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Author for correspondence:

Maciej Dunajski

e-mail: m.dunajski@damtp.cam.ac.uk

Skyrmions from gravitational instantons

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

We propose a construction of Skyrme fields from holonomy of the spin connection of gravitational instantons. The procedure is implemented for Atiyah–Hitchin and Taub–NUT instantons. The skyrmion resulting from the Taub–NUT is given explicitly on the space of orbits of a left translation inside the whole isometry group. The domain of the Taub–NUT skyrmion is a trivial circle bundle over the Poincaré disc. The position of the skyrmion depends on the Taub–NUT mass parameter, and its topological charge is equal to two.

1. Introduction

The Standard Model of elementary particles is a quantum gauge theory with a non-abelian gauge group $SU(3) \times SU(2) \times U(1)$. This model is at some level fundamental, and provides a complete field theory of interacting quarks. Thus, in principle, it should describe protons and neutrons. On the other hand, the Lagrangian underlying the Standard Model leads to nonlinear field equations, which has so far made it impossible to obtain exact results about the bound states describing the particles of the theory.

An alternative is to look for theories which ignore the internal structure of particles, and instead give an effective, low energy description of baryons. The Skyrme model [1] is an example of such theory, where baryons arise as solitons. The topological degree of these solitons is identified with the baryon number in a mechanism which naturally leads to a topological baryon number conservation.

In a recent paper, Atiyah *et al.* [2] (AMS) proposed a far reaching generalization of the Skyrme model which involves several topological invariants, and aims to give geometrical and topological interpretations to the electric charge, the baryon number and the lepton number. In the AMS model, static particles are described in terms of gravitational instantons—Riemannian four

manifolds which satisfy the Einstein equations and whose curvature is concentrated in a finite region of a space–time [3].

The electrically charged particles correspond to non-compact asymptotically locally flat (ALF) instantons (M, g) —complete four-dimensional Riemannian manifolds which solve the Einstein equations (possibly with cosmological constant) and approach S^1 bundle over S^2 at infinity. The first Chern class of the asymptotic $U(1)$ fibration gives the electric charge. Neutral particles correspond to compact instantons. In all cases, the baryon number has a topological origin and is identified with the signature of M .

The AMS model is inspired by the Atiyah–Manton approach [4] to the Skyrme model of baryons, where a static Skyrme field $U: \mathbb{R}^3 \rightarrow SU(2)$ with the boundary condition $U(\mathbf{x}) \rightarrow \mathbf{1}$ as $|\mathbf{x}| \rightarrow \infty$ arises from a holonomy of a Yang–Mills instanton on \mathbb{R}^4 along one of the directions. Thus, the physical three-space \mathbb{R}^3 is regarded as the space of orbits of a one-parameter group of conformal isometries of \mathbb{R}^4 . AMS use this as a motivation for their model, but in the AMS approach the three-space is (for the electrically charged particles) the base space of asymptotic circle fibrations.

The idea behind this paper is to use the AMS model as a motivation for relating particles to gravitational instantons, but then to proceed in a way analogous to the Atiyah–Manton construction to recover a skyrmion from a gravitational instanton. Thus, in our case, the three-space \mathcal{B} will arise as a quotient of M by a certain S^1 action. The $SU(2)$ instanton holonomy will be replaced by a holonomy of a spin connection on \mathbb{S}_+ , where $TM \otimes \mathbb{C} \cong \mathbb{S}_+ \otimes \mathbb{S}_-$, and \mathbb{S}_\pm are rank two complex vector bundles over M [5].

In §2, we shall reinterpret the $\mathfrak{su}(2)$ spin connection on \mathbb{S}_+ as the potential for a self-dual Yang–Mills field on the gravitational instanton background. In §3, we shall compute the holonomy of this potential along the orbits of an $SO(2)$ left-translation inside the whole isometry group of (M, g) , using the Atiyah–Hitchin (AH) and Taub–NUT instantons as examples. This will give rise to a skyrmion on the space of orbits \mathcal{B} of $SO(2)$ in M . We shall find the expression for the topological charge density, and compute this charge for the Taub–NUT skyrmion. In §4, we shall construct the Riemannian metric $h_{\mathcal{B}}$ on the three-dimensional domain \mathcal{B} of the skyrmion. In the case of Taub–NUT skyrmion, this metric is complete and describes a trivial circle fibration over the upper half-plane, with circular fibres of non-constant radius:

$$h_{\mathcal{B}} = g_{\mathbb{H}^2} + R^2 d\psi^2, \quad \text{where} \quad R^2 = \frac{1}{(\mu r + 1)^2 \sin^2 \theta + \cos^2 \theta}.$$

Here the constant parameter μ is the inverse mass in the Taub–NUT space, and $y = r \sin \theta > 0$, $x = r \cos \theta$ are coordinates on \mathbb{H}^2 with the hyperbolic metric $g_{\mathbb{H}^2} = y^{-2}(dx^2 + dy^2)$. In these coordinates, the skyrmion is given by

$$U = \exp \left(i\pi \left(\frac{r\mu}{r\mu + 1} \sin \theta (\cos \psi \tau_1 + \sin \psi \tau_2) - \frac{r\mu(r\mu + 2)}{(r\mu + 1)^2} \cos \theta \tau_3 \right) \right). \quad (1.1)$$

The skyrmion (1.1) is localized on the imaginary axis, around the point $(0, 5/(4\mu))$. We should again, at this point, emphasize the difference between our construction and the AMS approach. The ‘physical’ three-space in [2] admits an isometric $SO(3)$ action, whereas the three-dimensional Riemannian manifold $(\mathcal{B}, h_{\mathcal{B}})$ which supports the skyrmion admits only one isometry in the Taub–NUT case. This is because the generator of the left translation used in the construction of the quotient belongs to a two-dimensional abelian subalgebra inside the full Lie algebra of the isometry group $U(2)$. Thus, one Killing vector of the Taub–NUT space descends down to the quotient. In the AH case, the isometry algebra $SO(3)$ does not contain two-dimensional abelian subalgebras, and the quotient space \mathcal{B} does not admit any Killing vectors, or conformal Killing vectors.

2. Spin connection as gauge potential

Properties of a single particle are invariant with respect to ordinary rotations in three-space. Thus, the corresponding instanton should admit $SO(3)$ or its double cover $SU(2)$ as the group of isometries, i.e. the metric should take the form

$$g = f^2 dr^2 + (a_1 \eta_1)^2 + (a_2 \eta_2)^2 + (a_3 \eta_3)^2, \quad (2.1)$$

where η_i are the left invariant one-forms on $SU(2)$ such that

$$d\eta_1 = \eta_2 \wedge \eta_3, \quad d\eta_2 = \eta_3 \wedge \eta_1 \quad \text{and} \quad d\eta_3 = \eta_1 \wedge \eta_2$$

and (a_1, a_2, a_3, f) are functions of r . There is no loss of generality in this diagonal ansatz, as the induced metric can always be diagonalized on a surface of constant r , and then the Einstein equations imply [6] that the non-diagonal components are fixed (to zero) in the ‘evolution’ in r . The diffeomorphism freedom can be used to set $f = -a_2/r$.

In the AMS setup, the AH manifold [7] is a model for the proton and the self-dual Taub NUT manifold corresponds to the electron. Although the spin connection γ_- of (2.1) does not vanish in the invariant frame (2.1), the curvature of (\mathbb{S}_-, γ_-) is zero as both AH and Taub–NUT metrics have self-dual Riemann curvature. Thus, we shall consider the connection $\gamma = \gamma_+$ on \mathbb{S}_+ . It is best calculated using the self-dual two-forms [5]

$$\left. \begin{aligned} \Sigma_i &= \mathbf{e}_0 \wedge \mathbf{e}_i + \frac{1}{2} \varepsilon_{ijk} \mathbf{e}_j \wedge \mathbf{e}_k, \quad \text{where } i, j, k = 1, \dots, 3 \\ \text{and} \quad \mathbf{e}_0 &= f dr, \quad \mathbf{e}_1 = a_1 \eta_1, \quad \mathbf{e}_2 = a_2 \eta_2, \quad \mathbf{e}_3 = a_3 \eta_3. \end{aligned} \right\} \quad (2.2)$$

The spin connection coefficients γ_{ij} are skew-symmetric and are determined from the relations $d\Sigma_i + \gamma_{ij} \wedge \Sigma_j = 0$. We find

$$P_1 = f_1(r)\eta_1, \quad P_2 = f_2(r)\eta_2, \quad P_3 = f_3(r)\eta_3 \quad \text{and} \quad \gamma_{ij} = \varepsilon_{ijk} P_k, \quad (2.3)$$

where the functions $f_i(r)$ depend on the coefficients $a_i(r)$ and their derivatives.

- The Taub–NUT metric is a unique non-flat complete self-dual Einstein metric with isometric $SU(2)$ action such that the generic orbit is three-dimensional, and the $SU(2)$ action rotates the anti-self-dual two-forms. In this case,

$$a_1 = a_2 = r \sqrt{\epsilon + \frac{m}{r}} \quad \text{and} \quad a_3 = m \sqrt{\epsilon + \frac{m}{r}}^{-1},$$

where ϵ and m are constants. At $r=0$, the three-sphere of constant r collapses to a point—an example of a NUT singularity. The SD spin connection coefficients¹ give the Pope–Yuille instanton [8] (see [9] for a discussion on more general Yang–Mills instantons on self-dual ALF spaces)

$$f_1 = f_2 = -\frac{r\epsilon}{r\epsilon + m} \quad \text{and} \quad f_3 = \frac{r\epsilon(r\epsilon + 2m)}{(r\epsilon + m)^2}. \quad (2.4)$$

- The AH metric is a unique (up to taking a double covering) complete self-dual Einstein metric with isometric $SO(3)$ (rather than $SU(2)$) action such that the generic orbit is three dimensions, and the action rotates the anti-self-dual two-forms [7]. It is a metric on a moduli space of 2-monopoles with fixed centre. The $SO(3)$ isometric action can be traced back to the 2-monopole configuration, where the rotation group acts on the pair of unoriented spectral lines, and r is (a function of) an angle between these lines. The coordinate r parametrizes the orbits of $SO(3)$ in the moduli space. In this paper, we shall use the double cover of the moduli space of centred 2-monopoles, and, following [2], still call it the AH manifold.

¹For comparison, computing the ASD connection on \mathbb{S}_- would also give (2.3), but this time with $f_1 = f_2 = f_3 = 1$. The Maurer–Cartan equations on $SU(2)$ then imply that the curvature of this connection vanishes, and so the metric has self-dual Riemannian curvature.

In the case of the AH metric, we shall only need the asymptotic formulae. The coordinate r ranges between π and ∞ , and at $r = \pi$ the $\text{SO}(3)$ orbits collapse to a two-sphere (a bolt). For large r

$$a_1 = a_2 = r\sqrt{1 - \frac{2}{r}} + O(e^{-r}) \quad \text{and} \quad a_3 = -2\sqrt{1 - \frac{2}{r}}^{-1} + O(e^{-r}),$$

which leads to the asymptotic expressions

$$f_1 = f_2 = -\frac{r}{r-2} \quad \text{and} \quad f_3 = \frac{r(r-4)}{(r-2)^2}. \quad (2.5)$$

For r close to π , we have

$$a_1 = 2(r - \pi) + O((r - \pi)^2), \quad a_2 = \pi + \frac{1}{2}(r - \pi) + O((r - \pi)^2)$$

$$\text{and} \quad a_3 = -\pi + \frac{1}{2}(r - \pi) + O((r - \pi)^2),$$

which gives

$$f_1 = \frac{\pi - r}{\pi} - 3, \quad f_2 = \frac{\pi - r}{\pi} \quad \text{and} \quad f_3 = \frac{r - \pi}{\pi + r}. \quad (2.6)$$

To make contact with the ‘skyrmions from instantons’ ansatz of [4], we need to reinterpret the self-dual spin connection γ as $\mathfrak{su}(2)$ -valued gauge field A . This is done [10,11] by setting

$$A = P_1 \otimes \mathbf{t}_1 + P_2 \otimes \mathbf{t}_2 + P_3 \otimes \mathbf{t}_3, \quad (2.7)$$

where the matrices \mathbf{t}_i generate the Lie algebra $\mathfrak{su}(2)$ with the commutation relations $[\mathbf{t}_i, \mathbf{t}_j] = -(1/2)\varepsilon_{ijk}\mathbf{t}_k$, and the one-forms P_j are given by (2.3).

Topology of the Yang–Mills field is determined by the topology of the gravitational instanton (see [12] for a related construction where topology of an abelian vortex is determined by the topology of the underlying background surface). The topological charge of the Yang–Mills instanton is in general fractional, despite the action being finite. The relation between the Yang–Mills, and Einstein curvatures is easily expressed using the $\text{SO}(3)$ representation spaces:

$$A = P_i \otimes \mathbf{t}_i = \frac{1}{2}\varepsilon_{ijk}\gamma_{jk} \otimes \mathbf{t}_i$$

and

$$F = dA + A \wedge A = \frac{1}{2}\varepsilon_{ijk}R_{jk} \otimes \mathbf{t}_i, \quad \text{where } R_{ij} = d\gamma_{ij} + \gamma_{ik} \wedge \gamma_{jk}$$

is the Riemann curvature two-form of the gravitational instanton metric. Therefore, F is a self-dual Yang–Mills field

$$F = *F,$$

where asterisk (*) is the Hodge operator on the gravitational instanton background.

The two-form R_{ij} can be decomposed in terms of the Ricci tensor, Weyl tensor and Ricci scalar as $R_{ij} = W_{ijk}\Sigma_k + \Phi_{ijk}\Omega_k$, where Σ_k and Ω_k are basis of SD (see equation (2.2)) and ASD two-forms, respectively. The coefficients Φ_{ijk} have nine components corresponding to the trace-free Ricci tensor. The Bianchi identity $R_{ij} \wedge \Sigma_j = 0$ gives $W_{ijj} = 0$, so W_{ijk} can be further decomposed into a self-dual Weyl tensor (with five independent components), and the totally skew part $\Lambda\varepsilon_{ijk}$, where Λ is a multiple of the Ricci scalar. In the self-dual vacuum case, we have $\Phi_{ijk} = 0$, $\Lambda = 0$. Using the identities

$$\Sigma_i \wedge \Sigma_j = 2\delta_{ij}\text{vol}, \quad \text{Tr}(\mathbf{t}_i\mathbf{t}_j) = -\frac{1}{2}\delta_{ij}, \quad \varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

we find $\text{Tr}(F \wedge F) = -(\frac{1}{2})|W|^2\text{vol}$, where $|W|^2 = W_{ijk}W^{ijk}$.

In the case considered in this paper, where P_j are given by the one-forms (2.3) we find (with $\cdot = d/dr$)

$$\begin{aligned} F = & (\dot{f}_1 dr \wedge \eta_1 + (f_1 - f_2 f_3) \eta_2 \wedge \eta_3) \otimes \mathbf{t}_1 \\ & + (\dot{f}_2 dr \wedge \eta_2 + (f_2 - f_1 f_3) \eta_3 \wedge \eta_1) \otimes \mathbf{t}_2 \\ & + (\dot{f}_3 dr \wedge \eta_3 + (f_3 - f_1 f_2) \eta_1 \wedge \eta_2) \otimes \mathbf{t}_3. \end{aligned}$$

The instanton number is $k = -c_2$, where the Chern number is given by

$$c_2 = -\frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge F).$$

In our case,

$$\text{Tr}(F \wedge F) = \frac{d}{dr} \left(f_1 f_2 f_3 - \frac{1}{2} (f_1^2 + f_2^2 + f_3^2) \right) dr \wedge \eta_1 \wedge \eta_2 \wedge \eta_3,$$

and integration by parts gives $k_{\text{TN}} = 1$ and $k_{\text{AH}} = 2$. In evaluating the r -integrals, we took into account that the radial direction is oppositely oriented in the AH and the Taub–NUT cases. This is a consequence of a fact (carefully discussed in [2]) that the Taub–NUT metric is self-dual for the orientation which in the limit $\epsilon \rightarrow 0$ gives the standard orientation on \mathbb{C}^2 . The AH manifold on the other hand is self-dual for the orientation opposite to the complex orientations given by the underlying hyper-Kähler structure.

3. Skyrmions from spin connection holonomy

The Skyrme model is a nonlinear theory of pions in three space dimensions [1]. The model does not involve quarks and is to be regarded as a low energy, effective theory of QCD. A static skyrmion is a map

$$U: \mathbb{R}^3 \rightarrow \text{SU}(2)$$

satisfying the boundary conditions $U \rightarrow \mathbf{1}$ as $|\mathbf{x}| \rightarrow \infty$. The boundary conditions imply that U extends to a one-point compactification $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and thus U is partially classified by its integer topological degree taking values in $\pi_3(S^3)$. In the Skyrme model, this topological degree is identified with the baryon number, which by continuity is conserved under time evolution. The nonlinear field equations resulting from the Skyrme Lagrangian are not integrable and no explicit solutions are known. In contrast to other soliton models, the Bogomolny bound is not saturated in the Skyrme case, and the energy is always greater than the baryon number.

A good approximation of skyrmions is given by holonomy of $\text{SU}(2)$ instantons in \mathbb{R}^4 , computed along straight lines in one fixed direction [4]. Choosing the lines to be parallel to the $s = x^4$ axis gives the Atiyah–Manton ansatz

$$U(\mathbf{x}) = \mathcal{P} \exp \left(\int_{-\infty}^{\infty} A_4(\mathbf{x}, s) ds \right),$$

where A_4 is a component of the Yang–Mills instanton on \mathbb{R}^4 . The end points of each line should be identified with the north-pole of the four sphere compactification of \mathbb{R}^4 . The boundary conditions at $\mathbf{x} \rightarrow \infty$ are then satisfied as small circles on S^4 corresponding to straight lines shrink to a point (figure 1). Moreover, the instanton number of A is equal to the baryon number of the resulting skyrmion.

In this section, we shall adapt the Atiyah–Manton construction to gravitational instantons. While the underlying principle still applies, and holonomy of the spin connection along certain geodesics gives rise to a scalar group-valued field, the properties of the resulting skyrmion are very different. In particular, it is not defined on \mathbb{R}^3 which affects the boundary conditions. The topological degree can still be found, but we shall not interpret it as the baryon number as this is given by the signature of the underlying gravitational instanton. In particular, the baryon number is zero for the Taub–NUT skyrmion, but the Skyrme topological degree is not.

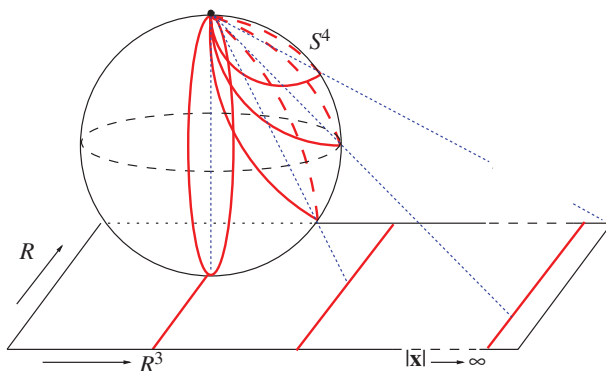


Figure 1. Boundary conditions from the instanton ansatz. (Online version in colour.)

Let $K = K^a \partial / \partial x^a$, $a = 0, \dots, 3$ be a vector field generating a one-parameter group of transformations of M with orbits Γ . The holonomy of the gauge field A along Γ arises from a solution to an ordinary differential equation $K^a D_a \Psi = 0$, where $D_a = \partial_a + A_a$, and $\Psi = \Psi(s, x^j)$ takes its value in the Lie group $SU(2)$. Here $K = \partial / \partial s$ and x^j are the coordinates on the space of orbits \mathcal{B} of K in M . If the trajectories Γ are non-compact, then one imposes the initial condition $\Psi(0, x^j) = \mathbf{1}$ at $s = -\infty$, and sets the Skyrme field to be $U(x^j) = \lim_{s \rightarrow \infty} \Psi(s, x^j)$. This gives

$$U = \mathcal{P} \exp \left(- \int_{\Gamma} A \right), \quad (3.1)$$

where \mathcal{P} denotes the s -ordering. In the case of Γ being a circle one breaks it up into an interval.

We now have to choose the curves Γ along which the holonomy is to be calculated. We shall need an explicit parametrization of the one-forms η_i in the metric (2.1)

$$\eta_2 + i\eta_1 = e^{-i\psi} (d\theta + i \sin \theta d\phi) \quad \text{and} \quad \eta_3 = d\psi + \cos \theta d\phi, \quad (3.2)$$

where to cover $SU(2) = S^3$ in the Taub–NUT case we require the ranges

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \quad \text{and} \quad 0 \leq \psi \leq 4\pi$$

so that $\int \eta_1 \wedge \eta_2 \wedge \eta_3 = -16\pi^2$. In the AH case take $0 \leq \psi \leq 2\pi$, and make an identification $(\theta, \phi, \psi) \cong (\pi - \theta, \phi + \pi, -\psi)$ so that $\int \eta_1 \wedge \eta_2 \wedge \eta_3 = -4\pi^2$.

One natural choice for Γ is the family of the asymptotic circles with the ψ coordinate varying, but this gives a trivial result as the resulting Skyrme field depends only on r , is abelian, and its topological charge vanishes. More generally, the holonomy should be calculated along the curves which are orbits of a Killing vector (or at least a conformal Killing vector) as otherwise the space of orbits \mathcal{B} of K in M does not admit a metric even up to scale. However, a metric on \mathcal{B} is necessary to compute the energy of the skyrmion. This rules out the asymptotic circles in the AH case.

We shall instead pick a left translation $SO(2)$ inside the isometry group $SO(3)$ (or its double cover $SU(2)$) of (2.1). Without loss of generality, we can always choose the Euler angles in (3.2) so that the generator of this left translation is the right invariant vector field $K = \partial / \partial \phi$. The S^1 fibres of \mathcal{B} have no points in common and the resulting skyrmion can only be defined up to conjugation. However, the preferred gauge has been fixed by choosing the $SO(3)$ or $SU(2)$ invariant frame (2.2) in which the spin connection components are proportional to the left-invariant one forms, and the coefficients only depend on the radial coordinate. This procedure is analogous to the one used by Atiyah & Sutcliffe [13]. The Yang–Mills connection resulting from our procedure is given in the radial gauge $A_r = 0$ as

$$A = f_1(r) \eta_1 \otimes \mathbf{t}_1 + f_2(r) \eta_2 \otimes \mathbf{t}_2 + f_3(r) \eta_3 \otimes \mathbf{t}_3.$$

The residual gauge freedom $A \rightarrow \rho A \rho^{-1} - d\rho \rho^{-1}$, where $\rho = \rho(\theta, \psi, \phi) \in \text{SU}(2)$ can either be fixed by demanding regularity of the point $r = 0$ (which is singled out as the fixed point of the isometry K) or by imposing the symmetry requirement

$$\mathcal{L}_{R_i} A = 0, \quad i = 1, 2, 3,$$

where the right-invariant Killing vector fields R_i generate left-translations, and thus preserve the left-invariant one-forms η_i , i.e. $\mathcal{L}_{R_i} \eta_j = 0$ and \mathcal{L} denotes the Lie derivative.

The anti-symmetric matrix $\nabla_a K_b$ has rank four at $r = 0$, where the norm of K vanishes. Thus, the Killing vector has an isolated fixed point $r = 0$, which is an anti-NUT in the terminology of [3]. Therefore, some care needs to be taken when constructing the metric on the three-dimensional domain of the skyrmion—this will be done in §4.

Restricting the left-invariant forms η_j to the ϕ -circles gives

$$\eta_j = n_j d\phi,$$

where the unit vector \mathbf{n} is given in the *unusual* spherical polar coordinates (ψ, θ) by

$$\mathbf{n} = (\cos \psi \sin \theta, \sin \psi \sin \theta, \cos \theta).$$

The integral (3.1) can be performed explicitly, as the component A_ϕ of the gauge field (2.7) does not depend on ϕ . This yields

$$U(r, \psi, \theta) = \exp \left(-i\pi \sum_{j=1}^3 f_j(r) n_j \tau_j \right), \quad (3.3)$$

where the Pauli matrices τ_j are related to the generators of $\mathfrak{su}(2)$ by $\mathbf{t}_j = (i/2)\tau_j$. The topological charge of the skyrmion is²

$$\begin{aligned} B &= -\frac{1}{24\pi^2} \int_{\mathcal{B}} \text{Tr}((U^{-1} dU)^3) \\ &= -\frac{1}{2\pi} \int \left(\frac{df_1}{dr} f_2 f_3 n_1^2 + f_1 \frac{df_2}{dr} f_3 n_2^2 + f_1 f_2 \frac{df_3}{dr} n_3^2 \right) \frac{\sin(\pi\kappa)^2}{\kappa^2} \sin \theta dr d\theta d\psi, \end{aligned} \quad (3.4)$$

where $\kappa = \sqrt{f_1^2 n_1^2 + f_2^2 n_2^2 + f_3^2 n_3^2}$.

Let us first consider the Taub–NUT case, where f_j are given by (2.4). The field U does not satisfy the boundary conditions usually expected from a skyrmion, as $U(0) = \mathbf{1}$, and $f_j \rightarrow (-1, -1, 1)$ as $r \rightarrow \infty$. It nevertheless gives rise to a well-defined constant group element at ∞ , as there $(f_1 n_1, f_2 n_2, f_3 n_3)$ tends to a unit vector and, setting $\mathbf{k} = (-n_1, -n_2, n_3)$, we get

$$\lim_{r \rightarrow \infty} U = \cos(-\pi) \mathbf{1} + i(\mathbf{k} \cdot \boldsymbol{\tau}) \sin(-\pi) = -\mathbf{1}.$$

The functions f_1 and f_2 monotonically decrease and f_3 monotonically increases. Thus—as the angle ψ varies between 0 and 4π —each element of the target space except $U = -\mathbf{1}$ has exactly two pre-images in the space of orbits on $\partial/\partial\phi$. Thus, the topological charge of the Taub–NUT skyrmion is

$$B_{\text{TN}} = 2.$$

This is confirmed by evaluating the integral (3.4). We stress that the value of this topological charge is intimately related to the period of the ψ coordinate. The density does not depend on ψ , so if the range is $\psi \in [0, k\pi]$, then $B_{\text{TN}} = k/2$.

²In the special case of the usual hedgehog ansatz where $f_1 = f_2 = f_3 = F(r)$, and $0 \leq \psi \leq 2\pi$ this formula reduces to the known expression

$$B = -2 \int_{r_0}^{\infty} \sin(\pi F)^2 \frac{dF}{dr} dr.$$

In the AH case $f_j \rightarrow (-1, -1, 1)$ when $r \rightarrow \infty$, and the Skyrme field tends to a constant group element -1 at infinity. We do not expect the skyrmion to have a constant value at $r = \pi$, as this corresponds to a bolt two-surface in the AH manifold. Formulae (2.6) give

$$U(r = \pi, \psi, \theta) = \exp(3i\pi \cos \psi \sin \theta \tau_1).$$

This skyrmion has a constant direction in $\mathfrak{su}(2)$ at the surface of the bolt, with magnitude varying along its boundary.

The degree of the AH skyrmion is still well defined, but we have so far failed in calculating it by direct integration. Instead we shall use the method of counting pre-images. Following [7,14], we consider the parametrization of the radial functions $a_i(r)$ by elliptic integrals. Set

$$\begin{aligned} r &= 2K \left(\sin \left(\frac{\beta}{2} \right) \right), \\ w_1 &= -r \frac{dr}{d\beta} \sin \beta - \frac{1}{2} r^2 (1 + \cos \beta), \\ w_2 &= -r \frac{dr}{d\beta} \sin \beta \\ \text{and} \quad w_3 &= -r \frac{dr}{d\beta} \sin \beta + \frac{1}{2} r^2 (1 - \cos \beta), \end{aligned}$$

where K is the elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \tau}} d\tau$$

so that when $\beta \in [0, \pi)$, then $r(\beta) \in [\pi, \infty)$ is a monotonically increasing function. The radial functions in the AH metric are

$$a_1 = \sqrt{\frac{w_2 w_3}{w_1}}, \quad a_2 = \sqrt{\frac{w_1 w_3}{w_2}} \quad \text{and} \quad a_3 = -\sqrt{\frac{w_1 w_2}{w_3}},$$

and we find the spin connection coefficients (2.3) to be

$$\begin{aligned} f_1 &= \frac{1}{2} \left(\frac{a_3}{a_2} + \frac{a_2}{a_3} - \frac{a_1^2}{a_2 a_3} \right) - \frac{r}{a_2} \frac{da_1}{dr}, \\ f_2 &= \frac{1}{2} \left(\frac{a_3}{a_1} + \frac{a_1}{a_3} - \frac{a_2^2}{a_1 a_3} \right) - \frac{r}{a_2} \frac{da_2}{dr} \\ \text{and} \quad f_3 &= \frac{1}{2} \left(\frac{a_1}{a_2} + \frac{a_2}{a_1} - \frac{a_3^2}{a_1 a_2} \right) - \frac{r}{a_2} \frac{da_3}{dr}. \end{aligned}$$

The graphs of $r(\beta), a_i(\beta), f_i(\beta)$ are shown on figure 2 (the graphs provided in [14] give a_i as functions of r). The functions f_1 and f_3 are monotonically increasing, and the function f_2 appears to have a local minimum near $\beta = 3.0635$. Thus, each point on $SU(2)$ except $U = -1$ has exactly one pre-image in \mathcal{B} , and the topological charge of the AH skyrmion is $B_{\text{AH}} = 1$.

4. Geometry of the Taub–NUT skyrmion

In both AH and Taub–NUT cases, the resulting skyrmion is defined on the space of orbits \mathcal{B} of the isometry $\phi \rightarrow \phi + \text{const}$ in (M, g) . This space has a natural conformal metric induced by (2.1). To find it, perform the standard Kaluza–Klein reduction on ϕ , simply by completing the square. Set

$$\Omega^2 = a_1^2 \cos \psi^2 \sin \theta^2 + a_2^2 \sin \psi^2 \sin \theta^2 + a_3^2 \cos \theta^2.$$

Then

$$g = h + \Omega^2 (d\phi + \Omega^{-2} \omega)^2,$$

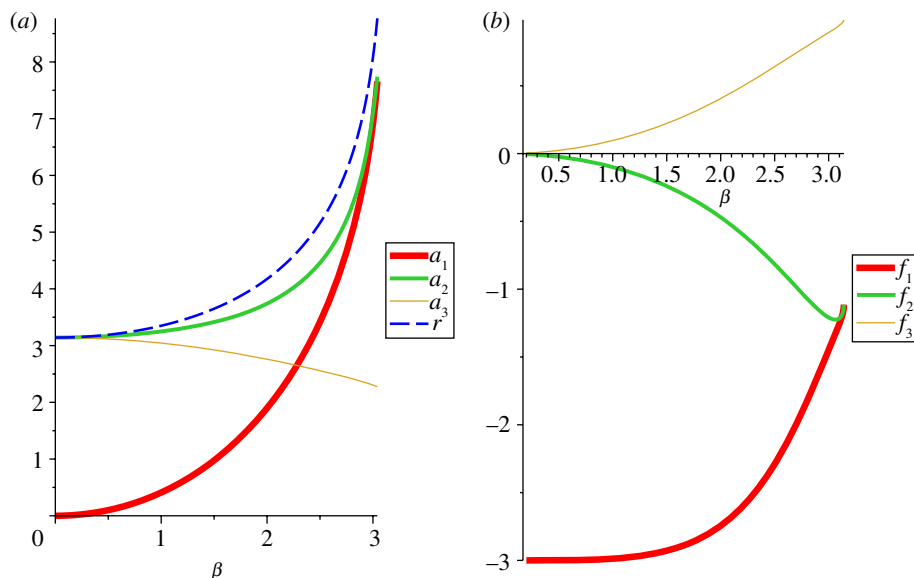


Figure 2. (a) Radial functions a_1, a_2, a_3 and r and (b) the spin connection coefficients as functions of $\beta \in [0, \pi)$. (Online version in colour.)

where (h, ω) are a metric and a one form, respectively, on the space of orbits of $\partial/\partial\phi$ given by

$$h = f^2 dr^2 + (a_1^2 \sin^2 \psi + a_2^2 \cos^2 \psi) d\theta^2 + a_3^2 d\psi^2 - \Omega^{-2} \omega^2$$

and

$$\omega = (a_2^2 - a_1^2) \sin \psi \cos \psi \sin \theta d\theta + a_3^2 \cos \theta d\psi.$$

The metric h is only defined up to scale on the space of orbits, and we can choose this scale freely.

One choice of the conformal factor which takes into account the range of the angular coordinate θ in the Taub–NUT case is³

$$h_{\mathcal{B}} = \frac{1}{a_1^2 \sin^2 \theta} h = g_{\mathbb{H}^2} + R^2 d\psi^2$$

This metric is defined on a trivial circle bundle over the hyperbolic plane. The vector $\partial/\partial\psi$ is an isometry of $h_{\mathcal{B}}$ which is a consequence of the fact that the right translations $\psi \rightarrow \psi + \text{const.}$ of the Taub–NUT space commute⁴ with the left translations $\phi \rightarrow \phi + \text{const.}$ In the upper half-plane model, where $x = r \cos \theta, y = r \sin \theta$ the hyperbolic metric $g_{\mathbb{H}^2}$ and the varying radius $2R$ (as ψ is between 0 and 4π) of the S^1 fibres are given by

$$g_{\mathbb{H}^2} = \frac{dx^2 + dy^2}{y^2} \quad \text{and} \quad R^2 = \frac{1}{(a_1/a_3)^2 \sin^2 \theta + \cos^2 \theta} = \frac{x^2 + y^2}{y^2 (\mu \sqrt{x^2 + y^2 + 1})^2 + x^2},$$

where $\mu = \epsilon/m$. The radius of the circles tends to two on the real line boundary of the upper half-plane, and shrinks away from the boundary. The metric is complete, as the radius does not vanish anywhere on \mathcal{B} . The density of the resulting skyrmion (3.3) attains its maximum at the y -axis in the upper half-plane model, where it is given by

$$\frac{\pi \mu y (\mu y + 2)}{(\mu y + 1)^2} \sin \left(\frac{\pi \mu y}{\mu y + 1} \right)^2.$$

³This conformal metric also admits a Weyl connection such that the Einstein–Weyl equations hold on \mathcal{B} . This is true for any conformal structure on the space of orbits of an isometry in a Riemannian manifold with self-dual Weyl curvature [5].

⁴This would not be the case for the AH metric, where the domain of the resulting skyrmion does not admit any isometries.

The location of the maximum is a root of the transcendental equation $\tan \hat{y} = \hat{y}^{-1}(\pi^2 - \hat{y}^2)$, where $\mu y = \pi/\hat{y} - 1$. The approximate solution is $y = 5/(4\mu)$, and the resulting maximal density is approximately 1.22 for all values of the parameter μ . The skyrmion density is independent on ψ . In the disc model of the hyperbolic space, the boundary of \mathcal{B} is a flat torus. Let the map $\mathbb{D} \rightarrow \mathbb{H}^2$ be given by

$$x + iy = \frac{z - i}{iz - 1},$$

where $|z| < 1$. The radius of fibres of $S^1 \rightarrow \mathcal{B} \rightarrow \mathbb{D}$ is discontinuous at the point $z = -i$ corresponding to $y = \infty$ on the boundary (figure 3b). The Ricci scalar of $h_{\mathcal{B}}$ is also discontinuous on the boundary, and equals -2 if $|z| = 1$ and $z \neq -i$, and -6 at $z = -i$ (figure 4). At the centre of the disc, the Ricci scalar depends on μ and is given by $-2(3\mu^2 + 3\mu + 1)/(\mu + 1)^2$. This does not cause a problem as the point $z = -i$ is not a part of the manifold \mathcal{B} . The density of the Taub–NUT skyrmion peaks on the diameter joining $z = -i$ and $z = i$. The skyrmion is located on this diameter at $z = i(4\mu - 5)/(4\mu + 5)$ (figure 5). At the centre of the disc, the skyrmion varies along the fibres according to

$$U = \exp\left(\frac{i\pi\mu}{(\mu + 1)(\cos \psi \tau_1 + \sin \psi \tau_2)}\right).$$

The point $r = 0$ corresponds to the point $z = i$ on the boundary of the disc, where $U = 1$. Note that this point has been removed from the domain of the skyrmion, as it is the fixed point (in four dimensions) of the isometry used to obtain the quotient \mathcal{B} . The boundary conditions $r \rightarrow \infty$ translate to $U = -1$ at the point $z = -i$ for all ψ .

5. Further remarks

We have constructed $SU(2)$ -valued Skyrme fields from a holonomy of a non-flat $\mathfrak{su}(2)$ spin connection on \mathbb{S}_+ corresponding to ALF gravitational instantons. The holonomy is calculated along orbits of an isometry generating a one-parameter subgroup $SO(2)$ of the full isometry group. In case of Taub–NUT, the Skyrme field carries a non-zero topological charge. This rules out the interpretation of the charge as the baryon number—which vanishes—but opens up a possibility of assigning other integral charges to particles in the AMS model. A lepton number is an obvious candidate,⁵ as no proposal of its topological interpretation has been put forward in [2].

A computation of the Skyrme field can be carried over for other gravitational instantons. To do it for the Fubini–Study metric on \mathbb{CP}^2 , one needs to express it in the Bouchiat–Gibbons form [15] adapted to the $SO(3)$ (rather than $U(2) \subset SU(3)$) action. This, with $r \in [0, \pi/2]$, leads to a connection (2.3) with

$$\begin{aligned} f_1 &= \frac{2 \sin(r + \pi/2) \cos r + \cos r^2}{\sin(r + \pi/2)}, \\ f_2 &= \frac{2 \cos(r/2 + \pi/4)^2 + 2 \sin r \cos(r/2 + \pi/4)^2 - \cos r^2}{2 \sin r \cos(r/2 + \pi/4)} \\ \text{and} \quad f_3 &= -\frac{2 \cos(r/2 + \pi/4)^2 + 2 \sin r \sin(r/2 + \pi/4)^2 + \cos r^2 - 2}{2 \sin r \sin(r/2 + \pi/2)}. \end{aligned}$$

The instanton number of the corresponding gauge field is fractional $k_{\mathbb{CP}^2} = 9/2$, which reflects the fact that \mathbb{CP}^2 does not admit a spin structure, and the gauge field resulting from the spin connection is not globally defined. The resulting Skyrme field behaves similarly to the AH case,

⁵It may be that some combination of the Skyrme charge, and the Euler number and the signature corresponds to the lepton number. An identification of the lepton number with $(B - 1)$ is consistent with the AMS proposal, but possibly too naive.

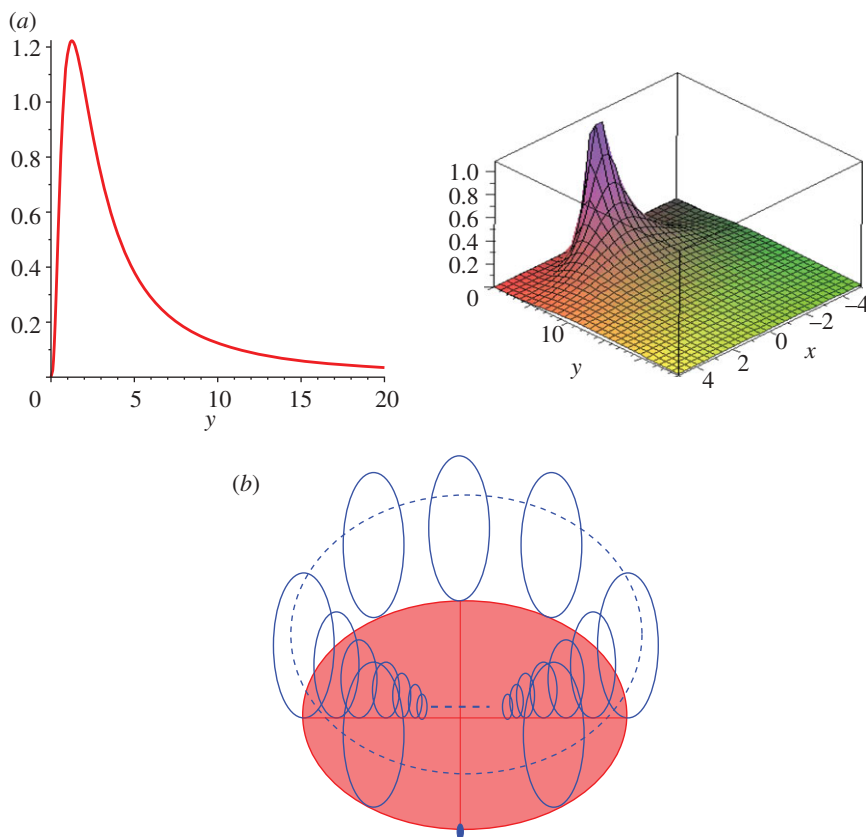


Figure 3. (a) Skymion density on the upper half plane. (b) Circle fibration over the Poincaré disc, with shrinking fibres. (Online version in colour.)

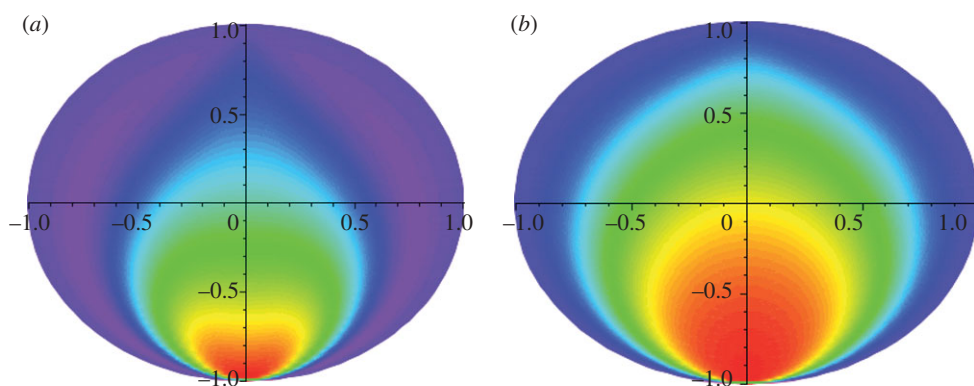


Figure 4. (a) Density plots of the Ricci scalar of $(\mathcal{B}, h_{\mathcal{B}})$ and (b) the radii of the circles in the fibration $S^1 \rightarrow \mathbb{D}$ for $\mu = 1$. (Online version in colour.)

as (f_1, f_2, f_3) equals $(3, 0, 0)$ at $r = 0$, and $(0, 0, 0)$ at $r = \pi/2$. While the number and the position of nuts and bolts might depend on the choice of the isometry— \mathbb{CP}^2 has three nuts for $\partial/\partial\phi$ and a nut and a bolt for $\partial/\partial\psi$ —the total number of nuts and bolts is constrained by a topological equality [3]

$$\sum \text{nut} + \sum \chi(\text{bolt}) = \chi(M)$$

and the r.h.s. is equal to 3 for \mathbb{CP}^2 .

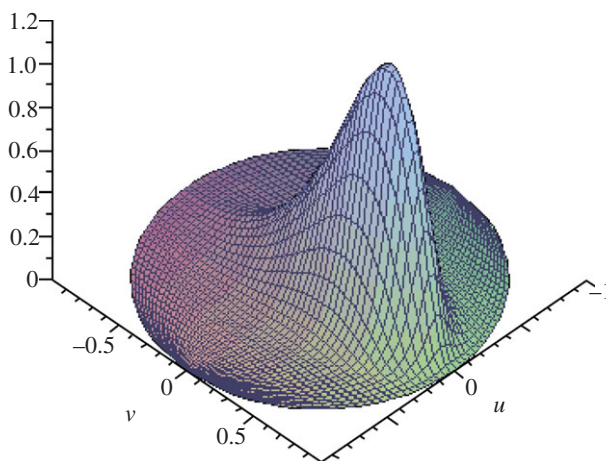


Figure 5. Density of the Taub–NUT skyrmion in the Poincaré disc model with $z = u + iv$, $\mu = 4$. (Online version in colour.)

We can also compute the Skyrme field for the Euclidean Schwarzschild metric. This gravitational instanton does not appear in [2], as its curvature is not self-dual. It is nevertheless possible that it can be used as a model for the neutrino in place of S^4 —the case for self-duality of the underlying Riemannian manifolds was not overwhelmingly strong in [2]. The topology of the Euclidean Schwarzschild is $\mathbb{R}^2 \times S^2$, with boundary $S^1 \times S^2$. Now there are two Yang–Mills fields constructed out of self-dual and anti-self-dual spin connections, and the Euler and Pontryagin numbers are linear combinations of the two instanton numbers. The Schwarzschild manifold has signature zero, which is compatible with AMS interpretation. The metric is asymptotically flat, and the asymptotic circle fibration is trivial. Thus, there is no associated electric charge.

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