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Physics Letters A 343 (2005) 129–132

PHYSICS LETTERS A

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A dispersionless integrable system associated to $\mathrm{Diff}(S^1)$ gauge theory

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Received 15 March 2005; received in revised form 26 May 2005; accepted 1 June 2005

Available online 13 June 2005

Communicated by A.P. Fordy

Abstract

A dispersionless integrable system underlying $(2+1)$-dimensional hyperCR Einstein–Weyl structures is obtained as a symmetry reduction of the anti-self-dual Yang–Mills equations with the gauge group $\mathrm{Diff}(S^1)$. Two special classes of solutions are obtained from well known soliton equations by embedding $SU(1,1)$ in $\mathrm{Diff}(S^1)$.

1. From ASDYM equations to Einstein–Weyl structures

The idea of allowing infinite-dimensional groups of diffeomorphisms of some manifold Σ as gauge groups provides a link between the Yang–Mills–Higgs theories on $\mathbb{R}^n$ and conformal gravity theories on $\mathbb{R}^n \times \Sigma$. The gauge-theoretic covariant derivatives and Higgs fields are reinterpreted as a frame of vector fields thus leading to a conformal structure [21]. This program has lead, among other things, to a dual description of certain two-dimensional integrable systems: as symmetry reductions of anti-self-dual Yang–Mills (ASDYM), or as special curved anti-self-dual conformal structures [6,7,16,22].

In this Letter we shall give the first example of a dispersionless integrable system in $2+1$ dimensions which fits into this framework (Theorem 1.1). As a spin-off we shall obtain a gauge-theoretic characterisation of hyperCR Einstein–Weyl spaces in $2+1$ dimensions (Theorem 1.2). We shall also construct two explicit new classes of solutions to the system (1.1) out of solutions to the nonlinear Schrödinger equation, and the Korteweg–de Vries equation (formulae (2.2) and (2.4)).

Consider a pair of quasi-linear PDEs

$$u_t + w_y + uw_x - wu_x = 0, \qquad u_y + w_x = 0, \quad (1.1)$$

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.06.014

for two real functions $u = u(x, y, t)$, $w = w(x, y, t)$. This integrable system has recently been used to characterise a class of Einstein–Weyl structures in $2 + 1$ dimensions [4]. It has also appeared in other contexts [8,17–19] as an example of $(2 + 1)$-dimensional dispersionless integrable models.

Eqs. (1.1) arise as compatibility conditions $[L, M] = 0$ of an overdetermined system of linear equations $L\Psi = M\Psi = 0$, where $\Psi = \Psi(x, y, t, \lambda)$ is a function, λ is a spectral parameter, and the Lax pair is given by

$$L = \partial_t - w\partial_x - \lambda\partial_y,$$
$$M = \partial_y + u\partial_x - \lambda\partial_x. \tag{1.2}$$

This should be contrasted with Lax pairs for other dispersionless integrable systems [1,10,13,14,20,23] which contain derivatives w.r.t. the spectral parameter.

The first equation in (1.1) resembles a flatness condition for a connection with the underlying Lie algebra $\mathrm{diff}(\Sigma)$, where $\Sigma = S^1$ or $\mathbb{R}$. The following result makes this interpretation precise.

Theorem 1.1. *The system (1.1) arises as a symmetry reduction of the anti-self-dual Yang–Mills equations in signature* $(2, 2)$ *with the infinite-dimensional gauge group* $\mathrm{Diff}(\Sigma)$ *and two commuting translational symmetries exactly one of which is null. Any such symmetry reduction is gauge equivalent to (1.1).*

Proof. Consider the flat metric of signature $(2, 2)$ on $\mathbb{R}^4$ which in double null coordinates $y^\mu = (t, z, \tilde{t}, \tilde{z})$ takes the form

$$\mathrm{d}s^2 = \mathrm{d}t\,\mathrm{d}\tilde{t} - \mathrm{d}z\,\mathrm{d}\tilde{z},$$

and choose the volume element $\mathrm{d}t \wedge \mathrm{d}\tilde{t} \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z}$. Let $A \in T^*\mathbb{R}^4 \otimes \mathfrak{g}$ be a connection one-form, and let F be its curvature two-form. Here $\mathfrak{g}$ is the Lie algebra of some (possibly infinite-dimensional) gauge group G. In a local trivialisation $A = A_\mu\,\mathrm{d}y^\mu$ and $F = (1/2)F_{\mu\nu}\,\mathrm{d}y^\mu \wedge \mathrm{d}y^\nu$, where $F_{\mu\nu} = [D_\mu, D_\nu]$ takes its values in $\mathfrak{g}$. Here $D_\mu = \partial_\mu - A_\mu$ is the covariant derivative. The connection is defined up to gauge transformations $A \to b^{-1}Ab - b^{-1}\,\mathrm{d}b$, where $b \in \mathrm{Map}(\mathbb{R}^4, G)$. The ASDYM equations on A_μ are $F = -*F$, or

$$F_{tz} = 0, \qquad F_{t\tilde{t}} - F_{z\tilde{z}} = 0, \qquad F_{\tilde{t}\tilde{z}} = 0.$$

These equations are equivalent to the commutativity of the Lax pair

$$L = D_t - \lambda D_{\tilde{z}}, \qquad M = D_z - \lambda D_{\tilde{t}}$$

for every value of the parameter λ.

We shall require that the connection possesses two commuting translational symmetries, one null and one non-null which in our coordinates are in $\partial_{\tilde{t}}$ and $\partial_{\tilde{y}}$ directions, where $z = y + \tilde{y}$, $\tilde{z} = y - \tilde{y}$. Choose a gauge such that $A_{\tilde{z}} = 0$ and one of the Higgs fields $\Phi = A_{\tilde{t}}$ is constant. The Lax pair has so far been reduced to

$$L = \partial_t - W - \lambda\partial_y, \qquad M = \partial_y - U - \lambda\Phi, \tag{1.3}$$

where $W = A_t$ and $U = A_z$ are functions of (y, t) with values in the Lie algebra $\mathfrak{g}$, and Φ is an element of $\mathfrak{g}$ which does not depend on (y, t). The reduced ASDYM equations are

$$\partial_y W - \partial_t U + [W, U] = 0, \qquad \partial_y U + [W, \Phi] = 0.$$

Now choose $G = \mathrm{Diff}(\Sigma)$, where Σ is some one-dimensional manifold, so that (U, W, Φ) become vector fields on Σ. We can choose a local coordinate x on Σ such that

$$\Phi = \partial_x,$$
$$W = w(x, y, t)\partial_x, \qquad U = -u(x, y, t)\partial_x, \tag{1.4}$$

where u, w are smooth functions on $\mathbb{R}^3$. The reduced Lax pair (1.3) is identical to (1.2) and the ASDYM equations reduce to the pair of PDEs (1.1). □

Recall that a Weyl structure on an n-dimensional manifold $\mathcal{W}$ consists of a torsion-free connection D and a conformal structure $[h]$ which is compatible with D in a sense that $Dh = \omega \otimes h$ for some one-form ω and $h \in [h]$. We say that a Weyl structure is Einstein–Weyl if the traceless part of the symmetrised Ricci tensor of D vanishes. The three-dimensional Einstein–Weyl structure is called hyperCR [3–5,9] if its mini-twistor space [11] is a holomorphic bundle over $\mathbb{CP}^1$.

In [4] it was demonstrated that if $n = 3$, and $[h]$ has signature $(+ + -)$ then all Lorentzian hyperCR Einstein–Weyl structures are locally of the form

$$h = (\mathrm{d}y + u\,\mathrm{d}t)^2 - 4(\mathrm{d}x + w\,\mathrm{d}t)\,\mathrm{d}t,$$
$$\omega = u_x\,\mathrm{d}y + (uu_x + 2u_y)\,\mathrm{d}t, \tag{1.5}$$

where u, w satisfy (1.1). This result combined with Theorem 1.1 yields the following coordinate independent characterisation of the hyperCR Einstein–Weyl condition.

Theorem 1.2. *The ASDYM equations in $2+2$ dimensions with two commuting translational symmetries one null and one non-null, and the gauge group $\mathrm{Diff}(\Sigma)$ are gauge-equivalent to the hyperCR Einstein–Weyl equations in $2+1$ dimensions.*

This is a Lorentzian analogue of a theorem proved in [2] in the Euclidean case. The readers should note that in [2] the result is formulated in terms of the Hitchin system, and not reductions of the ASDYM system.

2. Reductions to KdV and NLS

Reductions of the ASDYM equations with $G = SU(1,1)$ by two translations (one of which is null) lead to well-known integrable systems KdV, and NLS [15]. The group $SU(1,1)$ is a subgroup of $\mathrm{Diff}(S^1)$ which can be seen by considering the Möbius action of $SU(1,1)$

$$\zeta \to M(\zeta) = \frac{\alpha\zeta+\beta}{\bar\beta\zeta+\bar\alpha}, \qquad |\alpha|^2 - |\beta|^2 = 1,$$

on the unit disc. This restricts to the action on the circle as $|M(\zeta)| = 1$ if $|\zeta| = 1$. We should therefore expect that Eq. (1.1) contains KdV and NLS as its special cases (but not necessarily symmetry reduction). To find explicit classes of solutions to (1.1) out of solutions to KdV and NLS we proceed as follows. Consider the matrices

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the commutation relations

$$[\tau_+, \tau_-] = \tau_0,$$
$$[\tau_0, \tau_+] = 2\tau_+, \qquad [\tau_0, \tau_-] = -2\tau_-.$$

The NLS equation

$$i\phi_t = -\frac{1}{2}\phi_{yy} + \phi|\phi|^2, \quad \phi = \phi(y,t) \tag{2.1}$$

arises from the reduced Lax pair (1.3) with

$$W = \frac{1}{2i}\left(-|\phi|^2\tau_0 + \phi_y\tau_- - \bar\phi_y\tau_+\right),$$
$$U = -\phi\tau_- - \bar\phi\tau_+, \qquad \Phi = i\tau_0.$$

Now we replace the matrices by vector fields on Σ corresponding to the embedding of $su(1,1)$ in $\mathrm{diff}(\Sigma)$

$$\tau_+ \to \frac{1}{2i}e^{2ix}\frac{\partial}{\partial x}, \qquad \tau_- \to -\frac{1}{2i}e^{-2ix}\frac{\partial}{\partial x},$$
$$\tau_0 \to \frac{1}{i}\frac{\partial}{\partial x},$$

and read off the solution to (1.1) from (1.4)

$$u = \frac{1}{2i}\left(\bar\phi e^{2ix} - \phi e^{-2ix}\right),$$
$$w = \frac{1}{2}|\phi|^2 + \frac{1}{4}\left(e^{2ix}\bar\phi_y + e^{-2ix}\phi_y\right). \tag{2.2}$$

The second equation in (1.1) is satisfied identically, and the first is satisfied if $\phi(y,t)$ is a solution to the NLS equation (2.1).

Analogous procedure can be applied to the KdV equation

$$4v_t - v_{yyy} - 6vv_y = 0, \quad v = v(y,t). \tag{2.3}$$

The Lax pair for this equation is given by (1.3) with

$$W = q_y\tau_+ - \kappa\tau_- - \left(\frac{1}{2}q_{yy} + qq_y\right)\tau_0,$$
$$U = \tau_+ - q\tau_0 - \left(q_y + q^2\right)\tau_-, \qquad \Phi = \tau_-,$$

where

$$\kappa = \frac{1}{4}q_{yyy} + qq_{yy} + \frac{1}{2}{q_y}^2 + q^2 q_y,$$

and $v = 2q_y$. Now we choose x such that

$$\tau_+ \to -x^2\frac{\partial}{\partial x}, \qquad \tau_- \to \frac{\partial}{\partial x}, \qquad \tau_0 \to 2x\frac{\partial}{\partial x},$$

and read off the expressions for u and w

$$u = x^2 + 2xq + q_y + q^2,$$
$$w = -x^2 q_y - x(q_{yy} + 2qq_y) - \kappa. \tag{2.4}$$

The second equation in (1.1) holds identically, and the first is satisfied if v is a solution to (2.3).

In Refs. [17,18] the so-called 'universal hierarchy' was studied and a general procedure of constructing its differential reductions was proposed. The system (1.1) arises from the first two flows of this hierarchy, but it is not clear how the differential constraints imposed in [17,18] can be understood from the Diff(S^1) point of view. It would be interesting to see whether our reductions to NLS and KdV are 'differential' in the sense of the above references.

One remark is in place: there is a standard procedure [12] of constructing anti-self-dual conformal structures with symmetries out of EW structures in 3 or $2+1$ dimensions. The procedure is based on solving a linear generalised monopole equation on the EW background. Moreover, the hyperCR EW structures always lead to hyper-complex conformal structures with a tri-holomorphic Killing vector, and it is possible to choose a monopole such that there exist a Ricci-flat metric in the conformal class [9]. Any hyperCR EW (1.5) structure given in terms of KdV, or NLS potential by (2.4) or (2.2) will therefore lead to a $(++--)$ ASD Ricci-flat metric with a tri-holomorphic homothety. The explicit formulae for the metric in terms of solutions to (1.1) can be found in [4]. Another class of ASD Ricci-flat metrics has been constructed from KdV and NLS, by embedding $SU(1,1)$ in a Lie algebra of volume preserving transformations of the Poincaré disc [6]. These metrics generically do not admit any symmetries, and therefore are different from ours.

Acknowledgements

Both authors were partly supported by NATO grant PST.CLG.978984. M.D. is a member of the European Network in Geometry, Mathematical Physics and Applications. We wish to thank the anonymous referee for valuable comments.

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