Solitons from Geometry

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- Atiyah, Manton, Schroers. Geometric models of matter. `arXiv:1111.2934`. Proc. R. Soc. Lond **A** (2012).
  MD. Abelian vortices from Sinh–Gordon and Tzitzeica equations. `arXiv:1201.0105`. Phys. Lett. B. **710** (2012).
  MD. Skyrmions from gravitational instantons. `arXiv:1206.0016`. Proc. R. Soc. Lond. **A** (2013).

- *Here I will only say that I am emphatically in favour of the retention of the particle idea. Naturally, it is necessary to redefine what is meant.*

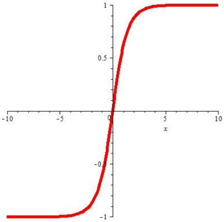
Max Born, Nobel Lecture, December 11, 1954.

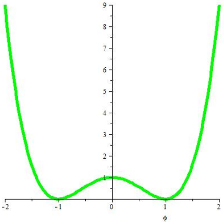
- Waves: Dispersion, diffraction, superpositions. QFT: Fourier transform, infinities, regularisation, . . . , string theory.
- Particles: mass, localisation. Solitons, classical non–linear field equations, topological charges, stability.

Solitons: Non–singular, static, finite energy solutions of the field equations.

$$L = \int_{\mathbb{R}} \Big(\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - U(\phi)\Big)dx, \quad \text{where} \quad \phi : \mathbb{R}^{1,1} \to \mathbb{R}.$$

- Stable vacuum: $U \geq 0$. Set $U^{-1}(0) = \{\phi_1, \phi_2, \dots, \phi_N\}$.
- Finite energy: $\phi(x,t) \to \phi_i$ as $|x| \to \infty$.
  - $\phi(x,t) = \phi_1 + \delta\phi(x,t)$. Perturbative Scalar boson $(\Box + m^2)\delta\phi = 0$.
  - Kink: $\phi \cong \phi_1$ as $x \to -\infty$. $\phi \cong \phi_2$ as $x \to \infty$.





Derrick's argument. Solitons on $\mathbb{R}^{d,1}$

Scalar field ϕ, gauge field $F = dA + A \wedge A$, derivarive $D\phi = d\phi + A\phi$.

- $E[\phi, A] = E_0 + E_p + E_F$

$$E_0 = \int_{\mathbb{R}^d} V(\phi) dx, \; E_p = \int_{\mathbb{R}^d} \mathrm{Pol}_p(D\phi) dx, \; E_F = \int_{\mathbb{R}^d} \mathrm{Tr}(F^2) dx.$$

- Spatial scalings $x \to \lambda x$.

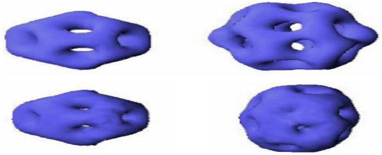
$$E_0 \to \lambda^{-d} E_0, \; E_p \to \lambda^{p-d} E_p, \; E_F \to \lambda^{4-d} E_F, \quad \frac{dE(\lambda)}{d\lambda}|_{\lambda=1} = 0.$$

1. $d = 1$, kinks with $E_0 = E_2$.
2. $d = 2$, vortices with E_0, E_2 and E_F.
3. $d = 3$, monopoles with E_2 and E_F, or skyrmions with E_2 and E_4.
4. $d = 4$, Yang–Mills instantons with E_F.

Solitons and Topology

- Soliton stability
  - Nontrivial topology ✓
  - Complete integrability
- Static vortices: Abelian gauge potential + Higgs field $\phi : \mathbb{R}^2 \to \mathbb{C}$.
  - Boundary conditions: $F = 0$ and $|\phi| = 1$ as $r \to \infty$.
  - $\phi_\infty : S^1 \to S^1$. Vortex number $\pi_1(S^1) = \mathbb{Z}$.
- Skyrmions: $U : \mathbb{R}^{3,1} \to SU(2)$.
  - Static+finite energy. $U : S^3 \to SU(2)$.
  - Topological baryon number $\pi_3(S^3) = \mathbb{Z}$.

$$B = -\frac{1}{24\pi^2}\int_{\mathbb{R}^3} \mathrm{Tr}[(U^{-1}dU)^3].$$



Abelian Higgs Model

- 2D Kähler manifold (Σ, g, ω). Hermitian line bundle $L \to \Sigma$ with $U(1)$ connection A and a global section C^∞ section $\phi : \Sigma \to L$.
- Ginsburg–Landau energy

$$E[A,\phi] = \frac{1}{2}\int_\Sigma (|D\phi|^2 + |F|^2 + \frac{1}{4}(1-|\phi|^2)^2)\text{vol}_\Sigma, \quad F = dA$$

- Bogomolny equations

$$\overline{D}\phi = 0, \quad F = \frac{1}{2}\omega(1-|\phi|^2)$$

- Taubes equation: $z = x + iy,\ g = \Omega(z,\bar{z})\, dzd\bar{z}$.
  Set $\phi = \exp(h/2 + i\chi)$, solve for A.

$$\Delta_0 h + \Omega - \Omega e^h = 0, \quad \text{where} \quad \Delta_0 = 4\partial_z\partial_{\bar{z}}.$$

Vortex from geometry. $\Delta_0 h + \Omega - \Omega e^h = 0$.

- Vortex number

$$N = \frac{1}{2\pi}\int_\Sigma B \, \mathsf{vol}_\Sigma, \quad h \sim 2N \log|z - z_0| + const + \dots \, .$$

- Vortices on $\mathbb{R}^2$, i. e. $\Omega = 1$. No explicit solutions.
- Make the Taubes equations integrable: $\Omega = \exp(-h/2)$.

$$\Delta_0(h/2) = \sinh(h/2), \quad \text{Sinh–Gordon equation.}$$

- Solitons from geometry: $L = T\Sigma, \quad SO(2) \cong U(1)$,
  Chern number=Euler characteristic.

Geometry of Sinh–Gordon vortex

- $g = e^{-h/2} dz d\bar{z}$. Constant mean curvature surface $\Sigma \subset \mathbb{R}^{2,1}$.
- Radial solutions, $h = h(r)$, s.t. $h \to 0$ as $r \to \infty$.
- Large r: Modified Bessel equation, $h \sim 8\lambda K_0(r)$ for a constant λ.
- Small r: Liouville equation, $h = 4\sigma \log r + \dots$ for a constant σ.
- In general, Painléve III equation.
- Theorem (MD, 2012)
  1. $\sigma(\lambda) = 2\pi^{-1} \arcsin(\pi\lambda)$ if $0 \le \lambda \le \pi^{-1}$.
  2. There exists a one–vortex solution with strength $4\sqrt{2}/\pi$.
  3. CMC surface with deficit angle π near $r = 0$.
- Uses isomonodromy theory, and connection formulae (Kitaev).
- One more integrable case: Tzitzeica equation and affine spheres.

Skyrmions

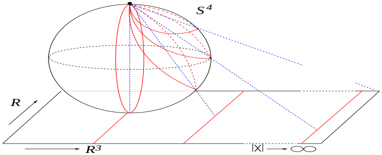
- Skyrme model of Baryons. Nonlinear pion field $U : \mathbb{R}^{3,1} \to SU(2)$.
- Energy of static skyrmions. Set $R_j = U^{-1}\partial_j U \in \mathfrak{su}(2)$.

$$E[U] = -\frac{1}{2}\int_{\mathbb{R}^3} \Big(\mathrm{Tr}(R_j R_j) + \frac{1}{8}(\mathrm{Tr}([R_i, R_j][R_i, R_j])\Big) d^3\mathbf{x} > 12\pi^2|B|.$$

- Finite energy, $U : {S^3}_\infty \to S^3$ and $B = \mathrm{deg}(U) \in \mathbb{Z}$.
- Skyrmions from instanton holonomy (Atiyah+ Manton, 1989).

$$U(\mathbf{x}) = \mathcal{P}\exp\Big(-\int_{\mathbb{R}} A_4(\mathbf{x}, \tau) d\tau.\Big)$$

Yang–Mills potential $A \in \mathfrak{su}(2)$ for a Yang–Mills field $F = dA + A \wedge A$ on $\mathbb{R}^4$. Baryon number = instanton number.



- Gravitational instantons = Riemannian, Einsten four manifolds (M, g) whose curvature is concentrated in a finite region of a space-time.
- Taub–NUT metric

$$g = \frac{r+m}{r} dr^2 + (r^2 + mr)(d\theta^2 + \sin\theta^2 d\phi^2) + \frac{rm^2}{r+m}(d\psi + \cos\theta d\phi)^2,$$

  where $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$.
  - Small r: $g \sim$ flat metric on $\mathbb{R}^4$.
  - Large r: Hopf bundle $S^3 \to S^2$, Chern class $= 1$.
  - Asymptotically locally flat (ALF): $M \sim S^1$ bundle over S^2 at ∞.
- Other gravitational instantons: Atiyah–Hitchin (ALF), Eguchi–Hanson (ALE), ... S^4, $K3$, $\mathbb{CP}^2$ (compact).

New Idea: particles as four–manifolds

- Particles = gravitational instantons. (Atiyah, Manton, Schroers).
- Charged particles = ALF instantons. Charge = Chern number of asymptotic S^1 bundle.
- Neutral particles = compact instantons.
- Baryon number = Hirzebruch signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M R_{ab} \wedge R_{ab}.$$

- Proton, neutron, electron, neutrino.
  Atiyah–Hitchin, $\mathbb{CP}^2$, Taub–NUT, S^4.

- $* : \Lambda^2(M) \to \Lambda^2(M), \quad *^2 = \mathsf{Id}$.
- $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$,

$$F = \frac{1}{2}(F + *F) + \frac{1}{2}(F - *F) = F_+ + F_-.$$

- $\Lambda^2 \cong \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.
- Spin connection $\gamma = \gamma_+ + \gamma_- \in \mathfrak{so}(4)$.
- AH, Taub–NUT. Riemann$_- = 0$.
- $\mathbb{CP}^2$, four–sphere. Weyl$_- = 0$.

- Skyrmion from the holonomy of the spin connection (MD, 2012).

$$\gamma_+ \in \mathfrak{su}(2) \rightarrow \text{self–dual YM on } (M, g) \rightarrow \text{Skyrmion on } M/SO(2)\ .$$

- $A = (1/2)\varepsilon_{ijk}\gamma_{ij} \otimes \mathbf{t}_k$, where $[\mathbf{t}_i, \mathbf{t}_j] = -\varepsilon_{ijk}\mathbf{t}_k$.
- AH, Taub–NUT, $\mathbb{CP}^2$ are $SO(3)$ invariant. Pick $SO(2) \subset SO(3)$.

$$U(r, \psi, \theta) = \exp\Big(-\pi \sum_{j=1}^{3} f_j(r) n_j \mathbf{t}_j \Big),$$

where $\mathbf{n} = (\cos\psi \sin\theta, \sin\psi\sin\theta, \cos\theta)$ and

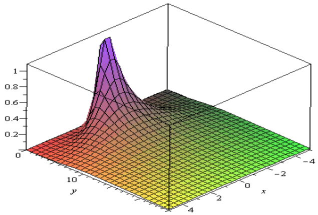
$$f_1 = f_2 = -\frac{r}{r+m}, \quad f_3 = \frac{r(r+2m)}{(r+m)^2}.$$

Geometry of Taub–NUT skyrmion

- Skyrmion on three–space $(\mathcal{B}, h) = M/SO(2) \cong \mathbb{H}^2 \times S^1$

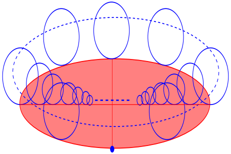
$$h = h_{\mathbb{H}^2} + R^2 d\psi^2, \quad R : \mathbb{H}^2 \to \mathbb{R}.$$

- Upper half plane: $h = \frac{dx^2+dy^2}{y^2}, R^2 = \frac{x^2+y^2}{y^2(m^{-1}\sqrt{x^2+y^2}+1)^2+x^2}$.
- Skyrmion density on the upper half plane, $y_{max} = 5m/4$.



- Poincare disc model $\mathbb{D} \to \mathbb{H}^2$

$$x + iy = \frac{z - i}{iz - 1}, \quad S^1 \to \mathcal{B} \to \mathbb{D}$$



- Ricci scalar (Fig. 4a) of $(\mathcal{B}, h)$ and the radii of the circles (Fig. 4b).

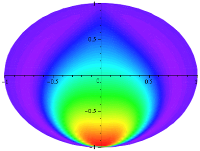


Fig 4a

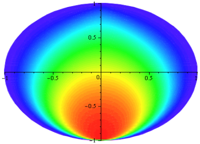
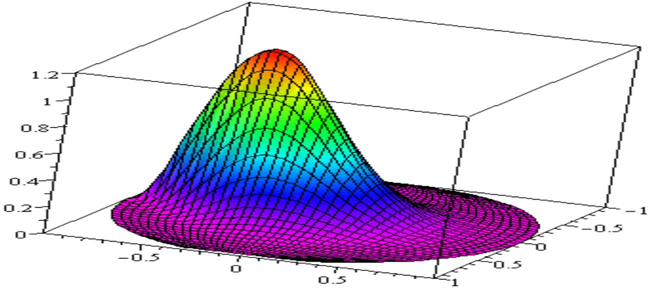


Fig 4b

Geometry of Taub–NUT skyrmion

Density of the Taub–NUT skyrmion in the Poincare disc model.



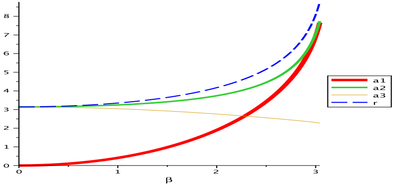
- $SO(3)$ invariant metrics. Let $a_i = a_i(r)$, and $d\eta_1 = \eta_2 \wedge \eta_3$ etc.

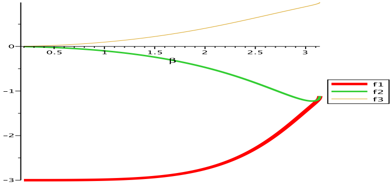
$$g = (a_2/r)^2 dr^2 + a_1 {\eta_1}^2 + a_2 {\eta_2}^2 + a_3 {\eta_3}^2,$$

- Self-dual spin connection

$$A = f_1(r)\eta_1 \otimes \mathbf{t}_1 + f_2(r)\eta_2 \otimes \mathbf{t}_2 + f_3(r)\eta_3 \otimes \mathbf{t}_3.$$

- Topological degree
  1. $B_{TN} = 2$ (integration).
  2. $B_{AH} = 1$. Set $r = 2\int_0^{\pi/2} \sqrt{1 - \sin{(\beta/2)}^2 \sin{(\tau)}^2}^{\,-2} d\tau$.





Outlook

- Solitons (vortices, skyrmions) induced by the geometry of space–time.
- Topological charges (vortex number, baryon number)= Characteristic classes the Levi–Civita connection (Euler characteristic, Hirzebruch signature).
- Explicit solutions.
- Particles as four–manifolds.

Thank You.