Neutral ASD conformal structures with null Killing vectors

Simon West

Girton College

Dissertation submitted for the degree of Doctor of Philosophy

Department of Applied Mathematics and Theoretical Physics

University of Cambridge

May 2007

ACKNOWLEDGEMENTS

I am very grateful to my supervisor, Maciej Dunajski, for his guidance, patience, and generosity with time and ideas. I also wish to thank my office-mate Tim Clifton for making lunchtime such a reliable source of enjoyment and relaxation. Finally, I thank my mother for her unwavering encouragement and support.

DECLARATION

The research presented in this thesis was carried out in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge between October 2003 and May 2007. This dissertation is my own work, except where explicit reference is made to the results of others. Some of the results were developed in collaboration with my supervisor Maciej Dunajski. The thesis is based on material that first appeared in the research papers [9] and [8].

This dissertation is not substantially the same as any that I have submitted, or am submitting, for a degree, diploma or other qualification at any other university.

Signed: .... Dated: ....

Neutral ASD conformal structures with null Killing vectors

Simon West

SUMMARY

We study four dimensional conformal structures in neutral signature whose Weyl tensor is anti-self-dual, and which possess a null conformal Killing vector. We show that the Killing vector gives rise to a natural foliation by totally null anti-self-dual surfaces, and that the leaf space inherits a projective structure. In the analytic case we show that the conformal structure and projective structure twistor spaces are related by dimensional reduction. We find a complete local classification that branches according to whether or not the Killing vector has twist. We study special types of metric within the conformal classes of the local classification, in particular Ricci-flat and pseudo-hyper-hermitian metrics, and find examples of conformal classes containing no Ricci-flat metrics. We give several explicit examples of the twistor space reduction that illustrate the general theory.

Contents

Chapter 1

Introduction

The subject of this thesis is a special class of four dimensional conformal structures in neutral signature, namely those whose Weyl tensor is anti-self-dual and which possess a null conformal Killing vector. In Section 1.4 of this chapter we will summarize the main results, but before doing so we put the topic into its proper context by providing some background material.

1.1 Anti-self-duality in four dimensional geometry

Let (M, g) be an oriented smooth four manifold with a metric, which may be indefinite. This is enough to induce a Hodge-$*$ operator which is a linear map from p-forms to $(4-p)$-forms. In particular, $*$ takes two-forms to two-forms. When g has Riemannian signature $++++$ or neutral signature $++--$, $*$ is an involution on two-forms and induces a decomposition

$$\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M \qquad (1.1)$$

into ± 1 eigenspaces. The splitting (1.1) is actually conformally invariant, i.e. any metric in the conformal class of g will induce the same splitting. The details are explained in Section 2.1.

Given a connection on a vector bundle over M, the curvature is a two-form with values in the Lie algebra of the structure group of the vector bundle. The Lie algebra will be some subalgebra of $gl(n, \mathbb{R})$, where n is the real dimension of the vector bundle. For example, the curvature of the Levi-Civita connection of a neutral metric on an oriented four manifold is a two-form with values in $so(2,2)$. As a two-form, the curvature decomposes under (1.1), and this is a fundamental feature of differential geometry in four dimensions. We now describe two distinct applications of this curvature decomposition.

The first application is gauge theory: a metric is fixed and one studies vector bundles with connection over M, using the splitting (1.1) to impose conditions on the curvature. The standard example is anti-self-dual (ASD) Yang-Mills theory, where one usually considers oriented complex Hermitian vector bundles with connection whose curvature is ASD. Another example is Seiberg-Witten theory, where one has a connection on a Hermitian line bundle whose self-dual curvature is required to satisfy an equation involving a parallel section of an associated vector bundle.

The second application is when one uses the splitting (1.1) to impose conditions on the Levi-Civita connection of the metric. This is fundamentally different from gauge theory, because the metric itself determines the splitting (1.1), and the connection is defined on the tangent bundle only. The most common example, and the one which this thesis is concerned with, is when the Weyl tensor is required to be ASD. The Weyl tensor is conformally invariant, and so is the splitting (1.1), so this is a well defined condition for a conformal class. We shall refer to a four dimensional metric/conformal class with ASD Weyl tensor as an *ASD metric/conformal class*.

In Riemannian signature these two basic situations have been intensively studied by both pure mathematicians and theoretical physicists, resulting in a wealth of interesting results and applications which we shall not attempt to survey here. We merely remark that in Riemannian signature the theory of elliptic PDEs can be applied, resulting in finite dimensional moduli spaces of solutions when suitable boundary conditions are imposed (for instance when working on compact manifolds).

In neutral signature, which will be our focus, one no longer has ellipticity; this results in behaviour that is fundamentally different from the Riemannian case. There are no finite dimensional moduli spaces of global solutions. Local rather than global behaviour becomes interesting, since the existence of 'time directions' allows reduction to hyperbolic and parabolic type equations. There are interesting links with integrable systems theory; many integrable systems are special cases of the neutral ASD Yang-Mills equations or ASD conformal structure equations. We shall not dwell on this, because it turns out that integrable systems do not play a role in our situation.

1.2 Twistor theory

Twistor theory encodes certain differential geometric objects into the holomorphic structure of complex manifolds. We now sketch the bare bones of the twistor theory that we need for the statement of the main results in the Section 1.4. Our presentation of twistor theory is biased in favour of neutral signature, and follows the original Penrose approach [25] rather than the Atiyah-Hitchin-Singer version [1], since the latter only applies to the

Riemannian case.

Suppose $(M, [g])$ is a neutral ASD conformal structure, and that there are local coordinates in which $[g]$ is real-analytic (generically such coordinates will not exist[1]). Then replacing the real coordinates with complex coordinates and the real-analytic functions in $[g]$ with holomorphic functions, we obtain the complexification $(M^{\mathbb{C}}, [g^{\mathbb{C}}])$. We call $[g^{\mathbb{C}}]$ a *holomorphic conformal structure*. Penrose [25] showed that the ASD condition is equivalent to the existence of a family of complex surfaces in $M^{\mathbb{C}}$, called *α-surfaces*. These are totally null and have self-dual tangent bi-vectors. There is a $\mathbb{CP}^1$ family of them through each point. The space of α-surfaces intersecting some suitable convex open set in $M^{\mathbb{C}}$ is the *twistor space* $\mathcal{PT}$, a complex three-manifold. $\mathcal{PT}$ contains a four parameter family of holomorphically embedded $\mathbb{CP}^1$s corresponding to the $\mathbb{CP}^1$s of α-surfaces through points in $M^{\mathbb{C}}$. $(M^{\mathbb{C}}, [g^{\mathbb{C}}])$ can, at least in principle, be recovered from $\mathcal{PT}$; to recover (M, g) from $\mathcal{PT}$ one requires extra data on $\mathcal{PT}$ called a *real structure*.

Another twistor correspondence we will need applies to two-dimensional projective structures [15]. A projective structure $[\Gamma]$ is an equivalence class of connections which share the same unparameterized geodesics. Suppose $(U, [\Gamma])$ is a two-dimensional projective structure that is real-analytic in some coordinate system, i.e. there is a connection $\Gamma \in [\Gamma]$ whose connection coefficients are real analytic. Complexifying as above, one obtains $(U^{\mathbb{C}}, [\Gamma^{\mathbb{C}}])$, a *holomorphic projective structure*. The space of complex unparameterized geodesics intersecting some suitable convex open set in $U^{\mathbb{C}}$ is the twistor space Z, a complex two-manifold. Z contains a two parameter family of holo-

[1]This can be seen as follows. Neutral ASD metrics exist with curvature scalars containing arbitrary functions. Choose the arbitrary function such that the curvature scalar is nonzero within some compact set and zero outside it. Such a function is not analytic in any coordinate system, so there is no coordinate system for which the metric is analytic.

morphically embedded $\mathbb{CP}^1$s corresponding to the $\mathbb{CP}^1$s of geodesics through points in $U^{\mathbb{C}}$. As in the conformal structure case, $(U^{\mathbb{C}}, [\Gamma^{\mathbb{C}}])$ can in principle be recovered from Z; again, to recover $(U, [\Gamma])$ requires a real structure.

1.3 ASD conformal structures with Killing vectors

Our work focuses on neutral ASD conformal structures with null conformal Killing vectors. The reason we focus on *null* conformal Killing vectors is that the non-null case is already well understood. We now briefly explain the non-null case, as it provides intuition for the null case and significantly influenced our work.

When the conformal Killing vector is non-null, the three-dimensional space of its trajectories inherits a non-degenerate conformal structure, simply by restricting the original conformal structure at each point to the orthogonal complement of the Killing vector. The Jones-Tod construction [17] (which applies in both Riemannian and neutral signature) is the fact that this induced conformal structure on the space of trajectories satisfies the Einstein-Weyl equations, as a consequence of the anti-self-duality of the four-dimensional conformal structure. The details are not important here; what is important is the general philosophy of taking a quotient using the Killing vector, and looking for geometrical structure on a lower dimensional space. We shall see that there is an analogue of the Jones-Tod construction for null Killing vectors. There is a fundamental difference however: the natural quotient in the null case is by *surfaces* containing the trajectories of the Killing vector, resulting in a two-dimensional base space. Of course, this can only be done in neutral signature, since there are no null vectors in Riemannian geometry.

1.4 Main results

We now present the main results of the thesis. Let $(M, [g], K)$ be a neutral ASD conformal structure with null conformal Killing vector. In this case, the conformal structure induced on the space of trajectories of K is degenerate. At first sight this does not seem promising. However, it turns out that K lies in an integrable two-plane distribution; quotienting by the integral surfaces gives a two-dimensional leaf space U. We find that U inherits a projective structure.

When $(M, [g], K)$ is real-analytic, the quickest route to the projective structure is by complexifying and appealing to twistor theory. In fact it is unlikely we would have discovered the role of two-dimensional projective structures without the twistor picture. The basic idea is that K induces a holomorphic vector field $\mathcal{K}$ on the twistor space $\mathcal{PT}$. $\mathcal{K}$ vanishes on a hypersurface $\mathcal{H} \subset \mathcal{PT}$, but one can show that the distribution it defines on $\mathcal{PT} - \mathcal{H}$ can be continued over $\mathcal{H}$. One can then quotient $\mathcal{PT}$ by the leaf space of the distribution, obtaining a complex manifold Z which is the twistor space of a projective structure.

Theorem 1. *Let $(M^{\mathbb{C}}, [g^{\mathbb{C}}])$ be a holomorphic ASD conformal structure, with twistor space $\mathcal{PT}$. Suppose there is a null conformal Killing vector $K^{\mathbb{C}}$. Then there is a holomorphic fibration $\mathcal{PT} \to Z$, where Z is the twistor space of a two dimensional projective structure.*

Theorem 1 does not tell us on which space the projective structure is defined. By analyzing the local geometrical structure of $(M^{\mathbb{C}}, [g^{\mathbb{C}}], K^{\mathbb{C}})$, one learns that it is defined on the leaf space of a foliation of $M^{\mathbb{C}}$ by anti-self-dual totally null surfaces (β-surfaces), as mentioned above. The projective structure geodesics can be related to the way the α-surfaces in $M^{\mathbb{C}}$ (which exist by virtue of anti-self-duality as explained in Section 1.2) intersect the β-surfaces. In fact this all works in the smooth real case, where generically there is no twistor space. Moreover, one can solve the equations and explicitly write down all ASD conformal structures with null conformal Killing vectors in terms of the underlying projective structure.

Theorem 2. *Let $(M,[g],K)$ be a smooth neutral signature ASD conformal structure with null conformal Killing vector. Then there exist local coordinates (t,x,y,z) and $g \in [g]$ such that $K = \partial_t$ and g has one of the following two forms, according to whether the twist $\mathbb{K} \wedge d\mathbb{K}$ vanishes or not ($\mathbb{K} := g(K,.)$):*

1. $\mathbb{K} \wedge d\mathbb{K} = 0$.

$$g = (dt + (zA_3 - Q)dy)(dy - \beta dx) -$$
$$(dz - (z(-\beta_y + A_1 + \beta A_2 + \beta^2 A_3))dx - (z(A_2 + 2\beta A_3) + P)dy)dx, \tag{1.2}$$

   where $A_1, A_2, A_3, \beta, Q, P$ are arbitrary functions of (x,y).

2. $\mathbb{K} \wedge d\mathbb{K} \neq 0$.

$$g = (dt + A_3\partial_z G dy + (A_2\partial_z G + 2A_3(z\partial_z G - G) - \partial_z\partial_y G)dx)(dy - zdx)$$
$$- \partial_z^2 G dx(dz - (A_0 + zA_1 + z^2A_2 + z^3A_3)dx), \tag{1.3}$$

   where A_0, A_1, A_2, A_3 are arbitrary functions of (x,y), and G is a function of (x,y,z) satisfying the following PDE:

$$(\partial_x + z\partial_y + (A_0 + zA_1 + z^2A_2 + z^3A_3)\partial_z)\partial_z^2 G = 0. \tag{1.4}$$

The functions $A_\alpha(x,y)$ in the metrics (1.2) and (1.3) determine projective structures on the two dimensional space U in the following way. A general projective structure corresponds to a second-order ODE

$$\frac{d^2y}{dx^2} = A_3(x,y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x,y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x,y)\Big(\frac{dy}{dx}\Big) + A_0(x,y). \quad (1.5)$$

In (1.3) all the $A_\alpha, \alpha = 0,1,2,3$ functions occur explicitly in the metric. In (1.2) the function A_0 does not explicitly occur. It is determined by the following equation:

$$A_0 = \beta_x + \beta\beta_y - \beta A_1 - \beta^2 A_2 - \beta^3 A_3, \quad (1.6)$$

as is shown in the proof of the theorem.

In the real-analytic case, when twistor spaces do exist for the conformal structure and projective structure in Theorem 2, then the twistor spaces are related as in Theorem 1.

Theorem 2 can be regarded as an analogue of the Jones-Tod correspondence for null Killing vectors. It works in the smooth case since it is proved using only local differential-geometric arguments; similarly the Jones-Tod correspondence can be proved in purely differential-geometric terms [17] which apply in the smooth neutral case, although it was motivated by twistor theory.

The two theorems above appeared in [9], and are the core results of the thesis. Having established them we proceed to special cases, such as existence of special metrics (Ricci-flat, pseudo-hyper-hermitian) within conformal classes and explicit twistor space constructions.

The thesis is structured as follows. Chapters 2 and 3 are introductory material. We explain the fundamental geometric structures that we need, namely neutral ASD conformal structures and projective structures, and set up the notation. The twistor correspondences are also reviewed. Chapter 4 is devoted to proving Theorem 1. This is done in two different ways. We also explain how it fits into a generalized picture due to Calderbank [5].

Chapter 5 is a proof of Theorem 2. This is actually independent of the previous twistor-theoretic chapter, and applies more generally, since it does not require analyticity. Having established the main theorems, the remaining chapters are devoted to special cases. In Chapter 6 we study special types of metric within the conformal classes that appear in Theorem 2. We find examples of Ricci-flat metrics, and show how to characterize pseudo-hyper-hermitian metrics in terms of the underlying projective structure. We also show how some previously known metrics fit into our picture. Chapter 7 is devoted to studying some twistor spaces explictly. We explain the flat case first, which is important as it provided important hints during the conception of this work, and then explain two curved examples in detail. We conclude in Chapter 8 with some open issues.

Chapter 2

Neutral ASD conformal structures

2.1 The Hodge-$*$ and self-duality

Suppose g is a non-degenerate inner product on the tangent bundle of an oriented four dimensional manifold M. We leave the signature of g unspecified for the moment. The inner product and orientation determine a volume form Ω on M, which is given in a local oriented basis $\mathbf{e}^a$ by

$$\Omega = 4!\sqrt{|det\ g_{ab}|}\ \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \wedge \mathbf{e}^4, \tag{2.1}$$

where $g = g_{ab}\ \mathbf{e}^a \odot \mathbf{e}^b$. The 4! factor is present so that when Ω is evaluated on an oriented orthonormal basis, the result is 1. The Hodge-$*$ operator restricted to two-forms is a linear operator $* : \Lambda^2 M \to \Lambda^2 M$, and is defined by

$$(*\omega)_{ab} = \frac{1}{2}\ \omega^{cd}\ \Omega_{cdab}, \tag{2.2}$$

for $\omega_{ab} = \omega_{[ab]} \in \Lambda^2 M$. In an oriented orthonormal basis the components of the volume form are $\Omega_{abcd} = \varepsilon_{abcd}$, where ε is the unique completely anti-symmetric symbol with $\varepsilon_{0123} = 1$. Evaluating $*^2$ in such a basis gives

$$(*^2\omega)_{ab} = \frac{1}{4}\,\omega_{cd}\varepsilon^{cdef}\varepsilon_{efab}, \tag{2.3}$$

where ε^{cdef} has been raised using the orthonormal form of g, which is a diagonal matrix with entries ±1. Let n be the number of positive entries in this matrix. Now applying the identity $\varepsilon^{cdef}\varepsilon_{efab} = 2(-1)^n(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c)$ gives

$$(*^2\omega)_{ab} = (-1)^n\,\omega_{ab}.$$

So when $n = 4$ or 2, the Hodge-$*$ is an involution on the six dimensional vector space of two-forms at each point, and by linear algebra we obtain a decomposition into three dimensional ±1 eigenspaces:

$$\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M. \tag{2.4}$$

Under a conformal transformation $g_{ab} \to cg_{ab}$ for a positive function c, the volume form (2.1) scales as $\Omega \to c^2\Omega$, and $g^{ab} \to c^{-1}g^{ab}$, so $*$ as defined in (2.2) is conformally invariant.

2.2 Spinors in neutral signature

From now on g will refer to a neutral four-metric. Existence of a neutral metric g on a four-manifold M requires a splitting

$$TM \cong T_+M \oplus T_-M,$$

where the summands are two-dimensional and are defined by the fact that g restricts to be positive/negative definite on them; this splitting imposes a topological restriction [14]. Further restrictions are imposed if we want spinors to be defined globally. We will work locally, ignoring these topological issues.

Let S, S' be real two-dimensional vector bundles, and fix volume forms $\epsilon \in \Gamma(S^* \wedge S^*)$, $\epsilon' \in \Gamma(S'^* \wedge S'^*)$ on them. Since we are working locally, all vector bundles are trivial, and we have $TM \cong S \otimes S'$ since both are four dimensional. We can then fix a map $TM \to S \otimes S'$ which is a vector space isomorphism at each point. This induces a map $T^*M \to S^* \otimes S'^*$. Extending these maps to all tensor products, they identify tensors on M with sections of tensor products of S and S'. To make contact with the neutral metric, we require

$$g \to \epsilon \otimes \epsilon' \tag{2.5}$$

under the linear map $T^*M \odot T^*M \to (S^* \otimes S'^*) \odot (S^* \otimes S'^*)$. This amounts to commutativity of the following diagram

$$\begin{tikzcd} TM \arrow[r] \arrow[d, "g(V,.)"'] & S \otimes S' \arrow[d, "\epsilon \otimes \epsilon'(V,.,.)"] \\ T^*M \arrow[r] & S^* \otimes S'^* \end{tikzcd}$$

Here $\epsilon \otimes \epsilon'(V, ., .)$ means contract V as a section of $S \otimes S'$ with the first and third entries of $\epsilon \otimes \epsilon'$ thought of as a section of $S^* \otimes S^* \otimes S'^* \otimes S'^*$. We now show how this works using local trivializations.

It is convenient in neutral signature to use a local coframe field $\{\mathbf{e}^{AA'}$, $A, A' = 0, 1\}$ in which the metric takes the form

$$g = 2(\mathbf{e}^{00'} \odot \mathbf{e}^{11'} - \mathbf{e}^{01'} \odot \mathbf{e}^{10'}); \tag{2.6}$$

this is always possible by linear algebra. Here $\mathbf{e}^{AA'} \odot \mathbf{e}^{BB'} := \frac{1}{2}(\mathbf{e}^{AA'} \otimes \mathbf{e}^{BB'} + \mathbf{e}^{BB'} \otimes \mathbf{e}^{AA'})$, so that if $\mathbf{e}_{AA'}$ is the dual basis of vectors satisfying $\mathbf{e}^{AA'}(\mathbf{e}_{BB'}) = \delta^A_B \delta^{A'}\delta_{B'}$ then for example $g(\mathbf{e}_{00'}, \mathbf{e}_{11'}) = g(\mathbf{e}_{11'}, \mathbf{e}_{00'}) = 1$. We will refer to such a frame field as a *Newman-Penrose tetrad.* The choice of such a frame is equivalent to fixing an isomorphism $TM \to M \times \mathbb{R}^4$ preserving the linear structure of the fibres, by means of the map $V = V^{AA'}\mathbf{e}_{AA'} \to V^{AA'}$ at each point of M.

Now trivialize S and S' by picking linearly independent sections $\{\iota, o \in \Gamma(S)\}$, $\{\iota', o' \in \Gamma(S')\}$, such that $\epsilon(\iota, o) = 1$. These trivialize S and S', i.e. $\mu \in \Gamma(S)$ can be expressed as $\mu = \mu^0 \iota + \mu^1 o$, and similarly for sections of S'. In other words these 'spin frames' fix isomorphisms $S \to M \times \mathbb{R}^2$, $S' \to M \times \mathbb{R}^2$ preserving linear structure of the fibres. We use capital letter for spinor indices, so

$$\nu^{A_1 \ldots A_j A'_1 \ldots A'_k}{}_{C_1 \ldots C_m C'_1 \ldots C'_n}(x)$$

is a section of $S^{\otimes j} S'^{\otimes k} S^{*\otimes m} S'^{*\otimes n}$ represented in a trivialization, where x denotes position on M. The volume forms are anti-symmetric matrices ϵ_{AB}, $\epsilon_{A'B'}$ with $\epsilon_{01} = \epsilon_{0'1'} = 1$ (note we suppress the prime on ϵ' when using indices, since the primed indices distinguish it from ϵ). These are used to raise and lower spinor indices, with the convention $\mu_B := \mu^A \epsilon_{AB}$ and similarly for primed spinors.

After trivializing the bundles in this way, the linear map $TM \to S \otimes S'$ simply sends $V = V^{AA'}\mathbf{e}_{AA'}$ to the section of $S \otimes S'$ defined by $V^{AA'}$. This map induces (2.5), as can be seen from rewriting (2.6) as

$$g = \epsilon_{AB}\epsilon_{A'B'} \; \mathbf{e}^{AA'} \otimes \mathbf{e}^{BB'}.$$

One can fix unique connections on S, S' (we shall denote both connections, as well as the Levi-Civita connection on TM, by ∇) by requiring covariant differentiation on the spin side to agree with covariant differentiation on the tensor side using the Levi-Civita connection, as well as requiring ϵ, ϵ' to be covariantly constant [26]. The connection coefficients $\Gamma_{AA'B}{}^{C}$, $\Gamma_{AA'B'}{}^{C'}$ in the spin frame described above are defined uniquely by the requirement

$$\nabla_{\mathbf{e}_{BB'}}\mathbf{e}_{AA'} = \Gamma_{BB'A}{}^{C}\mathbf{e}_{CA'} + \Gamma_{BB'}{}^{C'}{}_{A'}\mathbf{e}_{AC'},$$

together with the requirement that $\Gamma_{AA'[BC]} = \Gamma_{AA'[B'C']} = 0$, where $\Gamma_{AA'BC} := \Gamma_{AA'B}{}^{D}\epsilon_{DC}$, $\Gamma_{AA'B'C'} := \Gamma_{AA'B'}{}^{D'}\epsilon_{D'C'}$. Spinor fields are differentiated using these coefficients as follows:

$$\nabla_{BB'}\mu^{A} = \mathbf{e}_{BB'}(\mu^{A}) + \Gamma_{BB'C}{}^{A}\mu^{C}, \tag{2.7}$$

$$\nabla_{BB'}\mu_{A} = \mathbf{e}_{BB'}(\mu_{A}) - \Gamma_{BB'A}{}^{C}\mu_{C}, \tag{2.8}$$

and similarly for primed spinors.

The decomposition of two-forms (2.4) is simple in spinors. Let $\omega_{AA'BB'} = -\omega_{BB'AA'}$ be a two-form. We have

$$\omega_{AA'BB'} = \omega_{(AB)(A'B')} + \omega_{[AB](A'B')} + \omega_{(AB)[A'B']} + \omega_{[AB][A'B']}.$$

The first and last terms are not compatible with the index symmetry and must vanish. Also, wherever two anti-symmetrized indices appear we may replace with a multiple of ϵ or ϵ' since there is a unique anti-symmetric two-by-two matrix up to scale. Hence

$$\omega_{AA'BB'} = \psi_{AB}\epsilon_{A'B'} + \phi_{A'B'}\epsilon_{AB}, \tag{2.9}$$

where ψ_{AB}, $\phi_{A'B'}$ are symmetric in their indices.

Given a Newman-Penrose tetrad, with volume form $\Omega = 4!\ \mathbf{e}^{00'} \wedge \mathbf{e}^{10'} \wedge \mathbf{e}^{01'} \wedge \mathbf{e}^{11'}$, the spinor components of Ω are

$$\Omega_{AA'BB'CC'DD'} = \epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}.$$

This satisfies $\Omega_{00'10'01'11'} = 1$, and is antisymmetric on the four pairs of indices AA', BB', CC', DD' so must be the volume form.

So

$$
\begin{aligned}
(*\omega)_{AA'BB'} &= \frac{1}{2}(\psi^{CD}\epsilon^{C'D'} + \phi^{C'D'}\epsilon^{CD})(\epsilon_{CD}\epsilon_{AB}\epsilon_{C'A'}\epsilon_{D'B'} - \epsilon_{CA}\epsilon_{DB}\epsilon_{C'D'}\epsilon_{A'B'}) \\
&= (-\psi_{AB}\epsilon_{A'B'} + \phi_{A'B'}\epsilon_{AB}).
\end{aligned}
$$

Hence $\phi_{A'B'}\epsilon_{AB}$ is self-dual, $\psi_{AB}\epsilon_{A'B'}$ is anti-self-dual, and (2.9) is just (2.4) in spinors.

Finally, note that a vector V is null iff it decomposes as $V = \mu \otimes \nu$ for $\mu, \mu \in S, S'$, i.e. $V^{AA'} = \mu^A \nu^{A'}$ in components. This follows from the formula $g(V,V) = \det(V^{AA'}) = 0$ and linear algebra.

2.3 Alpha and beta planes

An α/β plane at a point $x \in M$ is a totally null plane (i.e. $g(V,W) = 0$ for any V, W in the plane) whose defining two-form is self-dual/anti-self-dual respectively. Such planes correspond to primed/unprimed spinors at x as follows. Let $\mu^A(x) \in S_x$. Consider the plane span$\{\mu^A \mathbf{e}_{AA'}, A' = 0, 1\}$. Let $V^{AA'} = \mu^A \iota^{A'}$ and $W^{AA'} = \mu^A \kappa^{A'}$ lie in this plane. Then

$$
g(V,W) = \epsilon_{AB}\epsilon_{A'B'}\mu^A \iota^{A'} \mu^B \kappa^{B'} = 0
$$

so it is totally null. It is defined by any two-form proportional to

$$
\begin{aligned}
V_{[a}W_{b]} &= \mu_A \iota_{A'} \mu_B \kappa_{B'} - \mu_B \iota_{B'} \mu_A \kappa_{A'} \\
&= c\ \mu_A \mu_B \epsilon_{A'B'},
\end{aligned}
$$

which is anti-self-dual. So $\mu^A(x)$ defines a β-plane. Similarly a primed spinor at x defines an α-plane.

An α/β surface is a surface whose tangent plane at each point is an α/β plane.

2.4 Anti-self-duality and Frobenius integrability

When expressed in terms of spinors, the decomposition of the Riemann tensor into irreducible components is very natural. One obtains ([26], pg. 236):

$$\begin{aligned} R_{abcd} = C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \tilde{C}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ + \frac{R}{12}(\epsilon_{AC}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}). \end{aligned} \quad (2.10)$$

Here $C_{ABCD}, \tilde{C}_{A'B'C'D'}$ are completely symmetric, and $\Phi_{ABC'D'}$ is symmetric on each pair of indices. R is the scalar curvature. The $C, \tilde{C}$ terms are the anti-self-dual and self-dual parts of the Weyl tensor. $\Phi_{AA'BB'}$ is proportional to the tracefree Ricci curvature:

$$\Phi_{AA'BB'} = -\frac{1}{2}(R_{ab} - \frac{1}{4}Rg_{ab}).$$

There is a formulation of the condition $C_{A'B'C'D'} = 0$ as a Frobenius integrability condition for self-dual null surfaces, due to Penrose [25], which we now describe using the spin bundle formalism. Denote the fibre coordinates of the spin bundle by $\pi^{A'}$, in a tetrad trivialization as described in the last section. Using the connection on S', the horizontal lifts of a Newman-Penrose tetrad $\mathbf{e}_{AA'}$ are

$$\tilde{\mathbf{e}}_{AA'} = \mathbf{e}_{AA'} - \Gamma_{AA'B'}{}^{C'}\pi^{B'}\frac{\partial}{\partial\pi^{C'}}. \quad (2.11)$$

We would like an expression for $[\tilde{\mathbf{e}}_{AA'}, \tilde{\mathbf{e}}_{BB'}]$. To find one, we will need the following formula ([26], pg. 247) relating curvature quantities to the derivatives of $\Gamma_{AA'C'}{}^{D'}$:

$$\begin{aligned}\mathbf{e}_{AA'}(\Gamma_{BB'E'}{}^{F'}) - \mathbf{e}_{BB'}(\Gamma_{AA'E'}{}^{F'}) &= \Gamma_{AA'E'}{}^{Q'}\Gamma_{BB'Q'}{}^{F'} - \Gamma_{BB'E'}{}^{Q'}\Gamma_{AA'Q'}{}^{F'} \\ &+ \Gamma_{AA'B'}{}^{Q'}\Gamma_{BQ'E'}{}^{F'} - \Gamma_{BB'A'}{}^{Q'}\Gamma_{AQ'E'}{}^{F'} \\ &+ \Gamma_{AA'B}{}^{Q}\Gamma_{QB'E'}{}^{F'} - \Gamma_{BB'A}{}^{Q}\Gamma_{QA'E'}{}^{F'} \\ + \epsilon_{AB}\epsilon^{F'Q'}\tilde{C}_{A'B'E'Q'} + \epsilon_{AB}(\epsilon_{A'E'}\epsilon_{B'}{}^{F'} &+ \epsilon_{A'}{}^{F'}\epsilon_{B'E'})\frac{R}{24} + \epsilon_{A'B'}\epsilon^{F'Q'}\Phi_{E'Q'AB}.\end{aligned}$$

Using this one obtains the following after some calculation:

$$\begin{aligned}[\tilde{\mathbf{e}}_{AA'}, \tilde{\mathbf{e}}_{BB'}] &= \Gamma_{AA'B}{}^{D}\tilde{\mathbf{e}}_{DB'} + \Gamma_{AA'B'}{}^{D'}\tilde{\mathbf{e}}_{BD'} - \Gamma_{BB'A}{}^{D}\tilde{\mathbf{e}}_{DA'} - \Gamma_{BB'A'}{}^{D'}\tilde{\mathbf{e}}_{AD'} \\ +(\epsilon_{AB}\epsilon^{F'Q'}\tilde{C}_{A'B'E'Q'} &+ \epsilon_{AB}(\epsilon_{A'E'}\epsilon_{B'}{}^{F'} + \epsilon_{A'}{}^{F'}\epsilon_{B'E'})\frac{R}{24} + \epsilon_{A'B'}\epsilon^{F'Q'}\Phi_{E'Q'AB})\pi^{E'}\frac{\partial}{\partial\pi^{F'}}.\end{aligned} \tag{2.12}$$

In turn, we can use this to show

$$\begin{aligned}[\pi^{A'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}] &= (\Gamma_{AA'B}{}^{D} - \Gamma_{BA'A}{}^{D})\pi^{A'}\pi^{B'}\tilde{\mathbf{e}}_{DB'} \\ &+ \pi^{A'}\pi^{B'}\epsilon_{AB}\epsilon^{F'Q'}\tilde{C}_{A'B'E'Q'}\pi^{E'}\frac{\partial}{\partial\pi^{F'}}.\end{aligned} \tag{2.13}$$

One can see from this that if $\tilde{C}_{A'B'C'D'} = 0$ then

$$\mathrm{span}\{L_A = \pi^{A'}\tilde{\mathbf{e}}_{AA'}, A = 0, 1\} \tag{2.14}$$

forms an integrable distribution, called the *twistor distribution*. Pushing down a leaf to M gives a surface whose tangent at each point is spanned by two-planes $\{\pi^{A'}(x)\mathbf{e}_{AA'}, A = 0, 1\}$, where $\pi^{A'}(x)$ is a section of S' over the surface in M. These are α-planes, as explained in 2.3, and the integral surfaces are called α-surfaces.

We can abstractly define the two-dimensional twistor distribution (2.14) on S' as follows. A point $s \in S'$ is determined by a primed spinor π at a point $x \in M$. The null vectors $\pi \otimes \mu$ for all unprimed spinors μ span an α-plane at x. Define the twistor distribution at s to be the subspace of horizontal vectors at s whose push-down to the base lies in this α-plane.

We have shown that anti-self-duality results in integrability for an α-surface through any α-plane. The converse is also true. Given an α-surface, it has a one parameter family of lifts to S', each corresponding to a solution of

$$\mu^{B'}\nabla_{BB'}\mu^{A'} = 0 \tag{2.15}$$

over the surface. A lift is a section of S' over the α-surface given by setting $\pi^{A'}(x) = \mu^{A'}$, and its tangent planes agree with the distribution $\pi^{A'}\mathbf{e}_{AA'}$ on S'. If there is an α-surface through any α-plane, then lift it to S' using (2.15). This gives a foliation of S' by surfaces with tangents of the form (2.14), and the fact that it is integrable implies $C_{A'B'C'D'} = 0$ using (2.13).

If we use the *projective* primed spin bundle $\mathbb{P}S'$ then each α-surface has a unique lift, given by the projective class of any spinor satisfying

$$\mu_{A'}\mu^{B'}\nabla_{BB'}\mu^{A'} = 0.$$

These lifts are tangent to the projectivization of L_A; since L_A are both homogeneous in the $\pi^{A'}$ fibre coordinates, they project to distributions on $\mathbb{P}S'$, resulting in an integrable two-dimensional distribution $\mathcal{L} \subset T\mathbb{P}S'$ which we also refer to as the twistor distribution. For a given ASD metric, the twistor distribution on $\mathbb{P}S'$ is the unique distribution that pushes down to the α-surfaces and is spanned by

$$L_A = \mathbf{e}_{A0'} + \lambda\mathbf{e}_{A1'} + f_A(x,\lambda)\frac{\partial}{\partial\lambda} \tag{2.16}$$

where f is cubic in λ, and λ is the affine coordinate $\pi^{1'}/\pi^{0'}$. The uniqueness follows because for a given α-surface the corresponding leaf is the section $\lambda(x)$ of $\mathbb{P}S'$ given by putting the tangents in the form $\mathbf{e}_{A0'} + \lambda(x)\mathbf{e}_{A1'}$. Note that (2.16) is in fact only an expression on one patch of $\mathbb{P}S'$, where $\pi^{0'} \neq 0$. It is cubic because the ∂_λ parts comes from the projectivization of the term $-\Gamma_{AA'B'}{}^{C'}\pi^{A'}\pi^{B'}\frac{\partial}{\partial\pi^{C'}}$ in the distribution (2.14). One calculates this projectivization by dividing the homogeneous version (2.14) by $\pi^{0'}$ to give an expression homogeneous of degree zero, and then pushing forward using $\lambda = \pi^{1'}/\pi^{0'}$. For example $\pi^{1'}\pi^{1'}\partial/\partial_{\pi^{0'}}$ projectivizes as follows:

$$\frac{\pi^{1'}\pi^{1'}}{\pi^{0'}}\frac{\partial}{\partial\pi^{0'}} \rightarrow \frac{\pi^{1'}\pi^{1'}}{\pi^{0'}}\frac{\partial\lambda}{\partial\pi^{0'}}\partial_\lambda = -\frac{(\pi^{1'})^3}{(\pi^{0'})^3}\partial_\lambda = -\lambda^3\partial_\lambda.$$

One obtains $\partial_\lambda, \lambda\partial_\lambda$ and $\lambda^2\partial_\lambda$ terms from projectivization of the other $\frac{\partial}{\partial\pi^{A'}}$ parts of (2.14), so f_A is cubic in λ.

Putting all this together we have the following reformulation of a famous result due to Penrose:

Theorem 3. *(Penrose [25]) Let $\boldsymbol{e}_{AA'}$ be four locally defined linearly independent vector fields on an open set $M \subset \mathbb{R}^4$. Then they form a Newman-Penrose tetrad for a neutral ASD metric if and only if there exist functions $f_A(x, \lambda)$, cubic in λ, such that*

$$span\{\boldsymbol{e}_{A0'} + \lambda\boldsymbol{e}_{A1'} + f_A\frac{\partial}{\partial\lambda}, A = 0, 1\} \tag{2.17}$$

is an integrable distribution on the affine patch of $M \times \mathbb{RP}^1$ with affine coordinate λ. The distribution (2.17) is the twistor distribution for the ASD metric, defined on an affine patch of $\mathbb{P}S'$.

2.5 Holomorphic ASD conformal structures

Everything in Sections 2.1 - 2.4 applies equally well to *holomorphic* conformal structures. A *holomorphic metric* $g^{\mathbb{C}}$ on a complex four-manifold $M^{\mathbb{C}}$ is a non-degenerate section of $T^*M^{\mathbb{C}} \odot T^*M^{\mathbb{C}}$, where $T^*M^{\mathbb{C}}$ is the holomorphic cotangent bundle. In local holomorphic coordinates z^i, a holomorphic metric just looks like a real metric, $g^{\mathbb{C}} = \Sigma\, g^{\mathbb{C}}_{ab}\, dz^i \odot dz^j$, where $g^{\mathbb{C}}_{ab}$ are holomorphic functions of the z^i. A holomorphic conformal structure is an equivalence class of holomorphic metrics up to multiplication by nonvanishing holomorphic functions. Given a real metric of any signature that is analytic in some coordinate system, one can obtain a holomorphic metric simply by replacing the real coordinates with complex ones and the real-analytic functions with holomorphic functions.

All the standard facts in real geometry, such as unique Levi-Civita connections, existence of geodesics, the Frobenius theorem, carry over directly into holomorphic geometry (details can be found in [20]). The spinor formalism explained in Section 2.2 applies to the holomorphic case simply by letting the spinors be complex, i.e. the spinor bundles are now two-complex-dimensional bundles $S^{\mathbb{C}}$ and $S'^{\mathbb{C}}$, and ϵ, ϵ' are holomorphic volume forms.

Suppose $(M^{\mathbb{C}}, [g^{\mathbb{C}}])$ is a holomorphic ASD conformal structure. Then Theorem 3 applies. There is a double fibration of $\mathbb{P}S'^{\mathbb{C}}$:

$$
\begin{array}{ccc}
 & \mathbb{P}S'^{\mathbb{C}} & \\
\swarrow & & \searrow \\
M^{\mathbb{C}} & & \mathcal{PT}
\end{array}
$$

where the left arrow is projection to $M^{\mathbb{C}}$, and the right arrow is the quotient by the leaves of the twistor distribution $\mathcal{L}$, giving the twistor space $\mathcal{PT}$, a complex three-manifold. Because $\mathcal{L}$ is transverse to each $\mathbb{CP}^1$ fibre of $\mathbb{P}S'^{\mathbb{C}}$, each fibre projects to a holomorphically embedded $\mathbb{CP}^1 \subset \mathcal{PT}$. Embedded curves in $\mathcal{PT}$ that arise in this way are called *twistor lines*. The normal bundle of a twistor line is the quotient of the normal bundle of its preimage in $\mathbb{P}S'^{\mathbb{C}}$ by the restriction of the twistor distribution to the preimage. We calculate this as follows.

A theorem of Birkhoff-Grothendieck states that holomorphic line bundles over $\mathbb{CP}^1$ are completely classified by their first Chern class; we denote the unique bundle of Chern class $n \in \mathbb{Z}$ by $\mathcal{O}(n)$, with $\mathcal{O} := \mathcal{O}(0)$ being the trivial bundle. The four vector fields (2.11) on $S'^{\mathbb{C}}$ are homogeneous of degree 0 in $\pi^{A'}$, so project to holomorphic vector fields on $\mathbb{P}S'^{\mathbb{C}}$. Restricting to a fibre, each is non-vanishing and transverse to the fibre, so the normal bundle of a fibre is the sum of four copies of the trivial bundle $\mathcal{O}$, since $\mathcal{O}$ is the only line bundle over $\mathbb{CP}^1$ with non-vanishing holomorphic sections.

The vector fields L_A from (2.14) are homogeneous of degree 1 in the $\pi^{A'}$ coordinates, so push down to one-dimensional holomorphic subbundles of $T\mathbb{P}S'^{\mathbb{C}}$, transverse to the fibres. To obtain a section of one of these subbundles, one must divide L_0 or L_1 by a homogeneous degree 1 polynomial in $\pi^{A'}$ coordinates. This gives a meromorphic section that blows up with order 1 at the single point at which the polynomial vanishes. Hence the subbundles are $\mathcal{O}(-1)$ by Birkhoff-Grothendieck. This gives an exact sequence of sheaves of sections of holomorphic bundles over a fibre:

$$0 \to \mathcal{O}(-1) \otimes \mathbb{C}^2 \to \mathcal{O} \otimes \mathbb{C}^4 \to N \to 0 \tag{2.18}$$

where N is the normal bundle of the image of the fibre in $\mathcal{PT}$. N is easily seen to be $\mathcal{O}(1) \oplus \mathcal{O}(1)$ as follows. Considering the form of L_0 in (2.16), the push-down of $\tilde{\mathbf{e}}_{01'}$ from $S^{\mathbb{C}}$ to $\mathbb{P}S'^{\mathbb{C}}$ defines a non-vanishing section of N, the quotient bundle, over the affine patch $\pi^{0'} \neq 0$. Likewise, the push down of $\tilde{\mathbf{e}}_{00'}$ defines a non-vanishing section of N on the affine patch $\pi^{1'} \neq 0$. On the overlap of these two patches we have $\tilde{\mathbf{e}}_{00'} \sim \tilde{\mathbf{e}}_{00'} - (\tilde{\mathbf{e}}_{00'} + \lambda\tilde{\mathbf{e}}_{01'}) = -\lambda\tilde{\mathbf{e}}_{01'}$, where $\sim$ is modulo L_A, as we are considering the quotient bundle of (2.18). Let $\tilde{\lambda} = \pi^{0'}/\pi^{1'} = 1/\lambda$ be the other affine coordinate. Then the two sections above define a trivialization of a line subbundle of N. A holomorphic section on the intersection of the two patches is given by $a(\lambda)\tilde{\mathbf{e}}_{01'}$ or $b(\tilde{\lambda})\tilde{\mathbf{e}}_{00'}$ with

a, b holomorphic functions on $\mathbb{C}^*$. For these to agree we require

$$a(\lambda)\mathbf{e}_{01'} \sim b(\tilde{\lambda})\tilde{\mathbf{e}}_{00'} \sim -\lambda b(\tilde{\lambda})\tilde{\mathbf{e}}_{01'},$$

so $a(\lambda) = -\lambda b(\tilde{\lambda})$. This is the transition function for $\mathcal{O}(1)$. Repeating the argument using $\tilde{\mathbf{e}}_{10'}$ and $\tilde{\mathbf{e}}_{11'}$ gives another $\mathcal{O}(1)$ subbundle, and we see that $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.

The power of twistor theory is that there is a converse to this. Let $\mathcal{PT}$ be a complex three-manifold with a holomorphically embedded $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Applying a theorem of Kodaira [19] shows that this $\mathbb{CP}^1$ belongs to a family of embedded $\mathbb{CP}^1$s parameterized by a complex manifold $M^{\mathbb{C}}$ of dimension 4 ($= \dim H^0(\mathbb{CP}^1, \mathcal{O}(1) \oplus \mathcal{O}(1))$). Holomorphic vectors at $x \in M^{\mathbb{C}}$ correspond to sections of the normal bundle of $\hat{x}$, the corresponding $\mathbb{CP}^1$, and null vectors are given by sections with a zero. This defines a conformal structure, because a global section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is given by $(a\pi^{0'} + b\pi^{1'}, c\pi^{0'} + d\pi^{1'})$ for homogeneous coordinates $\pi^{A'}$, $(a, b, c, d) \in \mathbb{C}^4$, and this can only be $(0, 0)$ when $ad - bc = 0$, which is a quadratic condition. In this case there is a zero at a single point, at $[\pi^{0'}, \pi^{1'}] = [-b, a]$ in non-homogeneous coordinates. The conformal structure is anti-self-dual, with α-surfaces defined by families of twistor lines through a fixed point in $\mathcal{PT}$.

In this picture, the α-surfaces are obtained as follows. Let $\hat{x} \subset \mathcal{PT}$ be the twistor line corresponding to a point $x \in M^{\mathbb{C}}$. Let $V \in T_xM$ be a null vector. We want to show that V lies in a unique α-surface through x. The corresponding section of the normal bundle of $\hat{x}$ has a zero at some point $p \in \mathcal{PT}$ because V is null. The α-surface corresponds to all the twistor lines that intersect $\hat{x}$ at p. There is a two-parameter family of sections that vanish at p, for instance if p is given by $[\pi^{0'}, \pi^{1'}] = [-b, a]$ as above then the sections

$$\alpha(a\pi^{0'} + b\pi^{1'}, \beta(a\pi^{0'} + b\pi^{1'}))$$

vanish at p; the two parameters are α and β. To show that these integrate to a two-parameter family of lines one must blow-up $\mathcal{PT}$ at p and use Kodaira theory; see [15] for details.

2.5.1 Reality conditions for split signature

In order to obtain a real split signature metric from a twistor space, we must be able to distinguish a four *real* parameter family of twistor lines, which we call real twistor lines, whose parameter space will be the four real dimensional manifold. In addition we require that given a line in this real family, the sections of the normal bundle that point to others in the family inherit a split signature conformal structure. As described above, a section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is defined by four complex numbers (a, b, c, d), with a quadratic form defined by $ad - bc$. If we restrict (a, b, c, d) to be real we obtain a real neutral signature quadratic form. A section of the normal bundle defined by real (a, b, c, d) with $ad - bc = 0$ vanishes at a point $[\pi^{0'}, \pi^{1'}] = [-b, a]$, and the locus of all such vanishing points is $\mathbb{RP}^1$, an equator of $\mathbb{CP}^1$.

To obtain a neutral signature conformal structure, each of the real twistor lines must be equipped with a choice of equator, such that sections of the normal bundle vanishing on the equator point to other real twistor lines nearby. This data is all encoded by the *real structure* σ of the twistor space. In the neutral signature context, σ is an anti-holomorphic involution $\mathcal{PT} \to \mathcal{PT}$ which maps a four real parameter family of twistor lines to themselves (the real twistor lines), and on these real lines restricts to a reflection fixing an equator.

Let us see how to construct σ, given the twistor space of a real-analytic neutral ASD metric. The complexificaton $M^{\mathbb{C}}$ of M is obtained by letting the real coordinates become complex. Suppose the complex coordinates are

(x, y, w, z). Then M is recovered as the set of fixed points of the anti-holomorphic involution $\tau : M^{\mathbb{C}} \to M^{\mathbb{C}}$ defined by

$$\tau(x, y, w, z) = (\bar{x}, \bar{y}, \bar{w}, \bar{z}).$$

Now τ maps α-surfaces to α-surfaces. To see this, suppose locally that α-surfaces are defined by

$$f_{a,b,c}(x, y, w, z) = 0, \quad g_{a,b,c}(x, y, w, z) = 0,$$

where f and g depend holomorphically on the three complex parameters a, b, c, since the space of α-surfaces is three dimensional. Now $f_{a,b,c}$ and $g_{a,b,c}$ are complexified versions of real analytic functions. So if one expresses them as power series, the coefficients will all be real. Therefore we have

$$f_{\bar{a},\bar{b},\bar{c}}(\bar{x}, \bar{y}, \bar{w}, \bar{z}) = 0, \quad g_{\bar{a},\bar{b},\bar{c}}(\bar{x}, \bar{y}, \bar{w}, \bar{z}) = 0,$$

so the α-surface defined by (a, b, c) gets mapped to the one defined by $(\bar{a}, \bar{b}, \bar{c})$. As an example take the flat case. Here α-surfaces are locally specified by (λ, σ, μ)[1] as follows:

$$f_{\lambda,\sigma,\mu} = \mu - x - \lambda y = 0, \quad g_{\lambda,\sigma,\mu} = \sigma - w - \lambda z = 0.$$

This includes all α-surfaces except those with $\lambda = \infty$. Under $(x, y, w, z) \to (\bar{x}, \bar{y}, \bar{w}, \bar{z})$, the α-surface defined by (λ, σ, μ) gets mapped to that defined by $(\bar{\lambda}, \bar{\sigma}, \bar{\mu})$. Therefore we have an anti-holomorphic involution $\sigma : \mathcal{PT} \to \mathcal{PT}$.

The fixed points of σ correspond to real α-surfaces through points in $M \subset M^{\mathbb{C}}$. Given a point $p \in M \subset M^{\mathbb{C}}$, any α-surface through p gets mapped under τ to another α-surface through p. Hence the corresponding

[1] We use Greek letters for the three parameters to agree with the conventions in Chapter 7.

line $\hat{p} \subset \mathbb{CP}^1$ is mapped to itself by σ. The lines $\hat{p}$ with $p \in M \subset M^{\mathbb{C}}$ are the real twistor lines. The α-surfaces in $M^{\mathbb{C}}$ containing the real α-surfaces in M, i.e. the complexifications of real α-surfaces, are fixed by τ, and the set of these through a point $p \in M \subset M^{\mathbb{C}}$ is an equator $\mathbb{RP}^1 \subset \mathbb{CP}^1$. In the $\mathbb{P}S'^{\mathbb{C}}$ picture, these lift to α-surfaces above p through points $[\pi^{0'}, \pi^{1'}]$ where the $\pi^{A'}$ are real, giving an $\mathbb{RP}^1$ of lifts. This $\mathbb{RP}^1$ of α-surfaces corresponds to the equator fixed by σ in the corresponding twistor line $\hat{p} \subset \mathcal{PT}$.

Given a twistor space $\mathcal{PT}$ with an antiholomorphic involution σ with the above properties, one can reconstruct a real neutral ASD conformal structure on the four real dimensional space of real twistor lines in $\mathcal{PT}$, by requiring the normal bundle sections of the real lines that vanish on the fixed equator to correspond to real null vectors. Null geodesics through a point p correspond to twistor lines intersecting $\hat{p}$ at a point on the fixed equator.

Up to holomorphism there are only two anti-holomorphic involutions of $\mathbb{CP}^1$: the one above fixing an equator, $[\pi^{0'}, \pi^{1'}] \to [\bar{\pi}^{0'}, \bar{\pi}^{1'}]$, and the antipodal map $[\pi^{0'}, \pi^{1'}] \to [-\bar{\pi}^{1'}, \bar{\pi}^{0'}]$, which is fixed point free. The latter is relevant for Riemannian twistor theory. The twistor space $\mathcal{PT}$ of a Riemannian ASD conformal structure possesses an antiholomorphic involution that fixes a four real parameter family of twistor lines and restricts to the antipodal map on them. The fact that there are no fixed points of σ restricted to real twistor lines corresponds to the fact that there are no null vectors in Riemannian geometry.

Chapter 3

Projective structures in two dimensions

3.1 Real projective structures

Let $(U, [\Gamma])$ be a local two dimensional real projective structure. That is, U is a local patch of $\mathbb{R}^2$, and $[\Gamma]$ is an equivalence class of torsion-free connections whose unparameterized geodesics are the same. Then in a local trivialization, equivalent torsion-free connections are related in the following way:

$$\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk} = a_j \delta^i_k + a_k \delta^i_j, \tag{3.1}$$

for functions a_i on U, and $i, j, k = 1, 2$. Note that this is a tensor equation since the difference between two connections is a tensor. The a_i are components of a one-form.

The geodesics satisfy the following ODE:

$$\frac{d^2 s^i}{dt^2} + \Gamma^i_{jk} \frac{ds^j}{dt} \frac{ds^k}{dt} = v \frac{ds^i}{dt},$$

where s^i are local coordinates on U, and t is a parameter, which is called affine if $v = 0$.

One can associate a second-order ODE to a projective structure by picking a connection in the equivalence class, choosing local coordinates $s^i = (x, y)$ say, and eliminating the parameter from the geodesic equations. The resulting equation determines the geodesics in terms of the local coordinates, without the parameter. The equation is as follows:

$$\frac{d^2y}{dx^2} = \Gamma^x_{yy}\Big(\frac{dy}{dx}\Big)^3 + (2\Gamma^x_{xy} - \Gamma^y_{yy})\Big(\frac{dy}{dx}\Big)^2 + (\Gamma^x_{xx} - 2\Gamma^y_{xy})\frac{dy}{dx} - \Gamma^y_{xx}. \qquad (3.2)$$

A general projective structure is therefore defined by a second-order ODE (1.5). In fact, two of the four functions A_0, A_1, A_2, A_3 can be eliminated by a coordinate transformation $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$ which introduces two arbitrary functions.

On TU, the horizontal lifts of $\partial/\partial s^i$ are defined by

$$S_i = \frac{\partial}{\partial s^i} - \Gamma^j_{ik} v^k \frac{\partial}{\partial v^j},$$

where v^i, $i = 1, 2$ are the fibre coordinates of TU. The geodesics on U lift to integral curves of the following spray on TU:

$$\Theta = v^i S_i = v^i \frac{\partial}{\partial s^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}, \qquad (3.3)$$

Now Θ is homogeneous of degree 1 in the v^i, so it projects to a one-dimensional distribution on $\mathbb{P}TU$, which we also denote by Θ. If λ is an affine coordinate on one patch of the $\mathbb{RP}^1$ factor, then the spray has the form

$$\Theta = \partial_x + \lambda\partial_y + (A_0(x, y) + \lambda A_1(x, y) + \lambda^2 A_2(x, y) + \lambda^3 A_3(x, y))\partial_\lambda. \qquad (3.4)$$

There is a unique curve in any direction through a point in U, so each curve has a unique lift to $\mathbb{P}TU$, and the lifted curves foliate $\mathbb{P}TU$.

To obtain (3.1) we argue as follows. If $\tilde{\Theta}$ is the spray corresponding to a different connection $\tilde{\Gamma}$, then Γ and $\tilde{\Gamma}$ are in the same projective class if Θ

and $\tilde{\Theta}$ push down to the same spray on PTU. This gives

$$\Theta - \tilde{\Theta} \propto v^i \frac{\partial}{\partial v^i},$$

from which (3.1) follows, using the fact that the connections are torsion-free (i.e. symmetric in their lower indices).

3.2 Holomorphic projective structures

If $U^{\mathbb{C}}$ is a local patch of $\mathbb{C}^2$ and $\Gamma^{\mathbb{C}}$ are holomorphic functions, one has a holomorphic connection, which gives rise to a *holomorphic projective structure* $(U^{\mathbb{C}}, [\Gamma^{\mathbb{C}}])$ using the holomorphic version of (3.1) in which the one-form a is holomorphic. Given a real-analytic projective structure, one can complexify by analytic continuation to obtain a holomorphic projective structure that will come equipped with a *reality structure* (see below).

Let $(U^{\mathbb{C}}, [\Gamma^{\mathbb{C}}])$ be a holomorphic projective structure. Then there is a double fibration of $\mathbb{P}TU^{\mathbb{C}}$:

$$\begin{array}{ccc} & \mathbb{P}TU^{\mathbb{C}} & \\ \swarrow & & \searrow \\ U^{\mathbb{C}} & & Z \end{array}$$

where the left arrow is projection to $U^{\mathbb{C}}$, and the right arrow is the quotient by the leaves of Θ, giving the twistor space Z, a complex two-manifold. Because Θ is transverse to each $\mathbb{CP}^1$ fibre of $\mathbb{P}TU^{\mathbb{C}}$, each fibre projects to a holomorphically embedded $\mathbb{CP}^1 \subset Z$. As in the ASD conformal structure case, embedded curves in Z that arise in this way are called twistor lines.

The normal bundle of a twistor line in Z is calculated using exactly the same method as that in Section 2.5 for an ASD conformal structures. The spray (3.4) is homogeneous of degree 1 in the v^i coordinates, and by the same reasoning as in Section 2.5 we obtain an exact sequence of sheaves on a $\mathbb{CP}^1$ fibre of $\mathbb{P}TU^{\mathbb{C}}$:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes \mathbb{C}^2 \to N \to 0 \tag{3.5}$$

where N is the normal bundle of the image of the fibre in Z. The same arguments show that $N \cong \mathcal{O}(1)$.

Again there is a converse. Let Z be a complex two-manifold with a holomorphically embedded $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(1)$. Kodaira's theorem shows that this $\mathbb{CP}^1$ belongs to a family of embedded $\mathbb{CP}^1$s parameterized by a complex manifold $U^{\mathbb{C}}$ of dimension 2 ($= \dim H^0(\mathbb{CP}^1, \mathcal{O}(1))$). A holomorphic vector $V \in T_u U^{\mathbb{C}}$ corresponds to a global section of the normal bundle $\mathcal{O}(1)$ of $\hat{u}$, the corresponding $\mathbb{CP}^1$. Such a section vanishes at a single point $p \in Z$. The geodesic of the projective structure through this direction is given by points in U corresponding to twistor lines in Z that intersect $\hat{u}$ at p. That there is a one-parameter family of such lines can be shown by blowing up Z at the vanishing point and using Kodaira theory, see [15].

3.2.1 Reality conditions for projective structures

In order to obtain a real projective structure from a twistor space, we must be able to distinguish a two real parameter family of twistor lines, which we call real twistor lines, whose parameter space will be the two real dimensional manifold. This is done by means of an anti-holomorphic involution $\sigma : Z \to Z$ that maps a two real parameter family of twistor lines to themselves, and fixes an equator of each line. The involution is obtained in exactly the same way as the one for the twistor space of a neutral ASD conformal structure explained in Section 2.5.1. That is, if (x, y) are complexified coordinates then the involution $(x, y) \to (\bar{x}, \bar{y})$ maps complex geodesics to complex geodesics, and generates the map σ. By the same reasoning as in Section 2.5.1, the involution σ leaves an equator on each real twistor line $\hat{p}$ unchanged, corresponding to the $\mathbb{RP}^1$ of complexified real geodesics through $p \in U \subset U^{\mathbb{C}}$. The real geodesics of the projective structure through p correspond to the twistor lines intersecting $\hat{p}$ at a point on its fixed equator.

The other possible anti-holomorphic involution of $\mathbb{CP}^1$, the antipodal map, does not play a role in the twistor theory of real two dimensional projective structures.

3.3 Flatness of projective structures

A projective structure is said to be *flat* if the corresponding second order ODE (1.5) can be transformed to the trivial ODE

$$\frac{d^2y}{dx^2} = 0 \tag{3.6}$$

by coordinate transformation $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$. The terminology comes from the fact that given any second order ODE one can construct a Cartan connection on a certain G-structure [4], and when this connection is flat the equation can be transformed to the trivial ODE (3.6). It turns out that a second order ODE must be of the form (1.5) to be flat, and in addition the functions A_0, A_1, A_2, A_3 must satisfy some PDEs. Defining

$$F(x, y, \lambda) = A_0(x, y) + \lambda A_1(x, y) + \lambda^2 A_2(x, y) + \lambda^3 A_3(x, y),$$

the following must hold [4]:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{11} - 4\frac{\mathrm{d}}{\mathrm{d}x}F_{01} - F_1\frac{\mathrm{d}}{\mathrm{d}x}F_{11} + 4F_1F_{01} - 3F_0F_{11} + 6F_{00} = 0, \tag{3.7}$$

where

$$F_0 = \frac{\partial F}{\partial y}, \qquad F_1 = \frac{\partial F}{\partial \lambda}, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y} + F\frac{\partial}{\partial \lambda}$$

This is a set of PDEs for the functions A_0, A_1, A_2, A_3.

Chapter 4

Null Killing vectors and twistor space reduction

In this chapter we show how the standard twistor constructions in Sections 2.5 and 3.2 are related when an ASD conformal structure possesses a null Killing vector.

4.1 Two foliations

Suppose g is a neutral metric, which is not required to be ASD at this point. A conformal Killing vector field K satisfies

$$\mathfrak{L}_K g = \eta g, \tag{4.1}$$

for some smooth function η. It follows that $\mathfrak{L}_K(e^c g) = (K(e^c) + e^c\eta)g$, so K is a conformal Killing vector for any conformally rescaled metric, and we can refer to K as a conformal Killing vector (C.K.V.) for the conformal structure $[g]$.

Now suppose g has a null conformal Killing vector K. We shall show that M is foliated in two different ways, by α and β surfaces, whose leaves intersect tangent to K. This is a property of the conformal structure $[g]$, since the Hodge-$*$ acting on 2-forms is conformally invariant.

The spinor form of the conformal Killing equation (4.1) is:

$$\nabla_a K_b = \phi_{A'B'}\epsilon_{AB} + \psi_{AB}\epsilon_{A'B'} + \frac{1}{2}\eta\epsilon_{AB}\epsilon_{A'B'}, \tag{4.2}$$

where $\phi_{A'B'}, \psi_{AB}$ are the self-dual and anti-self dual parts of the 2-form $\nabla_{[a}K_{b]}$.

Since K is null, we have $K = \iota \otimes o$, where ι is a section of S and o a section of S'. Choosing a null tetrad, we have $K^{AA'} = \iota^A o^{A'}$. These spinors are defined up to multiplication by a non-zero function α, since $K^{AA'} = \iota^A o^{A'} = (\alpha\iota^A)(o^{A'}/\alpha)$.

Lemma 1. *Let $K = \iota^A o^{A'} \boldsymbol{e}_{AA'}$ be a null conformal Killing vector. Then*

1. *The following algebraic identities hold:*

$$\iota^A \iota^B \psi_{AB} = 0, \tag{4.3}$$

$$o^{A'} o^{B'} \phi_{A'B'} = 0. \tag{4.4}$$

2. *ι^A and $o^{A'}$ satisfy*

$$\iota^A \iota^B \nabla_{BB'} \iota_A = 0, \tag{4.5}$$

$$o^{A'} o^{B'} \nabla_{BB'} o_{A'} = 0. \tag{4.6}$$

Proof. Using $K_{AA'} = \iota_A o_{A'}$, the Killing equation (4.2) becomes

$$o_{A'} \nabla_{BB'} \iota_A + \iota_A \nabla_{BB'} o_{A'} = \phi_{A'B'}\epsilon_{AB} + \psi_{AB}\epsilon_{A'B'} + \frac{1}{2}\eta\epsilon_{AB}\epsilon_{A'B'}. \tag{4.7}$$

Contracting both sides with $\iota^A o^{A'}$ gives

$$0 = o^{A'}\iota_B \phi_{A'B'} + \iota^A o_{B'} \psi_{AB} + \frac{1}{2}\eta \iota_B o_{B'}.$$

Multiplying by ι^B and $o^{B'}$ respectively leads to (4.3) and (4.4). To get (4.5) and (4.6), multiply (4.7) by $\iota^A\iota^B$ and $o^{A'}o^{B'}$, and use (4.3) and (4.4). $\Box$

Equations (4.5) and (4.6) are equivalent to the statement that the distributions spanned by $\iota^A\mathbf{e}_{AA'}$ and $o^{A'}\mathbf{e}_{AA'}$ are Frobenius integrable. So M is foliated in two different ways by α-surfaces and β-surfaces. It is clear that the α-surfaces and β-surfaces intersect on integral curves of K. Denote the β-surface distribution by $\mathcal{D}_\beta$; this will be used later.

4.2 Lift of K to $\mathbb{P}S'$

Let $(M, [g], K)$ be a holomorphic [1] ASD conformal structure with conformal Killing vector K (we abandon the superscript $^{\mathbb{C}}$ notation). The infinitesimal conformal isometry generated by K maps α-surfaces to α-surfaces, since the conformal structure g is preserved. So K gives rise to a canonically defined holomorphic vector field on $\mathcal{PT}$, which we denote by $\mathcal{K}$. When K is null, it is tangent to a two-parameter family of α-surfaces by Lemma 1. These α-surfaces are fixed by the infinitesimal motion generated by K, so $\mathcal{K}$ vanishes on a hypersurface $\mathcal{H} \subset \mathcal{PT}$.

It will be useful to express these facts using the double fibration picture of Section 2.5. We will need the following:

Proposition 1. *Let $K = K^{AA'}\boldsymbol{e}_{AA'}$ be a conformal Killing vector for an ASD metric g. Define a vector field $\tilde{K}$ on S' by*

$$\tilde{K} := K^{AA'}\tilde{\mathbf{e}}_{AA'} + \pi_{A'}\phi^{A'B'}\frac{\partial}{\partial \pi^{B'}} + \frac{1}{2}\eta\pi^{A'}\frac{\partial}{\partial \pi^{A'}}. \tag{4.8}$$

[1] Most of what we do in this section remains valid in the smooth case. However when we get to the proof of Theorem 1, holomorphicity is required.

Then this satisfies

$$[\tilde{K}, L_A] = (K^{BB'}\Gamma_{BB'A}{}^{D} - \psi_A{}^{D})L_D + \frac{3}{4}(\boldsymbol{e}_{AB'}\eta)\pi^{B'}\pi^{C'}\frac{\partial}{\partial \pi^{C'}}. \tag{4.9}$$

[Here the $\phi_{A'B'}, \psi_{AB}$ spinors come from the spinor form of the Killing equation (4.2).]

Proof. We will need the following identity:

$$K^a R_{abcd} = \nabla_b \nabla_c K_d - \frac{1}{2}(\eta_{,b}g_{cd} - \eta_{,c}g_{bd} + \eta_{,d}g_{bc}), \tag{4.10}$$

where η is the conformal factor appearing in (4.2).

One can calculate

$$[K^{AA'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}]$$

with the aid of (2.12). One obtains

$$\begin{aligned}
&[K^{AA'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}] = \\
&\qquad K^{AA'}\pi^{B'}(\Gamma_{AA'B}{}^{D}\tilde{\mathbf{e}}_{DB'} - \Gamma_{BB'A}{}^{D}\tilde{\mathbf{e}}_{DA'} - \Gamma_{BB'A'}{}^{D'}\tilde{\mathbf{e}}_{AD'}) + \\
&\quad K^{AA'}\pi^{B'}(\epsilon_{AB}(\epsilon_{A'E'}\epsilon_{B'}{}^{F'} + \epsilon_{A'}{}^{F'}\epsilon_{B'E'})\frac{R}{24} + \epsilon_{A'B'}\epsilon^{F'G'}\Phi_{E'G'AB})\pi^{E'})\frac{\partial}{\partial \pi^{F'}} \\
&\qquad\qquad - \pi^{B'}(\mathbf{e}_{BB'}K^{AA'})\tilde{\mathbf{e}}_{AA'}.
\end{aligned} \tag{4.11}$$

Using the spinor expression for the Riemann tensor (2.10), one can re-express the second term on the RHS as follows:

$$\begin{aligned}
&K^{AA'}\pi^{B'}(\epsilon_{AB}(\epsilon_{A'E'}\epsilon_{B'}{}^{F'} + \epsilon_{A'}{}^{F'}\epsilon_{B'E'})\frac{R}{24} + \epsilon_{A'B'}\epsilon^{F'G'}\phi_{E'G'AB})\pi^{E'})\frac{\partial}{\partial \pi^{F'}} \\
&\qquad = \frac{1}{2}K^{AA'}R_{AA'BB'EE'}{}^{EF'}\pi^{B'}\pi^{E'}\frac{\partial}{\partial \pi^{F'}}.
\end{aligned}$$

But now one can use (4.10) as follows:

$$
\begin{aligned}
\frac{1}{2}K^{AA'}R_{AA'BB'EE'}{}^{EF'}\pi^{B'}\pi^{E'}\frac{\partial}{\partial\pi^{F'}} &= \frac{1}{2}(\nabla_{BB'}(\psi_E{}^E\epsilon_{E'}{}^{F'} + \Phi_{E'}{}^{F'}\epsilon_E{}^E \\
&+ \frac{1}{2}\eta\epsilon_{E'}{}^{F'}\epsilon_E{}^E) - \frac{1}{2}(\mathbf{e}_{BB'}\eta)\epsilon^E{}_E\epsilon^{F'}{}_{E'} + \frac{1}{2}\epsilon^{F'G'}(\mathbf{e}_{GG'}\eta)\epsilon^G{}_B\epsilon_{E'B'} \\
&- \frac{1}{2}(\mathbf{e}_{BE'}\eta)\epsilon^{F'}{}_{B'})\pi^{E'}\pi^{B'}\frac{\partial}{\partial\pi^{F'}} \\
&= (\nabla_{BB'}\phi_{E'}{}^{F'} - \frac{1}{4}(\mathbf{e}_{BE'}\eta)\epsilon^{F'}{}_{B'})\pi^{E'}\pi^{B'}\frac{\partial}{\partial\pi^{F'}}
\end{aligned}
$$

Substituting this back into (4.11) and collecting terms gives:

$$
\begin{aligned}
[K^{AA'}\tilde{\mathbf{e}}_{AA'}, \pi^{B'}\tilde{\mathbf{e}}_{BB'}] &= (K^{AA'}\Gamma_{AA'B}{}^D - \psi_B{}^D)L_D \\
&- \pi^{B'}(\phi_{B'}{}^{A'}\epsilon_B{}^A + \frac{1}{2}\eta\epsilon_{B'}{}^{A'}\epsilon_B{}^A)\,\tilde{\mathbf{e}}_{AA'} + \pi^{B'}\pi^{E'}(\mathbf{e}_{BB'}\phi_{E'}{}^{F'} - \Gamma_{BB'E'}{}^{G'}\phi_{G'}{}^{F'} \\
&+ \Gamma_{BB'G'}{}^{F'}\phi_{E'}{}^{G'} - \frac{1}{4}(\mathbf{e}_{BE'}\eta)\epsilon^{F'}{}_{B'})\frac{\partial}{\partial\pi^{F'}}. \qquad (4.12)
\end{aligned}
$$

We wish to add a vertical term to $K^{AA'}\tilde{\mathbf{e}}_{AA'}$ which will cancel all the non-L_A terms on the RHS of (4.12), modulo a multiple of the Euler vector field, as this vanishes on projectivizing. A simple calculation shows that $\tilde{K}$ as defined in (4.8) does the trick. □

Since $\tilde{K}$ is weight zero in the $\pi^{A'}$ coordinates, it defines a vector field on $\mathbb{P}S'$, which we will also refer to as $\tilde{K}$ by abuse of notation. The last term on the right hand side of (4.9) is proportional to the Euler vector field, so does not contribute to $\tilde{K}$ on $\mathbb{P}S'$. Hence (4.9) shows that $\tilde{K}$ commutes with the twistor distribution $\mathcal{L}$ on $\mathbb{P}S'$. The vector field $\mathcal{K}$ on $\mathcal{PT}$ is the push-forward of $\tilde{K}$ to $\mathcal{PT}$, which is well defined because $\tilde{K}$ is Lie-derived along $\mathcal{L}$. The following lemma shows that when K is null, $\mathcal{K}$ vanishes on a hypersurface in $\mathcal{PT}$.

Lemma 2. *Let $K^{AA'} = \iota^A o^{A'}$ be a null conformal Killing vector for an ASD metric g, with $\tilde{K}$ defined as in Proposition 4.9. Then on the hypersurface $\mathcal{H} \subset \mathcal{PT}$ defined by $\{[\pi^{A'}] = [o^{A'}]\}$, $\tilde{K}$ lies in the twistor distribution.*

Proof. From (4.8) we have

$$\tilde{K} := \iota^A o^{A'} \tilde{\mathbf{e}}_{AA'} + \pi_{A'} \phi^{A'B'} \frac{\partial}{\partial \pi^{B'}} + \frac{1}{2} \eta \pi^{A'} \frac{\partial}{\partial \pi^{A'}}. \tag{4.13}$$

Now since the twistor distribution on $\mathbb{P}S'$ is the push down of $\pi^{A'} \tilde{\mathbf{e}}_{AA'}$ on S', it is clear that when $[\pi^{A'}] = [o^{A'}]$ the first term on the RHS of (4.13) lies in the twistor distribution. Now from (4.4), we obtain $o_{A'} \phi^{A'B'} \propto o^{B'}$. It follows that when $[\pi^{A'}] = [o^{A'}]$, the second term on the RHS of (4.13) is proportional to the Euler vector field $\pi^{A'} \frac{\partial}{\partial \pi^{A'}}$, so vanishes on pushing down to $\mathbb{P}S'$. The third term is a multiple of the Euler vector field everywhere, so is irrelevant. $\square$

The hypersurface $H \subset \mathbb{P}S'$ occurring in the previous lemma pushes down to the hypersurface $\mathcal{H} \subset \mathcal{PT}$, points of which correspond to the α-surfaces in the foliation defined by $o^{A'}$. The lemma shows that $\mathcal{K}$ vanishes on $\mathcal{H}$, as expected from the argument at the beginning of this section. It is clear that $\mathcal{K}$ is non-vanishing away from $\mathcal{H}$, since when $\{[\pi^{A'}] \neq [o^{A'}]\}$ on $\mathbb{P}S'$, the vector field $\tilde{K}$ is not tangent to the twistor distribution.

4.3 Beta surface lifts

Lemma 1 showed that a null C.K.V. gives rise to two foliations of M, one by α-surfaces and one by β-surfaces. Here we show that the β-surfaces lift to $\mathbb{P}S'$. Each β-surface has a one-parameter family of lifts, with a unique lift passing through each point in a fibre of $\mathbb{P}S'$. This contrasts with α-surface lifts which are unique.

Define a vector field

$$V = \iota^A L_A = \iota^A \pi^{A'} \tilde{\mathbf{e}}_{AA'}$$

on S'. This is weight one in the $\pi^{A'}$ coordinates, so gives a one dimensional distribution on $\mathbb{P}S'$ which restricts to the line bundle $\mathcal{O}(-1)$ on fibres, by the argument in Section 2.5. Together with span$\{\tilde{K}\}$, we get a two dimensional distribution on $\mathbb{P}S' - H$. We exclude H for the following reason. By the arguments in the proof of Lemma 4.8, $\tilde{K}$ reduces to the push-forward of $\iota^A o^{A'} \tilde{\mathbf{e}}_{AA'}$ on H, which lies in the span of the distribution defined by V. So the distribution defined by $\tilde{K}$ and V drops its rank from two to one on H.

The two dimensional distribution defined by $\{V, \tilde{K}\}$ on $\mathbb{P}S' - H$ pushes down to the β-plane distribution $\mathcal{D}_\beta$ on the base.

Lemma 3. *The two dimensional distribution on $\mathbb{P}S' - H$ determined by $\{V, \tilde{K}\}$ is integrable.*

Proof. We work on S' for convenience, and push down to $\mathbb{P}S'$ at the end. The distribution span$\{\tilde{K}, V\}$ on S' is two dimensional on S' when $\pi^{A'} o_{A'} \neq 0$.

$$
\begin{aligned}
[V, \tilde{K}] &= [\tilde{K}, \iota^C L_C] \\
&= \iota^C [\tilde{K}, L_C] + \tilde{K}(\iota^B) L_B \\
&= \iota^C ((K^{BB'} \Gamma_{BB'C}{}^D - \psi_C{}^D) L_D + \frac{3}{4} (\mathbf{e}_{CB'} \eta) \pi^{B'} \pi^{C'} \frac{\partial}{\partial \pi^{C'}}) \\
&\quad + K^{BB'} \mathbf{e}_{BB'} (\iota^C) L_C \\
&= (K^{BB'} \nabla_{BB'} \iota^C - \iota^D \psi_D{}^C) L_C + \# \Upsilon \\
&= (\iota^B o^{B'} \nabla_{BB'} \iota^C - \iota^D \psi_D{}^C) L_C + \# \Upsilon.
\end{aligned}
$$

From (4.3) we have $\iota^D \psi_D^C \propto \iota^C$, and from (4.5) we have $\iota^B o^{B'} \nabla_{BB'} \iota^C \propto \iota^C$. Hence the RHS is proportional to V, ignoring the irrelevant Euler vector field part. □

Next we will show that it is possible to continue this distribution over the hypersurface H so it is rank two on the whole of $\mathbb{P}S'$, and that the resulting distribution commutes on the hypersurface. It will then be possible to quotient $\mathbb{P}S'$ by the leaves of this distribution.

Lemma 4. *There is a two-dimensional integrable distribution $\mathcal{D}$ over $\mathbb{P}S'$, which on $\mathbb{P}S' - H$ is determined by $\{\tilde{K}, V\}$. Let ϱ be the projection $\mathbb{P}S' \to M$. Then for every $p \in \mathbb{P}S'$, we have $\varrho_*(\mathcal{D}\mid_p) = \mathcal{D}_\beta|_{\varrho(p)}$.*

Remark. One can think of $\mathcal{D}$ as a lift of the β-surfaces to $\mathbb{P}S'$, where each β-surface has a $\mathbb{CP}^1$ of lifts. Given a β-surface, take a $\mathbb{CP}^1$ fibre of $\mathbb{P}S'$ over some point on the surface. Then there is a unique leaf of $\mathcal{D}$ through each point in the fibre. By Lemma 4, each of these leaves projects to the β-surface under the projection $\mathbb{P}S' \to M$. So there is a $\mathbb{CP}^1$ of lifts of the β-surface.

Proof. Choose a spinor $\iota^{A'}$ satisfying $o^{A'}\iota_{A'} = 1$. Define the following (singular) vector field on S':

$$W = \frac{1}{\pi^{C'}o_{C'}}(V - (\pi^{D'}\iota_{D'})\tilde{K}). \tag{4.14}$$

This is weight zero in the $\pi^{A'}$, so defines a vector field on $\mathbb{P}S'$ by push-forward, which we shall also call W. We will now show that W is well defined even over $H \subset \mathbb{P}S'$, despite the $1/(\pi^{C'}o_{C'})$ factor in (4.14).

Without loss of generality, choose a tetrad such that

$$K = \iota^A o^{A'}\mathbf{e}_{AA'} = \mathbf{e}_{00'}.$$

That is, $\iota^A = (1,0)$, $o^{A'} = (1,0)$. Define $\lambda = \pi^{1'}/\pi^{0'}$ to be the coordinate on the $\pi^{0'} \neq 0$ patch of $\mathbb{CP}^1$, and extend this to a patch of $\mathbb{P}S'$; we call the patch $\mathcal{U}$. Then H lies entirely within $\mathcal{U}$ at $\lambda = 0$. We have the following expression for $\tilde{K}$, obtained by 'projectivizing' (4.8):

$$\begin{aligned}
\tilde{K} &= \tilde{\mathbf{e}}_{00'} + ({\phi_{0'}}^{1'} + \lambda({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda^2{\phi_{1'}}^{0'})\frac{\partial}{\partial\lambda} \\
&= \tilde{\mathbf{e}}_{00'} + (\lambda({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda^2{\phi_{1'}}^{0'})\frac{\partial}{\partial\lambda},
\end{aligned}$$

where ${\phi_{0'}}^{1'} = 0$ due to (4.4).

In the above conventions, we have $V = \pi^{A'}\tilde{\mathbf{e}}_{0A'}$. On $\mathcal{U} \subset \mathbb{P}S'$, the push forward of $\frac{1}{\pi^{C'}o_{C'}}V$ is

$$\frac{1}{\lambda}\tilde{\mathbf{e}}_{00'} + \tilde{\mathbf{e}}_{01'},$$

which is singular at H, corresponding to $\lambda = 0$. Choosing $\iota^{A'} = (0, -1)$, we then obtain the following expression for W on $\mathcal{U}$:

$$W = \frac{1}{\lambda}\tilde{\mathbf{e}}_{00'} + \tilde{\mathbf{e}}_{01'} - \frac{1}{\lambda}\tilde{K} = \tilde{\mathbf{e}}_{01'} - (({\phi_{1'}}^{1'} - {\phi_{0'}}^{0'}) + \lambda{\phi_{1'}}^{0'}))\frac{\partial}{\partial\lambda}.$$

This is a *non-singular* vector field on $\mathcal{U}$. By construction, away from H this lies in span$\{\tilde{K}, \tilde{V}\}$. Define $\mathcal{D}$ on $\mathcal{U}$ to be span$\{\tilde{K}, W\}$. This is clearly two-dimensional everywhere on $\mathcal{U}$. Note that W is also well defined over the other patch (i.e. at $\lambda = \infty$) so we can define $\mathcal{D}$ as span$\{\tilde{K}, W\}$ over the whole of $\mathbb{P}S'$.

We now want to show that $\mathcal{D}$ is integrable over H. We know (Lemma 3) that $\mathcal{D}$ is integrable away from H. Therefore on $\mathcal{U}$ we have

$$[\tilde{K}, W] = f\tilde{K} + gW + Y,$$

where f, g are holomorphic functions on $\mathcal{U}$ and Y is a holomorphic vector field vanishing on $\mathcal{U} - H$. But such a vector field must vanish, otherwise it is not even continuous, so is certainly not holomorphic.

The last part of the lemma is obvious, just from inspecting the coordinate expressions of $\tilde{K}$, W. $\Box$

Note that Lemma 4 applies equally well to the smooth case; in Section 5 we shall study it using local coordinates and find the general smooth solution.

We now have enough information to prove Theorem 1 from Section 1.4 (stated again below). There is a three dimensional integrable distribution $\mathcal{L} + \mathcal{D}$ on $\mathbb{P}S'$. It is three dimensional because $\mathcal{L}$ and $\mathcal{D}$ have a direction in common, which is the distribution defined by the push-forward to $\mathbb{P}S'$ of V on S'. Define Z to be the quotient of $\mathbb{P}S'$ by $\mathcal{L} + \mathcal{D}$.

Theorem 1. *Let $(M^{\mathbb{C}}, [g^{\mathbb{C}}])$ be a holomorphic ASD conformal structure, with twistor space $\mathcal{PT}$. Suppose there is a null conformal Killing vector $K^{\mathbb{C}}$. Then there is a holomorphic fibration $\mathcal{PT} \to Z$, where Z is the twistor space of a two dimensional projective structure, as defined in Section 3.2.*

Proof. $\mathcal{PT}$ is the quotient of $\mathbb{P}S'$ by $\mathcal{L}$. By Lemma 4, $\mathcal{L} + \mathcal{D}$ is an integrable distribution on $\mathbb{P}S'$, so there is a one-dimensional distribution on $\mathcal{PT}$; define Z to be the quotient of $\mathcal{PT}$ by this distribution. Equivalently, Z is the quotient of $\mathbb{P}S'$ by $\mathcal{L} + \mathcal{D}$. Since the distribution is holomorphic, there is a holomorphic fibration $\mathcal{PT} \to Z$. It remains to show that Z is a projective structure twistor space. Now $\mathcal{L}+\mathcal{D}$ is transverse to each $\mathbb{CP}^1$ fibre in $\mathbb{P}S'$, so each fibre projects to a holomorphically embedded $\mathbb{CP}^1 \subset Z$. We need to show that the normal bundle of such a $\mathbb{CP}^1$ is $\mathcal{O}(1)$.

The quotient of $\mathbb{P}S'$ by $\mathcal{L} + \mathcal{D}$ can be done by first quotienting by $\mathcal{D}$ and then quotienting by the residual one-dimensional distribution defined by $\mathcal{L}$. One can calculate the normal bundle of the projection of a fibre by considering a two-dimensional sub-bundle of $T\mathbb{P}S'|_{\mathbb{CP}^1}$, transverse to the $\mathbb{CP}^1$ fibre, and not in $\mathcal{D}$, with the residual $\mathcal{L}$ distribution as a sub-bundle. The required normal bundle is the quotient bundle of this two-dimensional distribution by the residual $\mathcal{L}$ distribution.

In the conventions of Lemma 4, the vector fields $\tilde{\mathbf{e}}_{10'}$, $\tilde{\mathbf{e}}_{11'}$ on $\mathbb{P}S'$, restricted to a fibre, do not lie in $\mathcal{D}$. The residual $\mathcal{L}$ distribution is simply the distribution defined by L_1. This is a linear combination of $\tilde{\mathbf{e}}_{10'}$ and $\tilde{\mathbf{e}}_{11'}$; it is $\mathrm{span}\{\tilde{\mathbf{e}}_{10'} + \lambda\tilde{\mathbf{e}}_{11'}\}$ on the $\pi^{0'} \neq 0$ patch, and $\mathrm{span}\{\tilde{\lambda}\tilde{\mathbf{e}}_{10'} + \tilde{\mathbf{e}}_{11'}\}$ on the $\pi^{1'} \neq 0$ patch. So $\mathrm{span}\{\tilde{\mathbf{e}}_{10'}, \tilde{\mathbf{e}}_{11'}\}$ is the two-dimensional sub-bundle of $T\mathbb{P}S'|_{\mathbb{CP}^1}$ described in the previous paragraph.

The normal bundle N of an image of a fibre under the quotient of $\mathcal{L} + \mathcal{D}$

fits into an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes \mathbb{C}^2 \to N \to 0,$$

where the $\mathcal{O} \otimes \mathbb{C}^2$ corresponds to the $\tilde{\mathbf{e}}_{10'}$, $\tilde{\mathbf{e}}_{11'}$ vectors restricted to the fibre. The $\mathcal{O}(-1)$ is the subdistribution of this vector bundle defined by L_1. From this exact sequence, the Chern class of N must be 1, so $N \cong \mathcal{O}(1)$ by the Birkhoff-Grothendieck theorem. Alternatively, one can see this by explictly coordinatizing N, using precisely the same method as that given under (2.18) in Section 2.5. Hence Z is a projective structure twistor space. $\square$

In fact we can canonically identify $\mathbb{P}S'/\mathcal{D}$ with $\mathbb{P}TU$, where U is the space of β-surfaces in M, as follows. Using the conventions of Lemma 4, the tangent planes to the β-surfaces in the base are spanned at each point by $\mathbf{e}_{00'}$, $\mathbf{e}_{01'}$. Now L_1 has the form $\mathbf{e}_{10'} + \lambda\mathbf{e}_{11'} + (\ldots)\partial_\lambda$, so at each point in the fibre above a point $x \in M$, L_1 pushes down to a different null direction transverse to the β-plane at x. Now suppose we take a lift of a β-surface Π, i.e. a leaf of $\mathcal{D}$ that projects down to Π. Push down L_1 at each point over this lift. This will give a vector field $\Theta = \mathbf{e}_{10'} + \lambda\mathbf{e}_{11'}$ over Π, where λ is now a function on the M.

We want to show that this determines a projective vector at the point $s \in U$ corresponding to $\Pi \subset M$. This means we require

$$[\mathbf{e}_{00'}, \Theta] \propto \Theta \ \mathrm{mod}\{\mathbf{e}_{00'}, \mathbf{e}_{01'}\}, \quad [\mathbf{e}_{01'}, \Theta] \propto \Theta \ \mathrm{mod}\{\mathbf{e}_{00'}, \mathbf{e}_{01'}\}.$$

But this follows from the fact that the distribution on $\mathbb{P}S'$ spanned by $\{\tilde{K}, W, L_1\}$ commutes. Hence to determine the projective vector corresponding to a leaf of $\mathcal{D}$, just choose a point on the leaf and push down L_1. Because of the above considerations, this direction will be independent of the choice of point on the leaf.

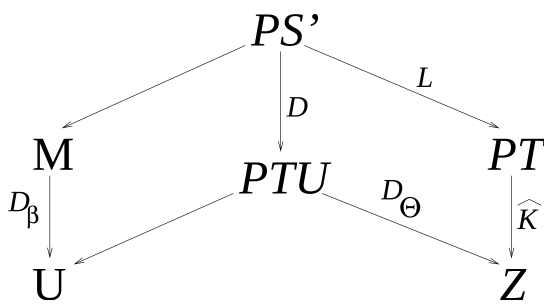

Figure 4.1: Relationship between foliation spaces.

What all this shows is that Z is the twistor space for a projective structure on U, the space of β-surfaces in M. Figure 1 illustrates the situation. Here p and q are the obvious projections. $\mathcal{D}_\beta$ represents the β-surface distribution on M. The $\widehat{\mathcal{K}}$ labelling the map from $\mathcal{PT}$ to Z requires some explanation. The vector field $\tilde{K}$ over PS' commutes with the twistor distribution (Lemma 1), so determines a vector field $\mathcal{K}$ on $\mathcal{PT}$. This vector field vanishes on a hypersurface $\mathcal{H} \subset \mathcal{PT}$, corresponding to the α-surfaces to which K is tangent; these are the α-surfaces appearing in Lemma 1. Now $\mathcal{K}$ on $\mathcal{PT}$ only depends on $\tilde{K}$ modulo $\mathcal{L}$. Lemma 4 shows that we can multiply $\tilde{K}$ modulo $\mathcal{L}$ by a meromorphic function $(1/\lambda)$ and obtain a vector field W commuting with the twistor distribution. This means that there is a one-dimensional distribution $\widehat{\mathcal{K}}$ over the whole of $\mathcal{PT}$ that never degenerates, and which agrees with span $\{\mathcal{K}\}$ on $\mathcal{PT} - \mathcal{H}$. The quotient of $\mathcal{PT}$ by this distribution gives Z, as illustrated in the diagram.

4.4 The divisor line bundle of $\mathcal{H}$

One can rephrase the above in terms of a divisor line bundle (some of the details in what follows are adapted from Calderbank [5]), leading to an alternative proof of Theorem 1. There is a holomorphic line bundle E over $\mathcal{PT}$

defined by the property that it has a meromorphic section ζ with a pole of order one on $\mathcal{H}$. This is the line bundle of the divisor [2] $-\mathcal{H}$. Then $\zeta \otimes \mathcal{K}$ defines a meromorphic section of $E \otimes T\mathcal{PT}$. Restricting to a twistor line, $\mathcal{K}$ gives a section of the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Also, E restricts to $\mathcal{O}(-1)$ because $\mathcal{H}$ intersects the twistor line at a point, and there is a section of E with a pole of order one on $\mathcal{H}$. So restricting this section to a twistor line gives a holomorphic section of the restriction of E to the line, with a pole of order one at a point. Hence by Birkhoff-Grothendieck $E \cong \mathcal{O}(-1)$. Therefore $\zeta \otimes \mathcal{K}$ restricts to a meromorphic section of $\mathcal{O} \oplus \mathcal{O}$ on a twistor line. This section is bounded away from the singular point, since $\mathcal{K}$ has a first order zero and ζ a first order pole, and non-zero, so must be non-vanishing everywhere on the line. Since this is true for every twistor line, $\zeta \otimes \mathcal{K}$ is a holomorphic, non-vanishing section of $E \otimes T\mathcal{PT}$. Equivalently, this is a non-vanishing section of $\mathrm{Hom}(E^*, T\mathcal{PT})$, i.e. an identification of a sub-bundle of $T\mathcal{PT}$ with E^*. Z is the quotient of $\mathcal{PT}$ by this sub-bundle. Note that this procedure for continuing the span of $\mathcal{K}$ over $\mathcal{H}$ is unique, so must be equivalent to the one explicitly constructed in Section 4.3.

In fact, the sub-bundle E^* must be transverse to twistor lines in $\mathcal{PT}$. This is because restricting the nonvanishing section of $\mathrm{Hom}(E^*, T\mathcal{PT})$ to a twistor line gives a nonvanishing section of $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathcal{O} \oplus \mathcal{O}$, where $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is the normal bundle of a line. All such sections are constant, so we get a non-vanishing section of the normal bundle, which means E^* is transverse to the line.

[2] In a complex manifold, a *divisor* is a formal sum of complex hypersurfaces $\Sigma_i n_i H_i$ where H_i are hypersurfaces. It is a standard fact, see [13] for details, that there is a corresponding line bundle, denoted $[\Sigma_i n_i H_i]$, with the property that there is a meromorphic section with zeros and poles on the hypersurfaces H_i, with orders specified by the numbers n_i. Positive n_i specifies a zero of order n_i, and negative n_i specifies a pole of order $|n_i|$.

Since the sub-bundle is transverse to all twistor lines, a twistor line projects to a line in Z. Now E^* restricts to $\mathcal{O}(1)$ on twistor lines. This gives the exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{O}(1) \otimes \mathbb{C}^2 \to \mathcal{O}(1) \to 0, \tag{4.15}$$

where the middle entry is the normal bundle of a twistor line in $\mathcal{PT}$, the left entry is the sub-bundle defined by the restriction of E^* (thought of as a subbundle of $T\mathcal{PT}$), and the final entry is the quotient. Thus, the normal bundle of a line in Z that is the projection of a twistor line in $\mathcal{PT}$ is $\mathcal{O}(1)$. This is the promised alternative proof of Theorem 1.

4.5 The relation between $\mathcal{PT}$ and Z

Here we investigate the quotient $\mathcal{PT} \to Z$ in more detail. We first note that there is a correspondence between β-planes at $x \in M$ and $\mathcal{O}(1)$ sub-bundles of the normal bundle of $\hat{x} \subset \mathcal{PT}$. This is seen as follows. A β-plane at x corresponds to a one-dimensional subspace of S_x. That is, it consists of vectors of the form $V^{AA'} = \mu^A \sigma^{A'}$ for $\sigma^{A'}$ varying and fixed non-zero μ^A. Now $V^{AA'}$ gives corresponds to a section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ by the map $V^{AA'} \to (V^{0A'}\pi_{A'}, V^{1A'}\pi_{A'})$, where $\pi_{A'}$ are the homogeneous coordinates of $\mathbb{CP}^1$. So sections corresponding to a β-plane are of the form

$$(\mu^0 \sigma^{A'} \pi_{A'}, \mu^1 \sigma^{A'} \pi_{A'}). \tag{4.16}$$

Clearly these sections define an $\mathcal{O}(1)$ sub-bundle $[a, b] = [\mu^0, \mu^1]$, where (a, b) are fibre coordinates of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Conversely, suppose we have an $\mathcal{O}(1)$ sub-bundle of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then a section of the sub-bundle will give a section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ with a single zero. This must be of the form (4.16) for some μ^A, and hence defines a β-plane.

In summary we have:

Lemma 5. *There is a one-one correspondence between β-planes at $x \in M$ and $\mathcal{O}(1)$ sub-bundles of the normal bundle of $\hat{x} \subset \mathcal{PT}$.*

In the previous section we showed that in the presence of a null C.K.V., there is a sub-bundle E^* of $T\mathcal{PT}$ that restricts to an $\mathcal{O}(1)$ sub-bundle of the normal bundle on twistor lines. It is easy to see that the β-plane to which K is tangent at $x \in M$ corresponds to the $\mathcal{O}(1)$ sub-bundle of the normal bundle obtained by restricting E^* to the twistor line $\hat{x}$. This is because $\mathcal{K}$, when restricted to $\hat{x}$, defines a section of the restriction of E^* to $\hat{x}$, and this section corresponds to K at x.

Now given E^* it is also easy to show that the corresponding β-plane distribution is integrable, using Kodaira deformation theory. Consider the union of trajectories of E^* intersecting a given twistor line $\hat{x}$. This is a divisor on $\mathcal{PT}$. Within this, $\hat{x}$ has normal bundle $\mathcal{O}(1)$ and hence by Kodaira theory there is a two-parameter family of nearby $\mathbb{CP}^1$s lying in the divisor, each of which intersects $\hat{x}$ at a point. The normal bundle of any of these nearby ones, within the divisor, is the sub-bundle of the total normal bundle determined by E^*, so corresponds to the β-plane to which K is tangent. Hence we have a β-surface distribution. Each β-surface corresponds in this way to a surface in $\mathcal{PT}$, the union of trajectories through any twistor line corresponding to a point on the β-surface.

Fig. 4.5 illustrates the situation. In M, a one parameter family of β-surfaces is shown, each of which intersects a one parameter family of α-surfaces, also shown. The β-surfaces correspond to a projective structure geodesic in U, shown at the bottom left.

The β-surfaces in M correspond to surfaces in $\mathcal{PT}$, as discussed above. These surfaces intersect at the dotted line, which corresponds to the one

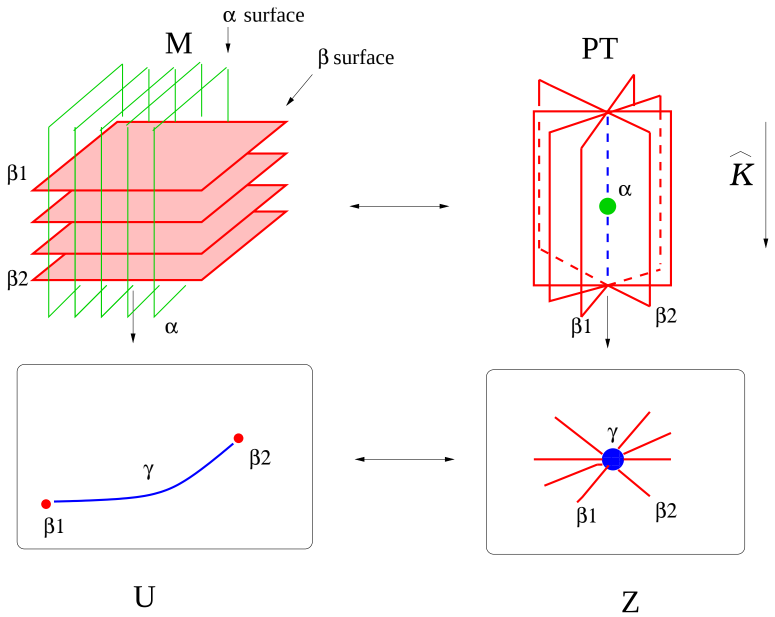

Figure 4.2: Relationship between M, U, $\mathcal{PT}$ and Z.

parameter family of α-surfaces in M. When we quotient $\mathcal{PT}$ by $\widehat{\mathcal{K}}$ to get Z, the surfaces become twistor lines in Z, and the dotted line becomes a point at which the twistor lines intersect; this is shown on the bottom right. This family of twistor lines intersecting at a point corresponds to the geodesic of the projective structure.

4.6 Calderbank's generalization

This picture was generalized by Calderbank in [5], who found a converse to Theorem 1. In this section we shall briefly explain his ideas; detailed proofs can be found in [5].

Suppose we start with a twistor space $\mathcal{PT}$ for an ASD conformal structure $(M, [g])$, and that $\mathcal{PT}$ has a holomorphic sub-bundle $\mathcal{B} \to T\mathcal{PT}$ of the tangent bundle, that is transverse to twistor lines, and restricts to an $\mathcal{O}(1)$

sub-bundle of the normal bundle on these lines. One can use the same Kodaira deformation argument as in Section 4.5 and Lemma 5 to show that there is a foliation of M by β-surfaces. This foliation need not be induced by a null C.K.V. in general. However, Calderbank shows that the foliation must be of a special type, which he calls an *anti-self-dual* β-surface foliation[3]. We shall now explain this.

A β-plane distribution is equivalent to a line sub-bundle $L \to S$, determined by a non-zero spinor ι^A up to scale. In order to treat this scale invariantly one introduces a connection[4] $\mathbf{D}$ on L^*. The sub-bundle $L \to S$ is equivalent to a section of $L^* \otimes S$, i.e. a line-bundle valued spinor. A change in the local trivialization of the line bundle is equivalent to a change in scale of the spinor. Now in a particular trivialization of L^* and S, denote the L^*-valued spinor by ι^A, and let the connection one-form of $\mathbf{D}$ be γ. Let $\nabla^{\mathbf{D}}$ be the connection on $L^* \otimes S$ that couples the connection ∇ on S to $\mathbf{D}$ on L^*. In the trivialisation, we have

$$\begin{aligned} \iota^A \iota^B \nabla^{\mathbf{D}}_{AA'} \iota_B &= \iota^A \iota^B (\nabla_{AA'} \iota_B + \gamma_{AA'} \iota_B) && (4.17) \\ &= \iota^A \iota^B \nabla_{AA'} \iota_B. && (4.18) \end{aligned}$$

Now the β-plane distribution is integrable iff $\iota^A \iota^B \nabla_{AA'} \iota_B = 0$, which by the above is equivalent to $\iota^A \iota^B \nabla^{\mathbf{D}}_{AA'} \iota_B = 0$. This last equation is equivalent to

$$\nabla^{\mathbf{D}}_{A'(A} \iota_{B)} = \nabla_{A'(A} \iota_{B)} + \gamma_{A'(A} \iota_{B)} = A_{A'(A} \iota_{B)} + \gamma_{A'(A} \iota_{B)},$$

for a one-form $A_{AA'}$. We have used the fact that $\iota^A \iota^B \nabla_{AA'} \iota_B = 0$ implies

$$\nabla_{A'(A} \iota_{B)} = A_{A'(A} \iota_{B)}. \tag{4.19}$$

[3]In [5] the terminology is a *self-dual* distribution, because his conventions are the reverse of ours.

[4]Unfortunately this corresponds to ∇ in [5], whilst our ∇ corresponds to his D. This is a consequence of our use of ∇ for the connections on S, S' induced by the Levi-Civita connection of g.

To see this simply expand $\nabla_{A'(A}\iota_{B)}$ using orthonormal bases for S and S':

$$\begin{aligned}\nabla_{A'(A}\iota_{B)} &= c_1\iota_{A'}\iota_A\iota_B + c_2\iota_{A'}\iota_{(A}o_{B)} + c_3\iota_{A'}o_Ao_B \\ &\quad + c_4o_{A'}\iota_A\iota_B + c_5o_{A'}\iota_{(A}o_{B)} + c_6o_{A'}o_Ao_B.\end{aligned}$$

Now using $\iota^A\iota^B\nabla_{AA'}\iota_B = 0$ we get $c_3 = c_6 = 0$. Then (4.19) follows, with

$$A_{A'A} = c_1\iota_{A'}\iota_A + c_2\iota_{A'}o_A + c_4o_{A'}\iota_A + c_5o_{A'}o_A.$$

Since changing the connection on L^* is achieved by adding an arbitrary one form to γ, there is a unique connection $\tilde{\mathbf{D}}$, with connection form $-A$, satisfying

$$\nabla^{\tilde{\mathbf{D}}}_{A'(A}\iota_{B)} = 0.$$

Calderbank calls $\tilde{\mathbf{D}}$ the *canonical connection* associated to a β-surface foliation, and if this connection has anti-self-dual curvature he calls the foliation an *anti-self-dual foliation*. In other words, dA is an anti-self-dual Maxwell field when the foliation is anti-self-dual. In more down-to-earth terms, a β-surface foliation is anti-self-dual if for any spinor ι^A determining it, the one-form $A_{AA'}$ defined by

$$\nabla_{A'(A}\iota_{B)} = A_{A'(A}\iota_{B)}$$

has the property that dA is anti-self-dual. This property doesn't depend on the choice of scale of ι_A because changing the scale results in A changing by df for a function f, so dA is unchanged. Calderbank argues that in the case that ι^A comes from a null C.K.V., this is satisfied.

In the case of a β-surface foliation that arises from a line sub-bundle $\mathcal{B}$, one obtains a connection on L^* using the Penrose-Ward transform of $\mathcal{B}\otimes\kappa^{1/4}$, where κ is the canonical bundle of $\mathcal{PT}$. This line bundle is trivial on twistor lines, so the Penrose-Ward transform gives rise to an ASD connection on a line bundle over M, which one can identify with L^*. One can then show that this connection is in fact canonical in the above sense, so the β-surface foliation is anti-self-dual.

Conversely, suppose one has an integrable β-plane distribution. By Lemma 5 this corresponds to an $\mathcal{O}(1)$ section of the normal bundle of each twistor line. To generate a sub-bundle of $T\mathcal{PT}$, these sub-bundles must agree where any two twistor lines intersect. Calderbank shows that this happens iff the β-surface distribution is anti-self-dual.

Chapter 5

The general solution

5.1 Coordinate choices adapted to the twistor distribution

In this chapter we introduce local coordinates in order to construct explicit metrics with null CKVs, the end result being Theorem 2. The material presented in Section 4.3 is useful as a guide in choosing coordinates that highlight the geometrical structures involved, such as the integrable β-surfaces and the projective structure. It turns out that one can essentially solve the anti-self-duality equations completely to find all neutral ASD conformal structures with null CKVs, once suitable coordinates have been chosen. One obtains explicit expressions for the twistor distribution, where the geometry presented in Chapter 4 is transparent. An interesting feature of these calculations is that they do not require any analyticity assumptions, and therefore apply to the general smooth case. This is similar to the fact that the Jones-Tod correspondence works for smooth neutral ASD metrics with non-null Killing vectors, even though the Penrose twistor correspondence only applies to the analytic case.

In the following calculations we will often use a shorthand for coordinate transformations: $t \to F(t,x,y,z)$ means define a new coordinate $\tilde{t} = F(t,x,y,z)$ and then relabel it t again. This avoids having to introduce new symbols for new coordinates. Partial derivatives will be denoted by subscripts, for example $F_z := \partial_z F$.

Let (M,g,K) be a smooth, neutral ASD metric with null conformal Killing vector. Choose a conformal factor such that K is a pure Killing vector; this can always be done since it amounts to solving an ODE. We will use the conventions of Lemma 4, that is we choose a Newman-Penrose tetrad with $K = \mathbf{e}_{00'}$, where the tangent planes to the β-surfaces are spanned by K and $\mathbf{e}_{01'}$. Now choose coordinates (t,z,x,y) in which $K = \partial_t$. Since K is pure Killing, in these coordinates the metric contains no functions of t. Hence $[K, \mathbf{e}_{01'}] = 0$ and we may take $\mathbf{e}_{01'} = \partial_z$. Then we have

$$\tilde{K} = \partial_t, \tag{5.1}$$

$$L_0 = \partial_t + \lambda\partial_z + f(x,y,z,\lambda)\partial_\lambda, \tag{5.2}$$

where L_0 is half of the twistor distribution (2.16). Note that f does not depend on t because it comes from connection coefficients, which do not depend on t since it does not occur in the metric. Also note that $\tilde{K} = \partial_t$, because L_0 and L_1 do not contain functions of t, so it commutes with both as required.

The next step is to eliminate the ∂_λ dependence in L_0. This is achieved by a Möbius transformation, $\lambda \to (\hat{\beta} + \hat{\delta}\lambda)/(\hat{\alpha} + \hat{\gamma}\lambda)$, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are functions on M. Now the new λ coordinate satisfies $\tilde{K}(\lambda) = L_0(\lambda) = 0$.

Therefore $\hat{\alpha}, \ldots, \hat{\delta}$ do not depend on t. This gives the following general form:

$$\tilde{K} = \partial_t, \tag{5.3}$$

$$L_0 = \alpha\, \partial_t + \beta\, \partial_z + \lambda(\gamma\, \partial_t + \delta\, \partial_z). \tag{5.4}$$

What we have done here is trivialize $\mathbb{P}S'$ by the requirement that λ is constant over any β-surface lift. One can see that this is possible invariantly as follows. Take a surface in M that is transverse to the β-surface foliation. On each fibre of $\mathbb{P}S'$ above this surface, choose a λ coordinate, smoothly varying along the surface. Now at each point in the surface, extend the coordinate along fibres above the unique β-surface through the point, by the requirement that it remain constant on the lifts of the β-surface. This gives a λ coordinate satisfying $L_0\lambda = 0$, hence of the form (5.4).

The projective structure now emerges naturally. Clearly (x, y) are the coordinates on the space of β-surfaces. The other vector in the twistor distribution has the form

$$\begin{aligned} L_1 := {} & J_0(x,y,z)\partial_x + J_1(x,y,z)\partial_y + \lambda(J_2(x,y,z)\partial_x + J_3(x,y,z)\partial_y) \\ & + (A_0(x,y,z) + \lambda A_1(x,y,z) + \lambda^2 A_2(x,y,z) + \lambda^3 A_3(x,y,z))\partial_\lambda \\ & + (C(x,y,z) + \lambda D(x,y,z))\partial_t + (E(x,y,z) + \lambda F(x,y,z))\partial_z. \end{aligned} \tag{5.5}$$

where $J_0 J_3 - J_1 J_2 \neq 0$. This is the most general possible form for L_1, where no functions depend on t since ∂_t is a Killing vector. The A_i functions multiplying ∂_λ are combinations of connection coefficients, which are partial derivatives of functions in the metric, and therefore because the functions in the metric do not have t dependence, neither do the functions A_i. The twistor distribution span$\{L_0, L_1\}$ is required to be integrable, by Theorem 3. This requires

$$\frac{(J_0)_z}{J_0} = \frac{(J_1)_z}{J_1} = \frac{(J_2)_z}{J_2} = \frac{(A_0)_z}{A_0} = \ldots = \frac{(A_3)_z}{A_3},$$

which means that all these functions share a common factor that depends on z, and when this is divided out one obtains functions of (x, y). Dividing L_1 by this common factor [1] results in an L_1 of the form

$$
\begin{aligned}
L_1 := J_0(x,y)\partial_x + J_1(x,y)\partial_y + \lambda(J_2(x,y)\partial_x + J_3(x,y)\partial_y) \\
+ (A_0(x,y) + \lambda A_1(x,y) + \lambda^2 A_2(x,y) + \lambda^3 A_3(x,y))\partial_\lambda \\
+ (C(x,y,z) + \lambda D(x,y,z))\partial_t + (E(x,y,z) + \lambda F(x,y,z))\partial_z, \quad (5.6)
\end{aligned}
$$

One now observes that the $\partial_x, \partial_y, \partial_\lambda$ terms precisely correspond to a projective structure spray on PTU. Since $\mathcal{D}$ is spanned by ∂_t, ∂_z, the quotient of L_1 by $\mathcal{D}$ gives a projective structure.

To put the projective structure spray occuring in (5.6) into the more standard form (3.4) (i.e. $J_0 = J_3 = 1, J_1 = J_2 = 0$) it is necessary to perform a Möbius transformation of λ depending on (x, y). Since this does not depend on t or z, the general form (5.4) of L_0 is unchanged by this, and we can assume that the projective structure spray in L_1 is of the form (3.4), which we shall do from now on.

We have found a general form that any $\{\tilde{K}, L_A\}$ can be put into. For it to give an ASD conformal structure, the L_A must commute modulo L_A. Imposing this gives equations for the unknown functions, which will lead us to the metrics appearing in Theorem 2.

First, it is convenient to change coordinates yet again, because together with conformal rescaling we can elimate three of the four functions in L_0. We may assume $\delta \neq 0$ (if $\delta = 0$ then $\beta \neq 0$, in which case perform the coordinate change $\lambda \to 1/\lambda$). Now change coordinates by $(t, x, y, z) \to (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) =$

[1] This corresponds to a conformal transformation of the metric. Since we are only interested in the conformal structure in this theorem, we will perform such operations freely and use the same symbol L_1.

$(t+j(x,y,z),x,y,k(x,y,z))$, where $k_z \neq 0$. This does not affect the general form (5.6) of L_1.

We obtain

$$L_0 = \alpha\partial_{\tilde{t}} + \beta(\frac{\partial k}{\partial z}\partial_{\tilde{z}} + \frac{\partial j}{\partial z}\partial_{\tilde{t}}) + \lambda(\gamma\partial_{\tilde{t}} + \delta(\frac{\partial k}{\partial z}\partial_{\tilde{z}} + \frac{\partial j}{\partial z}\partial_{\tilde{t}})).$$

Here we regard $\frac{\partial k}{\partial z}$ and $\frac{\partial j}{\partial z}$ as functions of $(\tilde{x},\tilde{y},\tilde{z})$. Now choose $j(x,y,z)$ such that the following equation is satisfied:

$$\frac{\partial j}{\partial z} = -\frac{\gamma}{\delta}.$$

The right hand side is not singular because $\delta \neq 0$. Then we have

$$L_0 = (\alpha - \frac{\beta\gamma}{\delta})\partial_{\tilde{t}} + \beta\frac{\partial k}{\partial z}\partial_{\tilde{z}} + \lambda\delta\frac{\partial k}{\partial z}\partial_{\tilde{z}}$$

Since the function $(\alpha - \frac{\beta\gamma}{\delta}) \neq 0$, we can divide by it. Finally one can choose $k(x,y,z)$ to satisfy

$$\frac{\partial k}{\partial z} = \frac{1}{\delta}(\alpha - \frac{\beta\gamma}{\delta}),$$

where the right hand side is non-vanishing.

Removing the tildes, we end up with

$$\tilde{K} = \partial_t, \tag{5.7}$$

$$L_0 = \partial_t - \beta(x,y,z)\partial_z + \lambda\partial_z, \tag{5.8}$$

$$\begin{aligned} L_1 = {} & \partial_x + \lambda\partial_y + (A_0(x,y) + \lambda A_1(x,y) + \lambda^2 A_2(x,y) \\ & +\lambda^3 A_3(x,y))\partial_\lambda + (C(x,y,z) + \lambda D(x,y,z))\partial_t + (E(x,y,z) + \lambda F(x,y,z))\partial_z. \end{aligned} \tag{5.9}$$

One can now read off an NP tetrad $\mathbf{e}_{AA'}$ for a metric $g \in [g]$ corresponding to the twistor distribution, using $L_A = \mathbf{e}_{A0'} + \lambda\mathbf{e}_{A1'} + f_A\partial_\lambda$. One finds that $\mathbb{K} \wedge d\mathbb{K} = \beta_z dx \wedge dy \wedge dz$, where $\mathbb{K} = g(\partial_t, .)$. Thus the twist of the Killing vector ∂_t vanishes iff β does not depend on z. Since existence of twist is a conformally invariant property [2], the cases $\beta_z = 0$ and $\beta_z \neq 0$ are genuinely distinct, not an artefact of our coordinate choices. We now analyse each in turn.

5.2 Twist-free case

We have $\beta_z = 0$. Calculating the commutator of L_0 and L_1 we obtain

$$\begin{aligned}[L_0, L_1] = \quad & (-\beta+\lambda)(C_z+\lambda D_z)\partial_t + (\beta_x + \lambda\beta_y - \beta E_z - \lambda\beta F_z + \lambda E_z + \lambda^2 F_z \\ & -(A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3))\partial_z. \qquad (5.10)\end{aligned}$$

Since we require $\{L_0, L_1\}$ to be integrable, this must be a multiple of L_0. We deduce that

$$[L_0, L_1] = (-\beta+\lambda)(C_z+\lambda D_z)L_0. \qquad (5.11)$$

Now comparing the ∂_z coefficients of (5.10) and (5.11) we get four equations, one for each power of λ.

The λ^3 equation is

$$D_z = -A_3 \rightarrow D(x,y,z) = -zA_3(x,y) + Q(x,y), \qquad (5.12)$$

where Q is arbitrary.

The λ^2 equation is

$$F_z - C_z = A_2 + 2\beta A_3. \qquad (5.13)$$

Substituting (5.12) into this and solving gives

$$F(x,y,z) - C(x,y,z) = z(A_2(x,y) + 2\beta(x,y)A_3(x,y)) + P(x,y), \qquad (5.14)$$

where P is arbitrary.

---

[2] Let $\mathbb{K} := g(K,.)$, and $\tilde{\mathbb{K}} := cg(K,.)$, where c is a conformal factor. Now $\tilde{\mathbb{K}} \wedge d\tilde{\mathbb{K}} = c\mathbb{K} \wedge (dc \wedge \mathbb{K} + cd\mathbb{K}) = c^2\mathbb{K} \wedge d\mathbb{K}$, so $\tilde{\mathbb{K}} \wedge d\tilde{\mathbb{K}} = 0$ iff $\mathbb{K} \wedge d\mathbb{K} = 0$.

The λ equation is

$$E_z + \beta C_z = -\beta_y + A_1 + \beta A_2 + \beta^2 A_3, \tag{5.15}$$

which integrates to gives

$$E(x, y, z) + \beta(x, y)C(x, y, z) = z(-\beta_y + A_1 + \beta A_2 + \beta^2 A_3) + R(x, y), \tag{5.16}$$

where R is arbitrary. Substituting D, F, E from (5.12), (5.14), (5.16) into L_1, we find the following:

$$\begin{aligned} L_1 &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3)\partial_\lambda \\ &+ \lambda(-zA_3 + Q)\partial_t + (z(-\beta_y + A_1 + \beta A_2 + \beta^2 A_3) + \lambda(z(A_2 + 2\beta A_3) + P))\partial_z, \end{aligned} \tag{5.17}$$

where P and Q are arbitrary functions of (x, y) and we have eliminated one arbitrary function by translating the z coordinate. There is one remaining equation to solve, corresponding to the λ^0 coefficient of ∂_z. This equation is as follows:

$$\beta_x + \beta\beta_y - A_0 - \beta A_1 - \beta^2 A_2 - \beta^3 A_3 = 0. \tag{5.18}$$

The metric (1.2) in Theorem 2 corresponds to the twistor distribution given by L_0, with $\beta_z = 0$, and (5.17). If $\beta(x, y)$ is regarded as defining a section of $\mathbb{P}TU$, then (5.18) says that this section is tangent to lifted geodesics of the projective structure. In terms of the base, a solution is given by a congruence of geodesics.

5.3 Twisting case

We have $\beta_z \neq 0$. We may perform a coordinate transformation $z \to \beta(x, y, z)$. This does not affect the general form (5.9) of L_1. Performing the coordinate change and dividing by β_z gives the following form for L_0:

$$L_0 = H(x, y, z)\partial_t - z\partial_z + \lambda\partial_z, \tag{5.19}$$

where H is a non-zero arbitrary function. Calculating the commutator gives

$$\begin{aligned}[L_0, L_1] &= ((-z+\lambda)(C_z + \lambda D_z) - (E + \lambda F)H_z)\partial_t \\ &\quad +((-z+\lambda)(E_z + \lambda F_z) - (E + \lambda F) \\ &\quad -(A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3))\partial_z.\end{aligned}$$

We require $[L_0, L_1] = \alpha L_0$ for some function $\alpha(x, y, z, \lambda)$, which is at most quadratic in λ, since otherwise αL_0 will contain powers of λ greater than three, and such terms do not occur in the commutator above. We make a replacement $L_1 \to L_1 - F L_0$, and analyze equations obtained from comparing coefficients of ∂_z, ∂_t. This puts L_1 in the form

$$\begin{aligned}L_1 &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3)\partial_\lambda \\ &\quad +(C + \lambda D)\partial_t + (A_0 + zA_1 + z^2 A_2 + z^3 A_3)\partial_z,\end{aligned}$$

where $C(x, y, z), D(x, y, z), H(x, y, z)$ satisfy

$$C_z - 2zD_z = -HA_2 + H_y. \tag{5.20}$$

$$D_z = -HA_3. \tag{5.21}$$

and

$$(\partial_x + z\partial_y + (A_0 + zA_1 + z^2 A_2 + z^3 A_3)\partial_z)H = 0. \tag{5.22}$$

The only things remaining now are to find expressions for C and D and construct the metric. In order to integrate equations (5.20) it is convenient to express $H(x, y, z)$ as the second derivative of another function $G(x, y, z)$, i.e. we set

$$H(x, y, z) = \frac{\partial^2 G}{\partial z^2}(x, y, z).$$

Then equations (5.20) and (5.21) integrate to give

$$
\begin{aligned}
C &= -G_zA_2 - 2A_3(zG_z - G) + G_{zy} + \rho(x,y), \\
D &= -G_zA_3 + \sigma(x,y),
\end{aligned}
$$

where ρ and σ are arbitrary functions. Notice that G has a 'gauge freedom' $G \to G + z\gamma(x,y) + \delta(x,y)$, since (1.4) will still be satisfied. Using this and a coordinate change $t \to t + \xi(x,y)$, one can set the functions ρ and σ to zero. The twistor distribution $\{L_0, L_1\}$ is now fully determined:

$$
\begin{aligned}
L_0 &= G_{zz}\partial_t - z\partial_z + \lambda\partial_z, \\
L_1 &= \partial_x + \lambda\partial_y + (A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3)\partial_\lambda \\
&\quad + (-G_zA_2 - 2A_3(zG_z - G) + G_{zy}) - \lambda(G_zA_3))\partial_t \\
&\quad + (A_0 + zA_1 + z^2A_2 + z^3A_3)\partial_z.
\end{aligned}
$$

The distribution is integrable iff G satisfies (1.4). Calculating the corresponding null tetrad gives the conformal structure (1.3) in Theorem 2. This completes the proof of Theorem 2.

5.4 Interpretation as a gauge theory

As explained by Calderbank [5], the calculations above can be attractively expressed in terms of a gauge theory. The idea is to consider a connection on a principal bundle over U, with fibre the diffeomorphism group of a surface. Let this surface have coordinates (t,z), while U has coordinates (x,y). The Lie algebra of the diffeomorphism group is the Lie algebra of vector fields on the (t,z) surface. The connection, being a Lie-algebra valued one-form on U, has the form $\alpha_x dx + \alpha_y dy$, where α_x, α_y are vector fields on the (t,z) surface (note that we are now using subscripts to denote components of a one-form, not partial derivatives). Explicitly, we have $\alpha_x = F(t,z,x,y)\partial_t + G(t,z,x,y)\partial_z$ and similarly for α_y. This connection is read off from the form (5.9) of L_1, by rewriting it as

$$\begin{aligned} L_1 = \partial_x + \alpha_x + \lambda(\partial_y + \alpha_y) \\ + (A_0(x,y) + \lambda A_1(x,y) + \lambda^2 A_2(x,y) + \lambda^3 A_3(x,y))\partial_\lambda. \end{aligned} \tag{5.23}$$

One can also rewrite the (5.8) form of L_0 as

$$L_0 = \phi_x + \lambda\phi_y,$$

where the ϕ_i are interpreted as the components of a Lie-algebra valued one-form on U. Now take a connection D in the projective class defined by the projective structure, and couple it to the gauge connection defined above. Let D^α denote this coupled connection. Now one can covariantly differentiate the one-form ϕ. Vector field commutators take the place of using matrix commutators, for example

$$D^\alpha_x\phi_x = \partial_x\phi_x + [\alpha_x, \phi_x] + \Gamma^x_{xx}\phi_x + \Gamma^y_{xx}\phi_y.$$

Then Calderbank shows that the equations that one obtains by requiring the twistor distribution to be integrable are given by

$$D^\alpha_{(a}\phi_{b)} = 0. \tag{5.24}$$

Moreover, he shows that these equations are projectively invariant, i.e. do not depend on the choice of connection within the projective class.

In fact this gauge theory formulation applies to the more general case of anti-self-dual β-surface foliations explained in Section 4.6. The essential point is that this more general case applies when α and ϕ are allowed to be functions of t. When this happens, one still obtains an integrable β-surface foliation for a neutral ASD metric (which is anti-self-dual in the terminology of Section 4.6), but ∂_t will no longer be a Killing vector (unless the t dependence can be removed by a Möbius transformation of λ, which is equivalent to a rotation of the tetrad and a conformal transformation).

Chapter 6

Special metrics

Theorem 2 provides a local form for any neutral ASD conformal structure with null CKV. It does not provide any information about metrics within a particular conformal class. In this chapter we will study examples of special types of metric within the conformal classes of the theorem, in particular pseudo-hyper-Kähler and pseudo-hyper-hermitian metrics. We also find examples of conformal classes possessing no Ricci-flat metrics. Finally, we show that neutral analogues of the well-known Lorentzian Fefferman metrics fits into our framework.

6.1 Pseudo-hyper-Kähler metrics

Consider a structure (M, I, S, T), where M is a 4-dimensional manifold and I, S, T are anti-commuting endomorphisms of the tangent bundle satisfying

$$S^2 = T^2 = \mathbf{1}, \quad I^2 = -\mathbf{1}, \quad ST = -TS = I. \tag{6.1}$$

Consider the hyperboloid of almost complex structures on M given by $aI + bS + cT$, for (a, b, c) satisfying $a^2 - b^2 - c^2 = 1$. If each of these almost complex structures is integrable, we call (M, I, S, T) a *pseudo-hypercomplex* manifold.

So far we have not introduced a metric. A natural restriction on a metric given a pseudo-hypercomplex structure is to require it to be hermitian with respect to each of the complex structures. This is equivalent to the requirement:

$$g(X,Y) = g(IX, IY) = -g(SX, SY) = -g(TX, TY), \qquad (6.2)$$

for all vectors X, Y. A metric satisying (6.2) must be neutral. To see this consider the endomorphism S, which squares to the identity. Its eigenspaces decompose into $+1$ and -1 parts. Any eigenvector must be null from (6.2). So choosing an eigenbasis one can find four null vectors, from which it follows that the metric is neutral. Given a pseudo-hypercomplex manifold, we call a metric satisfying (6.2) a pseudo-hyperhermitian metric.

Given a local pseudo-hypercomplex structure in four dimensions one can construct many pseudo-hyperhermitian metrics for it as follows. Take a vector field V and let (V, IV, SV, TV) be an orthonormal basis in which the metric has diagonal components $(1, 1, -1, -1)$. The fact that these vectors are linearly independent follows from (6.1). It is easy to check that (6.2) holds for any two vectors in the above basis, and hence by linearity for any (X, Y). By varying the length of V one obtains different metrics in the same conformal class. However, even the conformal class is not uniquely determined by the pseudo-hypercomplex structure. To see this take a vector W that is null for the metric specified by V, and form a new metric by the same procedure using W. Then W is not null in this new metric, so this metric must be in a different conformal class.

Now suppose we are given a pseudo-hypercomplex structure, and a pseudo-hyperhermitian metric. If the 2-forms

$$\omega_I(.,.) = g(.,I.), \quad \omega_S(.,.) = g(.,S.), \quad \omega_T(.,.) = g(.,T.), \tag{6.3}$$

are closed, g is called *pseudo-hyper-Kähler*.

It follows from similar arguments to those in standard Riemannian Kähler geometry that (I,S,T) are covariant constant, and hence so are $\omega_I, \omega_S, \omega_T$.

As in the Riemannian case, pseudo-hyperkähler metrics are equivalent to Ricci-flat anti-self-dual metrics. One can deduce this by showing that the 2-forms (6.3) are self-dual, and since they are also covariant constant there exists a basis of covariant constant primed spinors. Then using the spinor Ricci identities one can deduce anti-self-duality and Ricci-flatness.

6.1.1 Plebański tetrads

Plebański [27] discovered that any pseudo-hyper-Kähler metric can be expressed using a tetrad that involves a single function of four variables subject to a single PDE. We shall use this to find examples of pseudo-hyper-Kähler metrics with null CKVs. Plebański's result is the following:

Proposition. *Given any pseudo-hyper-Kähler metric, there are coordinates in which it is locally of the form*

$$g = dY(dT - \Theta_{XX}dY - \Theta_{TX}dZ) - dZ(dX + \Theta_{TT}dZ + \Theta_{TX}dY), \tag{6.4}$$

where $\Theta(T,X,Y,Z)$ *satisfies the equation*

$$\Theta_{TY} - \Theta_{ZX} + \Theta_{TT}\Theta_{XX} - \Theta_{TX}^2 = 0. \tag{6.5}$$

The metric (6.4) is in NP tetrad form. Let $o^{A'} = (1,0)$, $\iota^{A'} = (0,-1)$ be a basis of primed spinors, normalized so that $o^{A'}\iota_{A'} = 1$. It follows that

$\iota_{A'}o^{B'} - o_{A'}\iota^{B'} = \delta_{A'}{}^{B'}$. A basis of self-dual two-forms is given by

$$
\begin{aligned}
(\Sigma^{0'0'})_{AA'BB'} &= \iota_{A'}\iota_{B'}\epsilon_{AB}, &(6.6)\\
(\Sigma^{0'1'})_{AA'BB'} &= (\iota_{A'}o_{B'} + \iota_{B'}o_{A'})\epsilon_{AB}, &(6.7)\\
(\Sigma^{1'1'})_{AA'BB'} &= o_{A'}o_{B'}\epsilon_{AB}. &(6.8)
\end{aligned}
$$

It is easy to check using these spinor expressions that the following endomorphisms satisfy (6.1):

$$
\begin{aligned}
I_{AA'}{}^{BB'} &= (\Sigma^{0'0'})_{AA'}{}^{BB'} + (\Sigma^{1'1'})_{AA'}{}^{BB'}, &(6.9)\\
S_{AA'}{}^{BB'} &= (\Sigma^{0'0'})_{AA'}{}^{BB'} - (\Sigma^{1'1'})_{AA'}{}^{BB'}, &(6.10)\\
T_{AA'}{}^{BB'} &= (\Sigma^{0'1'})_{AA'}{}^{BB'}. &(6.11)
\end{aligned}
$$

Using equation (6.5) one can show that $d\Sigma^{A'B'} = 0$. Hence the metric (6.4) is indeed pseudo-hyper-Kähler.

The Cartan structure equations are

$$d\Sigma^{A'B'} = \Gamma^{(A'}{}_{C'} \wedge \Sigma^{B')C'}, \qquad (6.12)$$

where $\Gamma^{A'}{}_{C'}$ are connection one-forms for the primed spin connection. This gives $\Gamma^{A'}{}_{C'} = 0$ using $d\Sigma^{A'B'} = 0$. Hence the primed spin connection is flat. Moreover the spinors $\iota^{A'}$ and $o^{A'}$ are covariantly constant, and hence so are the two-forms $\Sigma^{A'B'}$ and the endomorphisms I, S, T.

Here we will consider pure and homothetic (constant conformal factor) null Killing vectors. If K is such a vector field, it satisfies

$$K^a R_{abcd} = \nabla_b \nabla_c K_d.$$

Using this together with the spinor expression (2.10) for the Riemann tensor and the Killing equation (4.2) gives

$$K^{AA'}C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} = \nabla_{BB'}(\phi_{C'D'}\epsilon_{CD} + \psi_{CD}\epsilon_{C'D'} + \frac{1}{2}\eta\epsilon_{CD}\epsilon_{C'D'}).$$

Contracting with ϵ^{CD} gives

$$\nabla_{BB'}\phi_{C'D'} = 0.$$

Thus in a Plebański tetrad the components of $\phi_{C'D'}$ are constant. Killing vectors come in three invariant classes:

1. $\phi_{B'C'} = 0$.

2. $\phi_{B'C'} \neq 0, \quad \det \phi_{B'C'} = 0$.

3. $\det \phi_{B'C'} \neq 0$.

These are invariant in the sense that given a Killing vector one of the three is true, regardless of which tetrad is being used.

We shall find examples of the first two for which the Heavenly equation can be solved explicitly, and we calculate the underlying projective structures for these cases. We will not attempt to classify all possibilities. Such a classification is attempted in [10], however it is known to be incomplete, as the reduction in [7] was not found.

Case 1: $\phi_{B'C'} = 0$

Let $K = \partial_T$. This is null and satisfies $d\mathbb{K} = 0$ where $\mathbb{K} := g(K, .)$. So when it is a Killing vector, it has $\phi_{B'C'} = 0$.

The Killing equations $\mathcal{L}_K g = 0$ give

$$\Theta_{TTT} = 0,$$
$$\Theta_{TTX} = 0,$$
$$\Theta_{TXX} = 0.$$

These integrate to:

$$\Theta = A(X, Y, Z) + TB(Y, Z) + T^2C(Y, Z) + XTD(Y, Z),$$

where the F_i are arbitrary functions. Substituting this into (6.5) we obtain

$$B_Y + 2TC_Y + XD_Y - TD_Z - A_{XZ} + 2CA_{XX} - D^2 = 0.$$

Since A, B, C, D are all independent of T, compare coefficients of T:

$$2C_Y - D_Z = 0,$$

which can be integrated using a potential function $H(Y, Z)$ such that

$$2C = H_Z, \quad D = -H_Y \ .$$

Substituting back into the metric we get

$$g = dY(dT - A_{XX}dY + H_Y dZ) - dZ(dX + H_Z dZ + H_Y dY). \qquad (6.13)$$

This can be simplified by changing coordinates $X \to X + H$ and $T \to T - G$ where $G = G(Y, Z)$ is arbitrary. Then we get

$$g = dY(dT + (G_Y - A_{XX})dY + (H_Y - G_Z)dZ) - dZdX.$$

Pick G such that $G_Z = -H_Y$. Finally, since the coefficient of dY^2 is just an arbitrary function of (X, Y, Z), call it $\tilde{\Theta}_{XX}$. The resulting metric is

$$g = dYdT - dZdX + \tilde{\Theta}_{XX}dY^2 \ .$$

This is in the same form as (6.4), with $\tilde{\Theta}_T = 0$. Equation 6.5 for this metric is simply

$$\tilde{\Theta}_{XZ} = 0, \qquad (6.14)$$

which has solution

$$\tilde{\Theta} = A(Y, Z) + B(X, Y).$$

Equation (6.14) is the reduction of (6.5) that we were looking for. $A(Y, Z)$ does not affect the form of the metric, so can be ignored. The final form of the metric is

$$g = dYdT - dWdX - Q(X, Y)dY^2, \qquad (6.15)$$

where $Q(X,Y)$ is arbitrary. The form of the null Killing vector was not changed by the coordinate transformations and remains ∂_T. This is simply the split-signature pp-wave metric, and is a special case of (1.2).

The local expression (6.15) can be used to find a class of global neutral metrics on certain compact four-manifolds. To see this we compactify the flat projective space $\mathbb{R}^2$, with (X,Y) coordinates, to two-dimensional torus $U = T^2$ with the projective structure coming from the flat metric. We choose $Q(X,Y)$ to be periodic in both variables. Both T and Z in are then also taken to be periodic, thus leading to $\hat{\pi} : M \longrightarrow U$, the holomorphic toric fibration over a torus. This leads to a commutative diagram

$$\begin{array}{ccc} & M & \\ T^2 & \downarrow & \searrow \hat{\pi}^* Q \\ & U & \xrightarrow{Q} \mathbb{R}. \end{array}$$

This example can be put into the framework of [18] and [11], where M is regarded as a primary Kodaira surface $\mathbb{C}^2/G$ and G is the fundamental group of M represented injectively in the group of complex affine transformations of $\mathbb{C}^2$. In this framework the Kähler structure on M is given by $\Omega_{flat} + i\partial\overline{\partial}(\hat{\pi}^* Q)$, where $(\partial, \Omega_{flat})$ is the flat Kähler structure on the Kodaira surface induced from $\mathbb{C}^2$.

Another example with $\phi_{B'C'} = 0$ is found in [2]. In that paper the authors find the following metric:

$$g = dpdt - \frac{1}{2}p^2 du(dv + H(p,u)du), \tag{6.16}$$

where $H(p,u)$ is arbitrary. They show that the Killing vector ∂_v has anti-self-dual $d\mathbb{K}$, i.e. it belongs to the class $\phi_{A'B'} = 0$. This is distinct from the pp-wave case above, because $d\mathbb{K} \neq 0$. Dividing by p^2 and redefining the

p coordinate one sees that the metric is conformal to the pp-wave metric, with the Killing vector mapping to the one above. Hence in this case the underlying projective structure is again flat. In fact they obtain the strong result that any anti-self-dual neutral metric with null Killing vector with anti-self-dual $d\mathbb{K} \neq 0$ is of the form (6.16), and therefore is Ricci-flat.

Case 2: $\phi_{B'C'} \neq 0$, $\det \phi_{B'C'} = 0$
Let $K = Y\partial_X + Z\partial_T$. We get $d\mathbb{K} = 2dZ \wedge dY$. In the Plebański tetrad (6.4) this is just $\mathbf{e}^{00'} \wedge \mathbf{e}^{10'}$. But this is a self-dual two-form. To see this consider the spinor version:

$$2dZ \wedge dY = 2\mathbf{e}^{00'} \wedge \mathbf{e}^{10'} = o_{A'}o_{B'}\epsilon_{AB}\mathbf{e}^{AA'} \otimes \mathbf{e}^{BB'}$$

where $o_{A'} = (1, 0)$, a covariantly constant spinor. So we have $\phi_{A'B'} = o_{A'}o_{B'}$, and therefore $\det \phi_{A'B'} = 0$ as desired.

The Killing equations for K give:

$$Y\Theta_{TTX} + Z\Theta_{TTT} = 0, \tag{6.17}$$

$$Y\Theta_{XXX} + Z\Theta_{XXT} = 0, \tag{6.18}$$

$$Y\Theta_{TXX} + Z\Theta_{TTX} = 0. \tag{6.19}$$

Integrating equation (6.17) w.r.t. (T, T), equation (6.18) w.r.t. (X, X) and equation (6.19) w.r.t. (T, X) gives

$$\begin{aligned} Y\Theta_X + Z\Theta_T &= T\,A(X, Y, Z) + B(X, Y, Z), \\ Y\Theta_X + Z\Theta_T &= X\,C(T, Y, Z) + D(T, Y, Z), \\ Y\Theta_X + Z\Theta_T &= E(X, Y, Z) + F(T, Y, Z). \end{aligned}$$

where $A, \ldots, F$ are arbitrary functions. These equations are only consistent if $A = C = 0$ and B, D, E, F are functions of (Y, Z) only. That is

$$Y\Theta_X + Z\Theta_T = P(Y, Z)\ .$$

Now Θ has a gauge freedom $\Theta \to \Theta + Q(Y,Z)$ for arbitrary Q since this doesn't change the metric. So we can choose a Q that will set $P = 0$. This can also be expressed as $K(\Theta) = 0$, and has solution $\Theta = \Theta(s, Y, Z)$ where $s = YT - ZX$. Equation (6.5) then conveniently simplifies to:

$$\Theta_{TY} - \Theta_{XZ} = 0 \ .$$

In terms of (s, Y, Z) this equation becomes

$$2\Theta_s + Y\Theta_{Ys} + Z\Theta_{Zs} + s\Theta_{ss} = 0 \ .$$

Letting $u = \Theta_s$ we obtain

$$2u + Yu_Y + Zu_Z + su_s = 0,$$

which is a linear first order PDE so can be solved by standard methods, to give

$$\Theta_s = \frac{1}{Y^2}\, F(\frac{Y}{Z}, \frac{s}{Y}) \ .$$

Integrate this w.r.t. s to get Θ:

$$\Theta = \frac{1}{Y}\, G(\frac{Y}{Z}, \frac{s}{Y}) \ + \alpha(Y, Z).$$

Here G is the integral of F w.r.t. its second argument, which is of course just another arbitrary function, and $\alpha(Y,Z)$ is the integration 'constant'. To find the metric we need to find second derivatives w.r.t. T and X:

$$\Theta_{TT} = \frac{1}{Y} G^{(0,2)}, \quad \Theta_{XX} = \frac{Z^2}{Y^3} G^{(0,2)}, \quad \Theta_{TX} = -\frac{Z}{Y^2} G^{(0,2)}.$$

Now $G^{(0,2)}$ is just another arbitrary function, call it J. The metric takes the form:

$$g = dYdT - dZdX - \frac{J}{Y}(dZ - \frac{Z}{Y}dY)^2.$$

This may be written more symmetrically with respect to (Y, Z) as:

$$g = dYdT - dZdX - \frac{H(\frac{Y}{YT-ZX}, \frac{Z}{YT-ZX})}{(YT-ZX)^3}(YdZ - ZdY)^2, \tag{6.20}$$

where H is arbitrary. This is a generalisation of the Sparling-Tod metric [28].

Using the following coordinate transformation

$$\begin{aligned} t &= -\frac{1}{2}(\frac{X}{Y} + \frac{T}{Z}), \\ z &= (YZ)^{-\frac{1}{2}}, \\ x &= \frac{YT - XZ}{(YZ)^{\frac{1}{2}}}, \\ y &= \log(\frac{Z}{Y}), \end{aligned} \tag{6.21}$$

whose inverse is

$$\begin{aligned} T &= \frac{1}{2}e^{y/2}(x - \frac{2t}{z}), \\ X &= -\frac{1}{2}e^{-y/2}(x + \frac{2t}{z}), \\ Y &= \frac{e^{-y/2}}{z}, \\ Z &= \frac{e^{y/2}}{z}, \end{aligned}$$

the metric (6.20) takes the following form:

$$g = \frac{1}{z^2}(dydt - dzdx + zA_3(x, y)dy^2), \tag{6.22}$$

where now the Killing vector is ∂_t. Multiplying by the conformal factor z^2, we get a special case of (1.2). The projective structure is non-trivial, unlike for the pp-wave above. The projective structure is special in that it depends on only one arbitrary function A_3.

We now show how the coordinate transformation above can be found systematically. The basic idea is to follow the transformations used in the proof of Theorem 2. The first step is to find coordinates on the two dimensional space of β-surfaces. The dual of the NP tetrad for the metric (6.20) is as follows:

$$\begin{aligned}
\mathbf{e}_{00'} &= \partial_T, \\
\mathbf{e}_{01'} &= \partial_Z - \frac{YZH}{(YT-ZX)^3}\partial_T + \frac{Y^2H}{(YT-ZX)^3}\partial_X, \\
\mathbf{e}_{10'} &= \partial_X, \\
\mathbf{e}_{11'} &= \partial_Y + \frac{Z^2H}{(YT-ZX)^3}\partial_T - \frac{YZH}{(YT-ZX)^3}\partial_X.
\end{aligned}$$

In this tetrad we have $K = Y\partial_X + Z\partial_T = \iota^A o^{A'}\mathbf{e}_{AA'}$ with $\iota^A = (Z, Y)$, $o^{A'} = (1, 0)$. Consider the vector $U = -\iota^A\iota^{A'}\mathbf{e}_{AA'} = Y\partial_Y + Z\partial_Z$, where $\iota^{A'} = (0, -1)$. This is linearly independent of K, and $\{K, U\}$ spans the integrable β-plane distribution defined by ι^A. They satisfy

$$[K, U] = -K, \tag{6.23}$$

so the distribution is integrable, as expected from Lemma 1. Coordinates on the space of β-surfaces are given by two independent functions which are annihilated by both K and U. It is easy to show that $\frac{Z}{YT-ZX}$ and $\frac{Y}{YT-ZX}$ satisfy these requirements. Moreover, these are the arguments of the arbitrary function H in (6.20).

In the proof of Theorem 2, coordinates (t, x, y, z) are chosen so that $K = \partial_t$, and the β-surface distribution is spanned by K and $V = \partial_z$. Clearly $[K, V] = 0$. The tetrad is chosen so that $K = \mathbf{e}_{00'}$, $V = \mathbf{e}_{01'}$. Therefore we need to transform (6.20) into the form

$$g = 2(\mathfrak{K} \odot \mathbf{e}_{11'} - \mathfrak{v} \odot \mathbf{e}_{10'}), \tag{6.24}$$

where K, V are the algebraic duals of $\mathfrak{K}$, $\mathfrak{v}$, and $[K, V] = 0$. Before doing so, we make some coordinate changes. We take the functions $s = \frac{Z}{YT-ZX}$

and $t = \frac{Y}{YT-ZX}$ as two new coordinates, as they are coordinates on the space of β-surfaces. The other two coordinates, a and b say, are determined by the requirement that $K = \partial_a$, $U = \partial_b - a\partial_a$. This is possible since $[K, U + aK] = [\partial_a, U + a\partial_a] = 0$ using (6.23), therefore by the Frobenius theorem one can find a coordinate system such that $U + a\partial_a = \partial_b$. The following choices of a and b work:

$$a = \frac{1}{2}(\frac{X}{Y} + \frac{T}{Z}), \quad (6.25)$$

$$b = \frac{1}{2}\log(-YZ). \quad (6.26)$$

In the new coordinates (a, b, s, t) the metric is

$$g = (da - e^{-b}J(s,t)(\frac{ds}{s} - \frac{dt}{t}))(e^{2b}(\frac{ds}{s} - \frac{dt}{t})) - db(\frac{1}{2}\frac{e^b}{t^{1/2}s^{1/2}}(\frac{ds}{s} + \frac{dt}{t})).$$

This is of the form (6.24), with $K = \partial_a$, $V = \partial_b$. The arbitrary function $J(s,t)$ is related to $H(s,t)$ by $J(s,t) = \frac{4stH(s,t)+1}{4s^{1/2}t^{1/2}}$. Then define

$$u = \log s + \log t,$$

$$v = \log s - \log t,$$

giving $du = \frac{ds}{s} + \frac{dt}{t}$, $dv = \frac{ds}{s} - \frac{dt}{t}$. Using these new coordinates the metric becomes

$$g = (da - e^{-b}J(u,v)dv)(e^{2b}dv) - db(\frac{1}{2}e^{-u/2+b}du).$$

Calculating the twistor distribution for this metric gives

$$L_0 = \partial_a + \lambda\partial_b + \lambda^2\partial_\lambda,$$

$$L_1 = 2e^{u/2-b}\partial_u + \lambda(e^{-2b}\partial_v + e^{-3b}J(u,v)\partial_a) + \lambda^3 e^{-3b}J(u,v)\partial_\lambda.$$

Now divide L_1 by e^{-b}; this is equivalent to multiplying the metric by the conformal factor e^{-b}. The Lax pair becomes

$$L_0 = \partial_a + \lambda\partial_b + \lambda^2\partial_\lambda,$$

$$L_1 = 2e^{u/2}\partial_u + \lambda(e^{-b}\partial_v + e^{-2b}J(u,v)\partial_a) + \lambda^3 e^{-2b}J(u,v)\partial_\lambda.$$

Now $\iota^A L_A$ is just L_0, so we now need a function of λ which is constant along the distribution spanned by $\{\tilde{K}, L_0\}$. This will eliminate the ∂_λ term in L_0, putting it into the canonical form of Theorem 2. Any function that doesn't depend on a is constant along $\tilde{K}$. As an ansatz, we let $\hat{\lambda} = \beta(b, u, v)\lambda$. We solve the PDE $L\hat{\lambda} = 0$ to get $\beta(b, u, v) = e^{-b} f(u, v)$ and set $f(u, v) = 1$. So $\hat{\lambda} = e^{-b}\lambda$.

So we change coordinates to $(\hat{a}, \hat{b}, \hat{u}, \hat{v}, \hat{\lambda}) = (a, b, u, v, e^b\lambda)$, which gives

$$\begin{aligned} L_0 &= \partial_{\hat{a}} + \hat{\lambda} e^{\hat{b}} \partial_{\hat{b}}, \\ L_1 &= 2e^{\hat{u}/2}\partial_{\hat{u}} + \hat{\lambda}(\partial_{\hat{v}} + e^{-\hat{b}} J(\hat{u}, \hat{v})\partial_{\hat{a}}) + \hat{\lambda}^3 J(\hat{u}, \hat{v})\partial_{\hat{\lambda}}. \end{aligned}$$

Finally we set $e^{\hat{b}}\partial_{\hat{b}} = \partial_{\hat{c}}$ by the coordinate change $\hat{c} = -e^{-\hat{b}}$, and rescale $\hat{u}$ so that $2e^{\hat{u}/2}\partial_{\hat{u}} = \partial_{\hat{w}}$ to get

$$\begin{aligned} L_0 &= \partial_{\hat{a}} + \hat{\lambda}\partial_{\hat{c}}, \\ L_1 &= \partial_{\hat{w}} + \hat{\lambda}(\partial_{\hat{v}} - \hat{c} J(\hat{w}, \hat{v})\partial_{\hat{a}}) + \hat{\lambda}^3 J(\hat{w}, \hat{v})\partial_{\hat{\lambda}}. \end{aligned}$$

This is the final and most simplified form of the Lax pair.

One can read off the projective structure spray from M to be

$$\Theta = \partial_{\hat{w}} + \hat{\lambda}\partial_{\hat{v}} + \hat{\lambda}^3 J(\hat{w}, \hat{v})\partial_{\hat{\lambda}}.$$

This projective structure is special in that it depends on only a single arbitrary function. It is flat iff $J_{\hat{w}} = 0$, as can be shown using (3.7). We can also read off the form of the conformal structure from the above Lax pair. Relabelling the coordinates and the arbitrary function gives

$$g = (dt + zA_3(x, y)dy)dy - dzdx. \tag{6.27}$$

This is in the form (6.24), and is a special case of (1.2), with $\beta = A_0 = A_1 = A_2 = P = Q = 0$. Following through all the coordinate transformations above, one obtains the total transformation (6.21), and the Ricci-flat metric (6.22), of which (6.27) is a conformal transformation.

Case 3: $\phi_{B'C'} \neq 0$, det $\phi_{A'B'} \neq 0$
Consider the vector field $K = T\partial_T + X\partial_X$. We have

$$d\mathbb{K} = dT \wedge dY - dX \wedge dZ = \mathbf{e}^{11'} \wedge \mathbf{e}^{00'} - \mathbf{e}^{01'} \wedge \mathbf{e}^{10'} = \phi_{A'B'}\epsilon_{AB}\mathbf{e}^{AA'} \otimes \mathbf{e}^{BB'},$$

where

$$\phi_{A'B'} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to show that K does not satisfy the pure Killing equations, but can be a conformal Killing vector. One finds that it is easy to solve the conformal Killing equation for Θ, but substitution into Equation (6.5) gives a complicated nonlinear equation for a function of three variables in this case. In principle, one should be able to put the metric into the form (1.3), since K has twist. However, we have not succeeded in finding the coordinate transformation that does this.

6.2 Pseudo-hyper-hermitian metrics

Pseudo-hyper-hermitian ASD metrics are described by the following:

Lemma. *[6] Given any local pseudo-hyper-hermitian ASD metric, there is an NP-tetrad $\boldsymbol{e}_{AA'}$ such that the twistor distribution is of the form*

$$L_0 = \boldsymbol{e}_{00'} + \lambda \boldsymbol{e}_{01'}, \quad L_1 = \boldsymbol{e}_{10'} + \lambda \boldsymbol{e}_{11'}.$$

This also applies to the complexified hyper-hermitian case, where there is a twistor space. Then since there are no ∂_λ in the twistor distribution, the lemma says that there is a holomorphic fibration $\mathcal{PT} \to \mathbb{CP}^1$.

It follows from the proof of the Lemma that pseudo-hypercomplex endomorphisms I, S, T are constructed by taking spinors $o^{A'} = (1,0)$, $\iota^{A'} = (0,-1)$ in a tetrad with no ∂_λ terms, and using the formulae (6.6)-(6.8) and (6.9)-(6.10).

Now suppose we have a *triholomorphic* null C.K.V. This is defined to be a C.K.V. that preserves each complex structure in the hyperboloid. That is,

$$\mathcal{L}_K I = 0, \quad \mathcal{L}_K S = 0, \quad \mathcal{L}_K T = 0.$$

It follows that

$$\mathcal{L}_K \Sigma^{A'B'} = -c\Sigma^{A'B'}, \tag{6.28}$$

where $\Sigma^{A'B'}$ are the self-dual two-forms defined by (6.6)-(6.8), and c is defined by $\mathcal{L}_K g = cg$. The $\Sigma^{AA'}$ are given explicitly by:

$$\begin{aligned}
\Sigma^{00'} &= 2\,\mathbf{e}^{00'} \wedge \mathbf{e}^{10'}, &(6.29)\\
\Sigma^{01'} &= 2(\mathbf{e}^{00'} \wedge \mathbf{e}^{11'} + \mathbf{e}^{01'} \wedge \mathbf{e}^{10'}), &(6.30)\\
\Sigma^{11'} &= 2\,\mathbf{e}^{01'} \wedge \mathbf{e}^{11'}. &(6.31)
\end{aligned}$$

Now we can use (6.28) to obtain restrictions on $\mathcal{L}_K \mathbf{e}^{AA'}$. For example, we have

$$\begin{aligned}
\frac{1}{2}\mathcal{L}_K \Sigma^{00'} &= \mathcal{L}_K(\mathbf{e}^{00'} \wedge \mathbf{e}^{10'}) \\
&= (\mathcal{L}_K \mathbf{e}^{00'}) \wedge \mathbf{e}^{10'} + \mathbf{e}^{00'} \wedge (\mathcal{L}_K \mathbf{e}^{10'}) \\
&= (f_{00'}\mathbf{e}^{00'} + f_{01'}\mathbf{e}^{01'} + f_{10'}\mathbf{e}^{10'} + f_{11'}\mathbf{e}^{11'}) \wedge \mathbf{e}^{10'} \\
&\quad + \mathbf{e}^{00'} \wedge (g_{00'}\mathbf{e}^{00'} + g_{01'}\mathbf{e}^{01'} + g_{10'}\mathbf{e}^{10'} + g_{11'}\mathbf{e}^{11'}) \\
&= (f_{00'} + g_{10'})\mathbf{e}^{00'} \wedge \mathbf{e}^{10'} + f_{01'}\mathbf{e}^{01'} \wedge \mathbf{e}^{10'} \\
&\quad + f_{11'}\mathbf{e}^{11'} \wedge \mathbf{e}^{10'} + g_{01'}\mathbf{e}^{00'} \wedge \mathbf{e}^{01'} + g_{11'}\mathbf{e}^{00'} \wedge \mathbf{e}^{11'},
\end{aligned}$$

for functions $f_{AA'}$, $g_{AA'}$. It follows from (6.28) that $f_{01'} = f_{11'} = g_{01'} = g_{11'} =$

0, so we get

$$\begin{aligned}
\mathcal{L}_K \mathbf{e}^{00'} &= f_{00'}\mathbf{e}^{00'} + f_{10'}\mathbf{e}^{10'}, &(6.32)\\
\mathcal{L}_K \mathbf{e}^{10'} &= g_{00'}\mathbf{e}^{00'} + g_{10'}\mathbf{e}^{10'}, &(6.33)
\end{aligned}$$

where we have absorbed the $\frac{1}{2}$ into the functions $f_{AA'}$, $g_{AA'}$. The same arguments for $\Sigma^{1'1'}$ give

$$\begin{aligned}
\mathcal{L}_K \mathbf{e}^{01'} &= h_{01'}\mathbf{e}^{01'} + h_{11'}\mathbf{e}^{11'},\\
\mathcal{L}_K \mathbf{e}^{11'} &= j_{01'}\mathbf{e}^{01'} + j_{11'}\mathbf{e}^{11'},
\end{aligned}$$

for functions $h_{01'}, h_{11'}$, $j_{01'}$, $j_{11'}$. We also have

$$f_{00'} + g_{10'} = h_{01'} + j_{11'}, \tag{6.34}$$

since both sides of this equation are equal to $-c/2$, using (6.28). Finally we consider (6.28) for $(\Sigma^{0'1'})$, using (6.32)-(6.34). This gives $h_{11'} = f_{10'}$, $j_{01'} = g_{00'}$, and

$$f_{00'} + j_{11'} = h_{01'} + g_{10'}. \tag{6.35}$$

Equations (6.34) and (6.35) imply $f_{00'} = h_{01'}$, $j_{11'} = g_{10'}$. In total then, we have the following:

$$\begin{aligned}
\mathcal{L}_K \mathbf{e}^{00'} &= f_{00'}\mathbf{e}^{00'} + f_{10'}\mathbf{e}^{10'},\\
\mathcal{L}_K \mathbf{e}^{01'} &= f_{00'}\mathbf{e}^{01'} + f_{10'}\mathbf{e}^{11'},\\
\mathcal{L}_K \mathbf{e}^{10'} &= g_{00'}\mathbf{e}^{00'} + g_{10'}\mathbf{e}^{10'},\\
\mathcal{L}_K \mathbf{e}^{11'} &= g_{00'}\mathbf{e}^{01'} + g_{10'}\mathbf{e}^{11'}.
\end{aligned}$$

Dualizing, one obtains

$$\begin{aligned}
\mathcal{L}_K \mathbf{e}_{00'} &= -f_{00'}\mathbf{e}_{00'} - g_{00'}\mathbf{e}_{10'}, &(6.36)\\
\mathcal{L}_K \mathbf{e}_{01'} &= -f_{00'}\mathbf{e}_{01'} - g_{00'}\mathbf{e}_{11'}, &(6.37)\\
\mathcal{L}_K \mathbf{e}_{10'} &= -f_{10'}\mathbf{e}_{00'} - g_{10'}\mathbf{e}_{10'}, &(6.38)\\
\mathcal{L}_K \mathbf{e}_{11'} &= -f_{10'}\mathbf{e}_{01'} - g_{10'}\mathbf{e}_{11'}. &(6.39)
\end{aligned}$$

Now let $\tilde{K}$ denote the lift of K to $\mathbb{P}S'$ in which there are no ∂_λ terms. One can use (6.36)-(6.39) to show that

$$[\tilde{K}, L_0] = -f_{00'}L_0 - g_{00'}L_1, \quad [\tilde{K}, L_1] = -f_{10'}L_0 - g_{10'}L_1. \tag{6.40}$$

Hence $\tilde{K}$ commutes with the twistor distribution, so it is the correct lift (4.8). Reversing the argument, one sees that if $\tilde{K}$ is the lift of a C.K.V. with no ∂_λ terms, then it is triholomorphic. We have proved the following

Lemma 6. *Let K be a triholomorphic C.K.V. for a pseudo-hyper-hermitian ASD metric. Then in a tetrad for which the twistor distribution contains no ∂_λ terms, the lift of K to $\mathbb{P}S'$ also contains no ∂_λ terms.*

We can use Lemma 6 to obtain a full classification in the case of a null triholomorphic Killing vector.

Proposition 2. *All pseudo-hyper-hermitian ASD metrics with triholomorphic null conformal Killing vectors are of the form (1.2) or (1.3) up to a conformal factor, where the corresponding ODE (1.5) is point equivalent to the derivative of a first order ODE.*

Proof. Let g be a pseudo-hyper-hermitian ASD metric, and K be a triholomorphic conformal Killing vector. Since g is ASD, it follows from Theorem 2 that there are coordinates such that, up to a conformal factor, g is of the form (1.2) or (1.3). From [6], it is possible to find a tetrad such that the twistor distribution has no ∂_λ terms. Now a change in tetrad corresponds to a Möbius transformation of λ. Since K is triholomorphic, its lift will have no ∂_λ terms, by Lemma 6. Therefore the Möbius transform does not depend on t, otherwise ∂_t will no longer Lie-derive the twistor distribution (one would have to add ∂_λ terms). Furthermore, the Möbius transformation does not depend on z, otherwise ∂_λ terms will be introduced into L_0. Hence there is a

Möbius transformation of λ, depending only on (x, y), such that the ∂_λ terms in L_1 are eliminated.

After this change in λ, the projective structure spray in L_1 will be of the following form:

$$\Theta = a\partial_x + b\partial_y + \lambda(c\partial_x + e\partial_y),$$

where a, b, c, e are functions of (x, y) with $ae - bc \neq 0$. Coordinate freedom $(x, y) \to (\hat{x}(x, y), \hat{y}(x, y))$ and scaling freedom (the projective structure is unchanged if Θ is multiplied by a non-zero function) allows us to set $a = 1$, $c = 0$, $e = 1$, giving $\Theta = \partial_x + (b + \lambda)\partial_y$. Now perform another Möbius transformation $\lambda \to b + \lambda$, which gives the following spray:

$$\partial_x + \lambda\partial_y + (b_x + \lambda b_y)\partial_\lambda. \tag{6.41}$$

This corresponds to the second-order ODE

$$\frac{d^2y}{dx^2} = A_1(x, y)\Big(\frac{dy}{dx}\Big) + A_0(x, y), \tag{6.42}$$

where $A_1 = \frac{\partial b}{\partial y}$, $A_0 = \frac{\partial b}{\partial x}$ for a function $b(x, y)$. This is the derivative of the general first-order ODE

$$\frac{dy}{dx} = b(x, y). \tag{6.43}$$

Hence the original projective structure is point-equivalent to the one corresponding to (6.42). $\square$

Note that if a (holomorphic) projective structure spray contains no ∂_λ terms, its twistor space fibres over $\mathbb{CP}^1$, since each integral curve can be labelled by the λ coordinate. So a by-product of the proof of the above proposition and the twistor correspondence for projective structures is the following

Proposition 3. *There is a one to one correspondence between holomorphic 2D projective structures s.t. the corresponding second order ODE is point equivalent to the derivative of a first order ODE, and complex surfaces which contain a holomorphic curve with normal bundle $\mathcal{O}(1)$ and fiber holomorphically over $\mathbb{CP}^1$.*

This is of interest purely as a statement about projective structures. Note that although all first order ODEs can be transformed to the trivial first order ODE $dy/dx = 0$ by coordinate transformation, this does not mean that the derivative of any such equation is flat, in the sense of Section 3.3. This can be shown by calculating the invariant (3.7) for (6.42) and showing that it does not necessarily vanish.

In [6], Plebański's formulation of the hyper-Kähler condition was generalized to the hyper-hermitian case, in the complexified setting. As in the hyper-Kähler case, this can be naturally adapted to neutral signature. The generalization states that any pseudo-hyper-hermitian metric has the local form

$$g = dY(dT + \frac{\partial\Theta_0}{\partial X}dY + \frac{\partial\Theta_0}{\partial T}dZ) - dZ(dX + \frac{\partial\Theta_1}{\partial X}dY + \frac{\partial\Theta_1}{\partial T}dZ), \quad (6.44)$$

where the two functions Θ_0, Θ_1 satisfy the following pair of PDEs:

$$\frac{\partial^2\Theta_A}{\partial T\partial Y} - \frac{\partial^2\Theta_A}{\partial X\partial Z} + \frac{\partial\Theta_0}{\partial X}\frac{\partial^2\Theta_A}{\partial T^2} - \frac{\partial\Theta_0}{\partial T}\frac{\partial^2\Theta_A}{\partial T\partial X} + \frac{\partial\Theta_1}{\partial X}\frac{\partial^2\Theta_A}{\partial T\partial X} - \frac{\partial\Theta_1}{\partial T}\frac{\partial^2\Theta_A}{\partial X^2} = 0, \quad (6.45)$$

for $A = 0, 1$. The endomorphisms I, S, T are formed in the same way as in the last section, as defined by (6.9), (6.10) and (6.11). They are integable as a consequence of (6.45).

To find a non-trivial example with a null Killing vector, impose $\partial\Theta_A/\partial T = 0$. Then ∂_T is twist-free null Killing. This is a generalization of the first case in the last section. Equations (6.45) reduce to

$$\begin{aligned} \frac{\partial^2\Theta_0}{\partial X\partial Z} &= 0, \quad \rightarrow \quad \frac{\partial\Theta_0}{\partial X} = Q(X,Y), \\ \frac{\partial^2\Theta_1}{\partial X\partial Z} &= 0, \quad \rightarrow \quad \frac{\partial\Theta_1}{\partial X} = R(X,Y). \end{aligned}$$

The metric (6.44) is then

$$g = dYdT - dZdX + Q(X,Y)dY^2 - R(X,Y)dYdZ.$$

This is not yet in our standard twist-free form (1.2). To put it in this form, perform the coordinate transformation $T \rightarrow T - R(X,Y)Z$. Changing uppercase coordinates to lowercase to agree with (1.2), the metric takes the form

$$g = (dt - zR_y dy)dy - (dz + zR_x dy)dy, \tag{6.46}$$

This is a special case of (1.2), with $\beta = A_0 = A_1 = P = Q = 0$, and $A_2 = -R_x$, $A_3 = -R_y$. It is a generalization of the neutral pp-wave (6.15), where now the projective structure is not flat. The projective structure spray corresponding to (6.46) is

$$\Theta = \partial_x + \lambda\partial_y - (\lambda^2 R_x + \lambda^3 R_y)\partial_\lambda. \tag{6.47}$$

Multiplying by $1/\lambda$ and performing the coordinate transformation $\lambda \rightarrow 1/\lambda$, one obtains

$$\Theta = \lambda\partial_x + \partial_y + (R_y + \lambda R_x)\partial_\lambda.$$

The corresponding ODE is

$$\frac{d^2x}{dy^2} = R_y + R_x\frac{dx}{dy}.$$

This is the derivative of the general first order ODE

$$\frac{dx}{dy} = R(x,y).$$

An explicit twistor construction of a hyper-hermitian metric is given later in Section 7.2.2.

6.3 Neutral Fefferman conformal structures

Consider the general neutral ASD metric with twisting null KV (1.3). If G_{zz} is simply a constant, then (1.4) is satisfied. So given any projective structure and setting $G_{zz} = 1$ we obtain a family of conformal structures with twist which reduce to the given projective structure. Solving for G gives

$$G = \frac{z^2}{2} + z\gamma(x,y) + \delta(x,y).$$

The corresponding metric takes the form

$$(dt + ((z+\gamma)A_3 + \sigma)dy + ((z+\gamma)A_2 + 2A_3(\frac{z^2}{2} - \delta) - \gamma_y + \rho)dx)(dy - zdx) \\ - (dz - (A_0 + zA_1 + z^2A_2 + z^3A_3)dx)dx, \quad (6.48)$$

where we have chosen not to eliminate σ and ρ. By direct calculation one can show that the ASD Weyl tensor has Petrov-Penrose type III or N, and it is type N precisely when the following hold:

$$\begin{aligned} \gamma A_3 + \sigma &= \frac{1}{3}A_2, \\ \gamma A_2 - 2A_3\delta - \gamma_y + \rho &= \frac{2}{3}A_1. \end{aligned}$$

One can always choose $\rho, \sigma, \gamma, \delta$ so that these are satisfied. In this case, the metric is the same as (31) in [24], with their Q cubic in p. These are neutral signature analogues of Fefferman conformal structures.

6.4 Generalized ASD pp-waves

Consider the general neutral ASD metric with non-twisting null Killing vector (1.2). It does not explicitly contain the function $A_0(x,y)$ of the projective structure. The metric is always ASD for any choice of β, A_1, A_2, A_3; one can regard (5.18) as giving $A_0(x, y)$ in terms of these functions. On the other hand, if one wants to specify A_0, then one must choose a solution of (5.18) for β. In the special case $A_0 = 0$, we have the solution $\beta = 0$. One then obtains the following metric:

$$g = (dt + (P + zA_2)dx + (Q + zA_3dy))dy - (dz + zA_1dx)dx, \tag{6.49}$$

which generalizes the ASD pp-wave.

As an aside, it is worth mentioning that different choices of $\beta(x, y)$ in (1.2) can give rise to different metrics. Suppose we choose the flat projective structure by setting $A_i = 0$, $i = 0, \ldots, 3$, so β satifies the equation (1.6) with these choices. By direct calculation one can show that the metric (1.2) is type III iff $\beta_{yy} \neq 0$, otherwise it is type N. So the conformal structures with $\beta_{yy} = 0$ and $\beta_{yy} \neq 0$ are genuinely distinct.

6.5 Conformal structures containing no Ricci-flat metrics

In this section we show that there are conformal structures of the form (1.2) which do not contain Ricci-flat metrics. Conformal structures of the form (1.3) with no Ricci-flat metrics are found using twistor methods in Section 7.2.2. First we discuss the Petrov-Penrose classification for the conformal structures (1.2) and (1.3).

Proposition 4. *Let $K^{AA'} = \iota^A o^{A'}$ be a null conformal Killing vector for ASD conformal structure. Then ι^A is a principal direction, that is*

$$\iota^A \iota^B \iota^C \iota^D C_{ABCD} = 0. \tag{6.50}$$

Moreover if the twist of K vanishes the conformal structure is of type III or N, that is

$$\iota^A\iota^B C_{ABCD} = 0. \tag{6.51}$$

Proof. From (4.3) we have

$$\nabla_{AA'}(\iota^C\iota^D\psi_{CD}) = 0.$$

Expanding this out we obtain

$$\iota^B\iota^C\nabla_{AA'}\psi_{BC} = -2\psi_{BC}\iota^C\nabla_{AA'}\iota^B = \iota_A\mu_{A'}, \tag{6.52}$$

for some spinor $\mu_{A'}$. The last equality follows from (4.3) and (4.5).

Now pick a conformal frame in which K is a pure Killing vector. The well known identity $\nabla_a\nabla_b K_c = R_{bcad}K^d$ implies

$$\nabla^{A'}_{A}\psi_{BC} = -2C^{D}_{\ ABC}K^{A'}_{D} - 2K^{B'}_{(A}\Phi^{A'}_{\ BC)B'} + \frac{1}{6}R\epsilon_{A(B}K^{A'}_{C)} - \frac{4}{3}\epsilon_{A(B}\Phi^{\ DD'A'}_{C)}K_{DD'}.$$

On contracting both sides by $\iota^A\iota^B\iota^C$ and using (6.52), all terms vanish except the term involving $C^{D}_{\ ABC}$, giving (6.50).

Now let us assume that K is non–twisting, i. e. $\mathbb{K}\wedge d\mathbb{K} = 0$ where $\mathbb{K} := g(K,)$. The Frobenius theorem implies the existence of functions P and Q such that $\mathbb{K} = PdQ$. We can now choose a conformal factor such that $d\mathbb{K} = 0$. Then K is covariantly constant ($\nabla_a K_b = 0$), and we deduce

$$\nabla_{AA'}\iota_B = A_{AA'}\iota_B, \tag{6.53}$$

$$\nabla_{AA'}o_{B'} = -A_{AA'}o_{B'}, \tag{6.54}$$

for some one-form $A_{AA'}$. Consider the spinor Ricci identity [26]

$$\triangle_{A'B'}o_{C'} = (C_{A'B'C'D'} - \frac{1}{12}R\epsilon_{D'(A'}\epsilon_{B')C'})o^{D'},$$

where $\triangle_{A'B'} = \nabla_{A(A'}\nabla^A_{B')}$. Substituting (6.54) into this and using $C_{A'B'C'D'} = 0$ gives

$$o_{C'}\nabla_{A(A'}A^A_{B')} = -\frac{1}{12}Ro_{(A'}\epsilon_{B')C'}.$$

By contracting with $o^{C'}$ we find $R = 0$. Now consider the Ricci identity

$$\triangle_{AB}\iota_C = (C_{ABCD} - \frac{1}{12}R\epsilon_{D(A}\epsilon_{B)C})\iota^D.$$

Substituting $R = 0$ and (6.53) into this gives

$$\iota_C\nabla_{A'(A}A^{A'}_{B)} = C_{ABCD}\iota^D.$$

Contracting this with ι^C gives (6.51), from which it follows that the curvature is type III or N. □

In the twisting case the algebraic type of the Weyl spinor can be general. This can be shown by using the following two scalar invariants [26]:

$$I = C_{ABCD}C^{ABCD}, \qquad J = C_{AB}{}^{CD}C_{CD}{}^{EF}C_{EF}{}^{AB}.$$

The condition for type III is $I = J = 0$, and for type II that $I^3 = 6J^2$. Now consider the metric (1.3), with the flat projective structure $A_i = 0, i = 0, \ldots, 3$. The function G_{zz} satisfies

$$(\partial_x + z\partial_y)G_{zz} = 0,$$

which is solved in general when G_{zz} is an arbitrary function of $(zx - y)$. Suppose G is given by:

$$G(x, y, z) = \frac{\mathbf{e}^{zx-y}}{x^2} + zB(x, y),$$

where $B(x, y)$ is arbitrary, so $G_{zz} = \mathbf{e}^{zx-y}$. Then the two scalar invariants are as follows:

$$I = -\frac{3}{2}xB_{yy}\mathbf{e}^{-3(zx-y)}, \tag{6.55}$$

$$J = \frac{3}{8}x(xB_{yyx} + 3B_{yy} + xzB_{yyy})\mathbf{e}^{-4(zx-y)}. \tag{6.56}$$

Therefore, from the conditions above, the metric is neither type II nor type III.

To find metrics that are not conformally Ricci-flat we use results of Szekeres [29]. Although these were derived for Lorentzian signature, they can also be applied to our ASD neutral signature case, essentially because the Weyl curvature is still made up of a single spinor $C_{abcd} = C_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}$ as in the Lorentzian case (of course in Lorentzian case it is complex hermitian, not real).

Consider the metric (6.49) with $A_1 = 0$. By direct calculation, one finds that C_{ABCD} is type N iff $(A_2)_x = 0$, otherwise it is type III. Now suppose it is type III, i.e. $(A_2)_x \neq 0$. The reason for this is that we can apply a result of Szekeres to obtain an obstruction to Ricci-flatness. It is shown in [29] that for types I, II, D or III, a necessary condition for existence of a Ricci-flat metric in the conformal class is the following tensor equation

$$-\frac{1}{2}C_{pqfh}C_{rs}{}^{fh}C_{abc}{}^{d}{}_{;d} + (C_{pq}{}^{df}C_{rsf}{}^{h}{}_{;h} + C_{rs}{}^{df}C_{pqf}{}^{h}{}_{;h}) = 0.$$

This is just the tensor version of the spinor identity (3.1), page 209 [29]. Calculating this one finds that $(A_2)_{xx}$ is an obstruction to its vanishing (we used MAPLE for the calculation). Therefore we have a class of non-conformally vacuum type III neutral ASD conformal structures with non-twisting null conformal Killing vectors.

Chapter 7

Twistor examples

Here we give examples of the twistor space reduction $\mathcal{PT} \to Z$ of Theorem 1. First we look at flat space where there are essentially only two cases to consider. As an aside, we also show how flat twistor space fibres over $\mathcal{O}(4)$. Then we present two curved twistor spaces, both of which fibre over $\mathcal{O}(1)$, the twistor space of the flat projective structure. One of these is already well known (the twistor space of the pp-wave metric), but the other is new.

7.1 Flat examples

The twistor space $\mathcal{PT}$ of the flat conformal structure defined on $\mathbb{C}^4$ is the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. It can be shown [30] that up to conformal transformations there are only two null CKVs in this case. Writing the flat metric as

$$g = dTdY - dXdZ,$$

the null CKVs are ∂_T and $T\partial_T + X\partial_X$. The first is twist-free whilst the second has twist. $\mathcal{PT}$ can be covered with two patches (λ, σ, μ), $(\tilde{\lambda}, \tilde{\sigma}, \tilde{\mu}) \in \mathbb{C}^3$, with transition functions $(\tilde{\lambda}, \tilde{\sigma}, \tilde{\mu}) = (1/\lambda, \sigma/\lambda, \mu/\lambda)$. The twistor distribution on one patch of $\mathbb{P}S'$ is

$$L_0 = \partial_T + \lambda\partial_Z, \tag{7.1}$$

$$L_1 = \partial_X + \lambda\partial_Y. \tag{7.2}$$

We take

$$\sigma = Z - \lambda T, \quad \mu = Y - \lambda X. \tag{7.3}$$

To calculate the transition functions of the holomorphic tangent bundle, write

$$\begin{aligned}\tilde{a}\partial_{\tilde{\lambda}} + \tilde{b}\partial_{\tilde{\sigma}} + \tilde{c}\partial_{\tilde{\mu}} &= a\partial_\lambda + b\partial_\sigma + c\partial_\mu \\ &= a(-\tilde{\lambda}^2\partial_{\tilde{\lambda}} - \tilde{\sigma}\tilde{\lambda}\partial_{\tilde{\sigma}} - \tilde{\mu}\tilde{\lambda}\partial_{\tilde{\mu}}) + b\tilde{\lambda}\partial_{\tilde{\sigma}} + c\tilde{\lambda}\partial_{\tilde{\mu}}.\end{aligned}$$

This gives the transition matrix

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{pmatrix} = \begin{pmatrix} -\tilde{\lambda}^2 & 0 & 0 \\ -\tilde{\sigma}\bar{\lambda} & \tilde{\lambda} & 0 \\ -\tilde{\mu}\bar{\lambda} & 0 & \tilde{\lambda} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{7.4}$$

7.1.1 Twist-free case, $K = \partial_T$

It is clear from (7.3) that ∂_T generates the vector field $\mathcal{K} = -\lambda\partial_\sigma = \partial_{\tilde{\sigma}}$ on twistor space, or in the above notation $b(\lambda) = -\lambda, \tilde{b}(\tilde{\lambda}) = -1$. The hypersurface $\mathcal{H}$ on which $\mathcal{K}$ vanishes is simply $\lambda = 0$. Now the divisor line bundle $[\mathcal{H}]$ is just the pull-back of $\mathcal{O}(1)$ to $\mathcal{PT}$, because if $e, \tilde{e}$ are the fibre coordinates on the two patches, with $\tilde{e} = e/\lambda$, then $\{e(\lambda) = \lambda, \tilde{e}(\tilde{\lambda}) = 1\}$ gives a holomorphic section vanishing to first order at $\lambda = 0$, as required. Dividing $\mathcal{K}$ by this section gives a non-vanishing section of $T\mathcal{PT} \otimes [-\mathcal{H}]$. This is spanned by ∂_σ in one patch, and $\partial_{\tilde{\sigma}}$ in the other, and so the quotient of $\mathcal{PT}$ by the trajectories simply removes the first $\mathcal{O}(1)$ factor, resulting in $Z = \mathcal{O}(1)$. The projective structure is the flat projective structure in (X, Y) that can be read off from (7.2), since all the conventions of Theorem 2 are satisfied. Twistor lines in $\mathcal{PT}$ are sections of $\mathcal{O}(1) \oplus \mathcal{O}(1)$, and it is clear that they are projected to sections of $\mathcal{O}(1)$, which are the twistor lines for the flat projective structure.

7.1.2 Twisting case, $K = T\partial_T + X\partial_X$

This case is slightly more subtle, because the twistor distribution as written above is not adapted to the conventions of Theorem 2. In fact the underlying projective structure is again flat. One way of showing this would be to transform the twistor distribution into the form appearing in Theorem 2 and then read off the projective structure. This is a special case of the calculation following the metric (6.20), with the arbitrary function in that metric vanishing. Here we will proceed in a different way that does not require coordinate transformation.

A simple calculation shows that the following lift of K commutes with the twistor distribution:

$$\tilde{K} = T\partial_T + X\partial_X - \lambda\partial_\lambda.$$

This satisfies $\tilde{K}(\lambda) = -\lambda$, $\tilde{K}(\sigma) = 0$, $\tilde{K}(\mu) = 0$, where λ, σ, μ are regarded as functions on $\mathbb{P}S'$. Therefore $\mathcal{K} = -\lambda\partial_\lambda = \tilde{\lambda}\partial_{\tilde{\lambda}} + \tilde{\sigma}\partial_{\tilde{\sigma}} + \tilde{\mu}\partial_{\tilde{\mu}}$. This vanishes at the union of a the hypersurface $\lambda = 0$ with the point $\tilde{\lambda} = \tilde{\sigma} = \tilde{\mu} = 0$. Since the hypersurface $\mathcal{H}$ is the same as in the non-twisting case, the divisor line bundle $[\mathcal{H}]$ is also the same. Dividing by λ, the section of $T\mathcal{PT} \otimes [-\mathcal{H}]$ in this case is spanned by ∂_λ on one patch and $\tilde{\lambda}\partial_{\tilde{\lambda}} + \tilde{\sigma}\partial_{\tilde{\sigma}} + \tilde{\mu}\partial_{\tilde{\mu}}$ on the other one (note that it still vanishes at the point $\tilde{\lambda} = \tilde{\sigma} = \tilde{\mu} = 0$, as this will be important below).

To determine the space of trajectories it is easiest to work with homogeneous coordinates, in which $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is the open set of $\mathbb{CP}^3$ defined by $[\omega^0, \omega^1, \pi_{0'}, \pi_{1'}]$ with $(\pi_{0'}, \pi_{1'}) \neq (0,0)$, and $\lambda = -\pi_{0'}/\pi_{1'}$, $\sigma = \omega^0/\pi_{1'}$, $\mu = \omega^1/\pi_{1'}$. Then $\mathcal{K}$ has the homogeneous form

$$-\pi_{0'}\partial_{\pi_{0'}}. \tag{7.5}$$

The projection of this to $\mathcal{PT}$ vanishes at $\pi_{0'} = 0$ and $(\omega^0, \omega^1, \pi_{1'}) = (0,0,0)$, where it becomes proportional to the Euler vector field. Therefore a trajectory in the non-projective space is specified by the constant values of $(\omega^0, \omega^1, \pi_{1'})$. Now a different trajectory defined by these values multiplied by some non-zero constant c projects down to the same trajectory in $\mathcal{PT}$, because a point $(\omega^0, \omega^1, \pi_{0'}, \pi_{1'})$ on the original trajectory projects down to the same point as $c(\omega^0, \omega^1, \pi_{0'}, \pi_{1'})$, which is certainly on the other trajectory as this consists of all possible $\pi_{0'}$ values, leaving the other variable constant.

Now removing the point $(\omega^0, \omega^1, \pi_{1'}) = (0,0,0)$ where $\mathcal{K}$ vanishes, the above argument shows that the space of trajectories is $\mathbb{CP}^2$, with homogeneous coordinates $[\omega^0, \omega^1, \pi_{1'}]$. We must now show that twistor lines in $\mathcal{PT}$ project to $\mathbb{CP}^1$s in $\mathbb{CP}^2$ with normal bundle $\mathcal{O}(1)$. The twistor lines in $\mathcal{PT}$ are defined by (7.3), and project down to lines in $\mathbb{CP}^2$ defined by

$$[\omega^0, \omega^1, \pi_{1'}] = [\pi_{1'}Z + \pi_{0'}T, \pi_{1'}Y + \pi_{0'}X, \pi_{1'}].$$

Since these are lines in $\mathbb{CP}^2$, they have normal bundle $\mathcal{O}(1)$ as desired. The lines are defined by the following homogeneous equation:

$$X\omega^0 - T\omega^1 - (XZ - TY)\pi_{1'} = 0.$$

So there is a two parameter family of them depending, when $X \neq 0$ say, on T/X and $(XZ - TY)/X$. Regarded as functions on M, it is easy to check that these are annihilated by K and $T\partial_Z + X\partial_Y$, which together span the β-surfaces, so they determine coordinates on the space of β-surfaces as expected.

7.1.3 A digression: flat twistor space as a bundle over $\mathcal{O}(4)$

As a contrast to the two cases above, we exhibit another way the flat twistor space can be regarded as a bundle over a complex surface. Here the surface is the total space of $\mathcal{O}(4)$. Let $\mathcal{PT} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, the flat twistor space.

We shall consider a nonvanishing sections of $T\mathcal{PT} \otimes \sigma^*\mathcal{O}(2)$, where $\sigma : \mathcal{PT} \to \mathbb{CP}^1$ is the obvious projection to the base. The transition matrix for $T\mathcal{PT} \otimes \sigma^*\mathcal{O}(2)$ is

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{pmatrix} = \begin{pmatrix} -\tilde{\lambda}^4 & 0 & 0 \\ -\tilde{\sigma}\tilde{\lambda}^3 & \tilde{\lambda}^3 & 0 \\ -\tilde{\mu}\tilde{\lambda}^3 & 0 & \tilde{\lambda}^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{7.6}$$

Take the global section given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda^3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \tilde{\lambda}^3 \end{pmatrix}.$$

This is clearly nonvanishing. It therefore defines a holomorphically varying one dimensional subbundle $\mathcal{D}$ of $T\mathcal{PT}$ (a one dimensional distribution in other tereminology). We now quotient $\mathcal{PT}$ by the leaves of this distribution.

In the U_0 patch a vector field lying in $\mathcal{D}$ is given by $\lambda^3\partial_\sigma + \partial_\mu$. The integral curves are as follows, using t for the parameter along them:

$$\begin{aligned} \lambda &= \lambda_0, \\ \sigma &= \lambda_0^3 t + \sigma_o, \\ \mu &= t, \end{aligned}$$

where they are chosen so that $(\lambda, \sigma) = (\lambda_0, \sigma_0)$ at $t = 0$. It is clear that any integral curve is transverse to any hypersurface of constant μ, and that each curve passes through such a hypersurface precisely once. So we can change coordinates on this patch to (λ_0, σ_0, t); one should regard these coordinates as specifying a trajectory (by λ_0 and σ_0) and a point along it (by t).

In the U_1 a vector field lying in $\mathcal{D}$ is $\partial_{\tilde{\sigma}} + \tilde{\lambda}^3 \partial_{\tilde{\mu}}$. The integral curves are as follows, using s for the parameter along them:

$$\begin{aligned}
\tilde{\lambda} &= \tilde{\lambda}_0, \\
\tilde{\sigma} &= s, \\
\tilde{\mu} &= \tilde{\lambda}_0^3 s + \tilde{\mu}_0,
\end{aligned}$$

and we change coordinates in this patch to $(\tilde{\lambda}_0, \tilde{\mu}_0, s)$, labelling a trajectory and a point along it as in the U_0 case.

We now have new coordinates (λ_0, σ_0, t) and $(\tilde{\lambda}_0, \tilde{\mu}_0, s)$ for the U_0 and U_1 patches respectively. We would like to determine the transition functions in these coordinates. They intersect at $\lambda_0, \tilde{\lambda}_0 \in \mathbb{C}^*$. Clearly $\lambda_0 = \frac{1}{\tilde{\lambda}_0}$. Now consider the trajectory defined by (λ_0, σ_0), where $\lambda_0 \in \mathbb{C}^*$. We need the corresponding value of $\tilde{\mu}_0$ for this trajectory. We have

$$\begin{aligned}
s &= \tilde{\sigma} \\
&= \frac{1}{\lambda}\sigma \\
&= \frac{1}{\lambda_0}(\lambda_0^3 t + \sigma_o) \\
&= \lambda_0^2 t + \frac{\sigma_0}{\lambda_0}. \qquad (7.7)
\end{aligned}$$

Now $\tilde{\mu}_0$ is the value of $\tilde{\mu}$ at $s = 0$, so at this point we have

$$0 = \lambda_0^2 t + \frac{\sigma_0}{\lambda_0} \quad \rightarrow \quad t = -\frac{\sigma_0}{\lambda_0^3}.$$

Hence

$$\mu = -\frac{\sigma_0}{\lambda_0^3},$$

and

$$\tilde{\mu}_0 = -\frac{1}{\tilde{\lambda}_0}\frac{\sigma_0}{\lambda_0^3} = -\frac{\sigma_0}{\lambda_0^4}.$$

Therefore the space of trajectories is covered by two patches coordinatized by $(\lambda_0, \sigma_0) \in \mathbb{C}^2$, $(\tilde{\lambda}_0, \tilde{\mu}_0) \in \mathbb{C}^2$, with transition functions $\tilde{\lambda}_0 = \frac{1}{\lambda_0}$, $\tilde{\mu}_0 = -\frac{\sigma_0}{\lambda_0^4}$ for $\lambda_0, \tilde{\lambda}_0 \in \mathbb{C}^*$. This is simply total space of $\mathcal{O}(4)$. So we have a holomorphic fibration $\mathcal{PT} \to \mathcal{O}(4)$.

The following relation between s and t was already established above in equation (7.7):

$$s = \lambda_0^2 t + \frac{\sigma_0}{\lambda_0}.$$

This exhibits $\mathcal{PT}$ as an affine line bundle over $\mathcal{O}(4)$. Denoting the total space of $\mathcal{O}(4)$ by $\mathcal{W}$, the underlying translation bundle of the affine bundle is $\pi^*\mathcal{O}(-2)$, where $\pi : \mathcal{O}(4) \to \mathbb{CP}^1$ is the map to the base. The function $\frac{\sigma_0}{\lambda_0}$ on the overlap can be interpreted as an element of the sheaf cohomology group $H^1(\mathcal{W}, \pi^*\mathcal{O}(-2))$.

7.2 Curved examples

Our general method of constructing curved examples consists of taking the total space of $\mathcal{O}(1)$, the twistor space of the flat projective structure on $\mathbb{C}^2$, and building twistor spaces $\mathcal{PT}$ over it. Let $\mathcal{B}$ be a holomorphic bundle over $\mathcal{O}(1)$ with one-dimensional fibres. Then if we want the total space of $\mathcal{B}$ to be an ASD conformal structure twistor space, we require the normal bundle of $\hat{x}$ in $\mathcal{B}_{\hat{x}}$ to be $\mathcal{O}(1)$. This does not guarantee that $\mathcal{B}$ will be the twistor space of a conformal structure with a null CKV; in general the conformal structure will have an ASD β-surface foliation (see Section 4.6). For a null CKV to exist, there must be a section of $T\mathcal{PT}$ that vanishes on a hypersurface $\mathcal{H}$ that intersects each twistor line once. This follows because by standard twistor theory, CKVs are in one-one correspondence with sections on $T\mathcal{PT}$ that are transverse to twistor lines, and when such a section vanishes once on each twistor line it must give a null CKV. In the examples we consider, such a section shall exist and we do in fact obtain conformal structures with null CKVs.

7.2.1 Non-twisting example using an affine bundle

In this section we will consider a Ricci-flat ASD metric with a tri-holomorphic null Killing vector. We now sketch some facts about the twistor spaces of Ricci-flat metrics. The following goes back to Penrose, though we use Hitchin's formulation [15]:

Theorem. *A Ricci-flat ASD metric is equivalent to a twistor space $\mathcal{PT}$ with the following properties:*

1. *A holomorphic fibration $\sigma : \mathcal{PT} \to \mathbb{CP}^1$*
2. *A family of holomorphic sections of the fibration σ, each with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$*
3. *An isomorphism $K_{\mathcal{PT}} \cong \sigma^*\mathcal{O}(-4)$*

In this situation, there is a nonvanishing section $\theta \in H^0(\Lambda^1\mathcal{PT} \otimes \sigma^*\mathcal{O}(2))$ given by $\pi^{A'}d\pi_{A'}$ where $\pi_{A'}$ are the homogeneous coordinates of the $\mathbb{CP}^1$ over which $\mathcal{PT}$ fibres. In non-homogeneous coordinates this is just $d\lambda$ in one patch and $d\tilde{\lambda}$ in the other. The tangent vectors along the fibres of σ are in the kernel of θ. The isomorphism $K_{\mathcal{PT}} \cong \sigma^*\mathcal{O}(-4)$ means $K_{\mathcal{PT}} \otimes \mathcal{O}(4)$ is trivial. So one can find a global volume form ρ twisted by $\mathcal{O}(4)$. One then obtains a holomorphic section

$$\varpi \in H^0(\mathcal{PT}, \Lambda^2 T_F^* \otimes \sigma^*\mathcal{O}(2)), \tag{7.8}$$

where T^*_F is the cotangent bundle to the fibres, by requiring $\theta \wedge \varpi = \rho$. The twisted two form ρ has a nice formulation on the correspondence space as follows. We will use the non-projective correspondence space $\mathcal{F}$ with fibre coordinates $\pi_{A'}$, which also correspond to the homogeneous coordinates of the $\mathbb{CP}^1$ over which $\mathcal{PT}$ fibres. The pullback of ϖ to $\mathcal{F}$ is a two-form homogeneous of degree two in the $\pi_{A'}$, $\Sigma = \pi_{A'}\pi_{B'}\Sigma^{A'B'}$, since ϖ is twisted by $\mathcal{O}(2)$. Then one can apply the following:

Proposition. *(Plebański [27], Gindikin [12]) If a two-form of the form*

$$\Sigma = \pi_{A'}\pi_{B'}\Sigma^{A'B'}$$

on the correspondence space satisfies

$$d_h\Sigma = 0, \qquad \Sigma \wedge \Sigma = 0 \tag{7.9}$$

where d_h is the exterior derivative holding $\pi_{A'}$ constant, then there exist one forms $\boldsymbol{e}^{AA'}$ related to $\Sigma^{A'B'}$ by

$$\boldsymbol{e}^{AA'} \wedge \boldsymbol{e}^{BB'} = \epsilon^{AB}\Sigma^{A'B'} + \epsilon^{A'B'}\Sigma^{AB}, \tag{7.10}$$

which are a tetrad for an ASD Ricci-flat metric.

One can show that (7.9) are satisfied by the two form constructed above, using the fact that it is Lie-derived along the twistor distribution. In the other direction, the Ricci-flat condition is equivalent to the existence of a covariantly constant basis of primed spinors [27]. Therefore the connection on S' must be flat, so there is a tetrad such that the primed connection coefficients vanish. In this tetrad, the twistor distribution on $\mathcal{F}$ has no $\frac{\partial}{\partial \pi_{A'}}$ components, so each leaf has unique $\pi_{A'}$ coordinates, and this gives the map $\sigma : \mathcal{PT} \to \mathbb{CP}^1$ of the Proposition above. Form the basis of ASD two-forms $\Sigma^{A'B'}$ using the formula (7.10); these are covariantly constant since they are constructed out of covariantly constant spinors. It follows that the two-form $\pi_{A'}\pi_{B'}\Sigma^{A'B'}$ on $\mathcal{F}$ is Lie-derived along the twistor distribution, and it descends to give ϖ, which is twisted by $\mathcal{O}(2)$ as Σ is quadratic in the $\pi_{A'}$ coordinates.

Now suppose we have a Ricci-flat metric with a triholomorphic null Killing vector. From the discussion in Section 6.2, we know that there is a tetrad in which the twistor distribution and the lift of K have no ∂_λ terms. At the twistor space level, this means that $\mathcal{K}$ is tangent to the fibres of $\mathcal{PT} \to \mathbb{CP}^1$, and it Lie-derives ϖ. Although ϖ is line-bundle valued, this statement is well defined, because sections of the line-bundle $\sigma^*\mathcal{O}(2)$ are constant along fibres, and $\mathcal{K}$ is tangent to the fibres.

Assume that the spinor $o^{A'}$, defined by $K = \iota^A o^{A'}\mathbf{e}_{AA'}$, is covariantly constant. This means that $\mathcal{K}$ vanishes on a fibre of $\sigma : \mathcal{PT} \to \mathbb{CP}^1$, and $[\mathcal{H}] \cong \sigma^*\mathcal{O}(1)$.

Now the discussion of Section 4.4 shows that $\zeta \otimes \mathcal{K}$ is a holomorphic section of $[-\mathcal{H}] \otimes T\mathcal{PT} \cong \sigma^*\mathcal{O}(-1) \otimes T\mathcal{PT}$. We can still say this Lie derives ϖ, as sections of $\sigma^*\mathcal{O}(-1)$ are constant on the fibres of $\mathcal{PT} \to \mathbb{CP}^1$. On the fibre at which $\zeta \otimes \mathcal{K}$ is non-zero but $\mathcal{K}$ vanishes, $\zeta \otimes \mathcal{K}$ will still Lie derive ϖ because the Lie derivative is holomorphic, and vanishes everywhere else on the twistor space.

This means that there is an $\sigma^*\mathcal{O}(n)$ valued Hamiltonian function h, for some n, such that if

$$\varpi(\zeta \otimes \mathcal{K}, .) = dh.$$

Here dh is the fibrewise exterior derivative of h, giving an $\mathcal{O}(n)$ valued one-form. Now ϖ is $\sigma^*\mathcal{O}(2)$-valued, and $\zeta \otimes \mathcal{K}$ is $\sigma^*\mathcal{O}(-1)$-valued, so h must be $\sigma^*\mathcal{O}(1)$ valued. Moreover, it is not constant on fibres. So it induces a holomorphic map $\mathcal{PT} \to \mathcal{O}(1)$, with fibres the trajectories of $\zeta \otimes \mathcal{K}$. Hence the underlying projective structure is flat. We have proved the following:

Proposition. *Let (M, g, K) be a Ricci-flat ASD metric with null triholomorphic Killing vector $K = \iota^A o^{A'} \boldsymbol{e}_{AA'}$, with $o^{A'}$ covariantly constant. Then the underlying projective structure is flat.*

The pp-wave metric (6.15) is an example of this. It may be the only possibility, but we will not investigate this question here. We now show how to construct the pp-wave metrics from their twistor spaces. This has been known since Ward's work [31], but we present it in a way that highlights the relationship with projective structures. We also use power series rather than contour integrals, to make everything very explicit.

We start with the total space of $\mathcal{O}(1)$ as the minitwistor space Z. The twistor lines are global holomorphic sections of $\mathcal{O}(1) \to \mathbb{CP}^1$. The flat twistor space $\mathcal{O}(1) \oplus \mathcal{O}(1)$ can be formed as follows. Consider the projection $\tau : \mathcal{O}(1) \to \mathbb{CP}^1$. Then $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is the pull-back bundle $\tau^*\mathcal{O}(1)$ over the total space of $\mathcal{O}(1)$. It is easy to check that this is the same as taking $K_Z^{-1/3}$ where K_Z is the canonical bundle of $Z = \mathcal{O}(1)$. To obtain curved twistor spaces, we can take affine bundles over $\mathcal{O}(1)$ modelled on $\tau^*\mathcal{O}(1)$. Explicitly, let (λ, μ) and $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{C}^2$ coordinatize $\mathcal{O}(1)$, with $(\tilde{\lambda}, \tilde{\mu}) = (1/\lambda, \mu/\lambda)$ on the overlap. Let $f(\lambda, \mu)$ be a holomorphic function on the overlap, i.e. for $\lambda \neq 0$. This defines a cohomology element $[f] \in H^1(\mathcal{O}(1), \tau^*\mathcal{O}(1))$. Then letting σ, $\tilde{\sigma} \in \mathbb{C}$ be fibre coordinates on the two patches, the affine bundle corresponding to $[f]$ has transition function

$$\tilde{\sigma} = \frac{\sigma}{\lambda} + f(\lambda, \mu). \tag{7.11}$$

The total space of this bundle has an obvious projection σ to $\mathbb{CP}^1$. Moreover, on the overlap the following holds:

$$d\tilde{\lambda} \wedge d\tilde{\mu} \wedge d\tilde{\sigma} = -\frac{1}{\lambda^4} d\lambda \wedge d\mu \wedge d\sigma.$$

It follows that $K_{\mathcal{PT}} \cong \sigma^*\mathcal{O}(-4)$. So the first and third parts of the theorem above are satisfied. The second part will be demonstrated below by explicitly constructing twistor lines. Therefore this is the twistor space of a Ricci-flat metric. Let the twistor lines in the base $\mathcal{O}(1)$ be given by the holomorphic sections, $\mu = X + \lambda Y$ for $(X, Y) \in \mathbb{C}^2$. Now restricting to one of these, we get

$$f(\lambda, X + \lambda Y) = h(X, Y, \lambda) - \tilde{h}(X, Y, \tilde{\lambda}),$$

where h, $\tilde{h}$ are holomorphic in λ and $\tilde{\lambda}$ respectively. Abstractly this is because when restricted to one of the sections, f defines an element in $H^1(\mathbb{CP}^1, \mathcal{O}(1))$, and this group vanishes. More concretely, h consists of the λ terms in the power series of the left hand side, and h consists of the $\tilde{\lambda}$ terms. The four parameter family of twistor lines is given by

$$\mu = X + \lambda Y, \quad \sigma = W + \lambda Z - \lambda h(X, Y, \lambda), \tag{7.12}$$

in one patch, and

$$\tilde{\mu} = \tilde{\lambda} X + Y, \tilde{\sigma} = \tilde{\lambda} W + Z - \tilde{h}(X, Y, \tilde{\lambda}). \tag{7.13}$$

in the other. It is easy to see that (7.11) is obeyed.

Using the identity $(\lambda \partial/\partial_X - \partial/\partial_Y) f(\lambda, X + \lambda Y) = 0$, we deduce

$$\lambda \frac{\partial h}{\partial X} - \frac{\partial h}{\partial Y} = \lambda \frac{\partial \tilde{h}}{\partial X} - \frac{\partial \tilde{h}}{\partial Y}. \tag{7.14}$$

Now expand $\tilde{h}$ in a power series:

$$\tilde{h} = \tilde{h}_0(X, Y) + \tilde{h}_1(X, Y)\tilde{\lambda} + \tilde{h}_2(X, Y)\tilde{\lambda}^2 + \dots$$

Now the right hand side of (7.14) is

$$\frac{\partial \tilde{h}_1}{\partial X} - \frac{\partial \tilde{h}_0}{\partial Y} + \lambda \frac{\partial \tilde{h}_0}{\partial X} + \frac{1}{\lambda}(\frac{\partial \tilde{h}_2}{\partial X} - \frac{\partial \tilde{h}_1}{\partial Y}) + \frac{1}{\lambda^2}(\frac{\partial \tilde{h}_3}{\partial X} - \frac{\partial \tilde{h}_2}{\partial Y}) + \dots$$

Now since the left hand side of (7.14) is holomorphic in λ, we see that

$$\frac{\partial \tilde{h}_i}{\partial X} - \frac{\partial \tilde{h}_{i-1}}{\partial Y} = 0$$

for $i \geq 2$. So we have

$$\begin{aligned} \lambda \frac{\partial h}{\partial X} - \frac{\partial h}{\partial Y} = \lambda \frac{\partial \tilde{h}}{\partial X} - \frac{\partial \tilde{h}}{\partial Y} &= \frac{\partial \tilde{h}_1}{\partial X} - \frac{\partial \tilde{h}_0}{\partial Y} + \lambda \frac{\partial \tilde{h}_0}{\partial X} \\ &= A(X,Y) + \lambda B(X,Y), \qquad (7.15) \end{aligned}$$

where $A(X,Y)$, $B(X,Y)$ are arbitrary functions.

To calculate the conformal structure one could take infinitesimal changes $(\delta X, \delta Y, \delta W, \delta Z)$ and find the quadratic condition for a $(\delta\tilde{\mu}, \delta\tilde{\sigma})$ to vanish. Rather than do this, we will pull back the form ϖ discussed above to obtain the ASD two-forms $\Sigma^{A'B}$, and thence a tetrad. The advantage of this is that the tetrad is guaranteed to give a Ricci-flat metric, by the arguments above.

The form ϖ is $d\mu \wedge d\sigma$ on one patch. Using (7.12) we can calculate the pull back to $\mathbb{P}S'$:

$$\Sigma = d_\lambda(X + \lambda Y) \wedge d_\lambda(W + \lambda Z - \lambda h(X, Y, \lambda)),$$

where d_λ denotes exterior differentiation keeping λ constant. This is because ϖ is a twisted section of $\Lambda^2 T^*_F \mathcal{PT}$, not $\Lambda^2 T^* \mathcal{PT}$. Expanding, we get

$$\begin{aligned} \Sigma &= dX \wedge dW + \lambda(dY \wedge dW + dX \wedge dZ) + \lambda^2 dY \wedge dZ \qquad (7.16) \\ &\quad -\lambda(\frac{\partial h}{\partial Y} - \lambda\frac{\partial h}{\partial X}) dX \wedge dY \\ &= dX \wedge dW + \lambda(dY \wedge dW + dX \wedge dZ + A(X,Y) dX \wedge dY) \\ &\quad +\lambda^2(dY \wedge dZ + B(X,Y) dX \wedge dY), \qquad (7.17) \end{aligned}$$

using (7.15). Now we have

$$\begin{aligned} \Sigma &= \Sigma^{00'} + 2\lambda \Sigma^{0'1'} + \lambda^2 \Sigma^{1'1'} \\ &= \mathbf{e}^{0'0'} \wedge \mathbf{e}^{10'} + \lambda(\mathbf{e}^{01'} \wedge \mathbf{e}^{10'} + \mathbf{e}^{00'} \wedge \mathbf{e}^{11'}) + \lambda^2 \mathbf{e}^{01'} \wedge \mathbf{e}^{11'}. \qquad (7.18) \end{aligned}$$

Comparing (7.17) with (7.18) we find that the NP tetrad form of the metric is

$$g = dX(dZ - B(X,Y)dZ) - dY(dW - A(X,Y)dX).$$

One can eliminate one of the arbitrary functions, for example by a translation $Z \to Z + F(X,Y)$. The resulting metric is just the pp-wave metric. There are in fact two obvious null triholomorphic Killing vectors, these are ∂_W and ∂_Z. It is easy to show that these correspond to the holomorphic vector fields ∂_σ and $\lambda\partial_\sigma$ on $\mathcal{PT}$.

7.2.2 Twisting example using a Ward bundle

In this section we construct a new twistor example that is hyper-hermitian. It is also never Ricci-flat unless it is flat, which is easily seen using twistor methods. The idea is as follows. Again, we take the minitwistor space Z to be the total space of $\mathcal{O}(1)$, the twistor space of the flat projective structure. Suppose we are given a 1-form ω on U. Regard ω as a holomorphic connection on a holomorphic line bundle $B \to U$. This gives rise to a holomorphic line bundle $E \to Z$, where the vector space over $z \in Z$ is the space of parallel sections of B over the geodesic in U corresponding to z. The *twistor lines* in Z are the two-parameter family of embedded $\mathbb{CP}^1$s, each corresponding to the set of geodesics through a single point in U. We denote the twistor line corresponding to a point $x \in U$ by $\hat{x}$. Now E restricted to a twistor line $\hat{x}$ is trivial, because to specify a parallel section of B through any geodesic through x, one need only know its value at x. This is a simple analogue of the Ward correspondence relating solutions of the anti-self-dual Yang-Mills equations on $\mathbb{C}^4$ to vector bundles over the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ that are trivial on twistor lines. The situation here is simpler since there are no PDEs involved; this is because there are no integrability conditions for a vector space of parallel sections to exist on a line. As with the Ward correspondence, the construction is reversible, i.e. given a holomorphic line bundle trivial on twistor lines one can find a connection on U to which it corresponds in the manner described above. We will not prove this here, as it is simply a case of mimicking the argument for the Ward correspondence [32].

Now to create the twistor space $\mathcal{PT}$, we must tensor E with a line bundle L so that $E \otimes L$ restricts to $\mathcal{O}(1)$ on the twistor lines in Z. Then the total space of $E \otimes L$ will have embedded $\mathbb{CP}^1$s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, so will be a twistor space for an ASD conformal structure. For L we choose the pull back of $\mathcal{O}(1)$ to the total space of $\mathcal{O}(1)$.

Let us now make the above explicit. We use the same coordinatization (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$ as in the previous section. Now suppose we have a line bundle $E \to Z = \mathcal{O}(1)$, that is trivial on holomorphic sections of $Z \to \mathbb{CP}^1$. Let σ, $\tilde{\sigma}$ be the fibre coordinates on the two patches, satisfying a transition relation $\tilde{\sigma} = F(\lambda, \mu)\sigma$, where $F(\lambda, \mu)$ is holomorphic and nonvanishing on the overlap, i.e. for $\lambda \in \mathbb{C} - \{0\}$, $\mu \in \mathbb{C}$. In sheaf terms, F is an element of $H^1(\mathcal{O}(1), \mathcal{O}^*)$. Now the short exact sequence of sheaves

$$0 \to \mathcal{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{7.19}$$

gives rise to a long exact sequence of sheaf cohomology groups, part of which is:

$$\ldots \to H^1(\mathcal{O}(1), \mathcal{Z}) \to H^1(\mathcal{O}(1), \mathcal{O}) \to H^1(\mathcal{O}(1), \mathcal{O}^*) \to H^2(\mathcal{O}(1), \mathcal{Z}) \to \ldots \tag{7.20}$$

The first term in (7.20) vanishes and the final term is $\mathcal{Z}$, by topological considerations. The final term gives the Chern class of the line bundle determined by the element of $H^1(\mathcal{O}(1), \mathcal{O}^*)$. This vanishes for E, since it is trivial on twistor lines. The third arrow in (7.19) is the exponential map. Together these facts imply that F can be written $F(\lambda,\mu) = e^{f(\lambda,\mu)}$, where $f(\lambda,\mu)$ is a holomorphic function on the overlap that may have zeros. After twisting by L, we obtain the following transition function for $E \otimes L$, again using σ, $\tilde{\sigma}$ as fibre coordinates:

$$\tilde{\sigma} = \frac{1}{\lambda} e^{f(\lambda,\mu)} \sigma. \tag{7.21}$$

We now have the twistor space, and can proceed to find twistor lines and calculate the conformal structure. Before doing this let us consider the structure of the twistor space in a bit more detail. A quick calculation gives

$$d\tilde{\lambda} \wedge d\tilde{\mu} \wedge d\tilde{\sigma} = -\frac{e^f}{\lambda^4} d\lambda \wedge d\mu \wedge d\sigma.$$

It follows that $K_{\mathcal{PT}} \cong E^{-1} \otimes \sigma^* \mathcal{O}(-4)$, where σ is the projection $\mathcal{PT} \to Z$. We only have $K_{\mathcal{PT}} \cong \sigma^* \mathcal{O}(-4)$, and hence a Ricci-flat metric, when E^{-1} is trivial. In that case we obtain the flat twistor space. So whenever E is non-trivial we obtain a conformal structure with no Ricci-flat metric. Such twistor spaces correspond to the conformal classes promised in Section 6.5; we calculate them explicitly below.

A standard result of twistor theory going back to Boyer [3] states that if the twistor space of an ASD conformal structure fibres holomorphically over $\mathbb{CP}^1$ then there is a hyperhermitian metric in the conformal structure. A proof in the complexified situation can be found in [6]. This applies to the twistor space constructed above, since there is an obvious fibration to $\mathbb{CP}^1$.

Now let us construct the twistor lines. The two parameter family in $\mathcal{O}(1)$ is given in one patch by $\mu(\lambda) = X\lambda + Y$, and in the other by $\tilde{\mu}(\tilde{\lambda}) = X + \tilde{\lambda} Y$. Restricting to one of these we can split f:

$$f(\lambda, X\lambda + Y) = h(X, Y, \lambda) - \tilde{h}(X, Y, \tilde{\lambda}), \tag{7.22}$$

where h and $\tilde{h}$ are functions on $U \times \mathbb{CP}^1$ holomorphic in λ and $\tilde{\lambda}$ respectively. For fixed (X, Y) there is then a further two parameter family of twistor lines, given by

$$\sigma(\lambda) = e^{-h(X,Y,\lambda)}(W - \lambda Z) \tag{7.23}$$

in one patch, and

$$\tilde{\sigma}(\tilde{\lambda}) = e^{-\tilde{h}(X,Y,\tilde{\lambda})}(\tilde{\lambda}W - Z). \tag{7.24}$$

It is easy to check that (7.21) is satisfied by (7.23) and (7.24).

By the same power series arguments as in the previous section, we obtain

$$(\frac{\partial}{\partial X} - \lambda\frac{\partial}{\partial Y})h = A(X,Y) + \lambda B(X,Y) \tag{7.25}$$

for arbitrary functions $A(X,Y)$, $B(X,Y)$.

To calculate the conformal structure one could take infinitesimal changes $(\delta X, \delta Y, \delta W, \delta Z)$ and find the quadratic condition for a $(\delta\tilde{\mu}, \delta\tilde{\sigma})$ to vanish, using (7.25). Rather than do this, we will use the method of Dunajski found in [6]. This has the advantage that it produces a tetrad for a hyperhermitian metric, so we need not worry about the conformal factor.

The isomorphism $K_{\mathcal{PT}} \cong E^{-1} \otimes \sigma^*\mathcal{O}(-4)$ means that there is a nonvanishing section ρ of $K_{\mathcal{PT}} \otimes E \otimes \mathcal{O}(4)$. We also have a nonvanishing section $\theta \in H^0(\Lambda^1\mathcal{PT} \otimes \sigma^*\mathcal{O}(2))$ given by $d\lambda$ in one patch and $d\tilde{\lambda}$ in the other, whose kernel at any point of $\mathcal{PT}$ is the tangent space of the fibre through that point. Similarly to the Ricci-flat case, these two differential forms give rise to a holomorphic section

$$\varpi \in H^0(\mathcal{PT}, \Lambda^2 T_F^* \otimes \sigma^*\mathcal{O}(2) \otimes E), \tag{7.26}$$

where T_F is the tangent bundle to the fibres, by the requirement that

$$\theta \wedge \varpi = \rho.$$

Notice that the difference between (7.26) and (7.8) is the extra twist by E. In the above coordinates, ϖ has the form $d\mu \wedge d\sigma$ on one patch, and $d\tilde{\mu} \wedge d\tilde{\sigma}$ on the other. Now pulling back $d\mu \wedge d\sigma$ to $\mathbb{P}S'$, where we keep λ constant for the same reason as in the Ricci-flat case above:

$$d_\lambda(\lambda X + Y) \wedge d_\lambda(e^{-h(X,Y,\lambda)}(W - \lambda Z)).$$

Using Dunajski's method we see that multiplying this by e^h gives a two-form Σ that is quadratic in λ, from which we can read off a tetrad for a hyperhermitian metric. We get

$$\begin{aligned}
\Sigma &= dY \wedge dW + \lambda(dX \wedge dW - dY \wedge dZ) + \lambda^2 dZ \wedge dX \qquad (7.27)\\
&\quad +(W + \lambda Z)(dX \wedge dY)\left(\frac{\partial h}{\partial X} - \lambda \frac{\partial h}{\partial Y}\right)\\
&= dY \wedge (dW - WAdX) \qquad (7.28)\\
&\quad +\lambda(dX \wedge dW - dY \wedge dZ + (ZA + WB)dX \wedge dY)\\
&\quad +\lambda^2 dX \wedge (-dZ + ZBdY), \qquad (7.29)
\end{aligned}$$

using (7.25). Using (7.18) to find a tetrad, we obtain the following hyperhermitian metric:

$$g = dXdW + dYdZ - (WdX + ZdY)(A(X,Y)dX + B(X,Y)dY). \quad (7.30)$$

This possesses the null conformal Killing vector $K = W\partial_W + Z\partial_Z$, which is twisting. The global holomorphic vector field on $\mathcal{PT}$ induced by K is $\sigma\partial_\sigma = \tilde{\sigma}\partial_{\tilde{\sigma}}$ where the equality holds on the intersection of the two coordinate patches. This vanishes on the hypersurface defined by $\sigma = 0$ in one patch and $\tilde{\sigma} = 0$ in the other, which intersects each twistor line at a single point, as expected.

The 1-form $AdX + BdY$ occuring in the metric (7.30) is the inverse Ward transform of $F \in H^1(\mathcal{O}(1), \mathcal{O}^*)$. It can be also interpreted as an ASD

Maxwell field on the background (7.30), using Dunajski's interpretation of hyperhermitian twistor spaces via the twisted photon construction.

To compare with (1.3) one must transform to coordinates (t, x, y, z) in which $K = \partial_t$. Dividing by a conformal factor W, transforming with $(t, x, y, z) = (\log W, Y, -X, Z/W)$, and then translating ϕ to eliminate an arbitrary one function of (x, y) gives

$$g = (dt + f(x, y)dx)(dy - zdx) - dzdx, \tag{7.31}$$

a special case of (1.3) with flat projective structure, and $G = z^2/2 - zC(x, y)$, where $f = \partial_y C$.

Chapter 8

Conclusion

We end by discussing some open issues. Despite the classification provided by Theorem 2, there are many local questions remaining. Perhaps the most significant is as follows. There is a theory of local scalar invariants of second order ODEs, and hence projective structures as a special case, going back to Cartan. The expression (3.7) is an example of one such local invariant. Now given a conformal structure one can construct local scalar invariants by contracting polynomials in the conformal curvature and its covariant derivatives. An obvious question is: what is the relationship between the projective structure invariants and the conformal structure invariants for ASD conformal structures with null CKV? One might hope to construct the projective structure invariants tensorially from the conformal structure invariants. Perhaps twistor methods could shed light on these questions. It is known [16] that the local invariants of an analytic projective structure can be expressed in terms of formal neighbourhoods of twistor lines. Presumably something similar holds for local conformal structure invariants, and one might be able to relate the two using the fibration of Theorem 1.

Another local question concerns the existence of special metrics within the conformal classes. Does existence of a particular type of metric, for instance an Einstein metric with non-vanishing Ricci-tensor, impose conditions on the local invariants of the underlying projective structure? We have already seen in Section 6.2 that for pseudo-hyper-hypermitian metrics with tri-holomorphic null Killing vector, the projective structure is of a special type. It is conceivable that twistor methods might shed light on such questions. For instance an Einstein metric results in a twistor space with extra structure, namely a holomorphic contact form, and combining this with the fibration of Theorem 1 might give information about the projective structure twistor space.

There are also some interesting global issues. The only compact example we have is the pp-wave metric, where the underlying projective structure is the flat one on a 2-torus. It would be nice to find compact examples with non-flat underlying projective structure.

There is a recent global twistor construction [21] for smooth neutral ASD conformal structures on $S^2 \times S^2$, whose null geodesics are periodic. This follows an earlier global twistor construction [22] for smooth projective structures on S^2 with periodic geodesics. In [23] it was shown in a special case that these two twistor constructions can be related by dimensional reduction, using the ideas of Theorems 1 and 2. In this special case, the projective structure in question is flat. It would be satisfying if one could demonstrate global dimensional reduction using an example with a non-flat underlying projective structure. Moreover there is a concrete way of approaching this. In [22], the general axisymmetric projective structure on S^2 with periodic geodesics is found explicitly. Plugging the resulting functions A_i into the conformal structures of Theorem 2, one might hope to find a conformal structure defined on $S^2 \times S^2$ with periodic null geodesics.

Bibliography

[1] Atiyah, M. F., Hitchin, N. J., and Singer, I. M. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A 362* (1978), 425–461.

[2] Barrett, J., Gibbons, G. W., Perry, M. J., Pope, C. N., and Ruback, P. J. Kleinian geometry and the N=2 superstring. *Int. J. Mod. Phys. A 9* (1994), 1457–1494.

[3] Boyer, C. P. A note on hyperhermitian four-manifolds. *Proc. Amer. Math. Soc. 102* (1988), 157–164.

[4] Bryant, R., Griffiths, P., and Hsu, L. Toward a Geometry of Differential Equations. In *Geometry, Topology, and Physics* (1995), Conf. Proc. Lecture Notes Geom. Topology, edited by S. -T. Yau, International Press, Cambridge, MA, pp. 1–76.

[5] Calderbank, D. M. J. Selfdual 4-manifolds, projective structures, and the Dunajski-West construction. *math.DG/0606754*.

[6] Dunajski, M. The twisted photon associated to hyper-hermitian four-manifolds. *J. Geom. Phys. 30* (1999), 266–281.

[7] Dunajski, M., Mason, L. J., and Tod, K. P. Einstein-Weyl geometry, the dKP equation and twistor theory. *J. Geom. Phys. 37* (2001), 63–93.

[8] Dunajski, M., and West, S. Anti-self-dual Conformal Structures in Neutral Signature. In *Recent developments in pseudo-Riemannian Geometry (to appear)*, E.S.I. Series on Mathematics and Physics.

[9] Dunajski, M., and West, S. Anti-self-dual conformal structures from projective structures. *Comm. Math. Phys. 272* (2007), 85–118.

[10] Finley, J. D., and Plebański, J. F. The classification of all H spaces admitting a killing vector. *J. Math. Phys. 20* (1979), 1938–1945.

[11] Fino, A., Pedersen, H., Poon, Y.-S., and Sorensen, M. W. Neutral Calabi-Yau structures on Kodaira manifolds. *Comm. Math. Phys. 248* (2004), 255–268.

[12] Gindikin, S. On one construction of hyperkähler metrics. *Funct. Anal. Appl 20* (1986), 82–136.

[13] Griffiths, P., and Harris, J. *Principles of Algebraic Geometry.* Wiley Classics Library. John Wiley and Sons, 1978.

[14] Hirzebruch, F., and Hopf, H. Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Ann. 136* (1958), 156–172.

[15] Hitchin, N. J. Complex manifolds and Einstein's equations. In *Twistor geometry and Non-Linear Systems* (1982), Lecture Notes in Math., 970, Springer, pp. 73–99.

[16] HURTUBISE, J. C., AND KAMRAN, N. Projective connections, double fibrations, and formal neighborhoods of lines. *Math. Ann. 292* (1992), 383–409.

[17] JONES, P. E., AND TOD, K. P. Minitwistor spaces and Einstein-Weyl spaces. *Class. Quant. Grav. 2* (1985), 565–577.

[18] KAMADA, H. Neutral hyper-Kähler structures on primary Kodaira surfaces. *Tsukuba J. Math. 23* (1999), 321–332.

[19] KODAIRA, K. A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds. *Annals of Math. 75* (1962), 146–162.

[20] LEBRUN, C. Spaces of Complex Null Geodesics in Complex-Riemannian geometry. *Trans. Am. Math. Soc. 278* (1983), 209–231.

[21] LEBRUN, C., AND MASON, L. J. Nonlinear Gravitons, Null Geodesics, and Holomorphic Disks. *Duke Math. J. (To appear)*.

[22] LEBRUN, C., AND MASON, L. J. Zoll Manifolds and Complex Surfaces. *J. Diff. Geom. 61* (2002), 453–535.

[23] NAKATA, F. Self-dual zollfrei conformal structures with alpha-surface foliation. *arXiv:math/0701116*.

[24] NUROWSKI, P., AND SPARLING, G. A. J. Three-dimensional cauchy-riemann structures and second-order ordinary differential equations. *Class. Quant. Grav. 20* (2003), 4995–5016.

[25] PENROSE, R. Nonlinear gravitons and curved twistor theory. *Gen. Rel. Grav. 7* (1976), 31–52.

[26] PENROSE, R., AND RINDLER, W. *Spinors and Space-time 1 & 2.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1988.

[27] PLEBAŃSKI, J. F. Some solutions of complex Einstein equations. *J. Math. Phys. 16* (1975), 2395–2402.

[28] SPARLING, G. A. J., AND TOD, K. P. An example of an H-Space. *J. Math. Phys. 22* (1981), 331–332.

[29] SZEKERES, P. Spaces conformal to a class of spaces in general relativity. *Proc. Roy. Soc. London Ser. A 274* (1963), 206–212.

[30] TAFEL, J., AND WÓJCIK, D. Null Killing vectors and reductions of the self-duality equations. *Nonlinearity 11* (1988), 835–844.

[31] WARD, R. S. A class of self-dual solutions of Einstein's equations. *Proc. Roy. Soc. London A363* (1978), 289–295.

[32] WARD, R. S., AND WELLS, R. O. *Twistor geometry and field theory.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1990.