{z} + \phi_{ab}dy^a dy^b - (\Phi A^2 + \phi_{33}B^2 + 2\chi AB),$$

where

$$\begin{aligned} A &= \frac{(\chi\phi_{a3} - \phi_{33}\phi_{ab}y^b)}{(\chi^2 - \Phi\phi_{33})}dy^a, \quad B = \frac{(\chi\phi_{ab}y^b - \Phi\phi_{a3})}{(\chi^2 - \Phi\phi_{33})}dy^a, \\ \chi &= \phi_{a3}y^a \quad \text{and} \quad \Phi = 1 + \phi_{ab}y^a y^b. \end{aligned}$$

Using the equation satisfied by ϕ_{ij}, $\det(\phi_{ij}) = 1$, the expression of g_4 can be simplified to

$$g_4 = e^{\psi} dz d\bar{z} + \frac{(\Phi_{ab} - \varepsilon_{ac}\varepsilon_{bd} y^c y^d)}{(\Phi_{cd} y^c y^d - \phi_{33})} dy^a dy^b, \tag{6.3.2}$$

where $\Phi_{ab} = \phi_{a3}\phi_{b3} - \phi_{ab}\phi_{33}$.

Again, imposing radial symmetry on ψ implies that a general g_4 is determined by the Painlevé III transcendents. However, since the two translational Killing vectors of the Calabi-Yau metric (6.3.1), $e^{-R}\frac{\partial}{\partial y^a}$ in the new coordinates, are not Lie derived along the orbits of $\{X, Y\}$, the metric (6.3.2) does not inherit a Killing vector.

In the special case when $\psi(s)$ satisfies the equation for elliptic functions (5.3.3) with $c \neq 0$, for $U = z^{-3}$, the metric (6.3.2) admits a Killing vector field coming from the lift of the radial symmetry, which takes the form

$$\frac{\partial}{\partial \theta} + c\left(y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2}\right)$$

in the 4-dimensional coordinates (see section 5.4.3). It is this case where one could explore if g_4 falls into the class of ASD Kähler metrics with Killing vectors of LeBrun, by seeing whether the elliptic equation (5.3.3) can arise as a special case of $SU(\infty)$-Toda field equation, see the discussion in section 6.3.

Finally, let us note that for this reduction the Kähler form of the Calabi-Yau metric (6.3.1) can be written in the adapted coordinates as

$$\omega = e^{2R}\left(\mu + p_3 dR \wedge e^{-R} dy + \alpha \wedge dR + \beta \wedge e^{-R} dy\right),$$

where μ, α and β are a two-form and two one-forms on the space of orbits of $\{X, Y\}$, and

$$\begin{aligned}
\mu &= \frac{1}{2} e^{\psi} (\bar{q}_a dz + q_a d\bar{z}) \wedge dy^a, \\
\alpha &= -p_a dy^a + \frac{1}{2} e^{\psi} (\bar{q}_a y^a dz + q_a y^a d\bar{z}), \\
\beta &= \frac{1}{2} e^{\psi} (\bar{q}_3 dz + q_3 d\bar{z}),
\end{aligned}$$

recalling $p_i, q_i, \bar{q}_i$, $i = 1, 2, 3$, from section 5.4.1 In particular, $d\mu = 0$. However, since μ is given in terms of $dz \wedge dy^a$ and $d\bar{z} \wedge dy^a$, it does not seem to be a Kähler form of the metric (6.3.2).

Reduction 3: $\{\frac{\partial}{\partial y^3},\ \frac{\partial}{\partial \theta} + m^i \frac{\partial}{\partial y^i}\}$

Here we only consider the Calabi-Yau metrics (6.3.1) that are determined from $\psi(s)$ satisfying the equation for elliptic functions (5.3.3), so we have the Killing vector field

$\frac{\partial}{\partial\theta} + m^i \frac{\partial}{\partial y^i}$. For concreteness we will set $c = 1$ in (5.3.3). Similar to reduction 2, one introduces new coordinates adapted to the symmetries $\hat{\theta}, \hat{y}^a, \hat{y}$, $a = 1, 2$, where

$$\hat{\theta} = \theta, \quad \hat{y}^1 = e^{-\theta} y^1, \quad \hat{y}^2 = e^{\theta} y^2, \quad \hat{y} = y^3.$$

The resulting 4-dimensional metric is given by

$$g_4 = dr^2 + r^2 e^{\psi} ds^2 + \frac{1}{T} \left(r^2 e^{\psi} s^2 (\phi_{a3}\phi_{b3} - \phi_{33}\phi_{ab}) - |\varepsilon^{ac}\varepsilon^{bd}| y^c y^d \right) dy^a dy^b,$$

where we have dropped the hat from the new coordinates and

$$T = (\pm)(\phi_{a3}\phi_{b3} - \phi_{33}\phi_{ab}) y^a y^b - \phi_{33} r^2 e^{\psi} s^2,$$

where a graded summation convection with $+$ when $a = b$ and $-$ when $a \neq b$ is used. Again, the metric only inherits one conformal Killing vector field $r\partial/\partial r + y^a \partial/\partial y^a$.

Chapter 7

Summary and Outlook

This thesis demonstrated how relations between anti-self-duality, integrability and soliton systems provide a rich and useful ground for the studies of various topics in mathematical physics. The work in the thesis was divided into four parts.

The aim of the first part was to give a topological explanation for the quantisation of the total energy of time-dependent multi-soliton solutions in the $U(N)$ integrable chiral model, also called the Ward model, in 2+1 dimensions [3]. The usual boundary condition imposed for finite energy implies only that at a fixed time a Ward soliton extends to a map from S^2 to $U(N)$. However, the homotopy group $\pi_2(U(N))$ is trivial, and hence it does not provide the required topological information. The plan was to exploit the integrability of the integrable chiral model, which is a model arising as a symmetry reduction of the $U(N)$ anti-self-dual Yang-Mills (ASDYM) equation in $2+2$ dimensions. As such, the Ward model inherits the Lax pair and solutions can be constructed from the associated extended solutions of the Lax pair. It was established in [9] that under a certain boundary condition, called the trivial scattering condition, the extended solutions at a fixed time can be regarded as maps from S^3 to $U(N)$, thus giving a topological invariant in $\pi_3(U(N)) = \mathbb{Z}$. Starting from this, we focused on a particular subclass of soliton solutions satisfying the trivial scattering condition, called time-dependent unitons. We proved that the total energy of a uniton solution is proportional to the winding number of the associated extended solution. This explains the quantisation of the energy. We also obtained an $SO(1,1)$ invariant expression relating the difference of the squares of the total energy and conserved momentum (in one spatial direction) to the square of the winding number of the extended solution. The proofs can be generalised to a more general class of solutions which satisfy the trivial scattering condition. However, in more general cases, the expressions for the total energy are not directly linked to the winding numbers of the extended solutions. To the best of our knowledge, this result is the first example of a topological mechanism leading to a classical energy quantisation of moving solitons. This work has been published in the Proceedings of the Royal Society A [1].

The second part was concerned with the twistor correspondence for the Ward model, or more generally, for the Yang-Mills-Higgs system in $2+1$ dimensions. The integrable chiral model is obtained from the Yang-Mills-Higgs system by a gauge fixing. By the twistor construction, solutions to the Yang-Mills-Higgs system correspond to holomorphic vector bundles E over the minitwistor space $T\mathbb{P}^1$. The field equation is defined on the $2+1$ dimensional Minkowski spacetime and the minitwistor space is non-compact. However, the setting admits a natural compactification, where the spacetime is compactified to $\mathbb{RP}^3$. The corresponding compactified twistor space $\overline{T\mathbb{P}^1}$ is obtained from $T\mathbb{P}^1$ by compactifying the $\mathbb{C}$-fibres to copies of $\mathbb{CP}^1$. We gave a detailed exposition of Ward's correspondence in the holomorphic setting between the compactified spacetime $\overline{M} \simeq \mathbb{CP}^3$ and the compactified minitwistor space $\overline{T\mathbb{P}^1}$, which is equivalent to a cone in another complex projective 3-space with blown-up vertex. The holomorphic vector bundles over the compactified twistor space have Chern numbers as topological invariants. The trivial scattering condition was actually stated by Ward in [9] as the criterion for the extension of the holomorphic vector bundles to the compactified twistor space. Moreover, in [9] the topological degree of the extended solutions to the Yang-Mills-Higgs Lax pair was identified with the second Chern number of the holomorphic vector bundles, although the proof was not given. As a starting point toward a proof of the identification, we explored a compactified double fibration from a correspondence space onto the compactified spacetime and the compactified twistor space. The correspondence space we considered is the blow up of a singular variety in the direct product of $\overline{M}$ and the $\mathbb{CP}^3$ where the cone lives. We then defined a restricted correspondence space by restricting the blow up to the region which fibres over an $\mathbb{RP}^2$-compactification of a spacelike surface $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$, and showed that it admits a surjective map to $\overline{T\mathbb{P}^1}$. We are interested in the restricted correspondence space because it gives rise to the non-compact domain $\mathbb{R}^2 \times S^1$ of the restricted extended solution $\psi(x, y, \theta)$ of which the topological degree is defined. The hope is to use the surjective map to pull back the vector bundle $E \to \overline{T\mathbb{P}^1}$ to the restricted correspondence space and obtain the expression for the second Chern number of the pulled-back bundle as an integral, which may be equivalent to the integral defining the third homotopy class of the extended solution. The study is left for future work.

The third study was motivated by recent work of Loftin, Yau and Zaslow [13], who gave a construction of semi-flat Calabi-Yau (CY) metrics, which arise in the context of mirror symmetry, from solutions of the affine sphere equation. Recall that the affine sphere equation is a second order partial differential equation for a real function of a complex variable z and its complex conjugate. The goal was to understand the integrability of the affine sphere equation using its close relation with the Tzitzéica equation, which is a well known integrable system. The two equations are equivalent under complexification: they are two real forms of a holomorphic equation which we called the holomorphic Tzitzéica equation. Based on the ansatz in [65], which shows that the Tzitzéica equation is a symmetry reduction of the ASDYM equation, we gave gauge-invariant characterisations of the affine sphere equation and the Tzitzéica equation as reductions of real forms of $SL(3,\mathbb{C})$ ASDYM equations by two translations, or equivalently as a special case of the Hitchin equation. The conditions turn out to constrain the Higgs fields and their covariant derivatives. A by-product of this work is a characterisation of the reduction of the ASDYM equation which gives rise to the $\mathbb{Z}_3$ reduction of a 2-dimensional Toda chain. Then we considered the affine sphere equation in the case where the holomorphic functional coefficient $U(z)$ in the equation is given by $U(z) = z^{-n}$. In such cases the equation admits radially symmetric solutions[1] which depend on $|z|$. We showed that imposing the radial symmetry reduces the elliptic affine sphere equation to the Painlevé III equation with special values of parameters. The only exception is $U(z) = z^{-3}$, where the resulting ordinary differential equation is solvable by elliptic functions. From the reduction, we derived a 3×3 isomonodromic Lax pair for the Painlevé III equation from the Lax pair of the ASDYM equation, also based on the ansatz in [65]. We then looked at the class of semi-flat CY metrics constructed by Loftin, Yau and Zaslow [13]. Given a solution to the affine sphere equation, the construction leads to an overdetermined linear system, which one needs to solve to obtain the metric. By exploring the aforementioned linear system, we wrote down a general expression of the semi-flat CY metrics explicitly in terms of solutions of the affine sphere equation. In addition, it was deduced that the metrics which are determined by the Painlevé III equation must be given by the Painlevé III transcendents, as opposed to non-transcendental solutions. Next, the isometries of the semi-flat CY metrics were analysed. Recall that the semi-flat CY metrics are regarded as metrics on the tangent bundle of a 3-dimensional affine manifold B. We considered the case where the metric is determined by a radially symmetric solution to the affine sphere equation with $U(z) = z^{-n}$. Then the 3-real dimensional based metric g_B admits a radial Killing symmetry. However, the radial Killing vector field cannot be lifted to the 6-real dimensional CY metric by a natural lift, which is the sum of the Killing vector of g_B and a vector field tangent to the fibres, except in the special case $U(z) = z^{-3}$

[1] Note that the semi-flat CY metrics with singularity structure required by Loftin, Yau and Zaslow are determined by the affine sphere equation with $U(z) = z^{-2}$.

where the metric is determined by an elliptic function. Lastly, we explored the twistor correspondence for the holomorphic Tzitzéica equation. We noted that the trivial (vanishing) solution of the holomorphic Tzitzéica equation gives rise to a non-flat ASDYM connection. Using the twistor correspondence for the ASDYM equation, we obtained the matrix transition function for the corresponding holomorphic vector bundle over the twistor space. This is intended to be a starting point toward the characterisations of the Tzitzéica equation and the affine equation in terms of the invariant vector bundles over the twistor space. Such a characterisation could facilitate the construction of solutions to the affine sphere equation by the twistor approach. The work in this part, except the analysis of the radial Killing symmetry of the semi-flat CY metric and the transition function for the vanishing solution of the holomorphic Tzitzéica equation, resulted in the article [2], which has appeared in the Communications in Mathematical Physics.

In the last part, we used the isomonodromic approach [20, 21] to study the holomorphic non-diagonal cohomogeneity-one metrics of Bianchi type V with anti-self-dual (ASD) Weyl tensor. In particular, we considered metrics which are conformal to complex Kähler metrics. The conformal classes considered are characterised by a particular section of the $\mathcal{O}(2)$ bundle over the curved twistor space $\mathcal{P}$. This section, which is defined by the generators of the isometry group, determines the Kähler form of the Kähler metric in each conformal class. We wrote down an explicit expression of any holomorphic cohomogeneity-one metric in the class, and gave some simple examples of ASD non-diagonal type V metric which are not conformally flat. The examples give an affirmative answer to the question raised by Tod [16] on the existence of such metrics. Moreover, we analysed all possible cases of Riemannian metrics arising from a Euclidean reality condition, and gave an example in terms of Bessel functions of order zero and one. We concluded that the Riemannian metrics which are non-conformally flat are either given by the example or are determined by Bessel functions of pure imaginary order. We noted that there are real metrics of ultrahyperbolic signature which are given in terms of elementary functions. The result on the Riemannian metrics can be used to check for the existence of a genuine ASD Kähler metric with group-invariant Kähler form of type V, and indeed of class B. This can be done by finding the conformal factor, which only depends on the coordinate of the space of orbits of the isometry group, to move from a Riemannian type V metric to a Kähler metric. It would also be interesting to check for the existence of hyperKähler metrics in the conformal classes we considered. This amounts to finding a Ricci-flat metric, since all Ricci-flat ASD metrics are hyperKähler.

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[2] John Howard Payne, 1791-1852.

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