On the Structure of Path Geometries and Null Geodesics in General Relativity

Stephen Casey[1]

Department of Applied Mathematics and Theoretical Physics,

University of Cambridge,

Wilberforce Road, Cambridge CB3 0WA, UK.

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[1]**Email:** sc581@cam.ac.uk

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Chapter 1

Introduction

Since the late 19th century, the study of differential equations has been underpinned by the ideas of geometric formalism. Associating a system of differential equations with a manifold structure allows us to view problems in a more intuitive way. Obstructions to solvability, for example, can be linked with notions of curvature or torsion on some geometric structure. Central to this paradigm evolution are the contributions of Lie on continuous symmetry groups and those of Cartan on exterior differential systems. The existence of invariance and symmetries provides us with substantial information on the integral curves or surfaces of a given system and so these ideas are of fundamental importance in the geometric approach. Mathematical physicists will have encountered the strength of this framework whether it be the symplectic structure that arises from Hamiltonian mechanics, Frobenius Theorem on solutions to PDE systems or, more recently, Penrose's twistor correspondence.

A *path geometry* on some open domain $U \subset \mathbb{R}^n$ is a set of unparametrised paths with the condition that there is exactly one passing through each point in any given direction. Regarding these curves as a solution to a system of $n-1$ second order ordinary differential equations (ODEs), one can give an alternative definition of the path geometry as an equivalence class of systems of second order ODEs. Two systems are considered equivalent if, locally, they can be mapped into each other by a change of dependent and independent variable. Such geometries play a broad and important role in mathematical physics where, for example, they arise as the trajectories of free particles moving under the influence of some force in some background metric. The general theory was first formalised by Douglas in [1] who showed how to construct the second order system representing an arbitrary path geometry.

$$\frac{d^2x^a}{dt^2} = f^a\left(t, x^b, \frac{dx^b}{dt}\right) \quad , \quad a, b = 1, \ldots, n.$$

One natural question that arises in this theory is, for a given system, does there exist a point transformation $x^i = x^i(\tilde{x}^j)$ and a reparametrisation $t = t(\tilde{t})$ such that

$$\frac{d^2\tilde{x}^a}{d\tilde{t}^2} = 0 \ , \ \ a = 1, \ldots, n$$

that is, the system is *trivial* for some choice of local coordinates and the integral curves correspond to straight lines? The answer to this question comes down to the construction of a set of invariants for an equivalence class of systems, the vanishing of which correspond to the given system being equivalent to a trivial one. This invariants can naturally be divided into two groups or *branches*, which we term as the *projective branch* and the *conformal branch*. The vanishing of the invariants in a given branch corresponds to the system having some extra geometric structure and we shall explore this complementary pair of structures in detail. In both cases, the underlying framework has important applications in mathematical physics.

In the first instance, we will consider path geometries which fall into the projective branch of the theory. The invariants associated to this aspect are termed *Fels* or *Wilczynski* invariants. This branch is so called for its relevance in the theory of projective geometry, a subject which traces its origins back to the fourth century when Pappus of Alexandria formulated his theorem concerning triplets of collinear points in the plane [2]. The beauty of this theorem is that it makes no reference to compass constructions, angles or lengths of any kind and so is invariant under projection. The subject was later given more attention by Desargues who introduced the notion of parallel lines meeting at a "point at infinity" [3] - a concept which allows for the modern construction of projective space in terms of homogeneous coordinates. Using the ideas of projection, he developed a number of theorems on the theory of conic sections and influenced later work by Pascal in this area. Projective geometry then remained relatively untouched until the 19th century when it was subjected to a more axiomatic approach. During this period, much time was spent on the study of geometric properties invariant under projective transformations, the simplest of which is the *cross-ratio*. Any triple of points in $\mathbb{RP}^1$ can be taken to any other triple of points via a projective transformation. The same is not true for a quadruple of points. The cross-ratio is the unique projective numeric invariant defined on an ordered quadruple of points in $\mathbb{RP}^1$. If we identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$ then it can be written as

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$

for real numbers z_i. In the latter part of the 19th century, attention was given to the differential invariants of projective transformations by people such as Darboux, Wilczynski and Halphen and the ideas from this time keep reappearing in many areas of mathematics. Most working mathematicians will have encountered the Schwarzian derivative at some time in their research. This operator, although unusual in appearance, occupies a very natural position in the framework of projective differential geometry (a notion which will be discussed later). A large amount of work at this time was due to Cartan who generalised the idea of an affine connection and, in particular, introduced the notion of the *projective connection* [4]. Much like the affine connection the projective connection also defines geodesics. These geodesics are not, however, affinely parametrised and so give us a purer notion of a path in $\mathbb{R}^n$, independent of parametrisation.

Here, we will concern ourselves with a problem attributed to Roger Liouville [5] which can be stated as follows:

Liouville's Problem

Given an open set $U \subset \mathbb{R}^n$ and a family of curves on U such that there is exactly one curve passing through each point in any given direction, can one find a metric on U whose unparametrised geodesics coincide with these curves?

It is here that we will use the structure endowed by the projective connection. Beginning with Douglas's prescription of path geometries in terms of systems of second order ODEs we obtain the unparametrised form

$$\frac{d^2y^i}{dx^2} = F^i\left(x, y^j, \frac{dy^j}{dx}\right) \qquad i, j = 2, \dots, n,$$

we outline necessary and sufficient conditions for a system of paths to arise as the unparametrised geodesics of some symmetric affine connection. To this end, we have the set of invariants, when $n \geq 3$,

$$S^i_{qrs} = F^i_{qrs} - \frac{6}{2n+2} F^m_{m(qr}\delta^i_{s)}.$$

which vanish if and only if the path geometry coincides with the set of geodesics of a projective connection (lower indices here represent differentiation with respect to the $\frac{dy^i}{dx}$ terms). This result is originally due to Fels [6] and, here, we give a short description of how to approach it. We say that two connections are *projectively equivalent* if they give rise to the same geodesics as unparametrised curves. Liouville outlined an algorithm for constructing explicit local obstructions to a projective equivalence class containing the Levi-Civita connection of some metric when $n = 2$ and, in this case, the necessary and sufficient conditions were found in [7]. In what follows, we use a similar procedure to construct local obstructions to metrisability for $n \geq 3$.

Following the work of Eastwood and Matveev [8], we show that the question of metrisability can be equated with the existence of solutions to a set of first-order linear homogeneous PDEs of the form

$$(\nabla_a \sigma^{bc})_0 = 0$$

where $(\dots)_0$ denotes the trace-free part and ∇_a is the *projective connection.* Using the procedure of *prolongation* to produce a closed system of Cauchy-Frobenius form, we can derive necessary conditions, algebraic in σ^{ab} for our system of second order ODEs to be metrisable. We analyze some interesting examples of systems of second order ODEs in dimension 3 and determine the conditions for their integral curves to arise as the unparametrised geodesics of a metric.

In 1865, Beltrami [9] discovered that two non-proportional metrics could give rise to the same unparametrised geodesics. In particular, he proved that, if g and $\tilde{g}$ are two Riemannian metrics on an n-dimensional manifold U then g having constant sectional curvature implies that $\tilde{g}$ has constant sectional curvature. The dimension of the space of metrics which give rise to the same unparametrised geodesics as g is termed the *degree of mobility* of g. We define a metric, g, to be *geodesically rigid* if its degree of mobility is 1. Essentially, this means that the only projectively equivalent metrics to g are those of the form cg where c is a constant. In particular, using the prolonged system, we will show that

Proposition 1.1. *The metrics Nil (3.24), Sol (3.27) and the generalised Berger sphere (3.33) ($a \neq 1$) are all geodesically rigid.*

Levi-Civita showed [10] that for all pairs of projectively equivalent non-proportional Riemannian metrics g and $\tilde{g}$, there exists some local coordinates such that g and $\tilde{g}$ take a specific form. Explicitly, he found a local normal form for all Riemannian metrics of degree of mobility 2. We shall discuss geodesic rigidity in terms of Levi-Civita's characterisation and show that our algorithmic procedure reproduces his result for a special case. The extensions of these normal forms has been recently extended to the pseudo-Riemannian case in [11] and [12]. These results allow us to address the following problem

Problem 1.2. *Given two projectively equivalent metrics g and $\tilde{g}$ such that g is conformally flat, under what circumstances is $\tilde{g}$ also conformally flat?*

This question is an attempt to generalize Beltrami's observation of the projective invariance of the condition that a metric has constant curvature. It has been considered in dimension four in [13] in terms of the holonomy of the connection. Here, we shall consider it in dimension three and place our answer in

the framework of degree of mobility.
We shall also frame the metrisability problem in a more geometric context using the technology of *tractor bundles*. The tractor technology is a useful framework for any type of problem involved with the identification of an underlying Cartan geometry as it concerns itself with an associated vector bundle of the Cartan connection. The method is due to Thomas [14] who also developed the theory in the conformal case [15] and a detailed modern exposition of this framework was discussed in [16]. The work herein, for the projective case, will be based largely on review material of the formalism used by Eastwood [17]. In particular, we have the following:

Proposition 1.3. *Covariantly constant sections of the tractor bundle with respect to the connection defined by (3.17) are in one-to-one correspondence with solutions of the equation (3.5).*

This will allow us to comment on the dimension of the space of solutions and the notion of geodesic rigidity.

The other set of path geometries which will be of interest in this work are those in the conformal branch. These systems of second order ODEs are characterized by the vanishing of the so called *Grossman invariants* which are complementary to those of Fels in describing systems locally diffeomorphic to a trivial one. The moduli space of solutions of path geometries in the conformal branch admit a Segré structure, as detailed in [18]. In our description of the path geometry, we will set up a correspondence between the domain $U \subset \mathbb{R}^n$ and the moduli space of solutions M reminiscent of Penrose's twistor correspondence. The case of particular interest will be systems in three dimensions. Here, the solution space M has dimension four and admits a natural conformal structure, where points of U correspond to totally isotropic two-dimensional surfaces in M, if and only if the Grossman invariants vanish. Additionally, the conformal structure will be anti-self-dual (ASD). We will often refer to such systems as *torsion-free* [18]. This work will be reminiscent of Penrose's non-linear graviton construction [19]. The idea here being that solutions to differential equations arising in mathematical physics may be encoded as natural structures in complex analytical geometry. In fact, the correspondence between a conformal structure of special type on M and the normal bundles of the *twistor lines* on the associated *twistor space* is just the complexified version of our picture and we will discuss the link in detail. It is now well understood, from the viewpoint of complexified twistor theory, how a conformal manifold $(M, [g])$ endowed with an additional structure such as Ricci-flatness or an Einstein metric may give rise to additional properties of the normal bundle on the associated twistor lines. However, if these curves are described by a system of second order ODEs (a path geometry) it is generally not known how this system may appear. In this respect, we have the following important result, proved in [20]:

Proposition 1.4. *If a given conformal structure $[g]$ on M contains a metric that is Ricci-flat then the corresponding torsion-free path geometry which describes the set of twistor lines is of the form*

$$y'' = 2\frac{\partial \Lambda}{\partial z'} \ , \ z'' = -2\frac{\partial \Lambda}{\partial y'}$$

where Λ is a special function determined by a solution of the heavenly equation and the conformal structure.

Every point symmetry of a given torsion-free path geometry gives rise to a symmetry of the conformal structure and vice versa. In Chapter 4, we will exploit this correspondence to construct ODE systems and conformal structures with a large number of symmetries (dimension 9 being the submaximal case). We will describe how, given an ASD conformal structure of signature (2,2), to construct a system of ODEs on the twistor space U whose integral curves are the twistor lines. Using this framework, we present an alternative derivation for the Grossman invariants assuming solely that the space of solutions be endowed with a conformal structure. We shall show how this construction proceeds in the Ricci-flat case and demonstrate that, in this case, the corresponding torsion-free system can be read off directly from the solution of Plebanski's heavenly equation. We shall explore the local isomorphism between groups of point symmetries of the path geometry and conformal symmetries of the corresponding conformal structure and use this to construct examples of systems with symmetry algebras of between dimension nine and two. A result of central importance here is the following,

Proposition 1.5. *Consider a torsion-free path geometry described by a system of two second order ODEs of the form*

$$y'' = 0 \ , \ \ z'' = B(y') = \sum_{k=0}^{\infty} \xi_k (y')^k.$$

If B is a quadratic function, then this system is also in the projective branch and diffeomorphic to a trivial one. Otherwise, the symmetry algebra has dimension six, seven or nine depending on the coefficients ξ_k.

At the end of Chapter 4, we shall explore the connection between systems of ODEs with vanishing Grossman invariants and unparametrised geodesics of Finsler structures of scalar flag curvature. This correspondence gives rise to yet another class of examples of torion-free systems of ODEs on U and hence, ASD conformal structures on the moduli space of solutions M. Some of these examples have gravitational analogues in the theory of plane-wave spacetimes, and the whole construction sheds new light on variational aspects of the Nonlinear Graviton Theorem.
The projective equivalence of metrics has seen many recent applications in the field of General Relativity. Four dimensional Lorentzian metrics which satisfy Einstein's equations provide accurate depictions of black holes and the timelike and null geodesics of such metrics correspond to the paths of freeling falling particles. Hence, one might hope to be able to reconstruct some essence of the local geometry of some patch of spacetime (perhaps the projective equivalence class of the Levi-Civita connection) by using the paths of particles. A suggestion of how this might be done experimentally was given in [11]. If, locally, the degree of mobility of a given spacetime metric is greater than one, it may happen that the metric structure cannot be pinpointed exactly, given

the equations governing the unparametrised geodesics. In [21], it was shown that a Friedmann-Lemaitre-Robertson-Walker metric with pure dust without a cosmological constant Λ was projectively equivalent to one with Λ, giving rise to the suggestion that dark energy is just a manifestation of the "wrong" choice of metric in a given projective equivalence class. However, it has been demonstrated in [22] that non-projectively invariant quantities in cosmology are also used to determine the local geometry, thereby making the cosmological constant physically meaningful.

A less obvious application of the notion of projective equivalence arises in the theory of static metrics. When trying to interpret the physical properties of a spacetime, of fundamental importance is the behaviour of null geodesics as these correspond to the trajectories of light rays. The majority of measurements made of the universe consist of observations of electromagnetic waves emitted in the distant past. The behaviour of light rays as they bend around the sun gave the first observational evidence for General Relativity and such gravitational lensing continues to be a significant branch of astronomy. For a given $n+1$-dimensional static Lorentzian spacetime, there is an n-dimensional Riemannian metric which can be constructed on the space of orbits of the time-like static Killing vector K which is called the optical metric. Null geodesics of the full spacetime g project down (along the time direction) to unparametrised geodesics of the optical metric h and the geodesic structure of h can be used to infer something about the conformal structure of g.

For example, the Schwarschild-deSitter solution, which describes an uncharged non-rotating black hole of mass m with cosmological constant Λ, can be written in local coordinates as

$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and admits static Killing vector $K = \frac{\partial}{\partial t}$. The optical metric associated to K on the space of orbits may be written as

$$h = \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^2} + \frac{r^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}(d\theta^2 + \sin^2\theta d\phi^2).$$

It was noted in [23] that the unparametrised geodesics of h are independent of Λ i.e, for two different values of Λ, the metrics h are projectively equivalent. It follows that the equations governing the dynamics of light rays in g are also invariant of the cosmological constant.

The idea of the projective equivalence of optical metrics is central to the work of Chapter 5. It is possible for a given metric to admit more than one timelike static Killing vector. Clearly, in this case, the definition of the optical metric is not unique - there will be different optical metrics associated to different hypersurface-orthogonal (HSO) timelike Killing vectors. Such metrics are termed *multistatic* and as to the general form of such metrics, we prove the following

Proposition 1.6. *Any generic multi-static metric is locally a warped product metric on $M = S_0 \times S_1$ given by*

$$g = e^{w}\gamma_0 + \gamma_1 \tag{1.1}$$

where (S_0, γ_0) is a two-dimensional Lorentzian manifold of constant curvature, (S_1, γ_1) is a two-dimensional Riemannian manifold and $w : S_1 \to \mathbb{R}$ is an arbitrary function.

If two Riemannian metrics h and $\tilde{h}$ arise as the optical metrics associated to different Killing vectors K and $\tilde{K}$ of a Lorentzian multistatic spacetime g, we say that h and $\tilde{h}$ are *optically equivalent*. A more formal definition is given herein and we will discuss the connection between optical equivalence and projective equivalence in Chapter 5. In particular, we have the following

Proposition 1.7. *If the curvature of γ_0 in (1.1) is non-zero, then the optical metrics associated with any two static Killing vectors are projectively equivalent. However, if γ_0 is flat then it is possible to construct examples where optically equivalent metrics are not projectively equivalent.*

So, optical equivalence does not imply projective equivalence, in general, but only does so under certain special circumstances.

The notion of the projective equivalence of optical metrics may also give rise to a correspondence picture between the path structure of seemingly unrelated geometries. This has seen some recent application in the theory of generalised higher-dimensional black holes. In [24], the authors consider the properties of null geodesics in Schwarzschild-Tangherlini spacetimes of $n+1$ dimensions. Here, the projection of any such curve to the space of orbits of the timelike Killing vector lies in a plane and coincides with an unparametrised geodesic of some two-dimensional optical metric. It is found that the cases $n=3$ and $n=6$ may be related by a conformal mapping as discussed by Bohlin [25] and Arnold [26]. This begs the question as to whether optical 2-metrics describing projected null geodesics are projectively related and, if not, how may this relationship be described?

We will explore this *Bohlin-Arnold duality* in depth in this context and argue that it does not give rise to the projective equivalence of metrics. Instead, we find that

Proposition 1.8. *The entire set of geodesics determined by the one-parameter family of metrics*

$$g_3(m) = \frac{dr^2}{\left(1-\frac{2m}{r}\right)^2} + \frac{r^2}{1-\frac{2m}{r}} d\phi^2,$$

parametrised by mass m, (which correspond to the optical 2-metrics of the $n=3$ case) can be mapped to those determined by the one-parameter family

$$g_3(m) = \frac{dr^2}{\left(1-\frac{2m}{r^4}\right)^2} + \frac{r^2}{1-\frac{2m}{r^4}} d\phi^2,$$

associated to the $n=6$ case.

Furthermore, this common system of paths is determined as the set of integral curves of some unique third order ODE for $u(\phi)$ with $u=1/r$. We will analyse the role of the cosmological constant for these spacetimes, and the optical metric construction, and show explicitly why it does not effect the equations governing the dynamics of light rays. This result expands on the observation in [23] and hints at a general dissociation of the cosmological constant from the conformal structure of a given static solution of Einstein's equations.

For $n = 3$, the projection of a zero energy light ray to the space of orbits of the static Killing vector is a cardioid and for the $n = 6$ case, it is a lemniscate of Bernoulli. The Bohlin-Arnold mapping we formulate herein does not associate these two curves but we shall discuss how the zero energy solutions fit into the duality and note that a more general mapping allows a correspondence between the set of zero energy curves for any two values of n.

A natural extension of this construction is to static black hole solutions which are also charged i.e, the Reissner-Nordstrom analogue in n dimensions. We consider the possibility of a similar notion of duality for such spacetimes and find that the additional charge parameter provides a constraint too restrictive to be applied for all values of n. An interesting correspondence does arise here between the values $n = \frac{3}{2}$ and $n = -2$ which, although does not have a clear interpretation in terms of the dimensions of black hole spacetimes, can be interpreted as arising from the set of trajectories of free particles moving under the influence of a central force of a particular restricted form.

The general overview given here gives us some idea of the use of the optical metric structure for a given static spacetime when considering the properties of its light rays. One may deduce several features about the conformal structure of a Lorentzian static metric from those of the geodesic structure of the optical metric which may be better understood. Recently, attempts have been made to generalise this geometric structure to spacetimes which do not necessarily admit a vector field which is static but which has some other particular property of interest. The null geodesics of the spacetime may then be projected to the space of orbits of this distinguished vector field for analysis where one may try to determine if the projected paths are then integral curves of some geometric structure on the hypersurface.

For example, if a given Lorentzian metric g is *stationary*, i.e, admits a timelike Killing vector field K, then there are two different structures on the space of orbits of K whose paths coincide with the projection of light rays. On one hand there is the set of unparametrised geodesics of a Finsler norm of Randers type which can be interpreted as the magnetic flow due to some one-form in the background of a curved metric. This Randers structure can be obtained by expressing the given stationary spacetime g in a particular set of coordinates. On the other hand, if we express g in *Painlevé-Gullstrand form* then we find that null geodesics also project down to the solutions of the Zermelo problem on the space of orbits of K with a specified background metric γ and wind vector $\mathbf{W}$. This problem can be described as follows:

Zermelo's Problem: Given a Riemannian metric γ on a manifold $\mathcal{B}$, what is the least time trajectory for a ship moving with constant speed in a wind $\mathbf{W}$?

Hence, there is a triality of structures in this picture where geometric properties of one can be deduced from those of others. A clear exposition of this picture is given in [23] and we shall also discuss some particular properties in Chapter 6.

We may also generalise this idea of the optical structure to metrics which admit a *timelike conformal retraction* Θ. In this case, the vector field has the property that the conformal structure on the space of orbits of Θ is preserved along its integral curves. In general, it is difficult to expose to what the projection of light rays along the direction of Θ may correspond and there is no "good choice" of local coordinates as in the static or stationary cases. In the Riemannian case, such metrics have arisen as supersymmetric solutions of the minimal $\mathcal{N} = 2$ gauged supergravity with anti-self-dual Maxwell field [27]. Moreover, when the anti-self-duality condition is relaxed in the case of positive cosmological constant, one obtains a solution, also admitting a conformal retraction, which is the Riemannian analogue of the well-known *Kastor-Traschen* metric [28],

$$g = -\frac{dT^2}{(V + cT)^2} + (V + cT)^2 h$$

where h is a Riemannian metric (which we take to be flat for simplicity), with coordinates x^a, $V = V(x^a)$ is a harmonic function and c is constant.

In Lorentzian signature, this metric is a time-dependent solution to the Einstein-

Maxwell equations which can be seen to describe an arbitrary number of dynamical charged black holes in a deSitter background. Here, we will use it as an analogue model in order to gain insight into the optical structure associated to spacetimes which admit a timelike conformal retraction. We will focus on the null geodesic structure and, in particular, it is found that light rays project down to the integral curves of a set of third order ODEs on the space of obrits of Θ. In particular, we find that the projected null geodesics form only a subset of the set of integral curves and this yields a freedom in how this third order system is defined.

We pay particular attention to the one-centre solution (a single black hole). In the limit as the cosmological constant tends to zero, our system of ODEs becomes that describing conformal circles of the flat metric (as described in [29]). We use this result to motivate the choice of ODE system to describe the projection of light rays and discuss the underlying numerology of the problem. The advantage of this new system is that it will allow us to give a physical interpretation to those integral curves which do not arise as projected null geodesics, coinciding with motion in the background magnetic field. This formulation also allows us to characterize those integral curves which are the projected light rays. The central result of this work is the following

Proposition 1.9. *If $c \neq 0$, the retraction projection of the set of null curves satisfying (6.14) for some value of λ coincides with the set of integral curves of*

$$\begin{aligned}\dddot{\boldsymbol{x}} &= -|\ddot{\boldsymbol{x}}|^2\dot{\boldsymbol{x}} - \frac{3}{2}\frac{\ddot{\boldsymbol{x}}.\nabla V}{|\dot{\boldsymbol{x}}\times\nabla V|^2}(\dot{\boldsymbol{x}}.\nabla V)\ddot{\boldsymbol{x}} - 2c(\dot{\boldsymbol{x}}\times(\nabla V\times\dot{\boldsymbol{x}}))\\ &+ \frac{\ddot{\boldsymbol{x}}.\nabla V}{|\dot{\boldsymbol{x}}\times\nabla V|^2}\left(\dot{\boldsymbol{x}}\times\left(\frac{d\nabla V}{ds}\times\dot{\boldsymbol{x}}\right)\right) + \frac{\ddot{\boldsymbol{x}}.(\dot{\boldsymbol{x}}\times\nabla V)}{|\dot{\boldsymbol{x}}\times\nabla V|^2}\Big[\left(\frac{\ddot{\boldsymbol{x}}.\nabla V}{|\dot{\boldsymbol{x}}\times\nabla V|^2}\right)(\dot{\boldsymbol{x}}.\nabla V)\ddot{\boldsymbol{x}}\times\nabla V\\ &+ ((\dot{\boldsymbol{x}}\times\nabla V).\ddot{\boldsymbol{x}})\dot{\boldsymbol{x}} + \frac{d}{ds}(\dot{\boldsymbol{x}}\times\nabla V)\Big].\end{aligned} \tag{1.2}$$

Furthermore, the projected null geodesics are precisely the integral curves of this system for which $\ddot{\boldsymbol{x}}.(\dot{\boldsymbol{x}}\times\nabla V) = 0$.

In the one centre case, there is a diffeomorphism which takes the Kastor-Traschen metric to that of the Reissner-Nordstrom deSitter metric with charge equal mass in some local set of coordinates. Analytic descriptions of the null geodesics for this spacetime are well documented, see for example [30]. This allows us to determine the projected null geodesics for Kastor-Traschen analytically. We give plots of some of these curves and discuss the horizon structure in both sets of coordinates.

Solutions of the Einstein-Maxwell equations of Kastor-Traschen correspond to physical situations when the harmonic function V is a sum of potentials inversely proportional to the distance from some fixed point i.e,

$$V = \sum_{\alpha=1}^{N} \frac{m_\alpha}{|\mathbf{x} - \mathbf{a}_\alpha|}.$$

The realisation of this solution is a set of N charge equal mass black holes with fixed centres and dynamic event horizons. We shall expand our investigation to the case when $N = 2$ where, for special initial conditions, the projection along the conformal retraction of a null geodesic will lie in a plane. For one such curve, we illustrate a connection between the null geodesics of the two-centre Kastor-Traschen metric and a third order system that arises in the analysis of the one-centre case. We also look at the perturbations away from this plane and give a strict condition for stability, the kind of calculation relevant from a physical point of view.

The work presented here gives some insight into the active area of path geometries particularly in the context of projective and conformal structures and the applications of such ideas to general relativity. There are many open problems presented herein, some of which will be discussed at the end, which make it a fruitful area for further research.

Chapter 2

Path Geometry

2.1 Description of the Path Geometry

Let us begin with an open domain U in $\mathbb{R}^n$ for $n \geq 3$. Henceforth, all results quoted will be for $n \geq 3$ and where appropriate, the corresponding results for $n = 2$ will be mentioned. Let x^a ($a = 1, \ldots, n$) be local coordinates on U, with (x^a, p^a) the corresponding coordinates on the tangent bundle TU. Then we define a *path geometry* on the domain U as a family of curves such that, given any $(x^a, p^a) \in TU$, there is exactly one curve on U passing through x^a in the direction p^a. A result of Douglas [1] tells us that a path geometry may be represented by a system of second order differential equations on U of the form

$$\frac{d^2x^a}{dt^2} = f^a\left(x^b, \frac{dx^b}{dt}\right) \quad a, b = 1, \ldots, n \tag{2.1}$$

where t is a parameter along each curve and the functions f^a are homogeneous of degree two in $\frac{dx^b}{dt}$. By construction, such a family of curves defines a flow on the tangent bundle TU which naturally gives rise to a spray

$$X = p^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial p^a}$$

We will say that two such sprays are *equivalent* if they give rise to the same family of unparametrised curves on U. Given that a curve in the path geometry passing through a point is fully described by the direction of the tangent vector at that point, then flows on TU determined by equivalent sprays project to the same foliation of the projective space $\mathbb{P}(TU)$. Hence, two sprays X, $\tilde{X}$ give rise to the same unparametrised curves on U if they differ by a multiple of the Euler vector field $(p^a \frac{\partial}{\partial p^a})$. This projection induces a geodesic spray Θ, the integral curves of which define a foliation by one-dimensional manifolds of the projectivised tangent bundle. Hence, we can think of $\mathbb{P}(TU)$ as a fibre bundle over some $(2n$-$2)$-dimensional space M with one-dimensional fibres given by these integral curves. This manifold M can be thought of as the space of solutions and, together with the canonical fibration of $\mathbb{P}(TU)$ over U, we obtain the double fibration picture

$$U \longleftarrow \mathbb{P}(TU) \longrightarrow M \tag{2.2}$$

which sets up a useful correspondence between M and U. In particular, points in M correspond to paths in U and points in U correspond to $(n$-$1)$-dimensional surfaces in M. We shall consider this correspondence in more detail later.

If we wish to discuss a purer notion of path geometry, independent of parametrisation, then Douglas's description (2.1) is inadequate but, due to the homogeneity properties of the functions f^a, we may eliminate the parameter t using the chain rule. To this end, let us rewrite the coordinates on U as $x^a = (x, y^2, \ldots, y^n) = (x, y^i)$ [**NB** I will use the convention here that roman indices from the start of the alphabet (e.g, $a, b, c, \ldots$) take values $1, \ldots, n$ whereas those from the middle of the alphabet (e.g, $i, j, k, \ldots$) take values $2, \ldots, n$]. Then, we have

$$\frac{dy^i}{dt} = \frac{dy^i}{dx}\frac{dx}{dt} \quad , \quad \frac{d^2y^i}{dt^2} = \frac{d^2y^i}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy^i}{dx}\frac{d^2x^i}{dt^2}$$

and the system (2.1) becomes

$$\frac{d^2y^i}{dx^2} = F^i\left(x, y^j, \frac{dy^j}{dx}\right), \qquad i, j = 2, \ldots, n \tag{2.3}$$

for some functions F^i. The system (2.3) is precisely what we need to give us a complete description of a path geometry in n dimensions and will henceforth be the main reference for establishing a geometric structure on U.

One of the most natural problems that arises in this area of study, and one which is of particular importance to how we proceed is the following

Problem Given a path geometry described by a system of second order ODEs (2.3), under what conditions does there exist a diffeomorphism $(x, y^i) \to (\tilde{x}, \tilde{y}^i)$ such that we have

$$\frac{d^2\tilde{y}^i}{d\tilde{x}^2} = 0 \ , \ \ i = 2, \ldots, n?$$

The answer to this question is well known and corresponds to the vanishing of a set of invariants which can be divided into two distinct groups. In one group, we have, what we shall name, the *Fels* invariants

$$S^i_{qrs} := F^i_{qrs} - \frac{6}{2n+2} F^m_{m(qr}\delta^i_{s)} \tag{2.4}$$

where, here, the lower indices on F indicate differentiation with respect to $p^i = \frac{dy^i}{dx}$.

Furthermore, if we define

$$T^i_j = -\frac{\partial F^i}{\partial y^j} - \frac{1}{4}\frac{\partial F^i}{\partial p^k}\frac{\partial F^k}{\partial p^j} + \frac{1}{2}\frac{d}{dx}\frac{\partial F^i}{\partial p^j},$$

then the other group comprises the *Grossman invariants* defined as

$$\tau^i_j := T^i_j - \frac{1}{n-1}\delta^i_j T^k_k \ , \ \ i, j, k = 2, \ldots, n. \tag{2.5}$$

A given path geometry (2.3) is then diffeomorphic to a trivial one if and only if

$$S^i_{qrs} \equiv 0 \ \text{and} \ \tau^i_j \equiv 0.$$

The set of invariants are naturally and conspicuously divided into two groups and vanishing of each of these groups separately selects a special subclass of three-dimensional path geometries. Let us say that a given path geometry belongs to the *projective branch* of the theory if the Fels invariants vanish. On the other hand, path geometries with vanishing Grossman invariants are said to lie in the *conformal branch*. Clearly, the only systems of second order ODEs (2.3) belonging to both branches are those diffeomorphic to the trivial system. The path geometries in each branch are endowed with some extra geometric structure which is important to mathematical physics and differential geometry in general and which we shall discuss in detail here.

2.2 Path Geometries of a Projective Structure

One important instance of a path geometry which will form a core part of this work is the subset of systems of parametrised differential equations of the form

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}(x)\frac{dx^b}{dt}\frac{dx^c}{dt} = \nu\left(\frac{dx^b}{dt}\right)\frac{dx^a}{dt}, \tag{2.6}$$

where ν is a function homogeneous of degree 1 in its arguments and $\Gamma^a_{bc} = \Gamma^a_{(bc)}$. The integral curves of this system coincide with the unparametrised geodesics of a symmetric affine connection Γ^a_{bc}. Furthermore, this system satisfies the homogeneity properties of (2.1) outlined by Douglas and hence, gives rise to a path geometry on U. Thus, saying that two connections Γ, $\tilde{\Gamma}$ give rise to the same unparametrised geodesics is equivalent to saying that the corresponding sprays on TU differ by a multiple of the Euler field. This equivalence may be reformulated, see [17], as

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b\Upsilon_c + \delta^a_c\Upsilon_b \tag{2.7}$$

where $\Upsilon = \Upsilon_a dx^a$ is a 1-form on U. We say that two connections are *projectively equivalent* if they give rise to the same unparametrised geodesics i.e, if (2.7) holds for some 1-form Υ. This equivalence relation splits the set of connections into equivalence classes which we term *projective structures*. Thus, an pertinent problem to address is the following:

Problem 2.1. *Given a path geometry described by the system of differential equations (2.3), determine conditions on the functions F^i such that the integral curves of (2.3) coincide with the unparametrised geodesics of some torsion-free connection on TU.*

This problem was solved by Fels [6] and here we give a brief outline of how to derive these conditions. If we begin with equation for unparametrised geodesics of a torsion-free connection Γ^a_{bc} (2.6) and eliminate the parameter t using the chain rule, the system reduces to one of $n-1$ second order ODEs of the form (2.3) with

$$\frac{d^2y^i}{dx^2} = F^i = \frac{dy^i}{dx}A_{jk}\frac{dy^j}{dx}\frac{dy^k}{dx} + B^i_{jk}\frac{dy^j}{dx}\frac{dy^k}{dx} + C^i_k\frac{dy^k}{dx} + D^i \ , \ i,j,k = 2,\dots,n \tag{2.8}$$

where

$$\begin{aligned} A_{jk} &= \Gamma^1_{jk} \\ B^i_{jk} &= \delta^i_j\Gamma^1_{1k} + \delta^i_k\Gamma^1_{1j} - \Gamma^i_{jk} \\ C^i_k &= \delta^i_k\Gamma^1_{11} - 2\Gamma^i_{1k} \\ D^i &= -\Gamma^i_{11}. \end{aligned}$$

This gives a sufficient condition for the integral curves of (2.3) to be the unparametrised geodesics of some connection. This condition can be written more succinctly as

$$F^i = A_{jk}p^ip^jp^k + B^i_{jk}p^jp^k + C^i_kp^k + D^i \tag{2.9}$$

where $p^i = \frac{dy^i}{dx}$, as before (we note here that A, B, C and D, defined by (2.9) are invariant under (2.7) i.e, these quantities are uniquely defined by a given projective structure). In fact, we can make an even stronger statement,

Theorem 2.1. *Given a system of ODEs of the form (2.3), its integral curves are the unparametrised geodesics of some connection if and only if there exist* A_{jk}, B^i_{jk}, C^i_k *and* D^i *such that (2.9) holds for all* $i = 2,\dots,n$.

Proof. The "only if" case has already been shown.

To prove the "if" case, it suffices to show that there exists a connection Γ such that the relations in (2.9) are satisfied. This is a convenient place to introduce the *projective connection*, defined by Thomas [31] as

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{n+1}\delta^a_b\Gamma^d_{dc} - \frac{1}{n+1}\delta^a_c\Gamma^d_{db}. \tag{2.10}$$

It can easily be checked that this definition is invariant under the transformation (2.7). Even more, the projective connection is uniquely defined by its projective structure and so, we will often use $[\Pi]$ to denote such an equivalence class.
Let us construct Π as follows,

$$\begin{aligned}
\Pi^i_{jk} &= \frac{1}{n+1}B^m_{mk}\delta^i_j + \frac{1}{n+1}B^m_{mj}\delta^i_k - B^i_{jk}, \\
\Pi^i_{j1} &= -\frac{1}{2}C^i_j + \frac{1}{2n+2}\delta^i_j C^m_m, \\
\Pi^i_{11} &= -D^i, \\
\Pi^1_{jk} &= A_{jk}, \\
\Pi^1_{j1} &= \frac{1}{n+1}B^m_{mj}, \\
\Pi^1_{11} &= \frac{1}{n+1}C^m_m. \qquad (2.11)
\end{aligned}$$

Then Π is a connection satisfying the relations (2.9). Hence, the unparametrised geodesics of the connection Π can be written in the form (2.9). □

We would prefer to reformulate this proposition in some coordinate invariant manner and, to this end, we have the following

Theorem 2.2. *A given path geometry on $U \subset \mathbb{R}^n$ corresponds to the set of unparametrised geodesics of some symmetric affine connection if and only if the Fels invariants (2.4) vanish that is, the path geometry lies in the projective branch* [1]*.*

2.3 Segré Structure on the Space of Solutions

On the other hand, we may consider the path geometries which lie in the conformal branch of the theory, i.e, the Grossman invariants (2.5) vanish. In this instance, we can exploit the double fibration picture (2.2). If a given path geometry on U is torsion-free then it endows the moduli space of solutions M with a *Segré structure*. The details of this correspondence were discussed in detail in

[1] If $n = 2$, i.e, a single ODE of the form $\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right)$, then the condition is $\frac{\partial^4 F}{\partial (y')^4} = 0$ where $y' = \frac{dy}{dx}$ - this result was known to Liouville [5]

[18] and we will only touch on the main features here.

Following from the work of Grossman, for positive integers k and m, let us define homogeneous coordinates Z^a_α (where $1 \leq a \leq m$, $1 \leq \alpha \leq k$) on the real projective space $\mathbb{RP}^{km-1}$. A *Segré variety* $\mathbb{S}(k-1, m-1)$ is then a subvariety of $\mathbb{RP}^{km-1}$ defined by the quadratic polynomial equations $Z^a_\alpha Z^b_\beta = Z^b_\alpha Z^a_\beta$. It is isomorphic to $\mathbb{RP}^k \times \mathbb{RP}^m$ and it carries a double ruling by subvarieties of dimension $k-1$ and $m-1$. There is a corresponding cone in the vector space over each point of $\mathbb{RP}^{km-1}$ defined by these quadratic polynomials called the *Segré cone*, $S(k, m)$. Then we have (see [18])

Definition 2.3. *A Segré structure of type (k, m) on a manifold $\mathcal{S}^{km}$ is a smoothly varying field of varieties $S_p(k, m) \subset T_p\mathcal{S}$ in the tangent spaces of $\mathcal{S}$, each linearly isomorphic to the Segré cone $S(k, m)$.*

Segré structures have been discussed in several different guises, with several different names. For a brief overview of the literature on this subject and some recent interesting work, see [32]. Given the canonical isomorphism above, we may define a *simple* vector as one of the form $\kappa \otimes \pi$ for $\kappa \in \mathbb{RP}^k$ and $\pi \in \mathbb{RP}^m$. Then, a plane spanned by simple vectors, tangent to a given point, with π fixed is called an α-plane (alternatively, the corresponding plane with κ fixed is termed a β-plane.) An immersed connected manifold $\Sigma \to M$ for which $T_p\Sigma$ is an α-plane for every point $p \in \Sigma$ is called a *proto-α surface* and is called an α-surface if Σ is maximal in the sense of inclusion.

Returning to the double fibration picture (2.2), it was shown in [18] that if a given path geometry on $U \subset \mathbb{R}^n$ has vanishing Grossman invariants then the moduli space of solutions M is endowed with a Segré structure of type $(n-1, 2)$ where each point in U corresponds to an α-surface of M. From the point of view of Twistor Theory, the most interesting case of this correspondence is when $n = 3$. Here, the definition of a Segré cone coincides with that of a null cone in the tangent space at a point and the Segré structure is a conformal structure. Thus, if a three dimensional path geometry on U described by a pair of second order ODEs is torsion-free then the space of solutions is endowed with a conformal structure which is fully defined by the set of two-dimensional α-surfaces. The torsion-free condition implies the existence of the maximal set of such α-surfaces each of which corresponds to a point in U.

The complexified version of this picture is reminiscent of some noteworthy twistor constructions of Penrose [19], Hitchin [33] and others. The idea of this procedure is to set up a duality between complex analytic geometry on one vector space and normal bundles of the corresponding twistor lines. We will discuss the twistorial point of view in more detail in Chapter 4 and we shall see how it fits into our framework.

Hence, there is some interesting geometric structure associated to each branch of the theory of path geometries and it is these structures that we use as the starting point for the rest of the discussion.

Chapter 3

Metrisability of Systems of Second Order ODEs

In the previous chapter, we determined a tensor S^i_{jkl}, the complete vanishing of which is a necessary and sufficient condition for the integral curves of (2.3) to be geodesics of some projective structure. We now pursue the problem of determining conditions for the geodesics of a given projective structure $[\Pi]$ to be compatible with those of a metric connection. In other words, we wish to determine necessary and sufficient conditions for the projective structure to contain the Levi-Civita connection of some metric g i.e,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{dc,b} + g_{bd,c} - g_{bc,d}) \tag{3.1}$$

for some $\Gamma \in [\Pi]$. It should be noted at this point that all our considerations are local and we are trying to determine local obstructions to metrisability. The problem of determining necessary and sufficient conditions for a given path geometry to coincide with the unparametrised geodesics of some metric is attributed to Roger Liouville [5] and was solved in dimension two in [7] - the authors determined a set of invariants for a projective structure, the vanishing of which coincide with the structure being metrisable.

So, let us assume that a given projective structure is metrisable. Then, using equations (3.1) and (2.10), we can write the projective connection in terms of the metric and it's first partial derivatives. This relationship is homogeneous but non-linear in general. Hence, we have defined a first order non-linear differential operator from the first jet space of symmetric two-forms on the tangent space of U to the space of projective structures on U which we write as follows:

$$\Omega_0 : J^1(S^2(T^*U)) \to J^0(Pr(U)). \tag{3.2}$$

In n dimensions the metric and its derivatives have a total of $\frac{n(n+1)^2}{2}$ components whereas the projective connection has $\frac{(n-1)(n)(n+2)}{2}$ components. This means that the map Ω_0 has kernel with rank $\frac{n(n+3)}{2}$. Hence, at zeroth order in the connection, it does not seem as though there should be an obstruction to a given projective structure being metrisable at a point. Differentiating the relations between the projective connection and the metric prolongs the operator Ω_0 to maps

$$\Omega_k : J^{k+1}(S^2(T^*U)) \to J^k(Pr(U)).$$

By calculating the rank of each jet space we can work out where the obstruction to metrisability occurs in dimension n. For $k = 1$ we calculate

$$\mathrm{rank}(J^2(S^2(T^*U))) = \frac{n(n+1)}{2}\left(1 + n + \frac{n(n+1)}{2}\right)$$

and

$$\mathrm{rank}(J^1(Pr(U))) = \frac{n(n-1)(n+2)}{2}(1+n).$$

Therefore, we get an obstruction to metrisability at first order whenever

$$\begin{aligned} \frac{n(n+1)}{2}\left(1 + n + \frac{n(n+1)}{2}\right) &\leq \frac{n(n-1)(n+2)}{2}(1+n) \\ \Rightarrow n + 1 + \frac{n(n+1)}{2} &\leq (n-1)(n+2) \\ \Rightarrow n &\geq 3 \end{aligned} \tag{3.3}$$

since $n + 2 \geq 0$. Thus, in dimension greater than two[1], there is already an obstruction to metrisability at first order in the connection which we may determine.

[1] $n = 2$ is really different in this sense as there are no obstructions up to order 5.

As stated previously, Ω_0 is nonlinear. It is possible, however, to express this map as a system of linear partial differential equations by choosing a new metric tensor

$$\sigma^{ab} = (\det(g))^{\frac{1}{n+1}} g^{ab}$$

or equivalently

$$g^{ab} = \det(\sigma)\sigma^{ab}. \tag{3.4}$$

This system may be written in the following form

$$(\nabla_a \sigma^{bc})_0 = 0 \tag{3.5}$$

where $(\ldots)_0$ denotes the trace-free part. We could also write this as

$$\nabla_a \sigma^{bc} - \frac{1}{n+1}\delta_a^b \nabla_d \sigma^{cd} - \frac{1}{n+1}\delta_a^c \nabla_d \sigma^{bd} = 0. \tag{3.6}$$

The covariant derivative here is that associated to the projective connection

$$\nabla_a \sigma^{bc} = \partial_a \sigma^{bc} + \Pi_{ta}^b \sigma^{tc} + \Pi_{ta}^c \sigma^{bt}.$$

In [8] the authors show that if ∇_a is a *special* torsion-free connection on the tangent bundle of an n-dimensional manifold U and there exists a metric tensor σ^{bc} satisfying (3.5) for this connection, then ∇_a is projectively equivalent to a metric connection - which turns out to be the Levi-Civita connection of the metric given by (3.4). The term special here means that there is a volume form on U, $\epsilon_{bc\ldots d}$ (unique up to scale) for which $\nabla_a \epsilon_{bc\ldots d} = 0$. As was noted in [17], the curvature tensor of any torsion-free connection ∇_a can be uniquely decomposed as

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = R_{ab}{}^c{}_d X^d = (W_{ab}{}^c{}_d + 2\delta^c_{[a} P_{b]d} + \beta_{ab}\delta^c_d) X^d$$

where $W_{[ab}{}^c{}_{d]} = 0$, $W_{ab}{}^c{}_d$ is totally trace-free and $\beta_{ab} = -2P_{[ab]}$. $W_{ab}{}^c{}_d$ is the *projective Weyl tensor*. Under a change of connection in the projective structure, we have

$$\tilde{W}_{ab}{}^c{}_d = W_{ab}{}^c{}_d \ , \ \ \tilde{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b \ , \ \ \tilde{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a}\Upsilon_{b]}$$

so that the Weyl tensor is projectively invariant (as one would expect by analogy with conformal structures). P_{ab} is called the *Schouten tensor*. Note that if the given connection comes from a metric, then P_{ab} is symmetric in its indices and $\beta_{ab} = 0$. When considered as a 2-form, the tensor β changes by an exact form under a change of connection in the projective class. Furthermore, the Bianchi identity $\nabla_{[a} R_{bc]}{}^{d}{}_{e} = 0$ implies that β is closed. Hence, there is a de Rham cohomology class which is a global obstruction to a projective structure containing a metric connection. Hence, we may restrict our attention to connections in a given projective structure for which $\beta_{ab} = 0$. In fact, β_{ab} can be interpreted as the curvature on volume forms so that restricting attention to the special connections in the projective structure will suffice for this condition to hold (and is the motivation for making such a restriction). There is a residual freedom in the transformation (2.7) via $\Upsilon_a = \nabla_a f$ for some arbitrary function f.

We have reduced the problem of determining whether or not a given projective structure is metrisable to a set of linear first-order partial differential equations with coefficients determined by the projective connection. Thus, if we can find a non-trivial solution of the system (3.5) then the projective structure is metrisable and the metric can be written as (3.4). Ideally, we would like to have $\det(\sigma) \neq 0$ at a point so we can then work in some open neighborhood.

3.1 Prolongation

The system of equations (3.5) is overdetermined as there are more equations than unknowns. We can derive necessary conditions for the existence of a solution of this system via the procedure of *prolongation* (in this example, we mimic the procedure of [8]). The idea here is to construct a closed system of Cauchy-Frobenius form (where all derivatives of unknowns are expressed as functions of unknowns) by repeatedly adding new unknowns for the unspecified derivatives and differentiating. We can then use integrability to derive some algebraic conditions that a solution to the system (3.5) must satisfy. As an example, for the case $n = 3$ we initially have 15 equations in our system but there are 18 first derivatives $\partial_a \sigma^{bc}$. Therefore, in order to specify all of the first derivatives at each point, we must add three unknowns μ^a to the system, constituting some vector field over U. In dimension n, we are required to add n unknowns and we may do so as follows:

$$\nabla_a \sigma^{ab} = (n+1)\mu^a. \tag{3.7}$$

We consider this as the defining equation of the unknowns μ^a. When we take the second derivative of the σ^{ab} terms, all but one of the derivatives of the μ^a terms can be specified. Adding the condition

$$\nabla_a \mu^a + P_{bd}\sigma^{bd} = n\rho \tag{3.8}$$

remedies this and gives us a closed system which may be written compactly as

$$\begin{aligned}
\nabla_a \sigma^{bc} &= \delta_a^b \mu^c + \delta_a^c \mu^b \\
\nabla_a \mu^b &= \delta_a^b \rho - P_{ac}\sigma^{bc} + \frac{1}{n} W_{ac}{}^b{}_d \sigma^{cd} \\
\nabla_a \rho &= -2P_{ab}\mu^b + \frac{4}{n} Y_{abc}\sigma^{bc}
\end{aligned} \tag{3.9}$$

where $Y_{abc} = \frac{1}{2}(\nabla_a P_{bc} - \nabla_b P_{ac})$ is the Cotton-York tensor. The connection ∇_a enjoys the curvature decomposition

$$(\nabla_a \nabla_b - \nabla_a \nabla_b)X^c = W_{ab}{}^c{}_d X^d + \delta_a^c P_{bd} X^d - \delta_b^c P_{ad} X^d. \tag{3.10}$$

Now, differentiating the first equation in (3.9), we find that

$$\begin{aligned}
(\nabla_a \nabla_b - \nabla_a \nabla_b)\sigma^{cd} &= 2\nabla_a \delta_b^{(c}\mu^{d)} - 2\nabla_b \delta_a^{(c}\mu^{d)} \\
&= 4\delta_{[a}^{(c}\delta_{b]}^{d)}\rho - 4P_{e[a}\delta_{b]}^{(c}\sigma^{d)e} - \frac{4}{n}\delta_{[a}^{(c}W_{b]e}{}^{d)}{}_f \sigma^{ef}.
\end{aligned}$$

Furthermore, the curvature condition (3.10) yields

$$(\nabla_a \nabla_b - \nabla_a \nabla_b)\sigma^{cd} = 2W_{ab}{}^{(c}{}_e \delta_f^{d)}\sigma^{ef} - 4P_{e[a}\delta_{b]}^{(c}\sigma^{d)e}.$$

Combining these last two equations gives us an algebraic obstruction to metrisability at first order in the connection

$$\Xi_{abef}^{cd}\sigma^{ef} \equiv \left(W_{ab}{}^{(c}{}_e \delta_f^{d)} + \frac{2}{n}\delta_{[a}^{(c}W_{b]e}{}^{d)}{}_f \right)\sigma^{ef} = 0. \tag{3.11}$$

Earlier counting led us to expect an obstruction to metrisability at this order. The symmetries of Ξ tell us that this represents a system of $\frac{n^4-5n^2+4}{4}$ equations in $\frac{n(n+1)}{2}$ unknowns for which a solution must exist for the projective structure to be metrisable.

Similarly, we can differentiate the second and third equations of (3.9) and equate with the corresponding curvature conditions from (3.10) to obtain

$$(n+3)W_{ab}{}^{c}{}_{d}\mu^{d} - S_{ab}{}^{c}{}_{de}\sigma^{de} = 0 \tag{3.12}$$

where

$$S_{ab}{}^{c}{}_{de} = 2\nabla_{[a}W_{b]d}{}^{c}{}_{e} - 8\delta^{c}_{[a}Y_{b]de} - 2nY_{abd}\delta^{c}_{e}$$

and

$$(n+3)Y_{abc}\mu^{c} - U_{abcd}\sigma^{cd} = 0 \tag{3.13}$$

where

$$U_{abcd} = P_{e[a}W_{b]c}{}^{e}{}_{d} + 2\nabla_{[a}Y_{b]cd}.$$

Conditions of this form were presented in [34] where the author also suggests an algorithm for determining whether a given projective structure is metrisable. We can extend this work and derive more necessary conditions at higher orders by differentiation. For example, from equation (3.11), we obtain the condition

$$\left(\nabla_j \Xi^{cd}_{abef}\right)\sigma^{ef} + \Xi^{cd}_{abjf}\mu^{f} + \Xi^{cd}_{abfj}\mu^{f} = 0. \tag{3.14}$$

Hence, we have generated a set of $n\left(\frac{n^4-5n^2+4}{4}\right)$ obstructions to metrisability at second order in the connection to add to the $\frac{n(n+1)}{2}$ we had before. These conditions are not independent, however, and, in particular, we have the following result:

Proposition 3.1. *Any given set of unknowns (σ^{ab}, μ^{a}) which satisfies the system (3.14) necessarily satisfies (3.12)*

Proof. Equation (3.14) may be written in full as

$$\left(\nabla_j W_{ab}{}^{(c}{}_{e}\right)\sigma^{d)e} + W_{ab}{}^{(c}{}_{e}\delta^{d)}_{j}\mu^{e} + W_{ab}{}^{(c}{}_{j}\mu^{d)} + \frac{2}{n}\left(\nabla_j \delta^{(c}_{[a}W_{b]e}{}^{d)}{}_{f}\right)\sigma^{ef} + \frac{4}{n}\delta^{(c}_{[a}W_{b](j}{}^{d)}{}_{f)}\mu^{f} = 0. \tag{3.15}$$

From the Bianchi identity, we derive the following conditions,

$$\nabla_{[a}W_{bc]}{}^{d}{}_{e} = 2\delta^{d}_{[a}Y_{bc]e} \ \ , \ \ \nabla_{a}W_{bc}{}^{a}{}_{e} = (2n-4)Y_{bce}.$$

Then, by contracting the j and c indices in (3.15) and applying the above identities, we find that

$$(n-2)Y_{abe}\sigma^{de} + \frac{1}{2}\left(6\delta^{d}_{[a}Y_{bj]e}\sigma^{je} - \nabla_{a}W_{bj}{}^{d}{}_{e}\sigma^{je} - \nabla_{b}W_{ja}{}^{d}{}_{e}\sigma^{je}\right)$$
$$+\frac{n^2+2n-3}{2n}W_{ab}{}^{d}{}_{e}\mu^{e} + \frac{1}{n}\nabla_{[a}W_{b]e}{}^{d}{}_{f}\sigma^{ef} + \frac{2n-4}{n}\delta^{d}_{[a}Y_{b]ef}\sigma^{ef} = 0.$$

If we rearrange the terms and multiply across by a factor of $\frac{2n}{n-1}$ we obtain

$$(n+3)W_{ab}{}^{c}{}_{d}\mu^{d} + 2nY_{abe}\sigma^{de} - 2\nabla_{[a}W_{b]j}{}^{d}{}_{e}\sigma^{je} + 8\delta^{d}_{[a}Y_{b]ef}\sigma^{ef} = 0$$

which is precisely the system (3.12). Hence, all necessary conditions for metrisability at first and second order in the connection are given by the equations (3.11) and (3.14). □

In section 3.3, we begin by following Nurowski's procedure and analyze the consequences of the conditions (3.11), (3.12) and (3.13) in some specific examples for $n = 3$. Essentially, if, for a given projective structure, a solution exists to these equations then we substitute that solution into (3.5) and determine when the resulting differential equations are satisfied. In one such example, we go further than [34] and construct an example for which there is a solution $(\sigma^{ab}, \mu^{a}, \rho)$ which satisfies (3.11), (3.12) and (3.13) but is not a solution of (3.14) illustrating the independence and necessity of these new conditions. We will also look at the notion of a metric being geodesically rigid and discuss the significance of this idea.

3.2 Tractor Bundles

The technology of tractor bundles has proven very useful in the areas of both conformal and projective differential geometry. Developed by Tracey Thomas [14] (the name tractor is a portmanteau of his name and the word vector) the

approach is to construct a particular associated vector bundle of the Cartan connection on which invariant differential operators can be constructed. An overview of the ideas in this field can be found in [16]. For the projective case, in particular, we also consider [17] and [8] which deals with the metrisability problem (most of the material here can be found in one of these papers).
As before, let us begin with an open set $U \subset \mathbb{R}^n$ and recall the relation for projectively equivalent connections (2.7). We can, equally, phrase this equivalence by the action of the covariant derivative on 1-forms,

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a$$

for some 1-form Υ_a or for X^a a vector field

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b + \Upsilon_c X^c \delta^b_a.$$

This relationship can be generalised for arbitrary tensor fields (see [17]). In particular, for a volume form σ,

$$\hat{\nabla}_a \sigma = \nabla_a \sigma - (n+1)\Upsilon_a \sigma$$

Let us define the line bundle of *projective densities* of weight w by $E(w) = (\Lambda^n)^{-w/(n+1)}$. Then, for any section of this line bundle σ we have

$$\hat{\nabla}_a \sigma = \nabla_a \sigma + w \Upsilon_a \sigma.$$

This definition of "weight" has a clear analogy with that seen in the conformal case [16], hence the terminology. Furthermore, let E_a represent the bundle of 1-forms and $E_a(w) = E_a \otimes E(w)$ and define other weighted tensor bundles similarly. With these definitions, we may construct a series of projectively invariant differential operators on weighted tensor bundles. For example, for $\sigma_a \in E_a(2)$, the symmetrised covariant derivative $\nabla_{(a}\sigma_{b)}$ is invariant under a change of connection in the projective structure. More importantly for us, if we let $\sigma^{ab} \in E^{(ab)}(-2)$ then the equation

$$(\nabla_a \sigma^{bc})_0 = 0 \tag{3.16}$$

is projectively invariant. This is something we should almost expect to be true as this system of equations arises from asking a question about the projective structure. From before, we know that, if ∇ is a special connection, this system of equations can be prolonged to give us the closed system (3.9) with $\mu^b \in E^b(-2)$ and $\rho \in E(-2)$. These tensors taken together form a section of the bundle

$$E^{(BC)} = E^{(bc)}(-2) \oplus E^b(-2) \oplus E(-2)$$

over U which we call the *tractor bundle*. Guided by equations (3.7) and (3.8) we decree that, under a change of connection in the projective class, sections of $E^{(BC)}$ change accordingly,

$$\widehat{\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix}} = \begin{pmatrix} \sigma^{bc} \\ \mu^b + \Upsilon_c \sigma^{bc} \\ \rho + 2\Upsilon_b \mu^b + \Upsilon_b \Upsilon_c \sigma^{bc} \end{pmatrix}.$$

There is a natural connection on this bundle whose definition is projectively invariant known as the *tractor connection* i.e, ∇_a satisfies

$$\widehat{\nabla}_a \widehat{\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix}} = \nabla_a \widehat{\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix}}.$$

For us, the tractor connection is

$$\nabla_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a^b \mu^c - \delta_a^c \mu^b \\ \nabla_a \mu^b - \delta_a^b \rho + P_{ac} \sigma^{bc} \\ \nabla_a \rho + 2P_{ab} \mu^b \end{pmatrix}$$

where the tensor Y_{abc} is the Cotton-York tensor. Alternatively, if we endow this bundle with the connection

$$D_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a^b \mu^c - \delta_a^c \mu^b \\ \nabla_a \mu^b - \delta_a^b \rho + P_{ac} \sigma^{bc} - \frac{1}{n} W_{ac}{}^b{}_d \sigma^{cd} \\ \nabla_a \rho + 2P_{ab} \mu^b - \frac{4}{n} Y_{abc} \sigma^{bc} \end{pmatrix} \tag{3.17}$$

then covariantly constant sections of $E^{(BC)}$, with respect to this connection, are in one-to-one correspondence with solutions of (3.5). This geometric picture immediately allows us to say the following:

Proposition 3.2. *The maximum degree of mobility of a metric in dimension n is $\frac{(n+1)(n+2)}{2}$. Furthermore, if we compute the curvature of this connection, this upper bound is achieved if and only if the projective structure is flat (i.e, the projective Weyl tensor vanishes or, in the case $n = 2$, the Cotton-York tensor vanishes) which implies that we have a metric of constant curvature.*

The second part of this proposition was shown in [8] using the tractor technology and is exactly the class of projectively equivalent metrics considered by Beltrami [9].

Surprisingly, D_a is not the tractor connection but solutions of the above equation correspond to solutions of

$$\nabla_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} - \frac{1}{n} \begin{pmatrix} 0 \\ W_{ac}{}^{b}{}_{d}\sigma^{cd} \\ 4Y_{abc}\sigma^{bc} \end{pmatrix} = 0. \tag{3.18}$$

The tractor connection is useful in this context as it simplifies the computation for the curvature of the connection D_a. More generally, the correspondence between the solutions of (3.5) and those of (3.18) may help us establish obstructions to metrisability in a more natural geometric way. A more complete overview on the topic and the underlying structure was given in [17].

Generalising Frobenius Theorem

We have reduced the problem of determining whether a given projective structure is metrisable to an overdetermined system of first-order homogeneous linear partial differential equations (3.5), the solutions of which correspond to parallel sections of the connection (3.17). Frobenius' Theorem gives us necessary and sufficient criteria for such a system to admit a maximal set of solutions (e.g, see [35]) or, in geometric terms, for the corresponding connection to admit the maximum number of parallel sections. Our work builds on this idea, asking the question

"How many parallel sections does a connection on a vector bundle have?"

This degree of mobility gives us some insight into the structure of the metrisability problem and some interesting results are already known. For example, in [36] the authors show that, in dimension 3, only metrics of constant curvature have degree of mobility ≥ 3 (i.e, the degree of mobility of a metric in 3 dimensions is either 1,2 or 10). We will use this property to our advantage in the next section.

3.3 Examples in Dimension Three

Here we consider some examples for $n = 3$ and demonstrate how this procedure works. Throughout we will use (x, y, z) in place of (x, y^1, y^2). We will also require that solutions be non-degenerate at a point.

Example 1

Consider the system of second order differential equations

$$\frac{d^2y}{dx^2} = F(x, y, z) \quad , \quad \frac{d^2z}{dx^2} = G(x, y, z). \tag{3.19}$$

Note here that this system is of the form (2.8) with $D^2 = F$, $D^3 = G$ and $A_{jk} = B^i_{jk} = C^i_k = 0$ and so, by Proposition 2.1, it's integral curves are the unparametrised geodesics of some projective structure. We would like to find necessary conditions on F and G such that the projective class contains a metric connection. By considering the equations (3.11) we obtain a non-degenerate solution only if:

$$F_z = 0 \quad , \quad G_y = 0 \quad , \quad F_y = G_z. \tag{3.20}$$

Thus, F and G must be of the form

$$\begin{aligned} F(x, y, z) &= yf(x) + p(x), \\ G(x, y, z) &= zf(x) + q(x) \end{aligned} \tag{3.21}$$

for the system to be metrisable. Here f, p and q are arbitrary functions of x. We must now determine if these conditions are sufficient for metrisability. To answer this, let us consider the case when the system (3.19) is equivalent to a pair of trivial ODEs i.e, there is a local coordinate transformation

$$(x, y, z) \to (\tilde{x}(x, y, z), \tilde{y}(x, y, z), \tilde{z}(x, y, z))$$

such that

$$\frac{d^2\tilde{y}}{d\tilde{x}^2} = 0 \quad , \quad \frac{d^2\tilde{z}}{d\tilde{x}^2} = 0.$$

As seen in Chapter 2, if a given system describes the unparametrised geodesics of some connection, then it will be equivalent to a trivial system if and only if the Grossman invariants vanish (2.5). In the case of the system (3.19) these invariants are F_z, G_y and $F_y - G_z$ i.e, if F and G are of the form (3.21), the given system is diffeomorphic to a trivial one. This is useful for us as the integral curves of trivial ODEs correspond to the unparametrised geodesics of metrics of constant curvature. Hence, we have established the following:

Proposition 3.3. *The integral curves of the system of second order ODEs (3.19) are the unparametrised geodesics of some metric if and only if (3.20) holds, in which case the system is diffeomorphic to a trivial one and hence, the metric has constant curvature.*

Example 2

Now let us consider the following system in three dimensions:

$$\frac{d^2y}{dx^2} = F(x, y, z)\left(\frac{dy}{dx}\right)^2 \quad , \quad \frac{d^2z}{dx^2} = G(x, y, z)\left(\frac{dz}{dx}\right)^2. \tag{3.22}$$

We note that this is also of the form (2.8). In this case, equations (3.11) yield three conditions. Firstly, we must have

$$F_x = 0 \quad , \quad G_x = 0.$$

The third condition is that some linear combination of F_z and G_y vanishes. If we impose σ is invertible in some open set, then this condition must be

$$F_z = G_y.$$

We can further break this solution into two categories:

1. $F_x = G_x = F_z = G_y = 0$,
2. $F_x = G_x = 0$, $F_z = G_y \neq 0$ and $\sigma^{12} = \sigma^{13} = \sigma^{22} = \sigma^{33} = 0$.

For the first solution we have $F = F(y)$ and $G = G(z)$. In this case, it is possible to make a coordinate transformation at the point

$$y \to \tilde{y} = \tilde{y}(y, x) \quad , \quad z \to \tilde{z} = \tilde{z}(z, x)$$

such that in the new coordinate system the ODEs become

$$\frac{d^2\tilde{y}}{dx^2} = 0 \quad , \quad \frac{d^2\tilde{z}}{dx^2} = 0.$$

We can again use the invariants (2.5) to confirm this.

The more interesting case is the second category of solutions. From these conditions we can rewrite F and G as

$$F = \frac{\partial H(y,z)}{\partial y} \quad , \quad G = \frac{\partial H(y,z)}{\partial z}$$

for some arbitrary $H = H(y, z)$. The second order conditions (3.12) in the prolongation procedure give us

$$\mu^2 = \frac{1}{4} G\sigma^{23} \ , \ \mu^3 = \frac{1}{4} G\sigma^{23}$$

which leads to the differential equations

$$\partial_y \sigma^{23} = \frac{1}{2} F\sigma^{23} \ , \ \partial_z \sigma^{23} = \frac{1}{2} G\sigma^{23}.$$

Solving these differential equations yields

$$\sigma^{23} = p(x) e^{\frac{1}{2} H(y,z)}$$

for p some arbitrary function of x. The third order conditions (3.13) are then automatically satisfied. We could now revert to the extra second order conditions (3.14) to derive extra constraints but it's simpler in this case to revert to the original metrisability equation (3.5). From this, we obtain the simplified partial differential equations (PDEs),

$$\partial_x \sigma^{11} = 0 \ , \ p'(x) = 0 \ , \ \partial_y \sigma^{11} = -\frac{1}{2}\frac{\partial H}{\partial y}\sigma^{11} \ , \ \partial_z \sigma^{11} = -\frac{1}{2}\frac{\partial H}{\partial z}\sigma^{11}$$

so that

$$\sigma^{11} = \alpha e^{-\frac{1}{2}H(y,z)} \ , \ \sigma^{23} = \beta e^{\frac{1}{2}H(y,z)}$$

for α and β constants. The solution for the metric whose Levi-Civita connection is in this projective structure is

$$g = -\frac{1}{\alpha^2\beta^2}dx^2 - \frac{2}{\alpha^3\beta}e^{-H(y,z)}dydz.$$

Since the projective structure does not see a rescaling of the metric we can equivalently write down any constant muliple of the metric g i.e,

$$g = dx^2 + ce^{-H(y,z)}dydz \tag{3.23}$$

for some constant c. Hence, if the integral curves of the system (3.22) are the unparametrised geodesics of a metric then either the system is equivalent to a trivial one or the metric is related to (3.23) by diffeomorphism and rescaling by a constant.

Remark: We saw that, in this example, the algebraic conditions (3.11), (3.12) and (3.13) were not sufficient for metrisability but that additional obstructions would appear at higher orders by differentiation. In this case, reverting to the original differential system was the simplest option but we shall look at another example where equation (3.14) will be useful.

Example 3 - Nil and Sol

In these next examples we discuss two of the eight Thurston geometries in the context of our problem. These geometric structures are central to the classification of compact 3-manifolds and, in some sense, describe fundamental geometries in three dimensions. We will see that these metrics are geodesically rigid. Starting with a Thurston metric, in locally defined coordinates, we can construct it's unparametrised geodesics. Then, using our prolongation procedure, we can determine whether or not there exists another metric which gives rise to the same geodesics as unparametrised curves.

Nil metric

Lets begin with the metric

$$g = dx^2 + dy^2 + (dz - xdy)^2. \tag{3.24}$$

We also recognise this as being the Kaluza-Klein metric in three dimensions. The motivation behind Kaluza-Klein theory is to try and obtain solutions to both Einstein's equations and Maxwell's equations on an n-dimensional manifold N by considering geodesic motion on a $n+1$-dimensional manifold U - the fibre bundle over N with $U(1)$ fibres. This approach makes sense, as we think of electromagnetism as a gauge theory with gauge group $U(1)$. Now, if we formulate this in the correct way, we can think of a gauge transformation as a coordinate transformation on the $n+1$-dimensional manifold. The metrics g, G on manifolds U, N, respectively, can be related as follows

$$g = G + (dz + A)^2$$

where A is the electromagnetic potential 1-form and z parametrises the extra spatial dimension. If we consider electromagnetism on a 2-dimensional manifold, endowed with the flat metric, such that the curvature 2-form $F = -dx \wedge dy$, then we could pick $A = -xdy$ (so that $F = dA$) and the three-dimensional Kaluza-Klein metric can be written as (3.24).

In this coordinate system, the unparametrised geodesic equations for the metric become

$$\begin{aligned} \frac{d^2y}{dx^2} &= -x\left(\frac{dy}{dx}\right)^3 + \left(\frac{dz}{dx}\right)\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + \left(\frac{dz}{dx}\right) \\ \frac{d^2z}{dx^2} &= \left(\frac{dz}{dx}\right)^2 - x\left(\frac{dz}{dx}\right)\left(\frac{dy}{dx}\right)^2 + (1-x^2)\left(\frac{dy}{dx}\right) + x\left(\frac{dz}{dx}\right). \end{aligned} \tag{3.25}$$

Now let's say that we are given this system of differential equations and would like to determine if it is metrisable. Using our procedure, we can obtain the most general form of a metric whose unparametrised geodesics are the integral curves of (3.25). From the integrability conditions (3.11), (3.12) and (3.13) we derive the following

$$\sigma^{12} = 0 \ , \ \sigma^{11} = \sigma^{22} \ , \ x\sigma^{11} = \sigma^{23} \ , \ \sigma^{13} = 0$$

$$\mu^1 = \mu^2 = \mu^3 = 0 \ , \ (1+x^2)\sigma^{11} = \sigma^{33}$$

and using the derivative relationship between μ^a and σ^{ab} we may write the matrix associated to σ

$$\sigma = q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix}$$

where q is a constant. Then, the metric whose Levi-Civita connection lies in the projective class is

$$g = \frac{1}{q^4}(dx^2 + dy^2 + (dz - xdy)^2).$$

We see that this is just a constant rescaling of the metric we started with, which we already know to be projectively equivalent.

In his paper [10], Levi-Civita characterises the normal forms of all Riemannian metrics on an open subset $U \subset M$ which are not geodesically rigid. Specifically, if g and $\tilde{g}$ are non-propotional geodesically equivalent Riemannian metrics then for all $x \in U$ there exist some coordinates $x_1, \ldots, x_n$ in a neighbourhood of x such that

$$\begin{aligned} g &= \pi_1 dx_1^2 + \pi_2 dx_2^2 + \ldots + \pi_n dx_n^2 \\ \tilde{g} &= \rho_1\pi_1 dx_1^2 + \rho_2\pi_2 dx_2^2 + \ldots + \rho_n\pi_n dx_n^2 \end{aligned} \tag{3.26}$$

where the functions π_a and ρ_a are given by

$$\begin{aligned} \pi_a &= (\lambda_a - \lambda_1)(\lambda_a - \lambda_2)\ldots(\lambda_a - \lambda_{i-1})(\lambda_a - \lambda_{i+1})\ldots(\lambda_a - \lambda_n) \\ \rho_a &= \frac{1}{\lambda_1\lambda_2\ldots\lambda_n}\frac{1}{\lambda_a} \end{aligned}$$

and, for each a, λ_a is a function of x_a. A more general form of this result can be found in [37].

From what we have shown above, the Kaluza-Klein metric is geodesically rigid, i.e, there are no non-proportional geodesically equivalent metrics so it will not fall into a Levi-Civita class that can be written in the form above. Even though Levi-Civita's characterisation of projectively equivalent metrics is useful, it is not known, in general, how to recognise such a metric invariantly.

Sol metric

Another example in the Thurston class worth considering is that of the Sol geometry. This geometry can be modelled, locally, by the metric (see e.g, [38])

$$g = dx^2 + e^x dy^2 + e^{-x} dz^2. \tag{3.27}$$

Physically, this geometry is useful for studying holography, as described in [39] it:- " results from the dimensional reduction of the decoupling limit of the D3-brane in a background B field."

As with the Kaluza-Klein metric, we may ask if this Riemannian metric is geodesically rigid. If not, then it should fall into the Levi-Civita class described by (3.26). Let us begin, as before, by constructing the unparametrised geodesics of this metric. They are the integral curves of

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{e^x}{2}\left(\frac{dy}{dx}\right)^3 + \frac{e^{-x}}{2}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 - \frac{dy}{dx} \\ \frac{d^2z}{dx^2} &= \frac{e^{-x}}{2}\left(\frac{dz}{dx}\right)^3 - \frac{e^x}{2}\left(\frac{dz}{dx}\right)\left(\frac{dy}{dx}\right)^2 + \frac{dz}{dx}. \end{aligned}$$

As before we try to determine all the conditions for which the integral curves of this system arise as the unparametrised geodesics of some metric i.e, all the conditions for the existence of a solution to (3.5). In this case, (3.11), (3.12) and (3.13) yield

$$\sigma^{12} = 0 \quad , \quad \sigma^{13} = 0 \quad , \quad e^x \sigma^{22} = e^{-x}\sigma^{33} \quad , \quad \sigma^{23} = 0$$

$$\partial_y \sigma^{22} = \partial_z \sigma^{33} = 0 \tag{3.28}$$

where, again, we use the definition for μ^a to produce (3.28). We can also derive extra conditions by combining the differential equations in (3.5) with (3.28). From this we get

$$\partial_x \sigma^{11} = \partial_y \sigma^{11} = \partial_z \sigma^{11} = 0$$

$$\partial_x \sigma^{22} = -\sigma^{22} \quad , \quad \sigma^{11} = e^x \sigma^{22}.$$

These conditions allow us to write the tensor σ^{ab} in matrix form as

$$\sigma^{ab} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-x} & 0 \\ 0 & 0 & e^{x} \end{pmatrix}$$

where α is a constant. This tells us that there is a metric connection in our projective class which is the Levi-Civita connection of the metric g_{ab} determined by equation (3.4). This can be written as

$$g = \frac{1}{\alpha^4}(dx^2 + e^x dy^2 + e^{-x} dz^2). \tag{3.29}$$

This metric is just a constant rescaling of (3.27). Hence, the Sol metric is geodesically rigid and does not belong to the Levi-Civita class (3.26).

Example 4 - Levi-Civita Normal Form

In the previous two examples we found that both metrics were geodesically rigid. Bearing this in mind, it would be a good idea to confirm that these results are not just some artefact of a restriction in our procedure. To check that the prolongation procedure is a good measure of geodesic rigidity we should begin with a metric in the Levi-Civita class and analyze it to see if there is another metric which is geodesically equivalent using the same methods as before. So, let us consider the first expression for g in (3.26) with

$$\lambda_1 = x \quad , \quad \lambda_2 = 1 \quad , \quad \lambda_3 = -1.$$

In this case,

$$g = (x^2 - 1)dx^2 + (2 - 2x)dy^2 + (2 + 2x)dz^2 \tag{3.30}$$

and the geodesic equations read

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{1-x^2}\left(-\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dx}\right) \\ \frac{d^2z}{dx^2} &= \frac{1}{1-x^2}\left(\left(\frac{dz}{dx}\right)^3 - \left(\frac{dz}{dx}\right)\left(\frac{dy}{dx}\right)^2 - \frac{dz}{dx}\right). \end{aligned} \tag{3.31}$$

Then, starting with this system, the first order conditions in the prolongation procedure yield

$$\sigma^{23} = 0 \quad , \quad \sigma^{12} = 0 \quad , \quad \sigma^{13} = 0 \quad , \quad \sigma^{11} + \sigma^{22} + \sigma^{33} = 0$$

$$\partial_y \sigma^{22} = \partial_z \sigma^{33} = 0 \quad , \quad (1 - x^2)\partial_x \sigma^{11} + \sigma^{22} + \sigma^{33} = 0.$$

There are some extra conditions gained from the higher order conditions. Instead, it is more convenient to look at (3.5). From this, we obtain the following

$$\partial_x \sigma^{22} = \frac{1}{1-x^2}\sigma^{22} \quad , \quad \partial_x \sigma^{33} = -\frac{1}{1-x^2}\sigma^{33}$$

$$\partial_y \sigma^{11} = \partial_z \sigma^{11} = \partial_z \sigma^{22} = \partial_y \sigma^{33} = 0.$$

A solution σ to these equations is of the form

$$\sigma^{ab} = \begin{pmatrix} -\alpha \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} - \beta \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & \alpha \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & \beta \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} \end{pmatrix}$$

where α and β are constants. The general metric whose Levi-Civita connection lies in the projective class of (3.30) can then be written as

$$\begin{aligned} g &= \frac{1}{\alpha\beta}\Big(\frac{1-x^2}{\alpha^2(1+x)^2 + \beta^2(1-x)^2 + 2\alpha\beta(1-x^2)}dx^2 \\ &- \frac{1-x}{(\alpha^2+\alpha\beta) + (\alpha^2-\alpha\beta)x}dy^2 - \frac{1+x}{(\alpha\beta+\beta^2)+(\alpha\beta-\beta^2)x}dz^2\Big). \end{aligned} \tag{3.32}$$

This metric reproduces the geodesics equations (3.31) for any choice of the constants α and β. Furthermore, we can see that it is more general than just a rescaling of the metric by some constant. If we choose $\alpha = \beta = \frac{1}{\sqrt{2}}$, we obtain the negative of the original metric (3.30). Even more interestingly, if we take $\alpha = -\beta = \frac{1}{\sqrt{2}}$, we reproduce the geodesically equivalent metric in the normal form given by the second equation of (3.26), as expected. This lack of geodesic rigidity can be quite a useful structure to have. More recently, a relationship has been noted between manifolds admitting projectively equivalent metrics and Liouville integrability of the corresponding geodesic flows. For example, in [40] the authors show that if metrics g and $\tilde{g}$ are projectively equivalent metrics on some n-dimensional manifold U which are non-proportional, then it is possible to construct n functionally independent pairwise commuting integrals of motion for the geodesic flow (of either metric).

Example 5 - Berger Sphere

The Berger sphere is a one-parameter family of metrics on S^3 obtained by taking the standard metric and deforming along fibres of the Hopf fibration. We can identify S^3 with the Lie group $SU(2)$ on which we can define a metric. So, let ω_1, ω_2, ω_3 be the canonical basis of 1-forms on $SU(2)$ satisfying

$$d\omega_a = \epsilon_{abc}\omega_b \wedge \omega_k,$$

where ϵ_{abc} is completely antisymmetric with $\epsilon_{123} = 1$. Then, the family of Berger metrics is defined as

$$g = a^2(\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 \tag{3.33}$$

parametrised by $a \neq 0$. Taking $a = 1$ returns us to the standard left-invariant metric on $SU(2)$. We may use our analysis to check for which values (if any) of a is this metric geodesically rigid. For $a = 1$ this is not true.[2] To tackle this problem, we can use Euler angles to parametrise the 1-forms in $SU(2)$ as follows

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}(d\psi + \cos\theta d\phi) \\
\sigma_2 &= \frac{1}{2}(\cos\psi d\theta + \sin\psi\sin\theta d\phi) \\
\sigma_3 &= \frac{1}{2}(\sin\psi d\theta - \cos\psi\sin\theta d\phi).
\end{aligned} \tag{3.34}$$

With this parametrisation, the Berger metric may be written in local coordinates

$$g = \frac{1}{4}(d\theta^2 + a^2 d\psi^2 + (a^2\cos^2\theta + \sin^2\theta)d\phi^2 + 2a^2\cos\theta d\psi d\phi)$$

[2]This result goes back to Beltrami [9] who was the first to observe that two non-proportional metrics can have the same geodesics. In particular, all surfaces of constant curvature are geodesically equivalent.

or, relabelling coordinates as (x, y, z) and rescaling,

$$g = dx^2 + a^2 dy^2 + (a^2 \cos^2 x + \sin^2 x)dz^2 + 2a^2 \cos x dy dz.$$

The geodesic equations of this metric are

$$\begin{aligned}
\frac{d^2y}{dx^2} &= a^2 \sin x \left(\frac{dy}{dx}\right)^2 \left(\frac{dz}{dx}\right) + (a^2 - 1)\cos x \sin x \left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 \\
&\quad -a^2 \cot x \left(\frac{dy}{dx}\right) + (\sin x - (a^2 - 2)\cos x \cot x)\left(\frac{dz}{dx}\right) \\
\frac{d^2z}{dx^2} &= a^2 \sin x \left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 + (a^2 - 1)\cos x \sin x \left(\frac{dz}{dx}\right)^3 \\
&\quad +a^2 \csc x \left(\frac{dy}{dx}\right) + (a^2 - 2)\cot x \left(\frac{dz}{dx}\right).
\end{aligned}$$

Beginning with this system and using the prolongation procedure we obtain the following whenever $a \neq 1$,

$$\sigma^{13} = \sigma^{12} = 0 \quad , \quad \sigma^{11} - \sin^2 x\, \sigma^{33} = 0$$

$$\cos x\, \sigma^{11} + \sin^2 x\, \sigma^{23} = 0 \quad , \quad \partial_y \sigma^{23} + \partial_z \sigma^{33} = 0 \tag{3.35}$$

plus some other more complicated conditions. Combining these with the equations (3.5) yields the following form for σ,

$$\sigma^{ab} = \alpha(\sin x)^{-\frac{3}{2}} \begin{pmatrix} \sin^2 x & 0 & 0 \\ 0 & \cos^2 x + \frac{1}{a^2}\sin^2 x & -\cos x \\ 0 & -\cos x & 1 \end{pmatrix}$$

for α a constant. Then the metric becomes

$$g = \frac{a^2}{\alpha^4}(dx^2 + a^2 dy^2 + (a^2 \cos^2 x + \sin^2 x)dz^2 + 2a^2 \cos x dy dz).$$

Of course, if $a = 1$ then the conditions in (3.35) don't appear (as there is a factor of $(a^2 - 1)$ multiplying each vanishing condition which we have neglected above). Hence, the metric is always geodesically rigid in the case $a \neq 1$.

Example 6 - Higher Order Obstructions

We will now construct an explicit example to illustrate how equation (3.14) produces obstructions to metrisability beyond (3.11), (3.12) and (3.13) thus justifying the expansion of our system of necessary conditions.

Consider the system of ODEs

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \alpha e^{x} + \beta\left(\frac{dy}{dx}\right) + \chi\left(\frac{dz}{dx}\right) + \Delta e^{-x}\left(\frac{dy}{dx}\right)^2 + \phi e^{-x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right) \\
&+ \gamma e^{-x}\left(\frac{dz}{dx}\right)^2 + \eta e^{-2x}\left(\frac{dy}{dx}\right)^3 + \xi e^{-2x}\left(\frac{dy}{dx}\right)^2\left(\frac{dz}{dx}\right) \\
&+ \varphi e^{-2x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 \\
\frac{d^2z}{dx^2} &= \kappa e^{x} + \lambda\left(\frac{dy}{dx}\right) + \mu\left(\frac{dz}{dx}\right) + \nu e^{-x}\left(\frac{dy}{dx}\right)^2 + \omega e^{-x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right) \\
&+ \theta e^{-x}\left(\frac{dz}{dx}\right)^2 + \eta e^{-2x}\left(\frac{dy}{dx}\right)^2\left(\frac{dz}{dx}\right) + \xi e^{-2x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 \\
&+ \varphi e^{-2x}\left(\frac{dz}{dx}\right)^3
\end{aligned}
\tag{3.36}
$$

where all the unknown coefficients here are constant. We notice that this system is of type (2.8) and, thus, its integral curves are the unparametrised geodesics of some projective structure. There is a one-parameter group of point symmetries of the system of ODEs which we call *projective symmetries* (symmetries which map unparametrised geodesics to unparametrised geodesics) given by the transformation

$$x \to x+\alpha \quad , \quad y \to ye^{\alpha} \quad , \quad z \to ze^{\alpha}.$$

Thus, we have a flow on TU generated by the vector field

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{3.37}$$

This is termed a *projective vector field* since its flow preserves the geodesic structure of U. We also have two more projective vector fields, namely $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. Together they form a Lie algebra of Bianchi V type. To check that there are no more such vector fields, it suffices (in this case) to consider the Lie derivative of the Weyl tensor along a generic vector field and impose that it vanishes. This must be so since the Weyl tensor is invariant under projective transformations. In any case, this appears to be an appropriate system for which the conditions derived at prolongation take a simple form. For general values of the coefficients the obstructions at each order are quite long and messy to write down. So let us look at a special case of this system, namely

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \alpha e^x + \chi\left(\frac{dz}{dx}\right) - 4e^{-x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right) \\
&+ e^{-x}\left(\frac{dz}{dx}\right)^2 + e^{-2x}\left(\frac{dy}{dx}\right)^2\left(\frac{dz}{dx}\right) \\
\frac{d^2z}{dx^2} &= \alpha e^x - \left(\frac{dy}{dx}\right) + e^{-x}\left(\frac{dy}{dx}\right)^2 \\
&- 4e^{-x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right) + e^{-2x}\left(\frac{dy}{dx}\right)\left(\frac{dz}{dx}\right)^2 . \qquad (3.38)
\end{aligned}
$$

For convenience we will say $\alpha \neq 0$ or 8 (see later). If we want a non-singular solution for the metric we quickly find that $\chi = -1$ is required by the set first order integrability conditions (3.11). To see this, we can compute the determinant

$$\left|\Xi^{cd}_{23ef}\right| = \frac{2}{243}e^{-12x}(1+\chi)^2$$

which is an obstruction to metrisability. Furthermore, assuming that $\chi = -1$, these conditions put the following restrictions on the tensor σ^{ab}:

$$\sigma^{22} = \sigma^{33} = 2e^x\sigma^{12} = 2e^x\sigma^{13}, \sigma^{23} = 4e^x\sigma^{12} - \frac{1}{2}e^{2x}\alpha\sigma^{11} \qquad (3.39)$$

where we have implemented the assumption that $\alpha \neq 0$. We see here that the metric tensor σ is determined by two functions of (x, y, z) and that, under the above assumptions, it is generally non-singular and the conditions (3.11) are satisfied. Continuing this procedure we can compute the second order integrability conditions in (3.12) to find

$$
\begin{aligned}
\mu^3 &= \mu^2 \\
\mu^2 &= \frac{1}{4}e^x\alpha\mu^1 - \frac{21}{48}e^x\alpha\sigma^{11} - \frac{1}{12}(\alpha-22)\sigma^{12} \\
\mu^1 &= \frac{(\alpha-4)(\alpha-12)}{3\alpha(\alpha-8)}e^{-x}\sigma^{12} + \frac{5\alpha+4}{12(\alpha-8)} \qquad (3.40)
\end{aligned}
$$

where we have taken $\alpha \neq 8$. Assuming the solution set given by (3.39) and (3.40), the conditions in (3.13) give

$$(\alpha-1)\sigma^{11} = 0$$

Let us first consider the solution $\alpha = 1$. Then, the general solution to our necessary conditions depends on two arbitrary functions of x, y and z (see (3.39)). The equations (3.14) then give the extra conditions that are only satisfied by setting

$$\sigma^{12} = \sigma^{11} = 0.$$

On the other hand, let us assume that α is still arbitrary and that $\sigma^{11} = 0$. In this case, we may still have a non-degenerate solution for the matrix σ which now depends on a single arbitrary function of x, y and z. Here, equations (3.14) yield $\sigma^{12} = 0$ i.e, a degenerate solution as before. Thus, the integral curves of (3.38) are not the geodesics of some metric for the values of α specified and this has been completely verified from the derived set of necessary algebraic conditions.

Example 7 - Egorov Example

As a final example of this procedure, let us consider the following path geometry

$$y'' = 2y(y')^2(z') \ , \ z'' = 2y(y')(z')^2.$$

This system has vanishing Fels invariants and so, integral curves of this system coincide with the unparametrised geodesics of some symmetric affine connection. Furthermore, as shown by Egorov [41], this system occupies an important place in the projective branch of the theory as it admits the submaximal number of point symmetries, i.e, amongst all systems in this branch which are not diffeomorphic to the trivial one, it admits the highest number of symmetries. The symmetry algebra here is an 8-dimensional subalgebra of $sl(4, \mathbb{R})$ spanned by the vector fields

$$\partial_x \ , \ \partial_z \ , \ y\partial_x \ , \ z\partial_x \ , \ 2x\partial_x + y\partial_y \ , \ x\partial_x + z\partial_z \ , \ yz\partial_x - \partial_y \ , \ y^3\partial_x - 3y\partial_z$$

Hence, there is a significant gap between the maximal and submaximal cases. The theory of such "gaps" in the context of Cartan geometries has been studied extensively, most recently in [42] and [43]. We will discuss the theory some

more when we consider path geometries in the conformal branch and make a comparison between the results.
For now, we may ask the question:

Does there exist a metric who unparametrised geodesics coincide with the integral curves of this path geometry?

From our theory, the first order conditions (3.11) for the corresponding projective structure are satisfied if and only if

$$\sigma^{12} = \sigma^{22} = \sigma^{23} = 0$$

which implies that the metric is degenerate. Hence, this system is not metrisable.

3.3.1 Generating Sufficient Conditions

In this chapter, we have derived a set of necessary algebraic conditions for a given path geometry to be metrisable using the curvature of the connection (3.17). Furthermore, we have shown how to generate conditions at higher orders by differentiation and, by means of an example, that the equations (3.14) are indeed new to the system. In theory, we may produce obstructions in this way ad infinitum but it is not clear, from the above work, at what point all sufficient conditions will be generated. Solving this problem completely would involve an aid from the theory of differential systems. For example, one thing that we do know is that if we differentiate and no new obstructions are generated (as above) then we may halt the procedure. The general case could then be considered in light of developments on the theory of geodesic mobility. However, this topic is subject to further reseacrh.

3.4 Projective Equivalence and Conformal flatness

In 1868, Beltrami [9] proved that, in a given number of dimensions, all metrics of constant curvature are projectively equivalent. Moreover, given two projectively equivalent metrics g and $\tilde{g}$, if g has constant curvature, then so does $\tilde{g}$. Upon seeing this idea, one might speculate on other properties of a metric which are invariant within a given projective class. The first question one may consider is the following:

Given two projectively equivalent metrics g and $\tilde{g}$ such that g is conformally flat, under what conditions is $\tilde{g}$ also conformally flat?

Given a metric g with Christoffel symbols Γ^a_{bc}, the corresponding connection components for the conformally equivalent $\bar{g} = e^{2\Upsilon} g$ are

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b - g^{ad} g_{bc} \Upsilon_d$$

which is similar to (2.7) but with an extra term added so it immediately appears as though conformal flatness shouldn't be an invariant within a projective structure. On the other hand if a given conformally flat metric g is geodesically rigid then all projectively equivalent metrics are just constant multiples of g and therefore must be conformally flat. Furthermore, if g is projectively flat (has vanishing projective Weyl tensor) then it is of constant curvature and so conformally flat and any projectively equivalent metric has constant curvature and also conformally flat. Hence, in the two extreme cases of geodesic mobility at least, the evidence is that conformal flatness is a projectively invariant property.

However, some recent work has been undertaken in [13] which shows that, in four dimensions, the above assertion is not generally true but can be made when the metric is not of general holonomy type. To my knowledge, no similar result has been postulated in three dimensions. To tackle this problem, we will use a different approach to [13] and consider the solution to a problem, originally proposed by Beltrami, to determine local normal forms of any pair of projectively equivalent metrics. In this area, there has been significant development, especially recently. As we have seen, Levi-Civita gave a complete characterisation of pairs of projectively equivalent Riemannian metrics [10]. In the pseudo-Riemannian case, the problem was solved in three dimensions by Petrov [44] and, in arbitrary dimensions, it was solved by Matveev and Bolsinov, see [11] and [12]. Here, we will use this general theory to our advantage and get to the heart of the problem, which are metrics g with degree of mobility two (since, in three dimensions, the degree of mobility of a metric may only be 1,2 or 10). We begin with a pair of projectively equivalent metrics g and $\tilde{g}$ which are projectively related but non-proportional. Then, these metrics may be represented in local coordinates in one of the normal forms given by either Levi-Civita or Matveev and Bolsinov. Then, we impose the condition of conformal flatness on g and determine necessary conditions for which $\tilde{g}$ is also conformally flat.

Locally, in three dimensions, the condition for conformal flatness coincides with the vanishing of the Cotton tensor

$$C_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4}(\nabla_b R g_{ac} - \nabla_c R g_{ab})$$

where R_{ab} is the Ricci tensor and R the Ricci scalar.

3.4.1 Riemannian Case

Let us assume that the degree of mobility of g is greater than 1 so that there exists a non-proportional equivalent metric $\tilde{g}$ in its projective class. As we have seen, for any pair of such metrics g and $\tilde{g}$, there exists a coordinate system

(x, y, z) so that these metrics may be written as

$$\begin{aligned}
g &= (\lambda(x)-\mu(y))(\lambda(x)-\nu(z))dx^2 + (\mu(y)-\lambda(x))(\mu(y)-\nu(z))dy^2 \\
&+ (\nu(z)-\lambda(x))(\nu(z)-\mu(y))dz^2 \\
\tilde{g} &= \frac{(\lambda(x)-\mu(y))(\lambda(x)-\nu(z))}{\lambda(x)^2\mu(y)\nu(z)}dx^2 + \frac{(\mu(y)-\lambda(x))(\mu(y)-\nu(z))}{\lambda(x)\mu(y)^2\nu(z)}dy^2 \\
&+ \frac{(\nu(z)-\lambda(x))(\nu(z)-\mu(y))}{\lambda(x)\mu(y)\nu(z)^2}dz^2
\end{aligned}$$

for some functions λ, μ, ν. Now the problem reduces to finding all metrics g for which the Cotton tensors of g and $\tilde{g}$ both vanish completely, i.e, $C_{abc} = \tilde{C}_{abc} = 0$. To make this easier to tackle, we may break it into three cases:

1. None of the functions λ, μ or ν is constant.

2. Exactly one of the functions λ, μ or ν is constant.

3. Exactly two of the functions λ, μ, ν are constants not equal to each other.

Obviously, if they are all constant, then the metric g is of constant curvature and has maximum degree of mobility.

Case 1

First consider the case where none of the functions $\lambda(x)$, $\mu(y)$ or $\nu(z)$ is a constant on U. Here, the Cotton tensor C_{abc} of the metric g has three independent components, e.g, C_{112}, C_{113} and C_{212} and if we require g to be conformally flat, then there are three necessary conditions to be satisfied. It transpires that one of these conditions (without loss of generality we choose C_{212}) can be deduced by taking a linear combination of one of the other conditions and its first derivative with respect to x. This means that for g to be conformally flat, there are really only two conditions we need to satisfy

$$C_{112} = 0 \ , \ C_{113} = 0. \tag{3.41}$$

The coefficient of $\lambda''(x)$ in C_{112} is non-vanishing by our starting assumption so we may treat $\lambda''(x)$ as an independent variable and solve the first equation in

(3.41) for it. Then, if we substitute this into the second equation of (3.41) we obtain a condition which is purely first order in λ. Note here that this procedure is undertaken in a very specific way as the Cotton tensor can contain third order terms. However, in this example, only C_{212} has a third derivative of λ and this is the one we have eliminated.

The resulting first order equation can be rewritten as

$$\lambda'(x)^2 = \sum_{m=0}^{5} a_m(y,z)\lambda(x)^m$$

where the functions a_m can be written in terms of μ, ν and their derivatives. Since λ is non-constant on U, each of the coefficients a_m must be constant for the right hand side to be a function of x only, i.e, we must have

$$\lambda'(x)^2 = A_5\lambda(x)^5 + A_4\lambda(x)^4 + A_3\lambda(x)^3 + A_2\lambda(x)^2 + A_1\lambda(x) + A_0$$

for some constants A_m. This conclusion leads to a system of six differential equations in y and z,

$$a_m(y,z) = A_m \ , \ \ m = 0, \ldots, 5.$$

Each of these equations is third order in y and z and the coefficient of $\mu'''(y)$ or $\nu'''(z)$ is necessarily non-vanishing in each so we can, for example, solve one of these conditions for $\mu'''(y)$ and use it to eliminate this term in the others leading to five conditions of *second* order in μ. Similarly, we can repeat this process to eliminate $\mu''(y)$ and produce four conditions, first order in μ. Each of these new equations is of the form

$$\mu'(y)^2 = \sum_{m=0}^{5} b_{k,m}(z)\mu(y)^m$$

for $k = 1, \ldots, 4$, where each $b_{k,m}(z)$ can be written in terms of $\nu(z)$ and its derivatives and, using the same argument as before, we obtain a set of 24 equations

$$b_{k,m}(z) = B_m = \text{constant}.$$

Again, with the same reasoning, we can eliminate the derivatives $\nu'''(z)$ and $\nu''(z)$ to obtain 22 equations of the form

$$(\nu'(z))^2 = c_{k,5}\nu(z)^5 + c_{k,4}\nu(z)^4 + c_{k,3}\nu(z)^3 + c_{k,2}\nu(z)^2 + c_{k,1}\nu(z) + c_{k,0}.$$

For each k, we then require

$$c_{k,m} = C_m = \text{constant}.$$

Solving these equations, we find that the metric g is conformally flat if and only if λ, μ and ν satisfy differential equations of the form

$$(\lambda'(x))^2 = \sum_{m=0}^{5} A_m\lambda(x)^m \ , \ (\mu'(y))^2 = \sum_{m=0}^{5} A_m\mu(y)^m \ , \ (\nu'(z))^2 = \sum_{m=0}^{5} A_m\nu(z)^m$$

where $A_0, \ldots, A_5$ are arbitrary constants.

Now let us compute the Cotton tensor $\tilde{C}_{abc}$ for the projectively equivalent metric $\tilde{g}$ and impose that the constraints above hold for λ, μ and ν. In this instance, we find that $\tilde{C}_{abc}$ vanishes completely only in the case $A_5 = 0$, that is g and $\tilde{g}$ are both conformally flat if and only if

$$(\lambda'(x))^2 = \sum_{m=0}^{4} A_m\lambda(x)^m \ , \ (\mu'(y))^2 = \sum_{m=0}^{4} A_m\mu(y)^m \ , \ (\nu'(z))^2 = \sum_{m=0}^{4} A_m\nu(z)^m.$$

If we now calculate the projective Weyl tensor for the metric g under these conditions and we find that it vanishes completely. Hence both g and $\tilde{g}$ are conformally flat if and only if g is projectively flat i.e, of constant curvature (so there are no examples here of degree of mobility two).

Cases 2 and 3

In the case, that at least one of the functions λ, μ or ν is a constant, the procedure runs in a similar fashion so we will not detail it here. In both cases, the overall result is the same as before. That is, we have the following theorem

Theorem 3.4. *Let g and $\tilde{g}$ be two projectively equivalent Riemannian metrics which are non-proportional. If g is conformally flat, then $\tilde{g}$ is also conformally flat if and only if g has constant curvature.*

3.4.2 Pseudo-Riemannian Case

To investigate the pseudo-Riemannian version of this problem in the same way, we need a local classification of such metrics in dimension three, similar to that provided by Levi-Civita. This problem had been solved by Petrov [44] but for our purposes, we will refer to some recent literature on how to construct the local normal forms of these metrics by "gluing" constructions, see especially [11] for more detail. The construction in general dimension was undertaken in [12] this year. The idea here is that when given a pair of geodesically equivalent metrics $(h_1, \tilde{h}_1)$ of dimension n_1 and another pair $(h_2, \tilde{h}_2)$ of dimension n_2, there exists a geodesically equivalent pair of metrics $(g, \tilde{g})$ of dimension $n_1 + n_2$ which is obtained by gluing together the lower-dimensional pairs in a certain way. More importantly, any pair of geodesically equivalent metrics can, in general, be "split" into component parts i.e, represented as a composition of metrics of lower dimension under this gluing operation. Those metrics which cannot be split into simpler parts are called "building blocks".
For example, in dimension 1, there is only one building block

$$h = dx^2 \ , \ \tilde{h} = X(x)dx^2$$

where $X(x)$ is some arbitrary function. We can create a pair of two-dimensional geodesically equivalent metrics g and $\tilde{g}$ by gluing together two 1-d building blocks. In particular, we put

$$h_1 = dx^2 \ , \ \tilde{h}_1 = \frac{1}{X(x)^2}dx^2 \ , \ h_2 = dy^2 \ , \ \tilde{h}_2 = \frac{1}{Y(y)^2}dy^2$$

and, combining these under the rules of the gluing construction, we have the pair of two-dimensional metrics

$$\begin{aligned} g &= (X(x) - Y(y))dx^2 + (Y(y) - X(x))dy^2 \\ \tilde{g} &= \frac{X(x) - Y(y)}{X(x)^2 Y(y)}dx^2 + \frac{Y(y) - X(x)}{X(x)Y(y)^2} \end{aligned}$$

which is the pair suggested by Dini [45]. We can generalise this construction to obtain Levi-Civita's normal form for a pair of projectively equivalent Riemannian metrics in n dimensions.

Higher Dimensional Building Blocks

In dimension 2, building blocks can be expressed in one of three normal forms. Apart from the trivial case, $\tilde{g} = cg$ for constant c we also have the following:

Complex-Liouville Case

$$g = \mathcal{I}(\Phi)dxdy,$$

$$\tilde{g} = -\left(\frac{\mathcal{I}(\Phi)}{\mathcal{I}(\Phi)^2+\mathcal{R}(\Phi)^2}\right)^2 dx^2 + 2\frac{\mathcal{I}(\Phi)\mathcal{R}(\Phi)}{\left(\mathcal{I}(\Phi)^2+\mathcal{R}(\Phi)^2\right)^2}dxdy + \left(\frac{\mathcal{I}(\Phi)}{\mathcal{I}(\Phi)^2+\mathcal{R}(\Phi)^2}\right)^2 dy^2$$

where Φ is an arbitrary holomorphic function of the complex variable $\zeta = x+iy$.

Jordan-block case

$$g = (1 + xY'(y))dxdy$$

$$\tilde{g} = \frac{1+xY'(y)}{Y(y)^4}\left(-2Y(y)dxdy + (1+xY'(y))dy^2\right)$$

where Y is an arbitrary function.

In 3 dimensions, building blocks may also take one of three normal forms. As well as the trivial case, there is an example due to Petrov [44], which generalises the Jordan block case:

Petrov case

$$\begin{aligned}
g &= \left(4y\lambda'(z)+2\right)dxdz + dy^2 + 2x\lambda'(z)dydz + x^2(\lambda'(z))^2dz^2, \\
\tilde{g} &= \frac{1}{\lambda(z)^6}\Big[\left(4y\lambda(z)^2\lambda'(z) + 2\lambda(z)^2\right)dxdz + \lambda(z)^2dy^2 \\
&- \left(4y\lambda z\lambda'(z) + 2\lambda(z) - 2x\lambda(z)^2\lambda'(z)\right)dydz \\
&+ \left(4y^2(\lambda'(z))^2 + 4y\lambda'(z) - 4xy\lambda(z)(\lambda'(z))^2\right)dz^2 \\
&+ \left(1 + x^2\lambda(z)^2(\lambda'(z))^2 - 2x\lambda(z)\lambda'(z)\right)dz^2\Big]
\end{aligned} \tag{3.42}$$

where the function λ is arbitrary.
The other normal form in three dimensions for such a projectively equivalent metric pair was essentially described by Eisenhart [46]

Eisenhart Case

$$\begin{aligned} g &= 2dzdx + h(y,z)_{11}dy^2 + 2h(y,z)_{12}dydz + h(y,z)_{22}dz^2 \\ \tilde{g} &= 2\alpha dzdx + \alpha h(y,z)_{11}dy^2 + 2\alpha h(y,z)_{12}dydz + (\beta + \alpha h(y,z)_{22})\, dz^2 \end{aligned} \tag{3.43}$$

where α and β are constants and h is an arbitrary two-dimensional metric. Therefore, there are essentially (up to some sign changes) just seven different normal forms for pseudo-Riemannian metrics in three dimensions:

- We may glue together three one-dimensional building blocks (=1),
- we may glue a one-dimensional building block to a two-dimensional one (=3),
- we may consider a three-dimensional building block (=3).

With respect to the current problem of the projective invariance of conformal flatness, the first case has been dealt with. Furthermore, we already know the answer for the generic (with degree of mobility = 1) and trivial three-dimensional building block and so there is no need to discuss it. This leaves us with five cases (three of which are of the 1+2 form and two which are 3-d building blocks) to consider and we will do this in separate parts. We will summarize here just the main results.

Metrics in the 1+2 Category

We'll refer to the different sections by the two-dimensional part.

Trivial Normal Form

By using the gluing construction we find that, in local coordinates, these metrics take the form

$$\begin{aligned} g &= \left(c^{-\frac{1}{3}} - Z(z)\right) h + \left(c^{-\frac{1}{3}} - Z(z)\right)^2 dz^2 \\ \tilde{g} &= -\frac{c}{Z(z)} \left(c^{-\frac{1}{3}} - Z(z)\right) h + \frac{c^{\frac{2}{3}}}{Z(z)^2} \left(c^{-\frac{1}{3}} - Z(z)\right)^2 dz^2 \end{aligned}$$

where c, $Z(z)$ and h are an arbitrary constant, function and a two-dimensional metric (with coordinates (x, y)), respectively.
In this case, we find that g is conformally flat if and only if h has constant curvature. This result can be shown by using isothermal coordinates to write

$$h = e^{w(x,y)}(dx^2 + dy^2)$$

where $w(x, y)$ is some arbitrary function. Then, the Cotton tensor vanishes identically if and only if

$$\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) e^{-w(x,y)} = \text{constant}$$

which is precisely the scalar curvature of h. In this instance, the Cotton tensor for $\tilde{g}$ also vanishes.
Interestingly, the projective Weyl tensor vanishes if and only if we additionally have

$$Z''(z) = \frac{R_h(1 - c^{\frac{1}{3}} Z(z))^3 + 2cZ'(z)^2}{c(Z(z) - c^{-\frac{1}{3}})}$$

where R_h is the curvature of h. Hence, in this case, we have many examples of metrics g of degree of mobility 2 for which g and all metrics projectively equivalent to g are conformally flat. On such example is

$$\begin{aligned} g &= (1-z)\,(dx^2+dy^2) + (1-z)^2\,dz^2, \\ \tilde{g} &= -\frac{1}{z}\,(1-z)\,(dx^2+dy^2) + \frac{1}{z^2}\,(1-z)^2\,dz^2. \end{aligned}$$

Complex-Liouville case

In local coordinates, these metrics take the form

$$\begin{aligned} g &= \frac{\mathcal{I}(\Phi)^2}{2^{4/3}}(dx^2 - dy^2) + \Big(2^{-1/3}\mathcal{I}(\Phi)\mathcal{R}(\Phi) - \mathcal{I}(\Phi)Z(z)\Big)\,dxdy \\ &\quad + \Big(Z(z)^2 - 2^{2/3}\mathcal{R}(\Phi)Z(z) + 2^{-2/3}\left(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2\right)\Big)\,dz^2 \\ \tilde{g} &= -\frac{\mathcal{I}(\Phi)^2}{(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2)^2}(dx^2 - dy^2) + \left(\frac{2\mathcal{R}(\Phi)\mathcal{I}(\Phi)}{(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2)^2} - \frac{2^{2/3}\mathcal{I}(\Phi)}{Z(z)(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2)}\right) dxdy \\ &\quad -\frac{2^{2/3}}{Z(z)^2(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2)}\Big(Z(z)^2 - 2^{2/3}\mathcal{R}(\Phi)Z(z) + 2^{-2/3}(\mathcal{R}(\Phi)^2 + \mathcal{I}(\Phi)^2)\Big). \qquad (3.44) \end{aligned}$$

As in the Riemannian case, there are several branches to consider here depending on the form of Φ and Z. In the generic case, for example, (where Z is not constant and $\Phi(\xi) = u(x,y) + iv(x,y)$ does not have the property that u and v are harmonic) g is conformally flat if and only if Φ and Z satisfy the differential equations

$$(Z'(z))^2 = \sum_{m=0}^{5} A_m Z(z)^m \ , \ (\Phi'(\zeta))^2 = \upsilon + i \sum_{m=0}^{5} \frac{2^{-m/3}}{8} A_m \Phi(\zeta)^m$$

for some constants A_m, υ. In this case, $\tilde{g}$ is also conformally flat if and only if $A_5 = 0$ but then g and $\tilde{g}$ are projectively flat. In the general case, this result also holds true but the computations are quite complicated to quote here. That is, we have the following

Proposition 3.5. *The projectively equivalent metrics g and $\tilde{g}$ in the Complex-Liouville class (3.44) are both conformally flat if and only if they are both projectively flat and hence, of constant curvature.*

Jordan Block Case

In local coordinates, the metrics here take the form

$$\begin{aligned} g &= -\left(2^{-\frac{1}{3}}Y(y) + Z(z)\right)(1 + xY'(y))\,dxdy - 2^{-\frac{4}{3}}\,(1 + xY'(y))^2\,dy^2 \\ &+ \left(Z(z) + 2^{-\frac{1}{3}}Y(y)\right)^2 dz^2, \\ \tilde{g} &= -\frac{2\left(2^{-\frac{1}{3}}Y(y) + Z(z)\right)(1 + xY'(y))}{Y(y)^3 Z(z)}dxdy + \frac{(1 + xY'(y))^2}{Y(y)^4}dy^2 \\ &- \frac{2^{\frac{2}{3}}}{Y(y)^2}\left(1 + \frac{2^{-\frac{1}{3}}Y(y)}{Z(z)}\right)^2 dz^2 \end{aligned}$$

for some arbitrary functions $Y(y)$ and $Z(z)$. By undertaking a similar procedure to that in the Riemannian case, we find that g is conformally flat if and only if one of the following holds:

$$Y(y) = \chi y + \psi \ , \ Z(z) = \zeta$$

where χ, ψ and ζ are constants **or**

$$Y(y) = \psi \ , \ Z'''(z) = -\frac{2^{\frac{5}{3}}\left(5Z'(z)^3 2^{\frac{5}{3}}\psi Z'(z)Z''(z) - 4Z(z)Z'(z)Z''(z)\right)}{\left(\psi + 2^{\frac{1}{3}}Z(z)\right)^2}$$

where ψ is constant.

In the first case, g is projectively flat and so, all projectively equivalent metrics are projectively flat and hence, conformally flat. In the second case, $\tilde{g}$ is also conformally flat if and only if $Z(z)$ satisfies

$$Z''(z) = \frac{2Z'(z)^2 P_1(Z(z))}{P_2(Z(z))}$$

where

$$\begin{aligned} P_1(Z(z)) &= 2^{\frac{2}{3}}\psi^6 + 12\psi^5 Z(z) + 30.2^{\frac{1}{3}}\psi^4 Z(z)^2 + 40.2^{\frac{2}{3}}\psi^3 Z(z)^3 \\ &+ 60\psi^2 Z(z)^4 + 24.2^{\frac{1}{3}}\psi Z(z)^5 + 4.2^{\frac{2}{3}}Z(z)^6, \\ P_2(Z(z)) &= 2^{\frac{1}{3}}\psi^7 + 7.2^{\frac{2}{3}}\psi^6 Z(z) + 42\psi^5 Z(z)^2 + 70.2^{\frac{1}{3}}\psi^4 Z(z)^3 \\ &+ 70.2^{\frac{2}{3}}\psi^3 Z(z)^4 + 84\psi^2 Z(z)^5 + 282^{\frac{1}{3}}\psi Z(z)^6 + 4.2^{\frac{2}{3}}Z(z)^7. \end{aligned}$$

Under this condition, the projective Weyl tensor vanishes and g and $\tilde{g}$ are both of constant curvature.

Three-Dimensional Building Blocks

Petrov Case

In this case, the metric g is conformally flat if and only if the function $\lambda(z)$ in (3.42) is constant and we have

$$g = dy^2 + 2dxdz$$

which is projectively flat (constant curvature).

Eisenhart Case

In the Eisenhart case, the metrics g and $\tilde{g}$ (3.43) are affinely equivalent so conformal flatness of one implies conformal flatness of the other. The Cotton tensor for g has just one non-vanishing component (in the coordinates given that is C_{323}) which is a third order PDE in the components of the two-dimensional metric h. This may be written as

$$2h_{11,z}^2 h_{11,y} + 2h_{11,y}^2 h_{22,y} - 4h_{11,y}^2 h_{12,z} - 2h_{11}^2 \left(h_{11,yzz} - 2h_{12,yyz} + h_{22,yyy}\right)$$

$$= h_{11} \left(2h_{11,z}h_{11,yz} + 2h_{11,y}h_{11,zz} - 2h_{12,z}h_{11,yy} + h_{22,y}h_{11,yy} - 6h_{12,yz}h_{11,y} + 3h_{22,yy}h_{11,y}\right)$$

where the y and z subscripts correspond to derivatives. On the other hand, we find that projective flatness of g is determined by a single condition of second order, namely

$$2h_{11} \left(h_{22,yy} + h_{11,zz} - 2h_{12,yz}\right) = h_{11,z}^2 + h_{11,y}h_{22,y} - 2h_{11,y}h_{12,z}.$$

Hence, in this class of examples there exist pairs of projectively equivalent metrics g and $\tilde{g}$ both of which are conformally flat but neither of which are projectively flat. In fact, if we let

$$K(y,z) = 2h_{11} \left(h_{22,yy} + h_{11,zz} - 2h_{12,yz}\right) - h_{11,z}^2 - h_{11,y}h_{22,y} + 2h_{11,y}h_{12,z}$$

then the two equations above are, respectively, equivalent to

$$\frac{\partial K}{\partial y} = 3\frac{h_{11,y}}{h_{11}}K \ , \ \ K = 0.$$

Hence, we wish to find metrics for which

$$K(y,z) = \kappa(z)h_{11}^3$$

for some arbitrary function $\kappa(z) \neq 0$. An example of a metric h which satisfies this condition is

$$h_{11} = e^z\rho(y) \ , \ \ h_{12} = h_{22} = 0$$

for $\rho(y)$ is an arbitrary function. The corresponding projectively equivalent metrics are

$$\begin{aligned} g &= 2dzdx + e^z\rho(y)dy^2 \\ \tilde{g} &= 2\alpha dzdx + \alpha e^z\rho(y)dy^2 + \beta dz^2 \end{aligned}$$

3.4.3 Main Result

Given the above analysis, we have the following result for the pseudo-Riemannian case

Theorem 3.6. *Let g and $\tilde{g}$ be two projectively equivalent three-dimensional Riemannian or pseudo-Riemannian metrics which are non-proportional. If g is conformally flat, then $\tilde{g}$ is also conformally flat if and only if g has constant curvature* ***or*** *can be written in local coordinates in one of the following forms:*

1. $$g = \left(c^{-\frac{1}{3}} - Z(z)\right) h + \left(c^{-\frac{1}{3}} - Z(z)\right)^2 dz^2$$

   where c and $Z(z)$ are an arbitrary constant and function, respectively, and h is a two-dimensional metric of constant curvature.

2. $$g = 2dzdx + h_{11}(y,z)dy^2 + 2h_{12}(y,z)dydz + h_{22}(y,z)dz^2$$

   where the functions h_{ab} satisfy the condition

   $$2h_{11}\left(h_{22,yy} + h_{11,zz} - 2h_{12,yz}\right) - h_{11,z}^2 - h_{11,y}h_{22,y} + 2h_{11,y}h_{12,z} = \kappa(z)h_{11}^3$$

   for some function $\kappa(z)$.

3.5 Non-Metrisable Extremal Curves

In his paper [47], Douglas tackles the problem of determining whether a system of paths defined by (2.3) can be identified with the totality of extremals of some variational problem

$$\int \phi\left(x, y^j, \frac{dy^j}{dx}\right) dx = \min.$$

He completely solves this problem in the three-dimensional case. Central to his solution is a 3×3 matrix of functions whose entries are determined by the F^i and their partial derivatives. Different systems fall into different categories determined by the rank of this matrix and subsequently various subcategories determined by relations the entries themselves must satisfy. An interesting question to ask is:

Question Does there exist a system of differential equations of the form (2.3) in some open set $U \subset \mathbb{R}^3$ such that its integral curves are the extremals of some variational problem but are not the unparametrised geodesics of some metric?

Unsurprisingly, the answer is "Yes". This is too be expected as not all solutions dynamical problems involving energy minimisation may be reformulated to a geodesic structure. We have all the information we need to show this and we do so by means of an example.

Proof. Consider the system of differential equations

$$\frac{d^2y}{dx^2} = F(x,y)\left(\frac{dy}{dx}\right)^2 \quad , \quad \frac{d^2z}{dx^2} = G(x,z)\left(\frac{dz}{dx}\right)^2 . \tag{3.45}$$

We notice that this is just a special case of (3.22). In Douglas's classification this system is of type IIa1 and, in [47], he demonstrates that the integral curves of a differential system which is more general than (3.45) are the extremal curves of some variational problem. He sets up a system of partial differential equations which controls this question and, in our case, we can solve to get

$$\phi = A\left(\frac{dy}{dx}\right)^2 \exp\left(-2\int_0^y F(x,\tilde{y})d\tilde{y}\right) + B\left(\frac{dz}{dx}\right)^2 \exp\left(-2\int_0^z G(x,\tilde{z})d\tilde{z}\right)$$

where A and B are constants. However, these curves will not be the geodesics of some metric if $F_x \neq 0$ or $G_x \neq 0$ as was shown in Example 2. $\square$

Chapter 4

Torsion-free Path Geometries

As discussed previously, for a given path geometry

$$\frac{d^2y^i}{dx^2} = F^i\left(x, y^j, \frac{dy^j}{dx}\right) = F^i(x, y^j, p^j) \ , \ \ i, j = 1, \ldots, n$$

there is an associated tensor

$$T^i_j = -\frac{\partial F^i}{\partial y^j} - \frac{1}{4}\frac{\partial F^i}{\partial p^k}\frac{\partial F^k}{\partial p^j} + \frac{1}{2}\frac{d}{dx}\frac{\partial F^i}{\partial p^j}.$$

We gave the result that if the Grossman invariants

$$\tau^i_j = T^i_j - \frac{1}{2}\delta^i_j T^k_k \tag{4.1}$$

vanish for a three-dimensional path geometry, then this is precisely the condition for the four-dimensional space of solutions M to be endowed with a conformal structure. As we also observed, these invariants form part of the conditions for a given path geometry to be locally diffeomorphic to a trivial system (straight lines). Where systems with vanishing Fels invariants are deemed to be in the *projective branch* of the set of path geometries, we termed those with vanishing Grossman invariants as being in the *conformal branch*. We also refer to such systems as being torsion-free.

In this chapter, we focus our attention on the conformal branch and, in particular, the double fibration picture (2.2). Here, the solution space M admits a Segré structure such that the $n-1$-dimensional submanifolds in M corresponding to points in U are α-surfaces. Of particular interest is the case $n = 3$,where the Segré structure is a conformal structure and α-surfaces are totally isotropic null two-dimensional surfaces. Systems of ODEs which lie both in the projective and conformal branches are diffeomorphic to the trivial system which has Lie point symmetry algebra $sl(4, \mathbb{R})$. The corresponding Lie group $PSL(4, \mathbb{R})$ acts projectively on $U \subset \mathbb{RP}^3$ preserving the unparametrised geodesics of the flat projective connection. An important group isomorphism underlying the double fibration picture is

$$PSL(4, \mathbb{R}) \sim SO(3, 3)$$

where $SO(3, 3)$ is the flat conformal structure on the solution space M. An interesting result which exploits this isomorphism, which we shall examine in this chapter, is that if the curvature of the conformal structure does not vanish, then the conformal symmetry group is a proper subgroup of $SO(3, 3)$ and conformal Killing vectors on M give rise to point symmetries of the corresponding path geometry on U. This is a result that we will exploit to construct conformal structures with a high number of symmetries using knowledge of the point symmetries of path geometries and vice versa.

4.1 Twistor Correspondence

The correspondence between three-dimensional path geometries on U and conformal structures on the space of solutions M is reminiscent of the classical twistor picture of Penrose [19]. In fact, we can make contact with this picture if we assume that the conformal structure $(M, [g])$ is real analytic and continue it to the complexified version. The motivation behind this mechanism is to relate natural structures in algebraic geometry with special systems of differential equations in mathematical physics. To begin with, for a given four-dimensional conformal structure $(M, [g])$ of signature $(2, 2)$ there is a canonical bundle isomorphism

$$TM \cong \mathbb{S} \otimes \mathbb{S}' \tag{4.2}$$

where $\mathbb{S}$ and $\mathbb{S}'$ are rank two vector bundles (called spin-bundles) over M with parallel symplectic structures ϵ, ϵ'. Under this isomorphism, the metric decomposes as follows

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \epsilon(v_1, v_2)\epsilon'(w_1, w_2)$$

where $v_1, v_2 \in \Gamma(\mathbb{S})$ and $w_1, w_2 \in \Gamma(\mathbb{S}')$. Then, null vectors of the conformal structure are those of the form $V = \kappa \otimes \pi$ and α-planes are those spanned by null vectors with π fixed and, as in the generalised Segré case, α-surfaces are defined as those two-dimensional surfaces with the tangent at each point being an α-plane.

Furthermore, for a given oriented 4-dimensional manifold (M, g) of signature $(2, 2)$, the Hodge-star operator $* : \Lambda^2 \to \Lambda^2$ is an involution on two forms and induces a decomposition into eigenspaces

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

which we call self-dual and anti-self-dual, respectively. Then, the Riemann curvature $R_{abcd} = R_{[ab][cd]} = R_{cdab}$ can be thought of as a section of symmetric endomorphisms of Λ^2, $\mathcal{R} : \Lambda^2 \to \Lambda^2$ which decomposes as follows:

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ + \frac{R}{12} & \phi \\ \hline \phi & C_- + \frac{R}{12} \end{array} \right)$$

where C_+ and C_- are the self-dual (SD) and anti-self-dual (ASD) parts of the conformal Weyl tensor, respectively, ϕ is the tracefree Ricci curvature and R is the scalar curvature. We say that the metric g is anti-self-dual if $C_+ = 0$ and, since the Weyl tensor is conformally invariant, this is actually a property of the conformal structure $[g]$.

These types of conformal structures are important to the theory of mathematical physics. In [19], a procedure was developed to describe a finite number

of physical gravitons (spin-2 massless Poincaré-covariant fields), which are solutions to the Einstein equations, without resorting to perturbative methods. In particular, a given ASD conformal structure describes a single graviton in a pure helicity state and so, is the fundamental example of this theory.
We may, similarly, consider the decomposition of the Riemann tensor under the canonical bundle isomorphism (4.2). Using indices $A, B, \ldots = 0, 1$ for the bundle $\mathbb{S}$ and $A', B', \ldots = 0, 1$ for $\mathbb{S}'$ we have

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \psi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &\quad + \phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ &\quad + \frac{R}{12}(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}) \end{aligned}$$

where ψ_{ABCD} and $\psi_{A'B'C'D'}$ are ASD and SD Weyl spinors which are symmetric in their indices and $\phi_{A'B'CD} = \phi_{(A'B')(CD)}$ is the traceless Ricci spinor. Then, a given conformal structure is anti-self-dual (or self-dual) if the SD (or ASD) Weyl spinor vanishes (again these objects are conformally invariant and hence well-defined for the conformal structure $(M, [g])$).
A seminal result of Penrose [19] in this area is the following:

Proposition 4.1. *A maximal three-dimensional family of α-surfaces exists in M if and only if the conformal structure $(M, [g])$ is anti-self-dual.*

Motivated by this result, let us the define the *twistor space* $\mathcal{T}$ as the space of α-surfaces in M i.e, each point in $\mathcal{T}$ corresponds to an α-surface. In the complexified version, $\mathcal{T}$ is a complex manifold and points in M corresponds to rational curves (copies of $\mathbb{CP}^1$). These are known as *twistor lines*. The embedding of a rational curve in $\mathcal{T}$ is described by its normal bundle, a holomorphic vector bundle of rank 2. The structure of such bundles is well understood. There is one such bundle or rank one denoted $\mathcal{O}(-1)$ and defined by

$$\mathcal{O}(-1) = \{(x, y) \in \mathbb{C}^2 \times \mathbb{CP}^1 : x \in y\}.$$

Other vector bundles can be constructed by taking duals, tensor products and sums. In particular,

$$\mathcal{O}(-1)^* = \mathcal{O}(1) \ , \ \mathcal{O}(m) = \mathcal{O}(1) \otimes \ldots \otimes \mathcal{O}(1).$$

Any holomorphic vector bundle over $\mathbb{CP}^1$ may then be written as

$$\mathcal{E} = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \ldots \oplus \mathcal{O}(m_k) \ , \ m_a \in \mathbb{Z}.$$

A result of Hitchin [33] is that if a given conformal structure $(M, [g])$ is ASD then the rational curves in $\mathcal{T}$ corresponding to points in M have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. On the other hand, a theorem of Kodaira [48] implies that, given a family of rational curves in some three-dimensional complex manifold $\mathcal{T}$ with normal bundle $Y = \mathcal{O}(1) \oplus \mathcal{O}(1)$, there exists a four-dimensional manifold M such that Y belongs to a locally complete family $\{Y_x : x \in M\}$ with $T_x M \cong H^0(Y_x, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbb{C}^4$. Then, there is a null cone at each point of M defined by set of sections of $H^0(Y_x, \mathcal{O}(1) \oplus \mathcal{O}(1))$ which vanish somewhere on Y_x.
Hence, in the twistor picture, there is a correspondence between ASD conformal structures $(M, [g])$ of signature (2,2) and path geometries on the twistor space $\mathcal{T}$ where points in $\mathcal{T}$ correspond to isotropic null 2-surfaces in M (α-surfaces) and points in M correspond to rational curves in $\mathcal{T}$ with normal bundle $\cong \mathcal{O}(1) \oplus \mathcal{O}(1)$. This allows us to relate an object in mathematical physics (the single graviton) to a well understood structure in complex analytical geometry. From our point of view, the aim is to describe these twistor lines by systems of second order ODEs in the form (2.3) which correspond to desired geometric structures on the correspondence space M. In this case, the twistor lines with specified normal bundle are precisely the integral curves of a path geometry with vanishing Grossman invariants.
Thus, we have the following result [18]:

Theorem 4.1. *There is a one-to-one correspondence between path geometries on U with*

$$\tau^i_j = T^i_j - \frac{1}{2}\delta^i_j T^k_k = 0 \tag{4.3}$$

and anti-self-dual conformal structures of signature (2,2) on the moduli space of solutions M with points in U corresponding to isotropic null 2-surfaces in M.

For the remainder of the discussion we will refer to U as the *twistor space*. A point $p \in M$ corresponds to an integral curve L_p of the corresponding path geometry (2.3) (a twistor line). Two points p_1 and p_2 in M are null separated (lie on the same α-surface) if and only if the curves L_{p_1} and L_{p_2} intersect at some point. The double fibration picture looks as follows:

$$U \leftarrow \mathcal{F} \rightarrow M$$

where, before, we had $\mathcal{F} = \mathbb{P}(TU)$ but also $\mathcal{F} \cong M \times \mathbb{RP}^1$. Then U arises as a quotient of $\mathcal{F}$ by a two-dimensional Frobenius integrable distribution described by vector fields L_0, L_1, called the Lax pair, which are horizontal lifts of vectors spanning an α-plane. The existence of such an integrable distribution is guaranteed by the ant-self-duality of $(M, [g])$. Then, if the conformal structure is given by

$$g = e^1 \odot e^2 - e^3 \odot e^4$$

for some one-forms $e^1, \dots, e^4$ with dual vector fields $E_1, \dots, E_4$,

$$L_0 = E_1 - \lambda E_3 + f_0 \frac{\partial}{\partial \lambda} \ , \ L_1 = E_4 - \lambda E_2 + f_1 \frac{\partial}{\partial \lambda}$$

where f_0 and f_1 are functions on $\mathcal{F} = M \times \mathbb{RP}^1$ making the Lax pair Frobenius integrable and λ parametrises the $\mathbb{RP}^1$.

4.2 From ASD Conformal Structures to Systems of ODEs

The twistorial picture allows us to link torsion-free systems of ODEs in three dimensions to ASD conformal structures of signature $(2, 2)$ in dimension four. Here, we show explicitly how, given a manifold with such a conformal structure $(M, [g])$, one might construct the corresponding system of ODEs. The idea here is to construct the real projective line parametrised by $\lambda \in \mathbb{RP}^1$ of real α-surfaces through a point $p \in M$, thereby utilising the underlying correspondence. Each curve depends on the coordinates in M of the point p and their union coincides with the integral curves of some second order system in three dimensions. The procedure involved to reproduce this system here is fully described in [20].

First, consider the Lax pair L_0 and L_1 on the ambient space $\mathcal{F} = M \times \mathbb{RP}^1$, as described above, and find three functions on $\mathcal{F}$ which satisfy

$$L_0 f = 0 \ , \ L_1 f = 0.$$

We name these functions $(\tilde{x}, \tilde{y}, \tilde{z})$ as they descend to the twistor space U where they provide a local coordinate system (x, y, z). A point in M then corresponds to a curve in U as desired. The pull back of the four-parameter family of curves to $\mathcal{F}$ can be parametrised as

$$\lambda \to (\tilde{x}(\lambda, p), \tilde{y}(\lambda, p), \tilde{z}(\lambda, p))$$

where $p = (\alpha, \beta, \gamma, \eta)$ is a point in M. Then, we use the implicit function theorem to solve the equation $x = \tilde{x}(\lambda, p)$ for λ and the relations

$$y = \tilde{y} \ , \ z = \tilde{z} \ , \ y' = \frac{\partial \tilde{y}}{\partial x} \ , \ z' = \frac{\partial \tilde{z}}{\partial x}$$

to express $(\alpha, \beta, \gamma, \eta)$ as functions of (y, z, y', z'). Differentiating once more and substituting $(\alpha, \beta, \gamma, \eta)$ gives a pair of second order ODEs (2.3) corresponding to a path geometry on U.

4.2.1 Ricci-flat case

As an example of this construction, consider an ASD conformal structure $(M, [g])$ of signature (2,2) which contains a Ricci-flat metric and let $(\alpha, \beta, \gamma, \eta)$ be local coordinates on M. A result of Plebański [49] is that any such metric g may be written locally as

$$g = d\alpha d\beta + d\gamma d\eta - \Theta_{\beta\beta} d\eta^2 - \Theta_{\gamma\gamma} d\alpha^2 + 2\Theta_{\beta\gamma} d\alpha d\eta \tag{4.4}$$

where subscripts indicate differentiation and $\Theta = \Theta(\alpha, \beta, \gamma, \eta)$ is a function satisfying the *second heavenly equation*

$$\Theta_{\alpha\beta} + \Theta_{\gamma\eta} + \Theta_{\beta\beta}\Theta_{\gamma\gamma} - \Theta_{\beta\gamma}^2 = 0. \tag{4.5}$$

In this case, the only non-vanishing part of the curvature is given by the ASD Weyl spinor

$$\psi_{ABCD} = \frac{\partial^4 \Theta}{\partial \alpha^A \partial \alpha^B \partial \alpha^C \partial \alpha^D} \ , \ A, B, C, D = 0, 1$$

where $(\alpha^0, \alpha^1) = (\gamma, -\beta)$. In the complexified version of the theory, the Ricci-flatness condition gives rise to additional structures on the twistor space U namely

1. A projection $\mu : U \to \mathbb{CP}^1$, such that the four parameter family of curves above are sections of μ.

2. A symplectic structure with values in $\mathcal{O}(2)$ on the fibers of μ.

We would like to represent this as some invariant condition on the pair of ODEs describing the path geometry. Firstly, the Lax pair for this system may be written as

$$\begin{aligned} L_0 &= \partial_\gamma - \lambda \left(\partial_\alpha - \Theta_{\beta\gamma} \partial_\gamma + \Theta_{\gamma\gamma} \partial_\beta \right), \\ L_1 &= \partial_\beta + \lambda \left(\partial_\eta + \Theta_{\beta\beta} \partial_\gamma - \Theta_{\beta\gamma} \partial_\beta \right). \end{aligned}$$

A curve $L_p \subset U$ corresponding to a point $p \in M$ is parametrised by choosing a two-dimensional fiber $\mu : U \to \mathbb{RP}^1$ and defining (α, η) to be the coordinates of the initial point of the curve and $(\gamma, -\beta)$ to be the tangent vector to the curve. In this way, the pull back of the curve is $\lambda \to (x = \lambda, y = \tilde{y}(\lambda, p), z = \tilde{z}(\lambda, p))$ where the functions $(\tilde{y}, \tilde{z})$ admit the following expansion which can be found in [50],

$$\begin{aligned} \tilde{y} &= \alpha + \lambda\gamma - \Theta_\beta \lambda^2 + \Theta_\eta \lambda^3 + \dots \\ \tilde{z} &= \eta - \lambda\beta - \Theta_\gamma \lambda^2 - \Theta_\alpha \lambda^3 + \dots . \end{aligned}$$

The terms of higher order in λ can be obtained by recursion of $L_A(\tilde{y}) = L_A(\tilde{z}) = 0$. In the next section, we will show how to derive the Grossman invariants from the ASD condition on the conformal structure and here, we may write the path geometry corresponding to a Ricci-flat conformal structure explicitly. Now, we investigate some specific examples which are of special note.

Example 1

Perhaps one of the simplest solutions to the second heavenly equation (4.5) is when Θ is a function of only one of its arguments, say γ. In this case, the non-vanishing part of the curvature is

$$\psi_{0000} = \frac{\partial^4 \Theta}{\partial \gamma^4},$$

so let us consider the example

$$\Theta = \frac{1}{4}\gamma^4.$$

The Lax pair here is

$$\begin{aligned} L_0 &= \partial_\gamma - \lambda\left(\partial_\alpha + 3\gamma^2\partial_\beta\right), \\ L_1 &= \partial_\beta + \lambda\partial_\eta \end{aligned}$$

and thus

$$\begin{aligned} \tilde{y} &= \alpha + \lambda\gamma, \\ \tilde{z} &= \eta - \lambda\beta - \gamma^3\lambda^2. \end{aligned}$$

The second order system describing the pull back of this curve is easy to compute. Since

$$y' = \gamma \ , \ y'' = 0 \ , \ z'' = -2\gamma^3$$

then

$$y'' = 0 \ , \ z'' = -2(y')^3. \tag{4.6}$$

This system has a nine-dimensional point symmetry algebra which is also the largest symmetry algebra of a non-trivial ASD conformal structure. We look at this in more detail later.

Example 2

This example, when analytically continued to Riemannian signature, is relevant in the theory of gravitational instantons. If we choose $\Theta = \Theta(\alpha, \beta, \gamma)$ then the second heavenly equation (4.5) reduces to a wave equation on a flat

(2 + 1)-dimensional background. If we let $\tau = \Theta_\beta$ and perform a Legendre transformation

$$H(\tau,\gamma,\alpha) := \tau\beta(\alpha,\gamma,\tau) - \Theta\left(\alpha,\beta(\alpha,\gamma,\tau),\gamma\right)$$

then with $\beta = H_\tau$ and $\Theta_\gamma = -H_\gamma$, the heavenly equation becomes

$$H_{\tau\alpha} + H_{\gamma\gamma} = 0. \tag{4.7}$$

The metric, in this case, is

$$\begin{aligned} g &= H_{\tau\tau}\left(\frac{1}{4}d\gamma^2 + d\alpha d\tau\right) - \frac{1}{H_{\tau\tau}}\left(d\eta - \frac{H_{\tau\tau}}{2}d\gamma + H_{\tau\gamma}d\alpha\right)^2 && (4.8)\\ &= V\left(\frac{1}{4}d\gamma^2 + d\alpha d\tau\right) - V^{-1}(d\eta + A)^2, && (4.9)\end{aligned}$$

where $V = H_{\tau\tau}$ and $A = H_{\tau\gamma}d\alpha - (H_{\tau\tau}/2)$ satisfy the monopole equation $\star dV = dA$ and $\star$ is the Hodge operator on $\mathbb{R}^{2,1}$ with its flat metric. Thus, this is an analytic continuation of the Gibbons-Hawking metric [51]. Since Θ_η vanishes, the whole series for $\tilde{y}$ truncates at second order. In particular,

$$\begin{aligned} \tilde{y} &= \alpha + \lambda\gamma - \lambda^2\tau \\ \tilde{z} &= \eta - \lambda H_\tau + \lambda^2 H_\gamma + \lambda^3 H_\alpha + \dots \end{aligned}$$

where $H = H(\alpha,\gamma,\tau)$ from which we can obtain the corresponding path geometry. An example, with $H(\alpha,\gamma,\tau) = \gamma\tau^2$, is

$$y'' = \frac{y'}{x} - \sqrt{\left(\frac{y'}{x}\right)^2 - \frac{2z'}{x}}, \tag{4.10}$$

$$z'' = \frac{1}{2}\left(\frac{y'}{x} - \sqrt{\left(\frac{y'}{x}\right)^2 - \frac{2z'}{x}}\right)^2. \tag{4.11}$$

The potential in the Gibbons-Hawking metric is linear in the flat coordinates on $\mathbb{R}^{2,1}$.

4.3 Grossman Invariants

Now let us describe how to derive the Grossman invariants (4.1) from the double fibration picture. This reverses the construction in the previous section and yields the technology required to construct the path geometry of integral curves on the twistor space of an ASD Ricci-flat conformal structure. The idea is to begin with an arbitrary ASD conformal structure and see the vanishing of the Grossman invariants of the corresponding path geometry arise as a necessary condition. Our procedure is analogous to the recursive construction of the Wilczynski invariants of a single nth order ODE [52].

Each point in U corresponds to an α-plane in M and the functions $(p^2, p^3) = \left(\frac{dy}{dx}, \frac{dz}{dx}\right)$ are null coordinates which are mutually orthogonal and so $a_2p^2+a_3p^3$ is null for arbitrary constants a_0, a_1. Furthermore, by differentiation, the one-form a_2dy+a_3dz, which for the sake of current convenience, I'll write as $a_2dy^2+a_3dy^3$ is orthogonal to $a_2p^2 + a_3p^3$.

In the derivation below, we shall regard $(y^i, p^i, x) = (y^2, y^3, p^2, p^3, x)$ as coordinates on the five-dimensional correspondence space $\mathcal{F} = \mathbb{P}(TU)$ from the double fibration picture (2.2), and define the degenerate metric g on this five-dimensional space. We then demand this quadratic form Lie derives up to scale along the total derivative d/dx and so gives a conformal structure on M. The metric g necessarily takes the form

$$g = \epsilon_{ij} dy^i dp^j + \phi_{ij} dy^i dy^j \tag{4.12}$$

where ϵ_{ij} is antisymmetric with $\epsilon_{23} = 1$ and $\phi_{ij} = \phi_{(ij)}$. The conformal structure of M is invariant along the fibres $\mathcal{F} \to M$ and therefore

$$\frac{dg}{dx} = \Omega^2 g$$

for some function Ω. Plugging in the expression (4.12) and comparing coefficients on both sides we obtain the equations

$$2\phi_{ij} + \frac{\partial F^k}{\partial p^j}\epsilon_{ik} = \Omega^2 \epsilon_{ij} \tag{4.13}$$

$$\frac{d\phi_{ij}}{dx} + \frac{\partial F^k}{\partial y^{(j}}\epsilon_{i)k} = \Omega^2 \epsilon_{ij}. \tag{4.14}$$

Taking the trace of (4.13) (using ϵ to raise and lower indices),

$$\frac{\partial F^k}{\partial p^k} = 2\Omega^2$$

and substituting back into (4.13) for Ω,

$$\phi_{ij} = -\frac{1}{2}\epsilon_{ik}\frac{\partial F^k}{\partial p^j} + \frac{1}{4}\epsilon_{ij}\frac{\partial F^k}{\partial p^k}.$$

Then, from equation (4.14), we obtain the Grossman invariants

$$-\frac{1}{2}\frac{\partial F^i}{\partial p^j} + \frac{\partial F^i}{\partial y^j} + \frac{1}{4}\frac{\partial F^i}{\partial p^k}\frac{\partial F^k}{\partial p^j} \sim \delta^i_j$$

where here we've used the fact that

$$\frac{\partial F^k}{\partial p^j}\frac{\partial F^i}{\partial p^k} - \frac{\partial F^k}{\partial p^k}\frac{\partial F^i}{\partial p^j} \sim \delta^i_j$$

to rewrite the third term. Note here that the expression for the conformal structure (4.12) actually gives a metric if $\Omega = 0$ i.e, $\frac{\partial F^k}{\partial p^k} = 0$ which implies

$$F^i = 2\epsilon^{ij}\frac{\partial \Lambda}{\partial p^j} \tag{4.15}$$

for some function Λ. The metric (4.12) then resembles the heavenly form of the Ricci-flat metric given by (4.4). The exact equivalence arises from evaluating (4.12) at $x = 0$ with coordinates $y^i = (\alpha, \eta)$, $p^i = (-\gamma, \beta)$ and $\Lambda(y^j, p^j) = -\Theta(\alpha, \beta, \gamma, \eta)$.

Example 3

One more example is given by

$$y'' = 0 \ , \ z'' = B(y') \tag{4.16}$$

for some arbitrary function B. The ASD conformal structure on the solution space M is type-N and Ricci flat. Notice here, that the system with submaximal point symmetry is a particular example.

This last example is Ricci flat. One may seek conditions on the functions F^i for this to be true more generally. We skip the details here but the following proposition was given in [20]:

Proposition 4.2. *Let $\Theta = \Theta(\alpha, \eta, \beta, \gamma)$ be a solution to the heavenly equation (4.5) which gives the Ricci-flat ASD metric (4.4). The corresponding system of*

ODEs with vanishing Grossman invariants is

$$y'' = 2\frac{\partial \Lambda}{\partial z'} \ , \ \ z'' = -2\frac{\partial \Lambda}{\partial y'}$$

where $\Lambda|_{x=x_0} = -\Theta\left(y(x_0), z(x_0), z'(x_0), -y'(x_0)\right)$ *and the* x*-dependence of* Λ *is determined by (4.14).*

4.4 Symmetries of Torsion-Free Path Geometries

In the twistor approach, we observed that if two points p_1 and p_2 in M are null separated then the corresponding twistor lines L_{p_1} and L_{p_2} intersect at a point. Then, null vectors in M can be obtained by considering two neighbouring curves in U and requiring that they intersect at exactly one point. It can be shown that any transformation which preserves points in U gives rise to one which preserves the conformal structure of M and vice versa.

Lemma 4.3. *There is a one-to-one correspondence between the conformal Killing vectors of a (2,2) ASD conformal structure* $(M, [g])$ *and point symmetries of the torsion-free system of ODEs whose integral curves are the corresponding twistor lines of* $(M, [g])$.

Proof. To show this, we again exploit the double fibration picture (2.2). Given a conformal Killing vector K of $(M, [g])$, we can lift it to a vector $\tilde{K}$ on the correspondence space $\mathcal{F}$, so that $[L_0, \tilde{K}] = 0$ and $[L_1, \tilde{K}] = 0$ where L_0 and L_1 are the Lax pair forming the twistor distribution and the commutators vanish modulo a linear combination of L_0 and L_1. This lift is given explicitly by $\tilde{K} = K + Q\partial_\lambda$ where Q is a quadratic polynomial in λ with coefficients depending on coordinates in M. The space U is a quotient of $\mathcal{F}$ spanned by L_0 and L_1 and so $\tilde{K}$ pulls back to a vector field $\mathcal{K}$ on U. Therefore, it generates a one-parameter group of transformations on U which take α-surfaces to α-surfaces in M. As K generates diffeomorphisms of M and integral curves of the associated path geometry in U correspond to points in M, then the action generated by $\mathcal{K}$ preserves the integral curves of the path geometry. Thus, it is a point symmetry.

Conversely, a point symmetry of the path geometry in U e.g, (2.3), corresponds to a transformation in M which maps α-surfaces to α-surfaces. and therefore, it gives rise to a conformal Killing vector on M. $\square$

In the trivial case, ($y'' = 0, z'' = 0$) the path geometry has point symmetry group $PSL(4, \mathbb{R})$ which is isomorphic to the symmetry group of the flat conformal structure $SO(3, 3)$. The sub-maximal case of a torsion free system of ODEs with nine-dimensional point symmetry algebra is (4.6) which corresponds to a Ricci-flat ASD conformal structure with only one non-vanishing component of the ASD Weyl tensor. Therefore, the 'gap' in (2,2) conformal geometry (the difference in dimension between the maximal and sub-maximal symmetry algebras) equals 6 = 15 - 9. We saw in Example 7 of Chapter 3 that, in the projective branch, following from Egorov's work [41], the submaximal case has 8-dimensional point symmetry algebra. Hence, by Lemma 4.3, the 'gap' in path geometries in three dimensions is also 6, and the submaximal case lies in the conformal branch. An account of the general theory of such 'gaps may be found in [43] and more recently in [42].

The system (4.6) is the unique (up to diffeomorphism) torsion-free path geometry with point symmetry algebra of sub-maximal size and we expect systems with 6,7 or 8 point symmetries to be also comparatively rare. To see explicitly why this occurs, consider the lift of an arbitrary path geometry

$$y'' = F(x, y, z, y', z') \ , \ \ z'' = G(x, y, z, y', z')$$

to the second jet bundle $J^2(U, \mathbb{R})$ which is a seven-dimensional manifold with local coordinates given by $(x, y, z, y', z', y'', z'')$. Any point symmetry of the path geometry can locally be described by some vector field χ on U which we can prolong to a vector field $\mathrm{pr}^{(2)}\chi$ over some open set in $J^2(U, \mathbb{R})$. Then the functions

$$\Delta^2 = y'' - F(x, y, z, y', z') \ , \ \ \Delta^3 = z'' - G(x, y, z, y', z')$$

are constant along $\mathrm{pr}^{(2)}\chi$.

Now suppose, such a path geometry admits a Lie point symmetry algebra of

dimension five, generated by five vector fields which prolong to an integral distribution $\tilde{L}$ of $J^2(U,\mathbb{R})$. In the generic case, the rank of $\tilde{L}$ is five and these vector fields will span the tangent bundle of some five-dimensional submanifold $\tilde{U} \subset J^2(U,\mathbb{R})$. Given that the codimension of $\tilde{U}$ is 2, we can construct two functions Δ^2, Δ^3 which are invariant with respect to the action of $\tilde{L}$, i.e, given a five-dimensional Lie algebra of vector fields L over U, there is no obvious obstruction to the existence of a path geometry with point symmetry algebra. Of course, the existence of such a path geometry depends on certain further regularity conditions. For example, there is no path geometry which has all the point symmetries $\partial_y, x\partial y, x^2\partial_y, x^3\partial_y, x^4\partial_y$, so caution is needed. It is sufficient to demand that the first prolongation of the vector fields to the first jet bundle $J^1(U,\mathbb{R})$ must not be contained within some four-dimensional submanifold of the tangent bundle $T(J^1(U,\mathbb{R}))$.
Lie algebras L of dimension six or greater will generically not give rise to invariant functions Δ^2, Δ^3, and there will be a constraint on finding non-trivial path geometries with point symmetry algebras of this size. In particular, we must require that the prolonged algebra forms a distribution of rank lower than six. If additionally, we impose that the path geometry be torsion-free (i.e, the Grossman invariants vanish) then this sudden decline of examples will be observed sooner, at dimension four rather than six. In the next section, we outline the prolongation procedure and present some examples of path geometries with dimensions 4,5,6,7,8,9 point symmetries together with some details of the Lie algebraic structure. Where possible we take advantage of the double fibration picture (2.2) to say something about the corresponding ASD conformal structure on the space of solutions.

4.4.1 Path geometries with large Symmetry Algebras

For a given path geometry in three dimensions

$$y'' = F(x,y,z,y',z') \ , \ \ z'' = G(x,y,z,y',z'),$$

the point symmetries may be found by the following well known procedure:

1. Let the generator of a point symmetry be

$$\chi = \chi_1\partial_x + \chi_2\partial_y + \chi_3\partial_z$$

where χ_a are functions of (x, y, z) for which we must solve.

2. Determine the first and second prolongations by

$$\eta_i^{(1)} = \frac{d\chi_i}{dx} - p^i\frac{d\chi_1}{dx} \quad , \quad \eta_i^{(2)} = \frac{d\eta_i^{(1)}}{dx} - q^i\frac{d\chi_1}{dx},$$

where $i = 2, 3$ and $q^i = \frac{d^2y^i}{dx^2}$.

3. The prolongation of the vector field χ to the second jet bundle $J^2(U, \mathbb{R})$ is given by

$$\mathrm{pr}^{(2)}(\chi) = \chi + \sum_{j=2}^{3}\eta_j^{(1)}\frac{\partial}{\partial p^j} + \sum_{j=2}^{3}\eta_j^{(2)}\frac{\partial}{\partial q^j}.$$

4. Then we determine the point symmetries by finding functions χ_i from

$$\mathrm{pr}^{(2)}(\chi)\left(\Delta^2\right)|_{\Delta^2=\Delta^3=0} = \mathrm{pr}^{(2)}(\chi)\left(\Delta^3\right)|_{\Delta^2=\Delta^3=0} = 0.$$

Special Ricci-Flat Case

Consider the system (4.16) with B real analytic. The submaximal torsion-free system (4.6) lies in this class so we might expect it to yield more examples of systems with a high number of point symmetries. The corresponding ASD conformal structure on the moduli space of solutions is given by

$$g = d\alpha d\beta + d\eta d\gamma - \frac{1}{2}B'(\gamma)d\alpha^2$$

and is Ricci-flat as it corresponds to (4.4) with $\Theta = (1/2)\int B(\gamma)d\gamma$.

Under these conditions, it is not difficult to show that the point symmetry algebra contains a six dimensional subalgebra $L_6 \subset sl(4, \mathbb{R})$. It transpires that L_6 is solvable and, in terms of point symmetries, we may write it down explicitly

$$L_6 = \mathrm{span}\{\mathbf{e}_1 = \partial_x, \mathbf{e}_2 = \partial_y, \mathbf{e}_3 = \partial_z, \mathbf{e}_4 = x\partial_y, \mathbf{e}_5 = z\partial_y, \mathbf{e}_6 = x\partial_x + 2y\partial_y + z\partial_z\}$$

However, as we have seen already, there will be some special cases of (4.16) for which the point symmetry algebra is larger but contains L_6 as a subalgebra.

Proposition 4.4. *Consider a system of two second order ODEs of the form (4.16) for some function B of the form*

$$B(y') = \sum_{k=0}^{\infty} \xi_k (y')^k.$$

If B is a quadratic function, then the system (4.16) is diffeomorphic to a trivial one (and the symmetry group is 15-dimensional). Otherwise, the symmetry algebra has dimension

$$12 - Rank(M_1) - Rank(M_2)$$

where M_1 is a matrix with rows

$$(\xi_k, \xi_{k+1}) \ , \ \ k \geq 3$$

and M_2 is a matrix with rows

$$(\xi_k \ \ (k-2)\xi_k \ \ (k-3)\xi_{k-1} \ \ -(k+1)\xi_{k+1}) \ , \ \ k \geq 3.$$

Proof. Without loss of generality, let us simplify the problem by making the diffeomorphism

$$z \to z + \frac{1}{2}\xi_0 x^2 + \xi_1 xy + \frac{1}{2}\xi_2 y^2$$

so that we obtain the system

$$y'' = 0 \ , \ \ z'' = \sum_{k=3}^{\infty} \xi_k (y')^k \tag{4.17}$$

which has the same number of point symmetries as the original.

For a given vector field,

$$\chi = \chi_1 \frac{\partial}{\partial x} + \chi_2 \frac{\partial}{\partial y} + \chi_3 \frac{\partial}{\partial z}$$

the expressions

$$\mathrm{pr}^{(2)}(\chi)\left(\Delta^2\right)|_{\Delta^2=\Delta^3=0} = 0 \ \text{ and } \ \mathrm{pr}^{(2)}(\chi)\left(\Delta^3\right)|_{\Delta^2=\Delta^3=0} = 0.$$

are real analytic in p^2 and p^3 with coefficients which are functions of x, y and z. For χ to be a symmetry of the system (4.17), each of these coefficients must vanish separately. This leads to the following system of differential equations for χ_1, χ_2 and χ_3:

$$\frac{\partial^2\chi_2}{\partial x^2}=\frac{\partial^2\chi_3}{\partial x^2}=\frac{\partial\chi_2}{\partial x\partial z}=\frac{\partial^2\chi_3}{\partial x\partial y}=\frac{\partial\chi_1}{\partial y\partial z}=\frac{\partial^2\chi_1}{\partial z^2}=\frac{\partial^2\chi_1}{\partial z^2}=0,$$

$$2\frac{\partial^2\chi_2}{\partial x\partial y}=\frac{\partial^2\chi_1}{\partial x^2}\ ,\ \frac{\partial^2\chi_2}{\partial y^2}=2\frac{\partial^2\chi_1}{\partial x\partial y}\ ,\ \frac{\partial^2\chi_2}{\partial y\partial z}=\frac{\partial^2\chi_1}{\partial x\partial z},$$

$$2\frac{\partial^2\chi_3}{\partial x\partial z}=\frac{\partial^2\chi_0}{\partial x^2}\ ,\ \frac{\partial^2\chi_3}{\partial y\partial z}=\frac{\partial^2\chi_1}{\partial x\partial y}\ ,\ \frac{\partial^2\chi_3}{\partial z^2}=2\frac{\partial^2\chi_1}{\partial x\partial z},$$

$$\frac{\partial^2\chi_3}{\partial y^2}=3\xi_3\frac{\partial\chi_2}{\partial x}\ ,\ \xi_3\frac{\partial\chi_2}{\partial z}=\frac{\partial^2\chi_1}{\partial y^2}\ ,\ \frac{\partial^2\chi_1}{\partial y^2}+3\xi_3\frac{\partial\chi_2}{\partial z}=0,$$

and for $k\geq 3$,

$$(k-3)\xi_k\frac{\partial\chi_1}{\partial z}-(k+1)\xi_{k+1}\frac{\partial\chi_2}{\partial z}\ ,\ \xi_{k+1}\frac{\partial\chi_2}{\partial z}=\xi_k\frac{\partial\chi_2}{\partial z},$$

$$(k-2)\xi_k\frac{\partial\chi_1}{\partial x}+\xi_k\frac{\partial\chi_3}{\partial z}+(k-3)\xi_{k-1}\frac{\partial\chi_1}{\partial y}-(k+1)\xi_{k+1}\frac{\partial\chi_2}{\partial x}-k\xi_k\frac{\partial\chi_2}{\partial y}=0.$$

Enforcing the system to be non-trivial (i.e, not all $\xi_k=0$) we obtain

$$\frac{\partial\chi_1}{\partial z}=\frac{\partial\chi_2}{\partial z}=\frac{\partial^2\chi_1}{\partial y^2}=0.$$

Then, the most general solution for χ is

$$\begin{aligned}
\chi_1 &= a_1x^2+a_2xy+(a_3+a_4)x+a_5y+a_6\\
\chi_2 &= a_1xy+a_7x+a_2y^2+2a_3y+a_8\\
\chi_3 &= a_1xz+a_9x+a_2yz+(a_3+a_{10})z+\frac{\xi_3}{2}a_1y^2+\left(\frac{3\xi_3}{2}a_7+a_{11}\right)y+a_{12}
\end{aligned}$$

where the a_m here are constants and, for all $k\geq 3$,

$$\xi_ka_2+\xi_{k+1}a_1=0$$

and

$$\xi_ka_{10}+(k-2)\xi_ka_4+(k-3)\xi_{k-1}a_5-(k+1)\xi_{k+1}a_7-(k+1)\xi_ka_3=0$$

and the result follows.

Thus, in the non-trivial case, the point symmetry algebra of (4.16) has at most dimension 9. A little more work shows us that systems of dimension 8 are not attainable for this example (if the rank of M_2 is 1, then the rank of M_2 is 2 and if the rank of M_1 is 2 then the rank of M_2 is at least 3.). □

Dimension 9

A system with point symmetry algebra of dimension 9 is given by (4.6)

$$y'' = 0 \ , \ \ z'' = -2(y')^2.$$

A quick check of the expressions for M_1 and M_2 shows that the system

$$y'' = 0 \ , \ \ z'' = \frac{(y')^3}{1-y'}$$

also admits a nine-dimensional point symmetry algebra but these systems can be shown to be diffeomorphic. The associated Lie algebra in the first case can be written as

$$L_9 = L_6 \oplus \mathrm{span}\{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9\}$$

with

$$\mathbf{e}_7 = 3y\partial_y + z\partial_z \ , \ \mathbf{e}_8 = \frac{3}{2}z^2\partial_y + x\partial_z \ , \ \mathbf{e}_9 = \frac{1}{2}x^2\partial_x + \left(\frac{1}{2}xy + \frac{1}{4}z^3\right)\partial_y + \frac{1}{2}xz\partial_z.$$

This has Levi-decomposition

$$L_9 = \tilde{L_6} \ltimes sl(2,\mathbb{R}) = \mathrm{span}\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8\} \ltimes \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_6 - \frac{1}{2}\mathbf{e}_7, \mathbf{e}_9\}.$$

Dimension 7

A path geometry with point symmetry algebra of dimension 7 is given by

$$y'' = 0 \ , \ \ z'' = (y')^k$$

for $k \geq 4$. This will be a solvable Lie algebra for any value of k and is obtained by adding the vector field $\mathbf{e}_7 = ky\partial_y + z\partial_z$ to L_6.

ASD Einstein - dimension 8

Consider the system

$$y'' = 0 \ , \ \ z'' = \frac{2(z')^2 y'}{zy' - 1}. \tag{4.18}$$

This system has made an appearance in the theory of chains in the homogeneous contact geometry [53]. The Grossman invariants vanish for (4.18) so it is in the conformal branch.

The corresponding conformal structure is found by demanding that two neighbouring intgeral curves in U intersect at one point - the condition which selects null vectors in M. In the complexified setting, this reduces the normal bundle of the integral curve L_p to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. To perform this calculation explicitly, solve (4.18) for (y, z) and demand that the equations $\delta y = 0$ and $\delta z = 0$ have a common solution. We apply this procedure to the integral curves

$$\tilde{y} = \alpha + \gamma x \ , \ \ \tilde{z} = \frac{1}{\gamma} + \frac{1}{\gamma^2(\eta - \beta x)}$$

where $\delta y = \sum_a \partial_a \tilde{y} \delta \alpha^a$, $\delta z = \sum_a \partial_a \tilde{z} \delta \alpha^a$, and $\alpha^a = (\alpha, \beta, \gamma, \eta)$. Then the discriminant condition for the two curves to intersect yields the metric

$$g = d\alpha d\beta + d\gamma d\eta + \beta^2 d\alpha^2 + \left(\eta^2 + \frac{2\eta}{\gamma}\right) d\gamma^2 + 2\left(\eta\beta + \frac{\beta}{\gamma}\right) d\alpha d\gamma$$

which is ASD and Einstein, with scalar curvature equal to -24. Both the system of ODEs and the metric have eight-dimensional symmetry group $SL(3, \mathbb{R})$ which acts isometrically on M. The homogeneous model for the space of solutions is $SL(3, \mathbb{R})/GL(2, \mathbb{R})$, which is a $(2, 2)$ real form of the Fubini-study metric on $\mathbb{CP}^2 = SU(3)/U(2)$.

Symmetry Algebra of Dimension 5

The example discussed in the Gibbons-Hawking context (4.11) has five-dimensional point symmetry algebra which we can write as

$$L_5 = \text{span}\{\partial_y, \partial_z, x\partial_x + y\partial_y, -\frac{1}{2}x\partial_x + z\partial_z, x^2\partial_y + 2y\partial_z\}.$$

The algebra L_5 is solvable and contains both Bianchi II and Bianchi V as three-dimensional subalgebras.

Sparling-Tod Solution

Another way to construct a torsion-free path geometry with a given point symmetry algebra is to determine an ASD metric with that conformal symmetry algebra and derive the corresponding system of ODEs governing twistor lines via the method provided in this chapter. The metric of [54] given by

$$g = d\alpha d\beta + d\eta d\gamma - \frac{2}{(\beta\alpha+\gamma\eta)^3}(\alpha d\eta - \eta d\alpha)^2$$

is an ASD Ricci-flat metric (4.4) with $\Theta = \frac{1}{\alpha\beta+\gamma\eta}$ and has five-dimensional conformal symmetry algebra. The conformal Killing vectors are

$$\mathbf{k}_1 = \eta\partial_\alpha - \beta\partial_\gamma, \mathbf{k}_2 = -2\alpha\partial_\alpha + \beta\partial_\beta - \gamma\partial_\gamma, \mathbf{k}_3 = \alpha\partial_\alpha + \eta\partial_\eta, \mathbf{k}_4 = -\gamma\partial_\beta + \alpha\partial_\eta, \mathbf{k}_5 = \eta\partial_\beta - \alpha\partial_\gamma.$$

We should note here that the resulting system of ODEs does not coincide with the previous example as the conformal symmetry algebra of the Sparling-Tod solution is not solvable. It has an $sl(2,\mathbb{R})$ subalgebra generated by $(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{k}_3, \mathbf{k}_4)$. Its Levi decomposition can be expressed as the semi-direct product of an $sl(2,\mathbb{R})$ with the 2-dimensional non-Abelian Lie algebra.

The system of ODEs can be read off directly using Theorem 4.2 and setting $\Lambda = -\Theta(y, z, z', -y')$. This yields $\Lambda = (y'z - yz')^{-1}$ and (4.15) gives

$$y'' = \frac{2y}{(y'z - yz')^2} \ , \ z'' = \frac{2z}{(y'z - yz')^2}.$$

The integral curves are

$$\tilde{y} = Ae^{kx} + Be^{-kx} \ , \ \tilde{z} = Ce^{kx} + De^{-kx} \ ,$$

where (A, B, C, D) are constants of integration and $k^2 = (AD - BC)^{-1}\sqrt{2}^{-1}$. The original metric can be recovered from the twistor lines by setting

$$\alpha = A + B \ , \ \eta = C + D \ , \ \gamma = k(B - A) \ , \ \beta = k(C - D).$$

Symmetry Algebra of dimension 4

. Consider the system

$$y'' = 0 \ , \ z'' = -\left(z' + \sqrt{(y')^2 - 1}\right)^2 . \tag{4.19}$$

This example was found by constructing the most general system of ODEs with Lie point symmetry algebra

$$L_4 = \mathrm{span}\{\partial_x, \partial_y, \partial_z, y\partial_x + x\partial_y\}$$

and imposing the torsion-free conditions. The algebra L_4 above is a particular realisation of the abstract algebra[1] $A_{4,1}$ as give by the authors in [55]. The integral curves of the system (4.19) are given by

$$\tilde{y} = \alpha + x\gamma \ , \ \tilde{z} = \log(x - \beta) - x\sqrt{\gamma^2 - 1} + \eta.$$

Following the procedure applied in the case with 8-dimensional point symmetry algebra, we find the ASD conformal structure

$$g = d\beta d\gamma + (d\alpha + \beta d\gamma)\left(d\eta + \frac{\gamma}{\sqrt{\gamma^2 - 1}}d\alpha\right).$$

This admits a null Killing vector $\partial/\partial\eta$ and thus fits into the classification of [56].

Symmetry algebra of dimension three or less

If the symmetry algebra has dimension 3, then it necessarily belongs to the Bianchi classification of three-dimensional Lie algebras. There are many examples in this case which are analytic continuations of Riemannian metrics. See [57] for a discussion of these examples. ASD conformal metrics with one or two symmetries can be found in the Gibbons-Hawking class (4.9). An axisymmetric solution H to the wave equation (4.7) gives a metric with two Killing vectors. Another example with two-dimensional symmetry will be discussed in the next section. A general solution of (4.7) with no symmetries gives a metric which only admits one Killing vector $\partial/\partial\eta$. Gravitational instantons of class D_k are examples of ASD conformal structures with no symmetries.

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[1] There are two other nonequivalent representations of this algebra as noted in [20]:

$$L_{4a} = \text{span}\{\partial_y, -x\partial_y, \partial_z, \partial_x - xz\partial_y\}$$

and

$$L_{4b} = \text{span}\{2\partial_x, \partial_z, -y^2\partial_x - y\partial_z, 2z\partial_x + \partial_y\}.$$

There is no torsion-free system with symmetry algebra L_{4b}, and L_{4a} appears as a subalgebra of a torsion-free system with seven-dimensional point symmetry algebra.

4.5 Finsler Structures with Scalar Flag Curvature

For a given n-dimensional domain $U \subset \mathbb{R}^n$ with coordinates x^a $(a = 1, \ldots, n)$, a *Finsler metric* is a positive continuous function $\mathcal{F} : U \to [0, \infty)$ such that

- $\mathcal{F}$ is smooth on $TU \backslash 0 = \{(x^a, p^a) \in TU | p \neq 0\}$,
- $\mathcal{F}(x^a, cp^a) = c\mathcal{F}(x^a, p^a)$ for $c > 0$,
- The tensor $f_{ab} = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial p^a \partial p^b}$ is positive definite for all $(x^a, p^a) \in TU \backslash 0$.

We shall consider the case $n = 3$ and set $x^a = (x, y, z)$ and $p^a = \dot{x}^a = \frac{dx^a}{dt}$ for some parameter t. Finsler geometry generalizes the notion of Riemannian geometry in that the norm on each tangent space $\mathcal{F}(x^a, \cdot)$ is not necessarily induced by a metric tensor. This makes these metrics useful in the study of problems involving paths of least time. The metric tensor f_{ab} allows us to define Finslerian geodesics, which are integral curves of the system

$$\ddot{x}^a + \gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \ , \ a, b, c = 1, \ldots, n$$

where

$$\gamma^a_{bc} = \frac{1}{2} f^{ad}(f_{dc,x^b} + f_{bd,x^c} - f_{bc,x^d}).$$

It was argued in [58] that, for $n \geq 2$, given any system of ODEs with vanishing Grossman invariants its integral curves arise as the set of unparametrised geodesics of a Finsler function of scalar flag curvature. In this context, the torsion-free path geometries are viewed as projective equivalence classes of *isotropic* sprays on TU. Given a spray

$$S = p^a \frac{\partial}{\partial p^a} \ , \ a = 1, \ldots, n$$

we can define its Riemann curvature by

$$R^d_{cab} = H_a\left(\Gamma^d_{bc}\right) - H_b\left(\Gamma^d_{ac}\right) + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}$$

where $\Gamma^a_{bc} = \frac{\partial^2\Gamma^a}{\partial p^b \partial p^c}$, etc and the H_a form a horizontal distribution determined by S:

$$H_a = \frac{\partial}{\partial x^a} - \Gamma^b_a \frac{\partial}{\partial p^b}.$$

Then the spray is said to be *isotropic* if its Jacobi endomorphism is given by

$$R^a_b = R^a_{cbd} p^c p^d = \rho \delta^a_b + \tau_b p^a$$

for some function ρ and covector τ_a. It was shown in [59], that for $n > 2$, a spray is isotropic if and only if it is projectively equivalent to one whose Riemann curvature vanishes (so-called R-flat) and in [60], that any R-flat spray arises as the geodesic spray of some Finsler function. More importantly, in [61] it is shown that the geodesic sprays of a Finsler function are isotropic if and only if the Finsler function has scalar flag curvature. Here, the flag curvature of a Finsler metric is defined in terms of its Riemann curvature by

$$K(x^a, p^a, v^a) = \frac{v^a(p^b R_{bacd} p^d) v^c}{f(p,p)f(v,v) - (f(p,v))^2}$$

where v is a section of the tangent bundle transverse to p and the indices are raised/lowered with the tensor f_{ab}. The flag curvature is scalar if

$$K(x^a, p^a, v^a) = K(x^a, p^a).$$

Thus, we could construct all systems of ODEs with vanishing Grossman invariants if we know how to characterize the Finsler functions of scalar flag curvature. Although a lot of work has been undertaken in this area, a complete characterization of such functions has not yet been achieved. Most success has come with a special type of Finsler function

$$\mathcal{F} = \sqrt{\alpha_{ab}(x^c) p^a p^b} + \beta_a(x^c) p^a \ \ , \ \ a, b, c = 1, \dots, n.$$

This is known as a *Randers metric*. One important aspect of Randers metrics which will be central to some of the work presented in Chapter 6 is that its set of unparametrised geodesics coincide with solutions of *Zermelo's problem* for some background metric h and wind vector $\mathbf{W}$. On the space of orbits of a stationary

Killing vector of some Lorentzian spacetime, we shall see that this duality is, in fact, one leg of a triality of geometric structures. For now, if we restrict our attention to Randers metrics of constant flag curvature, then we can use the following:

Theorem 4.5. *(Bao-Robles-Shen [62]) A Randers metric $\mathcal{F}$ has constant flag curvature if and only if the corresponding Zermelo data (h, W) satisfy the following:*

- *h is a Riemannian metric with with constant sectional curvature.*
- *W is a Killing vector or homothety of h.*

Here, the Randers data can be expressed in terms of the Zermelo data as follows [63]:

$$\alpha_{ab} = \frac{\lambda h_{ab} + W_a W_b}{\lambda^2} \ , \ \beta_a = -\frac{W_a}{\lambda}$$

where $\lambda = 1 - h_{ab}W^aW^b$ and $W_a = h_{ab}W^b$. This gives rise to a procedure for constructing torsion-free systems of ODEs from a three-dimensional Riemannian metric of constant sectional and a homothety of this metric. The geodesic spray coefficients of such systems was worked out in [63].

Example 4. Consider the Zermelo data

$$h = dx^2 + dy^2 + y^2dz^2 \ , \ W = \frac{\partial}{\partial z}.$$

The geodesics of the Randers metric associated to this Zermelo data are the integral curves of the systems of ODEs with vanishing Grossman invariants

$$\begin{aligned} y'' &= \frac{2yz'\sqrt{1 + (y')^2 + y^2((z')^2 - (y')^2 - 1)}}{y(y^2-1)} \\ &+ \frac{y\left(1 + (y')^2 + (z')^2y^2((z')^2 - (y')^2 - 1)\right)}{(y^2-1)^2}, \\ z'' &= \frac{2y'\left(z' + \sqrt{1 + (y')^2 + y^2((z')^2 - (y')^2 - 1)}\right)}{y(y^2-1)}. \end{aligned}$$

This system has two symmetries $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$.

We can also view this correspondence in the other direction i.e, given a system of ODEs with vanishing Grossman invariants, we can construct a Finsler metric of scalar flag curvature.

Example 5. To illustrate this point, let us consider the submaximal system (4.6) corresponding to ASD Ricci-flat pp waves with constant Weyl curvature. Then, using the procedure in [64], this system describes the unparametrised geodesics of the Finsler function

$$\mathcal{F} = \dot{x}\mathcal{G}\left(\frac{\dot{y}}{\dot{x}}, 2\frac{\dot{z}}{\dot{x}} - 2\frac{x\dot{y}^3}{\dot{x}^3} + \frac{6y\dot{y}^2}{\dot{x}^2}\right)$$

in some open domain of U where f_{ab} is positive definite, where $\mathcal{G}$ is any function of two variables. In particular, let $\mathcal{G}(x, y) = \sqrt{xy}$ and consider the Lagrangian $L = \frac{1}{2}\mathcal{F}^2$ in unparametrised form where $\dot{x} = 1$

$$L = 2y'z' - 2x(y')^4 + 6y(y')^3.$$

The Euler-Lagrange equations give (4.6). For this example, the flag curvature of $\mathcal{F}$ vanishes.

Similarly, other systems with vanishing Grossman invariants arise from a variational principle induced by the Finsler structure. Thus, all these systems fit into the formalism of [65].

Chapter 5

Optical Metrics and Projective Equivalence

Recently, the notion of projectively equivalent metrics has featured more prominently in the realm of relativistic physics. The paths of freely falling particles determine a projective structure in some open domain on a Lorentzian manifold and so, knowledge on the degree of mobility of a locally defined metric can tell us to what extent we can determine the metric structure from geodesic information. The problem of determining a spacetime metric from local geodesic data was discussed in [66] and [11]. Moreover, the geodesic mobility of Einstein metrics has been considered in [67] and [68]. Nurowski questioned the meaning of dark energy in [21] where he showed that different Robertson-Walker spacetimes can admit the same unparametrised geodesics and that experimental evidence of freely falling particles could not enable one to determine the existence of a energy momentum tensor with cosmological constant. From a physical point of view, the notion of determining the cosmological constant was cleared up in [22].

More importantly for the current discussion, the dynamics of light rays in Schwarzschild-deSitter spacetimes has been found to be independent of the cos-

mological constant, Λ. This property has been inferred as a consequence of projective equivalence of the corresponding *optical metrics*, see [23]. The optical metric is a useful geometric structure for studying the properties of light rays in a static spacetime. It may be thought of as the natural Riemannian geometry experienced by light rays and has also recently been used to give an alternative interpretation of black hole no-hair theorems [69].
Let (M, g) be a pseudo-Riemannian manifold with a metric of signature $(n, 1)$, where $n > 0$. The metric is called static if it admits a hypersurfaceorthogonal (HSO) timelike Killing vector K, i.e,

$$g(K, K) < 0 \ , \ K \wedge dK = 0 \ , \ \mathcal{L}_K g = 0$$

where $\mathcal{L}$ is the Lie derivative. Any such metric is locally of the form

$$g = V^2(-dt^2 + h) \tag{5.1}$$

where $h = h_{ij} dx^i dx^j$ is a Riemannian metric on the space of orbits Σ of $K = \partial/\partial t$ and $V = V(x^i)$ is a function on Σ. The metric h is called the optical metric of g and the motivation behind this terminology [70], [23], [69] comes from the fact that null geodesics of g project to unparametrised geodesics of h. This can be readily verified as null geodesics of g coincide with the null geodesics of $V^{-2}g$. The idea here is that properties of the conformal structure of g can be inferred from those of the geodesic structure of h. On immediate consequence, for example, is that g is conformally flat if and only if h has constant curvature.
In the four-dimensional Schwarzschild-deSitter case, we have

$$g = -\left(1 - \frac{2M}{r} - \Lambda r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \Lambda r^2} + r^2 \left(d\theta^2 + r^2 d\phi^2\right)$$

for mass parameter M and cosmological constant Λ. The optical metric is

$$h = \frac{dr^2}{\left(1 - \frac{2M}{r} - \Lambda r^2\right)^2} + \frac{r^2}{1 - \frac{2M}{r} - \Lambda r^2} \left(d\theta^2 + r^2 d\phi^2\right)$$

whose unparametrised geodesics are independent of Λ. Consequently, equations governing the dynamics of light rays of g are observed to be independent of the cosmological constant.

5.1 Projective Equivalence vs Optical Equivalence and Multi-static metrics

It is clear from the above discussion that an optical metric depends on the choice of a static timelike Killing vector. In the current context, three different equivalence classes of Riemannian metrics will play a role here. Let (Σ, h) and $(\bar{\Sigma}, \bar{h})$ be two n-dimensional Riemannian manifolds, and let $\rho : \Sigma \to \bar{\Sigma}$ be a diffeomorphism. The metrics h and $\bar{h}$ are

- *Equivalent*, if there exists ρ such that $\rho * \bar{h} = h$.
- *Projectively equivalent*, if there exists a ρ such that $\rho * \bar{h}$ and h share the unparametrised geodesics.
- *Optically equivalent*, if there exists a pseudo-Riemannian $(n+1)$-dimensional manifold M with two HSO Killing vectors K and $\bar{K}$ such that Σ and $\bar{\Sigma}$ are hyper-surfaces orthogonal to K and $\bar{K}$ respectively and $(h, \bar{h})$ are optical metrics of K and $\bar{K}$ respectively.

All equivalences we shall discuss are in fact local equivalences as ρ is only required to be a smooth map between some open sets. If two metrics are equivalent, they are also projectively equivalent, but the converse is not true in general. A less clear connection, which we shall consider in this chapter, is that between projective equivalence and optical equivalence. It turns out that the latter *almost always* implies the former. To enable the study of this connection we need to construct a spacetime which admits two non-equivalent optical metrics and thus, we have the following definition:

Definition 5.1. *A Lorentzian metric is called multi-static if it admits at least two non-proportional HSO timelike Killing vectors.*

We shall now classify local forms of pseudo-Riemannian multi-static structures (M, g) and for this purpose, let us assume that the dimension of M is four. Let (K, ξ) be two HSO timelike Killing vectors on M. We can choose a local coordinate system (**Note:** We shall use Greek indices $(\mu, \nu, \ldots)$ to run over 0,1,2,3 and Roman indices $(i, j, k, \ldots)$ to run over 1,2,3) $x^\mu = (t, x^i)$, such that the metric is given by (5.1) and $K = \partial/\partial t$. In this coordinate system

$$\xi = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i}$$

where $\xi^0, \ldots, \xi^3$ are functions of (x, t). From our assumptions it follows that not all ξ^i are identically zero (if they where, then the Killing equations $\nabla_0 \xi_0 = \nabla_{(i}\xi_{0)} = 0$ would imply $\xi^0 =$ constant thus contradicting our assumptions about the independence of K and ξ). Therefore there exists t_0 such that the projection of the restriction of ξ at the surface Σ given by $t = t_0$

$$\tilde{\xi} = \xi|_{t=t_0} \tag{5.2}$$

is a non-zero vector field. Furthermore, we can make the coordinate transformation $t \to t - t_0$ while preserving the form of the metric (5.1) so that $\tilde{\xi}^i = \xi^i|_{t=0}$. The HSO Killing equations for ξ imply that $\tilde{\xi}$ is a HSO Killing vector for $V^2 h$ and so there exists a function $r : \Sigma \to \mathbb{R}$ such that

$$V^2 h = e^w dr^2 + \gamma,$$

where $\tilde{\xi} = \partial/\partial r$, and (w, γ) are a function and a metric on a two-dimensional surface S_1 (the space of orbits of $\tilde{\xi}$ in Σ) which do not depend on r. We can use the isothermal coordinates (x, y) so that $\gamma = e^u(dx^2 + dy^2)$ and u, w are functions of (x, y). Thus the most general Lorentzian metric which admits more than one optical metric is locally of the form

$$g = -V^2 dt^2 + e^w dr^2 + e^u(dx^2 + dy^2), \tag{5.3}$$

where $V = V(r, x, y)$, $u = u(x, y)$ and $w = w(x, y)$. We note that the function V is not arbitrary - its form is restricted by the Killing equation for ξ.

Our next step is to classify the normal forms of ξ and thus read off the canonical forms of its optical metric $\bar{h}$ on some three-manifold $\bar{\Sigma}$ where $\bar{K} = \partial/\partial\bar{t}$ giving rise to $\bar{h}$ is the push forward of ξ under some local diffeomorphism between $\bar{\Sigma}$ and Σ. We shall make the additional genericity assumption

Definition 5.2. *A multistatic metric is called generic if the isometry group generated by any pair of HSO timelike Killing vectors (and their commutators) has two-dimensional orbits in M.*

The genericity assumption implies that for any t_0, the HSO Killing vector ξ restricted to the surface $t = t_0$ defined by K is proportional to a fixed vector field.

Proposition 5.3. *Any generic multistatic metric is locally a warped product metric on $M = S_0 \times S_1$ given by*

$$g = e^w \gamma_0 + \gamma_1 \tag{5.4}$$

where (S_0, γ_0) is a two-dimensional Lorentzian manifold whose curvature is constant, (S_1, γ_1) is a two-dimensional Riemannian manifold and $w : S_1 \to \mathbb{R}$ is an arbitrary function.

Proof. First we shall show that given a pair of HSO timelike Killing vectors (K, ξ), the genericity assumption implies existence of two functions (r, t) such that the metric takes the form (5.3), and

$$K = \frac{\partial}{\partial t} \ , \ \ \xi = \xi^0(t, r, x, y) + a(t)\frac{\partial}{\partial r} \tag{5.5}$$

where (x, y) are coordinates on the surface S_1 parametrising the 2D orbits in M, and a is a function which depends only on t. To prove this statement, note that the group generated by the Killing vectors and their commutators acts on M with twodimensional orbits so

$$[K, \xi] = pK + q\xi,$$

where p, q are functions on M. We need to show that there exists functions α, β such that

$$[\beta^{-1}(\xi - \alpha K), K] = 0, \tag{5.6}$$

as then the local existence of r, t will follow from the Frobenius theorem. Expanding the Lie bracket (5.6) and using (5.1) gives a pair of ODEs

$$K(\beta^{-1}) = \beta^{-1} q \ , \ \ K(\alpha\beta^{-1}) = \beta^{-1} p$$

The existence of α, β is a consequence of the Picard existence theorem applied to these ODEs and

$$K = \frac{\partial}{\partial t} \ , \ \xi = \alpha K + \beta \frac{\partial}{\partial r}.$$

Now consider the HSO Killing vector $\bar{\xi}$ given by (5.2) on the surface Σ of constant t. The Killing equations on Σ imply that $\beta = \beta(r,t)$ and that for any value of t_0 the resulting vector is proportional to the same Killing vector. Thus $\beta(r,t) = a(t)b(r)$. We now redefine the r coordinate to set $b(r) = 1$. This establishes (5.5). Therefore for any value of t_0,

$$\xi^i \frac{\partial}{\partial x^i}|_{t=t_0} \propto \frac{\partial}{\partial r}.$$

The Killing equations $\nabla_{(2}\xi_{0)} = 0 = \nabla_{(2}\xi_{0)}$ give $\xi^0 = \xi^0(t,r)$. Using this and equation (5.5) above, the hypersurface orthogonality conditions $\xi_{[0}\nabla_1\xi_{2]} = 0$ and $\xi_{[0}\nabla_1\xi_{3]} = 0$ yield

$$V^2(r,x,y) = v^2(r)e^{w(x,y)}$$

for some function $v(r)$. Hence the metric g may already be written as (5.4) where the two-dimensional metric γ_0 is given by

$$\gamma_0 = -v^2(r)dt^2 + dr^2.$$

The scalar curvature of this metric is

$$\kappa = -\frac{2v''(r)}{v(r)}. \tag{5.7}$$

This will be important later. The only remaining equations that need to be satisfied are the Killing conditions $\nabla_{(0}\xi_{0)} = 0$ and $\nabla_{(1}\xi_{0)} = 0$. These equations give

$$-v^2(r)\partial_t\xi^0 = v(r)\frac{dv(r)}{dr}a(t) \ , \ -v^2(r)\partial_r\xi^0 = -\frac{da(t)}{dt}.$$

Differentiating the first condition with respect to r and the second condition with respect to t and equating the mixed partial derivatives of ξ^0 yields

$$\frac{1}{a(t)}\frac{d^2a(t)}{dt^2} = \left(\frac{dv(r)}{dr}\right)^2 - v(r)\frac{d^2v(r)}{dr^2}.$$

The left hand side of the equation is a function of t only. Hence

$$\left(\frac{dv(r)}{dr}\right)^2 - v(r)\frac{d^2v(r)}{dr^2} = \Omega = \text{constant}.$$

Differentiating with respect to r, we find that

$$0 = v'(r)v''(r) - v(r)v'''(r) = \frac{v^2(r)}{2}\frac{\partial}{\partial r}\left(-\frac{2v''(r)}{v(r)}\right).$$

Hence, by (5.7), the curvature of γ_1 is constant. Furthermore, if the curvature is $\kappa \neq 0$ then we can set its absolute value to one by adding a constant to the function w. $\square$

5.1.1 Calculating the Optical Metrics

To determine the optical metrics resulting from (5.4) we need to consider three cases depending on the curvature of γ_0.

Zero Curvature Case

We can find local coordinates such that $\gamma_0 = -dt^2 + dr^2$, and the general HSO Killing vector of g becomes

$$\xi = (Ar + B)\frac{\partial}{\partial t} + (At + C)\frac{\partial}{\partial r}$$

for some constants A, B and C. If $A \neq 0$ we translate (r, t) by adding constants and rescale the Killing vector so that

$$\xi = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}.$$

Setting $t = \bar{r}\sinh(\bar{t}), r = \bar{r}\cosh(\bar{t})$ gives the optical metric of $\partial/\partial\bar{t}$

$$\bar{h} = \bar{r}^{-2}(d\bar{r}^2 + e^{-w}\gamma_1). \tag{5.8}$$

If $A = 0$, then a constant rescaling of t can be used to set $\xi = \cos\theta\partial_t + \sin\theta\partial_r$, where θ is a constant in a range which makes ξ timelike. The pseudoorthogonal transformation of (r, t) can now be used to set $\xi = \partial/\partial t$, so the optical metric in this case is

$$h = dr^2 + e^{-w}\gamma_1. \tag{5.9}$$

Anti deSitter Case

Now, let us consider the case where the metric has the form (5.4), where the constant curvature of γ_0 is negative. In the AdS_2 case we can choose local coordinates so that

$$\gamma_0 = \frac{-dt^2 + dr^2}{r^2}.$$

Both γ_0 and the resulting Lorentzian metric g have three Killing vectors generating $SL(2,\mathbb{R})$. In the chosen coordinates these vectors are

$$K_1 = \frac{\partial}{\partial t} \ , \ K_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \ , \ K_3 = \left(\frac{t^2+r^2}{2}\right)\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r},$$

and

$$[K_1, K_2] = K_1 \ , \ [K_2, K_3] = K_3 \ , \ [K_1, K_3] = K_2.$$

Furthermore, it is easy to show that any linear combination

$$\xi = AK_1 + BK_2 + CK_3$$

is an HSO Killing vector for the metric g, which is timelike in some open set to which we restrict our attention from now on.

Proposition 5.4. *For any timelike HSO Killing vector, ξ, of the metric (5.4), where γ_0 has negative constant curvature, the optical metric associated to ξ is diffeomorphic to*

$$\bar{h} = \frac{1}{(\phi + \bar{r}^2)^2} d\bar{r}^2 + \frac{e^{-w}}{\phi + \bar{r}^2}\gamma_1 \tag{5.10}$$

for some constant ϕ.

Proof. Let us first consider the HSO Killing vectors for which $C \neq 0$. Then, adding a constant to t we can set $B = 0$ without changing the metric. If $A = 0$ then divide ξ by $C/2$ to set $C = 2$. Otherwise, rescale (t, r) by the same constant factor to set $A = \pm C/2$ and then divide ξ by $C/2$. Thus, the resulting Killing vector can take one of three possible forms

$$\xi = (c + t^2 + r^2)\frac{\partial}{\partial t} + 2tr\frac{\partial}{\partial r}, \ \text{ where } \ c = 0, -1, 1.$$

We look for a coordinate transformation $(t, r) \to (\bar{t}, \bar{r})$ such that $\xi = \partial/\partial\bar{t}$.

- If $c = 1$, set
$$t = \frac{\sqrt{\bar{r}^2+4}\cos(2\bar{t})}{\bar{r}-\sqrt{\bar{r}^2+4}\sin(2\bar{t})} \quad , \quad r = \frac{2}{\bar{r}-\sqrt{\bar{r}^2+4}\sin(2\bar{t})}.$$
- If $c = -1$, set
$$t = \frac{\sqrt{\bar{r}^2-4}(1-e^{4\bar{t}})}{\sqrt{\bar{r}^2-4}(1+e^{4\bar{t}})-2\bar{r}e^{2\bar{t}}} \quad , \quad r = \frac{4e^{2\bar{t}}}{\sqrt{\bar{r}^2-4}(1+e^{4\bar{t}})-2\bar{r}e^{2\bar{t}}}.$$
- If $c = 0$, set
$$t = \frac{\bar{r}^2\bar{t}}{1-\bar{r}^2\bar{t}^2} \quad , \quad r = \frac{\bar{r}}{\bar{r}^2\bar{t}^2-1}.$$

This gives, in all three cases, $\gamma_0 = -(\bar{r}^2+4c)d\bar{t}^2+(\bar{r}^2+4c)^{-1}d\bar{r}^2$ and the optical metric (5.10) with $\phi = 4c$.

Now consider the case $C = 0$. Adding an appropriate constant to t sets $B = 0$ so that
$$\xi = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$$
Setting
$$t = \frac{\bar{r}}{\sqrt{\bar{r}^2-1}}e^{\bar{t}} \quad , \quad r = \frac{1}{\sqrt{\bar{r}^2-1}}e^{\bar{t}}$$
yields $\xi = \partial/\partial\bar{t}$ and $\gamma_0 = -(\bar{r}^2-1)d\bar{t}^2 + (\bar{r}^2-1)^{-1}d\bar{r}^2$. The optical metric in this case is (5.10) with $\phi = -1$.

Finally, suppose $C = B = 0$ so that $\xi = \frac{\partial}{\partial t}$. This gives the optical metric
$$\bar{h} = dr^2 + r^2e^{-w(x,y)}\gamma_1.$$
A coordinate transformation $r = \bar{r}^{-1}$ puts it in the form (5.10) with $\phi = 0$. Thus, we have covered all cases. $\square$

deSitter Case

In this case, γ_0 can be written in local coordinates as
$$\gamma_0 = \frac{-dt^2+dr^2}{t^2}.$$
This switches the role of r and t in the previous section. The general HSO timelike Killing vector on g is of the form
$$\xi = AK_1 + BK_2 + CK_3$$

where

$$K_1 = \frac{\partial}{\partial r} \ , \ K_2 = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} \ , \ K_3 = \left(\frac{t^2+r^2}{2}\right)\frac{\partial}{\partial r} + tr\frac{\partial}{\partial t}.$$

If $C \neq 0$, then adding a constant to r can be used to set $B = 0$. The resulting vector will be time-like (in a certain open set in M) only if $AC < 0$. In this case, we can rescale (r,t) by the same constant factor to set $A = -C/2$, so that

$$\xi = (-1+t^2+r^2)\frac{\partial}{\partial r} + 2tr\frac{\partial}{\partial t}.$$

A coordinate transformation

$$t = \frac{\sqrt{4-\bar{r}^2}(1+e^{4\bar{t}})}{\sqrt{4-\bar{r}^2}(1-e^{4\bar{t}})+2\bar{r}e^{2\bar{t}}} \ , \ r = -\frac{4e^{2\bar{t}}}{\sqrt{4-\bar{r}^2}(1-e^{4\bar{t}})+2\bar{r}e^{2\bar{t}}}$$

gives $\xi = \partial/\partial\bar{t}$ and

$$\gamma_0 = -(4-\bar{r}^2)d\bar{t}^2 + \frac{1}{4-\bar{r}^2}d\bar{r}^2$$

which is defined for $|\bar{r}| < 2$. The optical metric is

$$\bar{h} = \frac{1}{(4-\bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{4-\bar{r}^2}\gamma_1.$$

If $C = 0$ then adding an appropriate constant to r gives $\xi = K_2$. The transformation

$$t = \frac{\bar{r}}{\sqrt{1-\bar{r}^2}}e^{\bar{t}} \ , \ r = \frac{1}{\sqrt{1-\bar{r}^2}}e^{\bar{t}}$$

yields $\xi = \partial/\partial\bar{t}$ and $\gamma_0 = -(1-\bar{r}^2)d\bar{t}^2 + (1-\bar{r}^2)^{-1}d\bar{r}^2$. The optical metric in this case is

$$\bar{h} = \frac{1}{(1-\bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{1-\bar{r}^2}\gamma_1.$$

Finally if $C = B = 0$ then ξ is always spacelike and does not lead to an optical structure. Therefore, we have

Proposition 5.5. *For any timelike HSO Killing vector, ξ, of the metric (5.4), where the curvature of γ_0 is positive, the optical metric associated to ξ is diffeomorphic to*

$$\bar{h} = \frac{1}{(\phi-\bar{r}^2)^2}d\bar{r}^2 + \frac{e^{-w}}{\phi-\bar{r}^2}\gamma_1 \tag{5.11}$$

for some constant $\phi > 0$.

5.1.2 Projective Equivalence

Zero Curvature

We claim that $\bar{h}$ and h given by (5.8) and (5.9) respectively are not projectively equivalent even up to diffeomorphisms: The metric (5.9) admits a non-trivial affine equivalence, i. e. there exists a covariantly constant symmetric $(0, 2)$tensor h_1 that is not proportional to (5.9) (in our case $h_1 = dr^2$). A result of Levi Civita[1] [10] implies that (5.8) admits a non-affine geodesic equivalence i.e. there exists a geodesically equivalent metric that is not covariantly constant in the LeviCivita connection of (5.8). It is given by

$$h_2 = \frac{1}{\bar{r}^2+1}\left(\frac{\bar{r}^2}{\bar{r}^2+1}d\bar{r}^2 + e^{-w}\gamma_1\right).$$

Thus, if (5.8) and (5.9) were equivalent, there would exist at least three non-proportional metrics sharing the same geodesics. This in dimension three implies [67] that h has constant curvature and so it is flat.

Non-zero curvature

Let us first consider the case where γ_0 has negative curvature.

Proposition 5.6. *Let ξ_1 and ξ_2 be two time-like HSO Killing vectors for the metric g defined by (5.4) where γ_0 is AdS_2. Then, the optical metric associated to ξ_1 is projectively equivalent to the optical metric associated to ξ_2 after some diffeomorphism. Thus all optical metrics are equivalent to (5.10) with $\phi = 1$.*

Proof. Let us first consider (5.10). By Proposition 5.4, the optical metric associated to any time-like HSO Killing vector ξ is given, after diffeomorphism, by (5.10) for some constant ϕ. For Killing vectors ξ_1 an ξ_2, let h_1, h_2 be the

[1]The result of Levi-Civita is that the metrics

$$h = dr^2 + h(r)\gamma\ ,\ \text{and}\ \tilde{h} = \frac{1}{(\kappa f(r)+1)^2}dr^2 + \frac{f(r)}{\kappa f(r)+1}\gamma$$

are projectively equivalent for any constant κ. Here f is an arbitrary function of r and γ is an arbitrary $r$independent metric. The result holds in any dimension.

associated optical metrics written in the form (5.10) with corresponding constants ϕ_1 and ϕ_2, respectively. Let Γ^i_{jk}, $\bar{\Gamma}^i_{jk}$ be the connection components of the metric connection of h_1, h_2 respectively. Then, we know from the previous chapter that these metrics are projectively equivalent if and only if there exists a one-form $\Upsilon = \Upsilon_j dx^j$ such that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \Upsilon_k + \delta^i_k \Upsilon_j.$$

Working this out explicitly, we find that the one-form

$$\Upsilon = \frac{\bar{r}(\phi_2 - \phi_1)}{(\bar{r}^2 + \phi_1)(\bar{r}^2 + \phi_2)} d\bar{r}$$

satisfies this criteria. □

The same argument with

$$\Upsilon = \frac{\bar{r}(\phi_1 - \phi_2)}{(\bar{r}^2 - \phi_1)(\bar{r}^2 - \phi_2)} d\bar{r},$$

can be used in the dS_2 case, to show that any two optical metrics of the form (5.11) are projectively equivalent.

5.1.3 Ultra-Static Metrics

It transpires that, in the ultra-static case, (i.e, when $V = 1$), we can integrate the Killing equations directly without making the additional assumption of genericity. In this case, we find that the assumption that the given metric admits at least two non-proportional timelike HSO Killing vectors implies that it may locally be written in the form (5.4) with $w =$ constant and γ_0 flat. This is essentially the case considered by Sonego [71]. We shall take our analysis further than this by considering the optical metrics resulting from this construction.
In the adapted coordinate system, the Killing vector $\tilde{\xi}$ on Σ satisfies

$$(\xi^1, \xi^2, \xi^3)|_{t=0} = (1, 0, 0). \tag{5.12}$$

Now consider the Killing equations for ξ. Using $\Gamma^0_{ij} = 0$, we find that $\nabla_{(0}\xi_{0)} = 0$ and $\nabla_{(0}\xi_{i)} = 0$ imply

$$\partial_t \xi^0 = 0 \ , \ e^{w(x,y)} \partial_t \xi^1 = \partial_r \xi^0 \ , \ e^{u(x,y)} \partial_t \xi^2 = \partial_x \xi^0 \ , \ e^{u(x,y)} \partial_t \xi^3 = \partial_y \xi^0.$$

Integrating and using the initial conditions (5.12) gives

$$\xi^1 = e^{-w(x,y)}(\partial_r\xi^0)t + 1\ ,\ \xi^2 = e^{-u(x,y)}(\partial_x\xi^0)t\ ,\ \xi^3 = e^{-u(x,y)}(\partial_y\xi^0)t.$$

Now, let us consider the hypersurface orthogonality condition $\xi \wedge d\xi = 0$. We find

$$\begin{aligned} 0 &= \xi_{[0}\nabla_1\xi_{2]} \\ &= -\xi^0\left((\partial_r\partial_x\xi^0)t - (\partial_x\partial_r\xi^0)t - \frac{\partial w}{\partial x}e^w\right) \\ &\quad + \left((\partial_r\xi^0)t + e^w\right)(-2\partial_x\xi^0) + (\partial_x\xi^0)t(2\partial_r\xi^0). \end{aligned}$$

This, together with a similar condition resulting from $\xi_{[0}\nabla_1\xi_{3]} = 0$ implies, after some algebra,

$$\xi^0 = \rho(r)e^{\frac{1}{2}w(x,y)}. \tag{5.13}$$

The rest of the hypersurface orthogonality conditions are then satisfied automatically. The remaining Killing equations will yield conditions for $\rho(r)$ as follows: Equation (5.13) and $\nabla_{(2}\xi_{3)} = 0$ give

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right)\right]. \tag{5.14}$$

Similarly, the Killing conditions $\nabla_{(2}\xi_{2)} = 0 = \nabla_{(3}\xi_{3)}$ give

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)\right], \\ \frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right)\right]. \end{aligned} \tag{5.15}$$

The Killing equations $\nabla_{(1}\xi_{2)} = 0 = \nabla_{(1}\xi_{3)}$ are now satisfied and the condition $\nabla_{(1}\xi_{1)} = 0$ gives

$$\frac{\partial^2\rho(r)}{\partial r^2} = -\frac{1}{4}e^{w-u}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\rho(r).$$

The left hand side of this equation depends only on r so the quantity

$$\mu^2 \equiv \frac{1}{4}e^{w-u}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{5.16}$$

is a constant. The theory then splits into two cases depending on whether or not this constant μ vanishes.

Case 1: $\mu \neq 0$

Solving (5.16) for u and substituting the partial derivatives of u into (5.14) and (5.15) gives, after some algebra,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) = 0.$$

This means that the function $e^{w/2}$ is harmonic, thus $e^{w(x,y)/2} = G(\zeta) + \bar{G(\zeta)}$, where G is holomorphic in $\zeta = x + iy$. A coordinate transformation

$$X = \frac{2}{\mu}\mathcal{R}(G)\cos(\mu r)\ ,\ Y = \frac{2}{\mu}\mathcal{R}(G)\cos(\mu r)\ ,\ Z = \frac{2}{\mu}\mathcal{I}(G)\ ,\ T = t$$

yields the Minkowski metric $g = -dT^2 + dX^2 + dY^2 + dZ^2$.

Case 2: $\mu = 0$

In this case, the condition (5.16) implies that $w(x, y)$ is a constant so that the metric (5.3), after rescaling r, becomes

$$g = -dt^2 + dr^2 + \gamma_1,$$

where $\gamma_1 = e^u(dx^2 + dy^2)$. We also have $\rho = Ar + B$ and given the initial conditions, the Killing vector ξ may be written as

$$\xi = (Ar + Be^{\frac{1}{2}w})\frac{\partial}{\partial t} + (At + e^{\frac{1}{2}w})\frac{\partial}{\partial r}.$$

If $A \neq 0$, we translate (r, t) by adding constants and rescale the Killing vector so that

$$\xi = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}.$$

Setting $t = \bar{r}\sinh(\bar{t}), r = \bar{r}\cosh(\bar{t})$ gives

$$g = \bar{r}^2(-d\bar{t}^2 + \bar{h}),$$

where

$$\bar{h} = \bar{r}^{-2}(d\bar{r}^2 + \gamma_1)$$

is the optical metric associated to the Killing vector $\partial/\partial\bar{t}$.
If $A = 0$ then a constant rescaling of t can be used to set $\xi = \cos\theta\partial_t + \sin\theta\partial_r$, where θ is a constant angle in a range which makes ξ timelike. The pseudo-orthogonal transformation of (r, t) may now be used to set $\xi = \partial/\partial t$, so the optical metric in this case is

$$h = dr^2 + \gamma_1.$$

5.2 Optical 2-metric of the Schwarzschild-Tangherlini Spacetimes

We can use the notion of the optical metric to further analyze the remarkable properties of the null geodesic structure of any static Lorentzian spacetime. In [24], the authors consider the properties of null geodesics in Schwarzschild-Tangherlini spacetimes of $n+1$ dimensions. Here, the projection of any such curve to the space of orbits of the timelike Killing vector lies in a plane and coincides with an unparametrised geodesic of a two-dimensional optical metric. It is seen that the cases $n = 3$ and $n = 6$ may be related by a conformal mapping due to Bohlin [25] and Arnold [26]. This begs the question as to whether the optical 2-metrics in these cases are projectively related and, if not, how can the relationship described?
Here, we explore the Bohlin-Arnold duality in this context. We analyse the role of the cosmological constant for these spacetimes and discuss how the zero energy solutions fit in to the duality. We also consider the possibility of a similar notion of duality for Reissner-Nordstrom spacetimes in $n+1$ dimensions.

5.2.1 Null Geodesics and Optical Metrics

Ricci-flat black holes in $n+1$ dimensions can be described by the Schwarzschild-Tangherlini (ST) metric [72]

$$g_{ST} = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{n-1}^2, \tag{5.17}$$

where $d\Omega_n^2$ is the round metric on the unit $(n-1)$-sphere and

$$\Delta = 1 - \frac{2M_n}{r^{n-2}}.$$

If we project a null geodesic of this metric to the space of orbits of the Killing vector $\frac{\partial}{\partial t}$, we find that it lies entirely in a plane through the origin. Endowing this plane with polar coordinates (r, ϕ) and setting $u = \frac{1}{r}$ this motion is described by the differential equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = 2M_n u^n + \frac{1}{b^2} \tag{5.18}$$

where b is a constant impact parameter. Null geodesics of g_{ST} may then be mapped into the motion of a non-relativistic particle moving in an attractive central force

$$F \propto \frac{1}{r^{n+1}} \Leftrightarrow V \propto \frac{1}{r^n}.$$

Thus, one may use results from dynamics to discuss the optics of black holes (see [73] for a recent application of this idea). Alternatively, equation (5.18) describes unparametrised geodesics of the optical 2-metric:

$$ds_{o_n}^2 = \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta} d\phi^2 \tag{5.19}$$

with $0 \le \phi \le 2\pi$, and projected null geodesics of the metric g_{ST} precisely coincide with the totality of unparametrised geodesics described by (5.19) on each plane through the origin.

Remark: According to [74] and [75], the cases $n = 3, 4, 6$ are integrable and may be solved in terms of elliptic functions. For a recent discussion, see [76]. The case $n = 3$ admits a special solution in the form of a *cardioid*. The case $n = 6$ admits a special solution of the form of a *Lemniscate of Bernoulli* with node at the singularity and which touches the horizon.

5.2.2 Bohlin-Arnold Duality

The following equivalence between dynamical systems in the plane is due to Arnold [26] but it has its origin in a paper due to Bohlin [25]. He introduces the complex coordinate $\zeta = x + iy$ and uses the Jacobi Principle, according to which, at fixed energy per unit mass $\mathcal{E}$, the paths described by (5.18) will be unparametrised geodesics of the metric

$$ds^2_{\text{Jacobi}} = (2\mathcal{E} - V(x,y))\, d\zeta d\bar{\zeta}. \tag{5.20}$$

Now consider a similar system in the complex $w = u + iv$ plane with Jacobi metric

$$ds^2_{\text{Jacobi}} = \left(2\tilde{\mathcal{E}} - \tilde{V}(x,y)\right) dw d\bar{w}.$$

The two systems will coincide under pullback by the conformal map

$$w = f(\zeta)$$

if

$$V = -|f'(\zeta)|^2 \tilde{\mathcal{E}} \ , \ \mathcal{E} = -|f'(w)|^2 \tilde{V}.$$

Let us consider only conformal maps of the form $w = \zeta^p$ for now. For these maps, one finds that $V \propto r^{2p-2}$ and $\tilde{V} \propto r^{\frac{2-2p}{p}}$ will work (setting $p \to \frac{1}{p}$ merely interchanges the role of V and $\tilde{V}$). Furthermore, such expressions for the potential are physically interesting from the perspective of the classical orbit. $p = 1$ gives a trivial case, but some other cases are of special note

- $p = -1$, i.e, *inversion* which is the self-dual case with

$$V \propto \tilde{V} \propto \frac{1}{r^4}.$$

- $p = 2$ takes the simple harmonic oscillator to the Kepler problem

$$V \propto r^2 \ , \ \tilde{V} \propto \frac{1}{r}.$$

- $p = -\frac{1}{2}$, this example will be at the core of our discussion and gives

$$V \propto \frac{1}{r^3} \ , \ \tilde{V} \propto \frac{1}{r^6}.$$

Another useful way to look at things is to note that if

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = Au^\alpha + B,$$

for some constants A, B and α, then

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = Ar^{4-\alpha} + Br^4.$$

This seems as though it would be particularly relevant for $\alpha = 4$. We will come back to the interpretation of this formulation later.

The duality between black holes in 3+1 and 6+1 dimensions is interesting. One example of this is the two pairs of special solutions

$$au = \frac{\cosh\phi - 2}{\cosh\phi + 1} \text{ or } \frac{\cosh\phi + 2}{\cosh\phi - 1} \quad \Rightarrow F \propto \frac{1}{r^4}.$$

$$a^2u^2 = \frac{\cosh 2\phi - 1}{\cosh 2\phi + 2} \text{ or } \frac{\cosh 2\phi + 1}{\cosh 2\phi - 2} \quad \Rightarrow F \propto \frac{1}{r^7}.$$

The first goes to the second under the replacement

$$(au, \phi) \to (a^2u^2, 2\phi),$$

which is precisely Bohlin-Arnold duality.

The special zero energy solutions are not related in the same way. However, we will see why this is the case in what follows.

5.2.3 Duality and Projective Equivalence

Now that we have demonstrated the Bohlin-Arnold duality, we may probe it a bit further. The results of the previous section seem to hint at the notion of projective equivalence. In particular, it appears that the metrics (5.19) for $n = 3$ and $n = 6$ may give rise to the same geodesics as unparametrised curves. Here, we present an argument for why that is not the case and explore the true consequences of the duality.

The family of metrics projectively equivalent to (5.19) can be completely determined for arbitrary $n \geq 3$, using the results of Chapter 3 (obviously for $n = 2$, the metric (5.19) is flat):

$$g_n = \frac{dr^2}{a\left(a - \frac{2aM_n}{r^{n-2}} + cr^2\right)^2} + \frac{r^2}{a^2\left(a - \frac{2aM_n}{r^{n-2}} + cr^2\right)} d\phi^2 \tag{5.21}$$

for constants a and c (i.e, degree of mobility 2).

This more general metric (5.21) is also familiar from a physical point of view. It is the optical metric of an ST black hole with cosmological constant $\Lambda = -\frac{n(n-1)}{2}c$. It is clear from (5.21) that, in the given set of coordinates, the metrics for $n = 3$ and $n = 6$ are not projectively equivalent but, of course, may be after a coordinate transformation. In general, determining if such a diffeomorphism exists is a difficult problem. However, we can consider the case where just the r-coordinate is transformed - this is the type of transformation suggested by the Bohlin-Arnold duality. If we consider such a coordinate change for the $n = 3$ metric (5.21) which results in the $n = 6$ case, given by $r = F(\tilde{r})$, then we find that it is impossible to transform both the dr^2 and $d\phi^2$ terms simultaneously in the appropriate way.

Hence, it seems that these metrics are not projectively equivalent even after a diffeomorphism.

5.2.4 Probing the Duality

The equation for unparametrised geodesics of (5.19) is

$$r'' - \frac{2(r')^2}{r} + \frac{nM_n}{r^{n-3}} - r = 0$$

where ' represents differentiation with respect to ϕ. Alternatively, we can express everything in terms of $u = \frac{1}{r}$:

$$u'' + u = nM_n u^{n-1} \tag{5.22}$$

or integrating once

$$(u')^2 + u^2 = 2M_n u^n + \frac{1}{b^2}, \tag{5.23}$$

where b is a constant impact parameter, as before. In this form, we can expose the correspondence between the $n = 3$ and $n = 6$ cases. Specifically, let $n = 3$

in (5.23) and make the transformation

$$u = \tilde{u}^p = \tilde{u}^{-2} \quad , \quad \phi = p\tilde{\phi} = -2\tilde{\phi}. \tag{5.24}$$

Then (5.23) becomes

$$(\tilde{u}')^2 + \tilde{u}^2 = 2M_3 + \frac{1}{b^2}\tilde{u}^6$$

whose integral curves coincide with geodesics of the $n = 6$ metric (5.19) with mass parameter $2M_6 = \frac{1}{b^2}$ and impact parameter $\frac{1}{\tilde{b}^2} = 2M_3$. From this analysis, we get a clearer picture of what is happening. The effect of the transformation is to switch the roles of the mass (fixed) and the impact parameter (constant of integration). There do not exist two mass values M_3 and M_6 so that the totality of the geodesics from one metric will be mapped into those of the other. However, if we consider the mass term as a variable integration parameter, putting it on the same footing as b, then we see how the duality works.

In particular, the entire set of geodesics determined by the one-parameter family of metrics

$$g_3(m) = \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2} + \frac{r^2}{1 - \frac{2m}{r}}d\phi^2$$

can be mapped into those determined by the one-parameter family

$$g_6(m) = \frac{dr^2}{\left(1 - \frac{2m}{r^4}\right)^2} + \frac{r^2}{1 - \frac{2m}{r^4}}d\phi^2.$$

To make this idea more clear for a general n, we can recognize this collection of geodesics as the integral curves of a 3rd order differential equation (thus turning the mass term into a constant of integration) which can be constructed as follows:

From (5.22)

$$u^{1-n}u'' + u^{2-n} = nM_n$$

and by differentiating, we obtain

$$u''' + (1-n)\frac{1}{u}(u')(u'') + (2-n)u' = 0.$$

Again, this equation for $n = 3$ can be mapped into the $n = 6$ equation via the change of coordinates

$$u = \tilde{u}^{-2} \quad , \quad \phi = -2\tilde{\phi}.$$

Remark: This procedure will work for any value of n as long as we pick $p = -\frac{2}{n-2}$ which is the transformation implied by the Bohlin-Arnold duality.

5.2.5 Zero Energy Solutions

As said before, the zero energy solution for $n = 3$ is a cardioid and for the $n = 6$ case it is a Lemniscate of Bernoulli. These solutions do not get directly mapped onto each other but we can try to determine the dual curves. To obtain them, first note that the zero energy geodesic coincides with the solution of (5.23) for which $\frac{1}{b^2} \to 0$. Clearly, for any value of n, this gives rise to a dual curve with vanishing mass parameter which is just a projected light ray in the Minkowski case (a straight line). Thus, the zero energy solutions in the $n = 3$ and $n = 6$ cases with equal mass can be mapped onto each other but not directly via Bohlin-Arnold. Indeed, the zero energy curves of the equal mass black holes for any two values of n can be mapped to each other in this way.

5.2.6 Special Conformal Transformation

In [77], the Bohlin-Arnold duality of forces is also uncovered as a diffeomorphism of the complex plane which corresponds to a conformal transformation in real coordinates. In 2 dimensions, all real metrics are conformally flat so, from this point of view, it does not seem that this transformation is particularly special. However, by viewing it as a function on the complex plane we restrict our attention to special types of conformal transformation, which takes account of the underlying geometry.

Furthermore, if we wish to retain the Jacobi form of the metric, as in (5.20) such that the roles of the energy $\mathcal{E}$ and the potential $V(x, y)$ are switched, then we must have the transformation in the form $f(z) = z^p$ and this provides a map between geometries with potentials of the form $V \propto r^p$. Hence, the duality map lies in a special category of conformal transformations.

5.3 Reissner-Nordstrom metrics and Duality

One question that arises from the above work is whether a similar notion of duality exists in the case of charged black holes. To answer this, first note that the projection of a null geodesic of the metric (5.17) for an arbitrary function $\Delta = \Delta(r)$ will lie in a plane, due to the inherent spherical symmetry of (5.17), and will coincide with an unparametrised geodesic of the optical metric (5.23) on that plane. The differential equation describing unparametrised geodesics of (5.23) for general $\Delta = \Delta(r)$ is

$$(u')^2 = -u^2\Delta + \frac{1}{b^2} \tag{5.25}$$

where we have chosen the constant of integration to match up with the definition of the impact parameter from before.

It is clear from this equation that if we modify Δ by adding an r^2 term then the set of unparametrised geodesics will be unchanged. This highlights the fact that the dynamics of light rays in such static space-times will be invariant with respect to the addition of a cosmological constant.

For the Reissner-Nordstrom metric in n dimensions

$$\Delta = 1 - \frac{2M_n}{r^{n-2}} + \frac{Q_n^2}{r^{2n-4}}$$

so that the equation for unparametrised geodesics of the optical metric becomes

$$(u')^2 + u^2 = 2M_n u^n - Q_n^2 u^{2n-2} + \frac{1}{b^2}. \tag{5.26}$$

Now let us produce a new equation by making the change of coordinates

$$u = \tilde{u}^p \quad , \quad \phi = p\tilde{\phi}$$

Then (5.26) becomes

$$(\tilde{u}')^2 + \tilde{u}^2 = 2M_n \tilde{u}^{(n-2)p+2} - Q_n^2 \tilde{u}^{(2n-4)p+2} + \frac{1}{b^2}\tilde{u}^{2-2p}. \tag{5.27}$$

If there is a duality as in the ST case, then we must be able to put this equation in the form of (5.26) for some value of n.

Two of the exponents in (5.27) will be equal only for $n = 0, 1$ or 2. Otherwise, we require that one of the new terms takes the place of the impact parameter i.e, one of the exponents vanishes. Since $p = 1$ is trivial, this means that we have two cases to consider

$$p = \frac{2}{2-n} \quad \text{or} \quad p = \frac{2}{4-2n}.$$

In the first case, (5.27) becomes

$$(\tilde{u}')^2 + \tilde{u}^2 = 2M_n - Q_n^2 \tilde{u}^{-2} + \frac{1}{b^2} \tilde{u}^{\frac{2n}{n-2}}$$

This equation resembles (5.26) only when $n = 0$ (where the duality is trivial) or when $\frac{2n}{n-2} = -6$. Hence, there is a non-trivial duality between the cases $n = -2$ and $n = \frac{3}{2}$.

When $p = \frac{2}{4-2n}$, (5.27) becomes

$$(\tilde{u}')^2 + \tilde{u}^2 = 2M_n \tilde{u} - Q_n^2 + \frac{1}{b^2} \tilde{u}^{\frac{2n-2}{n-2}}$$

which in the form of (5.26) only for $n = 1$ (trivial) and $n = -2$ (the duality from before).

In summary, we've obtained the following dual solutions:

- $n = 0$

  There is a duality between the $n = 0$ case and the zero energy Reissner-Nordstrom solution for any value of p where $M_0 + \frac{1}{2b^2}$ is the new mass parameter and Q_n is the charge. Furthermore, this reduces to the ST solution for $n = 1$ where $M_0 + \frac{1}{2b^2}$ becomes the mass parameter and $-Q_n^2$ is the integration constant/impact parameter.

- $n = 1$

  Similarly, for $n = 1$, we obtain a duality with the zero energy R-N solution for any value of p where M_1 is the new mass parameter and $\sqrt{Q_n^2 - \frac{1}{b^2}}$ is the charge. This reduces to the ST for $n = -2$.

- $n = 2$

  This is the flat case with $\Delta =$ constant and the solutions can be mapped into the zero energy solution of any ST projected null geodesic by an appropriate choice of p.

- Finally, there is a duality between the cases $n = \frac{3}{2}$ and $n = -2$ where the roles of the mass, charge and impact parameters of the former are interchanged with the charge, impact parameter and mass, respectively, of the latter. However, this does not correspond to the dynamics of light rays in some optical 2-metric.

Even though none of these cases gives rise to a duality that is interesting from the optical metric point of view, we can still view the trajectories as describing particles moving in a central force of the form

$$F = \frac{\alpha}{r^{n+1}} + \frac{\beta}{r^{2n-1}}$$

for constants α and β making them still physically relevant.
Yet again, it is clear from the expressions (5.26) and (5.27) that the zero energy solutions can be mapped into each other for any two values of n by appropriate choice of p.

Chapter 6

Conformal Retraction and the Kastor-Traschen metric

In the previous chapter, we demonstrated how the light rays of any static Lorentzian spacetime project to the unparametrised geodesics of the optical metric associated to the timelike static Killing vector. This structure allows us to draw conclusions about the null geodesic structure of static metrics, invariantly. As stated previously, this formalism has important applications in general relativity, e.g, [23] and [69]. A recent idea has been to generalise this notion in order to study of the behaviour of null geodesics of different classes of spacetimes which admit a particular type of timelike vector field.
Significant progress has been made in the case of *stationary metrics* [78], where there are two distinguished geometric structures on the space of orbits $\mathcal{B}$ of the timelike Killing vector field. To obtain these pictures, we must broaden our scope from the Riemannian geometry of the optical metric discussed in the static case to Finslerian geometry as introduced previously. Any stationary metric is locally form

$$g = -V^2(x^i)(dt + \omega_j(x^i)dx^j)^2 + h_{jk}dx^j dx^k \ \ , \ \ i, j, k = 1, \ldots, n$$

where V and ω are some arbitrary function and one-form, respectively, h is an n-dimensional Riemannian metric and the stationary Killing vector is $K = \partial/\partial t$. On one hand, the light rays of the stationary metric project down to solutions of the Zermelo navigation problem on $\mathcal{B}$ which can be stated as follows:

Zermelo's problem: Given a Riemannian metric γ_{ij} on $\mathcal{B}$, what is the least time trajectory for a ship moving with constant speed in a wind $\mathbf{W}$?

To see this explicitly, we can rewrite the metric g in *Painlevé-Gullstrand form* [79], [80] by choosing

$$\gamma_{ij} = \frac{1}{1 + V^2 g^{rs}\omega_r\omega_s} \quad , \quad W^i = V^2 g^{ij}\omega_j .$$

Then we have

$$g = \frac{V^2}{1 - \gamma_{ij}W^iW^j}\left[-dt^2 + \gamma_{ij}(dx^i - W^i dt)(dx^j - W^j dt)\right]$$

The conformal properties of g are encoded in the metric in square brackets and it can be shown that light rays of this metric project down to solutions of the Zermelo problem with background metric γ_{ij} and wind vector $\mathbf{W}$. Moreover, metrics of this form (is the square brackets) have applications in theoretical physics, in particular being used as analogue models of black holes and also to describe electromagnetic waves propagating in moving media.

Alternatively, from the work in Chapter 4, we know there is a duality between solutions of the Zermelo problem and unparametrised geodesics of a Finsler metric of Randers form. Indeed, if we choose

$$\alpha_{ij} = V^{-2}h_{ij} \ \ , \ \ \alpha^{ij} = V^2 h^{ij} \ \ , \ \ \beta_i = -\omega_i ,$$

then our static metric becomes

$$g = V^2\left[-(dt - \beta_j dx^j)^2 + \alpha_{jk}dx^j dx^k\right].$$

From this perspective, it is clear that null geodesics of g will project down to unparametrised geodesics of the Randers metric

$$\mathcal{F} = \sqrt{\alpha_{ij}(x)dx^i dx^j} + \beta_i(x)dx^i \ \ , \ \ i, j = 1, \ldots, n.$$

If $\beta_i = -\omega_i = 0$ we make contact with the static case and the Randers norm is just the norm of a metric. Physically, the integral curves of this system correspond to the magnetic flow due to $d\beta$ on the curved manifold $\{\mathcal{B}, \alpha_{ij}\}$.

Hence, overall we have a triality of structures between the integral curves of Randers structures and Zermelo structures of dimension n and the null curves of stationary Lorentzian metrics of one dimension higher. As before, conformal properties of the metric g may be inferred from properties of the corresponding structures on the space of orbits. For example, it is shown in [23] that the stationary metric g is conformally flat if and only if the corresponding Randers metric has constant flag curvature.

In this chapter, we generalise this work to another class of metrics - those which admit a timelike *conformal retraction*, i.e, there exists a hypersurface-orthogonal timelike vector field, Θ, for which the conformal structure on its space of orbits is preserved along the integral curves of Θ. In general, it is not straightforward to obtain a similar structure on the space of orbits of Θ which mimics the projection of the null geodesics of g. In the Riemannian case, such metrics have arisen as supersymmetric solutions of minimal $\mathcal{N} = 2$ gauged supergravity with anti-self-dual Maxwell field [27]. Moreover, when the anti-self-duality condition is relaxed in the case of positive cosmological constant, one obtains a solution, also admitting a conformal retraction, which is the Riemannian analogue of the well-known *Kastor-Traschen* metric [28].

In Lorentzian signature, this metric is a time-dependent solution to the Einstein-Maxwell equations which can be seen to describe an arbitrary number of dynamical charged black holes in a deSitter background. Here, we focus on the null geodesic structure of these metrics to gain some relevant insight into the general case and in particular, we will see that light rays project down to integral curves of a system of third order ordinary differential equations (ODEs) on the space of orbits of Θ.

We will pay particular attention to the one-centre solution (single black hole). We show that, in the limit as the cosmological constant tends to zero, our system of ODEs becomes that describing conformal circles of the flat metric (as

described in [29]). We use this result to motivate the discussion on the numerology of the problem and construct a new third order system in three dimensions. This formulation also allows us to characterize the projected light rays.
There is a well understood diffeomorphism between the extreme Reissner-Nordstrom deSitter metric and the Kastor-Traschen metric which we shall make us of to derive analytic expressions for the projection of null geodesics. We give plots of some of these curves and discuss the horizon structure in both sets of coordinates.
Included near the end is a discussion of the Kastor-Traschen solution with two centres. For one such curve, we will illustrate a connection between the null geodesics of the two-centre Kastor-Traschen metric and a third order system that arises in the analysis of the one-centre case. We also look at the perturbations away from this plane and give a strict condition for stability.
Throughout, we refer to the projection of geodesics onto the space of orbits of the conformal retraction Θ as the *retraction projection* in order to avoid ambiguity.

6.1 Conformal Retraction in the Kastor-Traschen metric

The Kastor-Traschen metrics are a class of time dependent solutions to the Einstein-Maxwell equations with positive cosmological constant Λ [28]. In local coordinates, these metrics may be written as

$$g = -\frac{dT^2}{(V+cT)^2} + (V+cT)^2 h \tag{6.1}$$

with Maxwell 1-form

$$A = \frac{dT}{V+cT},$$

where $V = V(x)$ is a harmonic function on the spatial coordinates, $c = \pm\sqrt{\frac{\Lambda}{3}}$ is a constant and

$$h = h_{ij}dx^i dx^j \quad , \quad i,j = 1,2,3$$

is the flat 3-dimensional Riemannian metric (which we express in Cartesian coordinates for now).

Notice here that since V is a harmonic function there is a freedom in defining the electromagnetic field tensor $F = dA$ so that the Einstein-Maxwell equations are still satisfied i.e, we can write

$$F = \frac{1}{\sqrt{1+\nu^2}}\left(dA - \nu\epsilon_{ijk}h^{il}\frac{\partial V}{\partial x^l}dx^j \wedge dx^k\right) \quad , \quad i,j,k = 1,2,3 \tag{6.2}$$

where ν is a constant and ϵ_{ijk} is totally anti-symmetric in its indices with $\epsilon_{123} = 1$. This allows us to introduce a magnetic field $\mathbf{B} \propto \nabla V$ into our definition of the Kastor-Traschen solution, a notion which will be useful later.

In the limit as $c \to 0$, these metrics reduce to the well-known Majumdar-Papapetrou solutions [81], [82]. For the M-P metrics, it can be shown that solutions with

$$V = \sum_{\alpha=1}^{N} \frac{m_\alpha}{|\mathbf{x} - \mathbf{w}_\alpha|},$$

where $|\mathbf{x} - \mathbf{w}_\alpha| = (h_{jk}(x^j - w_\alpha^j)(x^k - w_\alpha^k))^{1/2}$ and $\mathbf{w}_\alpha$ is a fixed vector for each α, can be analytically extended to be interpreted as a system of charge equal mass black holes [83]. In this system, the gravitational forces on each black hole are balanced by the electrostatic forces. However, when $c \neq 0$, the black holes are dynamic and can be observed to coalesce [28].

As noted in the introduction, an interesting property of the K-T metrics is that they admit a timelike *conformal retraction* Θ. In other words,

$$\mathcal{L}_\Theta H_{\mu\nu} = fH_{\mu\nu} + \Theta_{(\mu}C_{\nu)}$$

where

$$H_{\mu\nu} = g_{\mu\nu} - \frac{\Theta_\mu\Theta_\nu}{\Theta^2} \quad , \quad \Theta^2 := g_{\mu\nu}\Theta^\mu\Theta^\nu$$

and f and C are an arbitrary function and one-form, respectively. Greek indices here run over the values 0,1,2,3 and we raise and lower indices using the metric $g_{\mu\nu}$. In general, am $(n+1)$-dimensional metric g which admits such a timelike conformal retraction can be written in local coordinates in the form

$$g = -V^2(T,x)dT^2 + 2B_i(T,x)dTdx^i + \left(\gamma_{ij}(x)e^{\phi(T,x)} - \frac{B_iB_j}{V^2}\right)dx^idx^j \ , \ i,j = 1,\dots,n$$

for time coordinate T, where V and ϕ are arbitrary functions, B_i is an arbitrary one-form and γ is a metric of dimension n. The conformal retraction is $\Theta = \frac{\partial}{\partial T}$ as it is in (6.1). Furthermore, in the Kastor-Traschen case, the tensor H is given by

$$H_{0\mu} = 0 \quad , \quad H_{\mu\nu} = g_{\mu\nu} \quad \text{otherwise.}$$

The Lie derivative of this tensor is easy to compute

$$(\mathcal{L}_\Theta H)_{\mu\nu} = \Theta^\lambda H_{\mu\nu,\lambda} + H_{\lambda\nu}\Theta^\lambda_{,\mu} + H_{\mu\lambda}\Theta^\lambda_{,\nu} = \frac{\partial}{\partial T} H_{\mu\nu} = 2c(V + cT)H_{\mu\nu}.$$

Hence, in this case, our function f and one-form C are given by

$$f = 2c(V + cT) \quad , \quad C = 0.$$

Now, let us decree that under a change of metric $\tilde{g} = \Omega^2 g$, the choice of conformal retraction remains unchanged i.e, $\tilde{\Theta} = \Theta = \frac{\partial}{\partial T}$. Then, $\tilde{H}_{\mu\nu} = \Omega^2 H_{\mu\nu}$ and

$$(\mathcal{L}_{\tilde{\Theta}}\tilde{H})_{\mu\nu} = (\mathcal{L}_\Theta \Omega^2 H)_{\mu\nu} = (\mathcal{L}_\Theta \Omega^2) H_{\mu\nu} + \Omega^2(\mathcal{L}_\Theta H)_{\mu\nu} = (f + 2\mathcal{L}_\Theta \log\Omega)\tilde{H}_{\mu\nu} + \Omega^2 \Theta_{(\mu}C_{\nu)}.$$

Hence,

$$\tilde{f} = f + 2\mathcal{L}_\Theta \log\Omega \quad , \quad \tilde{C} = \Omega^2 C.$$

and, for our example, $C = 0$ for any choice of metric in the conformal class of g.

6.2 Projection of Null geodesics with arc-length parametrisation

Since we are considering only the null geodesic structure of the Kastor-Traschen metrics, we may as well begin with a conformally rescaled version of (6.1) - this will ease the computation a little. So, let us take a new definition of g

$$g \to \frac{1}{(V + cT)^2} g = -\frac{dT^2}{(V + cT)^4} + h_{jk} dx^j dx^k.$$

For this metric, we can calculate the Christoffel symbols

$$\Gamma^i_{00} = -\frac{2}{(V+cT)^5} h^{ij} \frac{\partial V}{\partial x^j} \quad , \quad \Gamma^i_{0j} = 0 = \Gamma^i_{jk}.$$

Hence, the geodesic equations for the spatial components may be written as

$$\ddot{x}^i - \frac{2}{(V+cT)^5} h^{il} \frac{\partial V}{\partial x^l} \dot{T}^2 = F(s)\dot{x}^i \quad ; \quad \dot{} = \frac{d}{ds}$$

where $F(s)$ is some function of the curve parameter s. If we assume that our geodesics are null, then we can rewrite this equation as

$$\ddot{x}^i - \frac{2}{V+cT} h^{il} \frac{\partial V}{\partial x^l} h_{jk}\dot{x}^j\dot{x}^k = F(s)\dot{x}^i. \tag{6.3}$$

Now let us impose the condition that s be the arc-length parameter for the metric h, i.e, $h_{jk}\dot{x}^j\dot{x}^k = 1$.

$$\Rightarrow 0 = h_{jk}\dot{x}^j\ddot{x}^k = \frac{2}{V+cT}\frac{\partial V}{\partial x^k}\dot{x}^k + F(s)$$
$$\Rightarrow F(s) = -\frac{2}{V+cT}\frac{\partial V}{\partial x^k}\dot{x}^k.$$

Hence, given that we use the arc-length parametrisation, the equation for null geodesics (6.3) may be written as

$$\ddot{x}^i = \frac{2}{V+cT}\left(h^{il}\frac{\partial V}{\partial x^l} - \frac{\partial V}{\partial x^k}\dot{x}^k\dot{x}^i\right). \tag{6.4}$$

In the case when h in (6.1) is non-flat, we can derive an analogous expression

$$\ddot{x}^i + \hat{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k = \frac{2}{V+cT}\left(h^{il}\frac{\partial V}{\partial x^l} - \frac{\partial V}{\partial x^k}\dot{x}^k\dot{x}^i\right) \tag{6.5}$$

where $\hat{\Gamma}^i_{jk}$ are the connection components of the metric h. Two properties that we get from equation (6.4) are

$$h_{jk}\ddot{x}^j\ddot{x}^k = \frac{2}{V+cT}\frac{\partial V}{\partial x^k}\ddot{x}^k \tag{6.6}$$
$$\ddot{x}^k\frac{\partial V}{\partial x^k} = \frac{2}{V+cT}\left(\left(\frac{\partial V}{\partial x^k}\right)\left(\frac{\partial V}{\partial x_k}\right) - \left(\frac{\partial V}{\partial x^k}\dot{x}^k\right)^2\right) \tag{6.7}$$

Now, if we differentiate (6.4), we get the following

$$\begin{aligned}\dddot{x}^i &= -\frac{2}{(V+cT)^2}\left(\frac{\partial V}{\partial x^k}\dot{x}^k + c\dot{T}\right)\left(h^{il}\frac{\partial V}{\partial x^l} - \frac{\partial V}{\partial x^k}\dot{x}^k\dot{x}^i\right)\\ &+ \frac{2}{V+cT}\left(h^{il}\frac{\partial^2 V}{\partial x^l\partial x^m}\dot{x}^m - \frac{\partial^2 V}{\partial x^k\partial x^l}\dot{x}^k\dot{x}^l\dot{x}^i - \frac{\partial V}{\partial x^k}\ddot{x}^k\dot{x}^i - \frac{\partial V}{\partial x^k}\dot{x}^k\ddot{x}^i\right).\end{aligned}$$

Then, using the null condition to eliminate $\dot{T}$ and equations (6.4), (6.6) and (6.7) to eliminate the T parameter, we can rewrite this in the form:

$$\begin{aligned} \dddot{x}^i &= -h_{jk}\ddot{x}^j\ddot{x}^k\dot{x}^i - \frac{3\ddot{x}^k\frac{\partial V}{\partial x^k}}{2\left(\frac{\partial V}{\partial x^k}\frac{\partial V}{\partial x_k} - \left(\frac{\partial V}{\partial x^k}\dot{x}^k\right)^2\right)}\frac{\partial V}{\partial x^l}\dot{x}^l\ddot{x}^i - 2c\left(h^{il}\frac{\partial V}{\partial x^l} - \frac{\partial V}{\partial x^k}\dot{x}^k\dot{x}^i\right) \\ &+ \frac{2}{V+cT}\left(h^{il}\frac{\partial^2 V}{\partial x^l \partial x^m}\dot{x}^m - \frac{\partial^2 V}{\partial x^k \partial x^l}\dot{x}^k\dot{x}^l\dot{x}^i\right). \end{aligned} \tag{6.8}$$

Here, we can also eliminate the factor of $\frac{2}{V+cT}$ by using (6.7) so that we really have a system of third order ODEs completely dependent on the spatial coordinates of g only. However, it is useful to keep it in the form above for the computation in the next section.

6.3 One-Centre Case

As was described in [28], a spacetime containing N charged black holes with masses m_α ($\alpha = 1, \ldots, N$) and charges $q_\alpha = m_\alpha$ in a deSitter background can be represented by equation (6.1) with

$$V = \sum_{\alpha=1}^{N} \frac{m_\alpha}{|\mathbf{x} - \mathbf{w}_\alpha|}$$

where $\mathbf{w}_\alpha$ is a fixed vector for each α. It is easily verified that V is a harmonic function.

In this section, we look at the case $N = 1$ where the black hole is situated at the origin. So, put $V = \frac{m}{|\mathbf{x}|}$. With this definition, we can obtain the following identities:

$$\frac{\partial V}{\partial x^j} = -\frac{m}{|\mathbf{x}|^3}h_{jl}x^l \quad , \quad \frac{\partial^2 V}{\partial x^j \partial x^k} = \frac{3m}{|\mathbf{x}|^5}h_{jl}x^j h_{km}x^m - \frac{m}{|\mathbf{x}|^3}h_{jk}.$$

In particular, equation (6.4) becomes

$$\ddot{x}^i = \frac{2}{V+cT}\left(-\frac{m}{|\mathbf{x}|^3}x^i + \frac{m}{|\mathbf{x}|^3}(\mathbf{x}.\dot{\mathbf{x}})\dot{x}^i\right), \tag{6.9}$$

where $\mathbf{x}.\dot{\mathbf{x}} \equiv h_{jk}x^j\dot{x}^k$ and all subsequent dot products are taken with respect to the metric h unless otherwise stated. This allows us to reduce the last term of

(6.8), that is

$$
\begin{aligned}
& \frac{2}{V+cT}\left(h^{il}\frac{\partial^2 V}{\partial x^l \partial x^m}\dot{x}^m - \frac{\partial^2 V}{\partial x^k \partial x^l}\dot{x}^k\dot{x}^l\dot{x}^i\right) \\
= \; & \frac{2}{V+cT}\left(\frac{3m}{|\mathbf{x}|^5}(\mathbf{x}.\dot{\mathbf{x}})x^i - \frac{3m}{|\mathbf{x}|^5}(\mathbf{x}.\dot{\mathbf{x}})^2\dot{x}^i\right) = -\frac{3}{|\mathbf{x}|^2}(\mathbf{x}.\dot{\mathbf{x}})\ddot{x}^i.
\end{aligned}
$$

With this simplification in mind, we can rewrite our system of third order ODEs (6.8) as

$$
\dddot{x}^i = -|\ddot{\mathbf{x}}|^2\dot{x}^i + \frac{2mc}{|\mathbf{x}|^3}\left(x^i - (\mathbf{x}.\dot{\mathbf{x}})\dot{x}^i\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}|^2 - (\mathbf{x}.\dot{\mathbf{x}})^2)}\right)\ddot{x}^i. \tag{6.10}
$$

Any null geodesic of the Kastor-Traschen metric g will project down to an integral curve of this system of third-order ODEs.

6.3.1 Conformal Circles

As $c \to 0$, it's obvious that the second term on the right-hand side of (6.10) vanishes. However, using the second equation in (6.7) with $c = 0$ and $V = \frac{m}{|\mathbf{x}|}$, we also find that

$$
\mathbf{x}.\ddot{\mathbf{x}} = 2\left(\frac{(\mathbf{x}.\dot{\mathbf{x}})^2}{|\mathbf{x}|^2} - 1\right)
$$

and the third term vanishes. To see the vanishing of the third term explicitly occuring with the vanishing of c, it seems we need to reintroduce the time coordinate T, in some way. For example, using (6.7) we can write (6.10) as

$$
\dddot{x}^i = -|\ddot{\mathbf{x}}|^2\dot{x}^i + \frac{2mc}{|\mathbf{x}|^3}\left(x^i - (\mathbf{x}.\dot{\mathbf{x}})\dot{x}^i\right) + 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{cT|\mathbf{x}|(\mathbf{x}.\ddot{\mathbf{x}})}{2m(|\mathbf{x}|^2 - (\mathbf{x}.\dot{\mathbf{x}})^2)}\right)\ddot{x}^i.
$$

Hence, as $c \to 0$, null geodesics satisfy

$$
\dddot{x}^i = -|\ddot{\mathbf{x}}|^2\dot{x}^i. \tag{6.11}
$$

We shall see that this system (6.11) occupies a central role in the theory of conformally flat manifolds.

In general, given a conformal structure $[\tilde{h}]$ on an n-dimensional manifold, there is a distinguished family of curves, known as the *conformal circles*. These curves arise as the integral curves of a system of third order ODEs, see [84]. To write this system down, let us choose a metric $\tilde{h}$ in the conformal class with torsion-free connection $\tilde{\Gamma}^i_{jk} = \tilde{\Gamma}^i_{(jk)}$, Ricci tensor R_{jk} and scalar curvature R. Then, the Schouten tensor P_{jk} is defined as

$$P_{jk} = -\frac{1}{n-2}\left(R_{jk} - \frac{R}{2(n-1)}\tilde{h}_{jk}\right).$$

Furthermore, define the vector components $U^i = \dot{x}^i$ and $A^i = \ddot{x}^i + \tilde{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k$ so that $\mathbf{U}.\mathbf{A} = \tilde{h}_{jk}U^jA^k$, etc, and scalar products coincide with our previous definitions when $\tilde{h}$ is flat. Then, a curve is a conformal circle of $[\tilde{h}]$ if it satisfies

$$\frac{dA^i}{ds} + \tilde{\Gamma}^i_{jk}A^jU^k = \frac{3\mathbf{U}.\mathbf{A}}{|\mathbf{U}|^2}A^i - \frac{3|\mathbf{A}|^2}{2|\mathbf{U}|^2}U^i + |\mathbf{U}|^2U^jP^i_j - 2P_{jk}U^jU^kU^i \tag{6.12}$$

where there is no restriction on the parameter s. Equation (6.12) is invariant with respect to conformal transformations, $\tilde{h} \to \Omega^2\tilde{h}$, and so, conformal circles are defined invariantly by any metric in the conformal class. These curves have arisen in a physical context in [85] where the authors have used them to discuss the asymptotics of Einstein's equations. Furthermore, properties of "conformal geodesics" (lifts of conformal circles to the bundle of second order frames over the manifold endowed with the conformal Cartan connection) in vacuum and warped-product spacetimes have been studied in [86].

It is shown in [84] that the conformal circles of a given conformal manifold can be equally defined as the set of integral curves of the system of ODEs

$$\frac{dA^i}{ds} + \tilde{\Gamma}^i_{jk}A^jU^k = -|\mathbf{A}|^2U^i + U^jP^i_j - P_{jk}U^jU^kU^i \tag{6.13}$$

where, here, s is required to be the arc-length parameter of the metric $\tilde{h}$. This formulation was originally given by Yano in [29] and is more useful for our purposes. If we now let $\tilde{h}$ be the flat Riemannian metric, equation (6.13) reduces to (6.11).

Hence, as $c \to 0$ in the one-centre Kastor-Traschen metric, null geodesics project down to conformal circles of the flat metric in three dimensions. It is easily verified, that the set of integral curves of (6.11) in three dimensions is precisely the set of all circles in $\mathbb{R}^3$.

We should note here that we get the same result when we let $m \to 0$. Again, the second term on the right-hand side of (6.10) obviously vanishes and (6.9) reduces to

$$\ddot{x}^i = 0,$$

so the third term also vanishes.

6.4 Characterisation of Null geodesics

In the preceding sections, we have determined a system of third-order ODEs which the projected null geodesics of g along the conformal retraction Θ must satisfy. However, it is not clear that, given an integral curve of (6.8), it will necessarily be the projection of some light ray of g. In fact, we can show this not to be the case and it transpires that the third order ODE system (6.8) is not uniquely defined. In this section, we discuss this point and construct a new third order system in three dimensions for which the integral curves constitute a retraction projection of a special set of null curves of the Kastor-Traschen metric, which have a physical interpretation. The projected null geodesics form a subset of these curves which we can characterize.

For example, let us consider the case $c \to 0$ for the one-centre metric. Here, we found that the integral curves of (6.11) will be the set of circles in $\mathbb{R}^3$.
However, as $c \to 0$, the metric g becomes static with static Killing vector $\Theta = \frac{\partial}{\partial T}$ and it is well known that the null geodesics of this metric project down to the unparametrised geodesics of the associated optical metric

$$h_{\mathrm{opt}} = \left(\frac{m}{|\mathbf{x}|}\right)^4 h_{jk} dx^j dx^k.$$

One can check that the unparametrised geodesics of this metric will be precisely the set of circles in $\mathbb{R}^3$ which pass through the origin. Hence, only a subset of the integral curves of (6.11) will coincide with the projected null geodesics of g. Note that this is also the case for $m \to 0$ where projected null geodesics are described by $\ddot{x}^i = 0$ - straight lines. We can check the general numerology here to see what happens.

Firstly, we note that the set of unparametrised geodesics of an arbitrary metric $\hat{g}$ on some open set $U \subset \mathbb{R}^n$ will lift to a foliation, by the geodesic spray, of the projectivised tangent bundle $\mathbb{P}(TU)$ which we can think of as a 1-dimensional fibration over some $(2n-2)$-dimensional space, Z, with each point in Z coinciding with a unique geodesic in U. Hence, the set of unparametrised geodesics of an n-dimensional manifold constitute a $(2n-2)$-parameter family of curves. Taking the specific example of the Kastor-Traschen metric g, we have $n = 4$ and so the number of parameters describing unparametrised geodesics is 6. Invoking the null condition, we see that the retraction projection of unparametrised null geodesics will, in general, constitute a 5-parameter family of curves in $\mathbb{R}^3$ - in the special case where the conformal retraction is a static Killing vector, this is a 4-parameter family.

On the other hand, let us consider a set of curves on some open set $U \subset \mathbb{R}^n$ described by a system of third order ODEs. If we write this in an unparametrised way - as a set of $(n-1)$ third-order ODEs in terms of one of the coordinates - then we see that the integral curves of this system will lift to a foliation of the jet bundle $J^2(U, \mathbb{R})$ which is $(3n-2)$-dimensional.
Hence the unparametrised integral curves of a system of third-order ODEs in n dimensions constitutes a $(3n-3)$-parameter family of paths. When $n = 3$, for example, we will have a 6-parameter family of such curves which is consistent with our results above.

Hence, the set of projected null geodesics of the Kastor-Traschen metric, g, will constitute a 5-dimensional subset of the 6-dimensional family of unparametrised integral curves of (6.8) (except in the static case).
So, a natural question arises: Given that we construct a third order system (such as (6.8)) for which the projected null geodesics form a proper subset of the set of integral curves then to what do the other integral curves correspond?

The system (6.8) does not help us to answer this question but we can derive a different third order system which will. For convenience and clarity on this point, let us write our system of equations describing null geodesics of the Kastor-Traschen metric (6.4) in three-dimensional vector notation i.e,

$$\ddot{\mathbf{x}} = \frac{2}{V(\mathbf{x}) + cT}\left(\nabla V - (\nabla V.\dot{\mathbf{x}})\dot{\mathbf{x}}\right) = \frac{2}{V + cT}\left(\dot{\mathbf{x}} \times (\nabla V \times \dot{\mathbf{x}})\right).$$

Now let us consider a modification of this equation by adding an orthogonal term on the right-hand side, that is

$$\ddot{\mathbf{x}} = \frac{2}{V + cT}\left(\dot{\mathbf{x}} \times (\nabla V \times \dot{\mathbf{x}})\right) + \lambda(\dot{\mathbf{x}} \times \nabla V) \tag{6.14}$$

where λ is a constant. Clearly, null geodesics satisfy this equation for $\lambda = 0$. More interestingly, there is a six-parameter family of curves which satisfy this equation for some value of λ. Hence, we might expect these curves to be the integral curves of some third order system in three dimensions which is independent of λ.

First from (6.14), we can derive the following equations by taking specific scalar products:

$$\begin{aligned} |\ddot{\mathbf{x}}|^2 &= \frac{2}{V + cT}\nabla V.\ddot{\mathbf{x}} + \lambda(\dot{\mathbf{x}} \times \nabla V).\ddot{\mathbf{x}} \\ \ddot{\mathbf{x}}.\nabla V &= \frac{2}{V + cT}|\nabla V \times \dot{\mathbf{x}}|^2 \\ \lambda &= \frac{\ddot{\mathbf{x}}.(\dot{\mathbf{x}} \times \nabla V)}{|\dot{\mathbf{x}} \times \nabla V|^2}. \end{aligned} \tag{6.15}$$

Differentiating (6.14) and simplifying using the first two equations of (6.15), we derive the system of third order ODEs

$$\begin{aligned} \dddot{\mathbf{x}} &= -|\ddot{\mathbf{x}}|^2\dot{\mathbf{x}} - \frac{3}{2}\frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}(\dot{\mathbf{x}}.\nabla V)\ddot{\mathbf{x}} - 2c(\dot{\mathbf{x}} \times (\nabla V \times \dot{\mathbf{x}})) + \frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}\left(\dot{\mathbf{x}} \times \left(\frac{d\nabla V}{ds} \times \dot{\mathbf{x}}\right)\right) \\ &+ \lambda\left[\left(\frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}\right)(\dot{\mathbf{x}}.\nabla V)\ddot{\mathbf{x}} \times \nabla V + ((\dot{\mathbf{x}} \times \nabla V).\ddot{\mathbf{x}})\dot{\mathbf{x}} + \ddot{\mathbf{x}} \times \nabla V + \dot{\mathbf{x}} \times \frac{d\nabla V}{ds}\right]. \end{aligned}$$

For $\lambda = 0$, this system of equations reduces to (6.8) with the $\frac{2}{V+cT}$ term replaced using (6.7) as expected. We can eliminate λ from this equation using the third equation of (6.15). Notice then that the vanishing of λ coincides with

$\ddot{\mathbf{x}}.(\dot{\mathbf{x}} \times \nabla V) = 0$ i.e, the vectors $\ddot{\mathbf{x}}, \dot{\mathbf{x}}$ and ∇V lie in the same plane. In particular, for the one centre case, the projections of null geodesics lie in a plane through the origin (centre). Overall, we have the following result

Proposition 6.1 *If $c \neq 0$, the retraction projection of the set of null curves satisfying (6.14) for some value of λ coincides with the set of integral curves of*

$$\begin{aligned}
\dddot{\mathbf{x}} &= -|\ddot{\mathbf{x}}|^2\dot{\mathbf{x}} - \frac{3}{2}\frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}(\dot{\mathbf{x}}.\nabla V)\ddot{\mathbf{x}} - 2c(\dot{\mathbf{x}} \times (\nabla V \times \dot{\mathbf{x}})) \\
&+ \frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}\left(\dot{\mathbf{x}} \times \left(\frac{d\nabla V}{ds} \times \dot{\mathbf{x}}\right)\right) + \frac{\ddot{\mathbf{x}}.(\dot{\mathbf{x}} \times \nabla V)}{|\dot{\mathbf{x}} \times \nabla V|^2}\Big[\left(\frac{\ddot{\mathbf{x}}.\nabla V}{|\dot{\mathbf{x}} \times \nabla V|^2}\right)(\dot{\mathbf{x}}.\nabla V)\ddot{\mathbf{x}} \times \nabla V \\
&+ ((\dot{\mathbf{x}} \times \nabla V).\ddot{\mathbf{x}})\dot{\mathbf{x}} + \frac{d}{ds}(\dot{\mathbf{x}} \times \nabla V)\Big]. \qquad (6.16)
\end{aligned}$$

Furthermore, the projected null geodesics are precisely the integral curves of this system for which $\ddot{\mathbf{x}}.(\dot{\mathbf{x}} \times \nabla V) = 0$.

Proof. As we have shown, any integral curve of (6.14) satisfies (6.16). To verify the reverse inclusion, we just need to consider the initial data unique to one integral curve γ of (6.16) which will be given by seven parameters - three for initial position, two for initial unit velocity and two for initial acceleration (perpendicular to the velocity vector). By varying the values of T and λ, it is clear that there is an integral curve of (6.14) with the same initial data and its projection necessarily coincides with γ. $\square$

The proposition doesn't work for $c = 0$ as we cannot use T as a parameter for the initial acceleration data in the above proof.

Magnetic Flow

The addition of this extra λ term may seem a little *ad hoc* here but is actually a sensible choice when we see the proof of this proposition i.e, we need to add a term orthogonal to $\dot{\mathbf{x}}$ but not in the direction of $\dot{\mathbf{x}} \times (\nabla V \times \dot{\mathbf{x}})$. Furthermore, this system of equations (6.14) can be interpreted as describing a magnetic flow in the background of the Kastor-Traschen metric with magnetic field $\mathbf{B} \propto \nabla V$.

This is precisely the magnetic field we saw in (6.2) when discussing the freedom in the 2-form F and so these additional integral curves occupy a significant role in the geometry of the Kastor-Traschen metric.

6.4.1 A Solution in the One-Centre Case

Let $\varphi = 4mc$ and define a curve in the plane $x^3 = 0$ by

$$x^i(s) = \left(\varphi s \cos\left(\frac{\sqrt{1-\varphi^2}}{\varphi}\log(\varphi s)\right), \varphi s \sin\left(\frac{\sqrt{1-\varphi^2}}{\varphi}\log(\varphi s)\right), 0\right). \quad (6.17)$$

Then this curve satisfies $h_{jk}\dot{x}^j\dot{x}^k = 1$ and is an integral curve of the system of ODEs (6.16) for $V = \frac{m}{|\mathbf{x}|}$. Furthermore, since it lies on a plane through the origin, we know, by Proposition 6.1, that it must be the retraction projection of a null geodesic of g.

We can plot this curve in the plane and realise that it is just a reparametrised logarithmic spiral. This example is motivated by work in section (WHAT?)

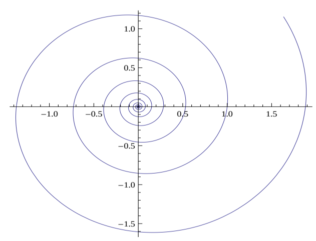

Figure 6.1: Logarithmic Spiral with $\varphi = 0.1$

where we will show how to derive analytic expressions for the projected null geodesics of the one-centre K-T metric and thus, the integral curves of (6.10) which lie on a plane through the origin. We should note here, that the limit $c \to 0$, for this example, is ill-defined. By L'Hopital's Rule, both expressions in (6.17) tend to zero, in this limit, for any value of s.

6.5 Geodesics obtained from Extremal RNdS

In this section, we will make use of a diffeomorphism between the one-centre Kastor-Traschen and the extremal Reissner-Nordstrom deSitter metrics. The advantage of this is that analytic solutions for the null geodesic equations of the RNdS metrics are well known [30]. We will show how to derive these solutions which will enable us to obtain analytic solutions in the Kastor-Traschen coordinates and plot the retraction projection of the null geodesics in some cases.

6.5.1 RNdS Transformation

We begin with the special Kastor-Traschen metric with potential $V = \frac{m}{|\mathbf{x}|}$ and now use spherical polar coordinates ($|\mathbf{x}| = R$) to represent the flat metric h i.e,

$$g = -\frac{1}{\left(\frac{m}{R} + cT\right)^2} dT^2 + \left(\frac{m}{R} + cT\right)^2 (dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{6.18}$$

Assuming that $c \neq 0$ we can make the coordinate transformation

$$R = e^{-cS} \quad , \quad T = \frac{r - m}{c} e^{cS}. \tag{6.19}$$

If we choose t such that $dt = dS + \frac{r-m}{c\Delta_r} dr$, where $\Delta_r = (r-m)^2 - c^2 r^4$, then the metric becomes

$$g = -\frac{\Delta_r}{r^2} dt^2 + \frac{r^2}{\Delta_r} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.20}$$

The analogous coordinate transformation in the Riemannian case was given in [27]. Here, the resulting metric is the Reissner-Nordstrom deSitter spacetime in the extremal case with charge $Q = m$ and $c = \pm\sqrt{\frac{\Lambda}{3}}$ where Λ is the cosmological constant. From now on, we will refer to t as *static time* in order to distinguish it from the time T. We also take $c > 0$ as the definition of (6.20) is invariant with respect to $c \to -c$ and we can compensate for it in the Kastor-Traschen case by sending $T \to -T$.

Null and timelike geodesics of black hole spacetimes with cosmological constant were studied extensively in [30]. In particular, the author discussed the different types of orbits possible for Reissner-Nordstrom metrics and showed how to derive analytic formulae for the geodesics. The RNdS metric admits the timelike static Killing vector $\frac{\partial}{\partial t}$ and we can plot the projection of the null geodesics to the space of orbits of $\frac{\partial}{\partial t}$, analogous to what was done in [30], which we call the *static projection*. We then use this information to plot the null geodesics in the Kastor-Traschen metric projected along the conformal retraction Θ, which, by Proposition 6.1, will be solutions of the system (6.16) and lie on a plane through the origin.

Null geodesics of the RNdS metric (6.20) satisfy

$$-\frac{\Delta_r}{r^2}\dot{t}^2 + \frac{r^2}{\Delta_r}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 0. \tag{6.21}$$

The Euler-Lagrange equation for θ gives

$$\frac{d}{ds}(2r^2\dot{\theta}) = 2r^2\sin\theta\cos\theta\dot{\phi}^2,$$

where s parameterizes the curves. By a choice of axes, we can set the initial conditions to be $\dot{\theta} = 0$, $\theta = \frac{\pi}{2}$, which results in motion in the equatorial plane - this coincides with the results of Proposition 6.1. Similarly, from the Euler-Lagrange equation for ϕ and t we find that

$$\begin{aligned} r^2\dot{\phi} &= \Phi \\ -\frac{2\Delta_r}{r^2}\dot{t} &= -2E \end{aligned}$$

where Φ and E are constants. Let us focus our attention on non-radial geodesics and assume that $\Phi > 0$ so that the geodesics are traced out in the direction of increasing ϕ. The null equation (6.21) implies that

$$-\frac{r^2}{\Delta_r}E^2 + \frac{r^2}{\Delta_r}\dot{r}^2 + \frac{\Phi^2}{r^2} = 0$$

or

$$\dot{r}^2 = E^2 - \frac{\Delta_r}{r^4}\Phi^2 \equiv E^2 - V_{eff}. \tag{6.22}$$

Here V_{eff} is the effective potential of the system which we can plot as a function of r.

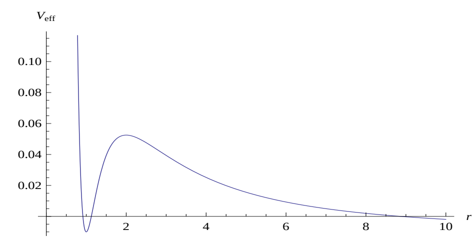

Figure 6.2: Plot of V_{eff} as a function of r

We note here that physically acceptable regions for null geodesics are those for which $E^2 \geq V_{eff}$. From the diagram, it is clear that we have two kinds of orbits - *bound orbits* (where r oscillates between two boundary values) and unbound *flyby orbits* (where r starts at ∞, then approaches a periapsis and goes back to ∞). This is consistent with the results of [30] and we use the same terminology.

6.5.2 Circular Orbits

As was observed in [87], the extremum values of V_{eff} occur at $r = m$ and $r = 2m$, independent of the cosmological constant (in this case, independent of c), resulting in circular orbits. The minimum of this function will not always be non-negative and, since $E^2 \geq 0$, it will not be attained for some values of c. Indeed, unless $c = 0$, we will not get a circular orbit at $r = m$. However, the transformation for R and T does not behave well in this limit and therefore does not allow us to see what this circular orbit corresponds to in the Kastor-Traschen coordinates.

On the other hand, we will observe a circular orbit at $r = 2m$ as long as $4mc \leq 1$ (beyond this, the local maximum drops below the axis in Figure 6.2). For this solution, there will be some factors of c in the static time variable t and hence, when we make the coordinate transformation to R and T, the orbit in the retraction projection will be dependent on c. In fact, when we do this, the resulting orbit is just essentially a constant multiple of that given section 6.4.1 - a logarithmic spiral in some hyperplane of the space of orbits of the conformal retraction Θ, which passes through the origin.

6.5.3 Horizon Structure and Nature of Orbits

Clearly, for $E^2 > V_{max}$, all orbits are unbound in the RNdS coordinates whereas for $E^2 < V_{max}$, we can get both bound and unbound orbits. If $m = 0$ then this graph becomes $V_{eff} = \frac{1}{r^2} + const$, and there are no bound orbits.
Using the chain rule and the E-L equation for ϕ, we can rewrite equation (6.22) as

$$\left(\frac{dr}{d\phi}\right)^2 = (\kappa^2 + c^2)r^4 - (r-m)^2 \tag{6.23}$$

where $\kappa = \frac{E}{\Phi}$. Also,

$$\frac{dt}{d\phi} = \kappa \frac{r^4}{\Delta_r}.$$

Note here that roots of Δ_r (which we inevitably cross for some orbits) will cause infinities in $\frac{dt}{d\phi}$ and in t itself - this will lead to a null geodesic tracing out a finite path in an infinite amount of static time (see analysis). Radii at which $V_{eff} = 0$, or equally $\Delta_r = 0$, correspond to horizons. In particular, if:

- $4mc < 1$; there are three horizons (two black hole horizons and one cosmological horizon) with $r_{bh-} < r_{bh+} < r_{ch}$. The geometry is static for $r < r_{bh-}$, $r_{bh+} < r < r_{ch}$ and corresponds to a black hole in a de-Sitter universe.

- $4mc > 1$; there is only one cosmological horizon. The geometry is static for $r < r_{ch}$ and corresponds to a naked singularity in a de-Sitter universe.

The metric (6.20) has a singularity at $r = 0$ which is covered by the Cauchy horizon $r = r_{bh-}$ in the case $4mc < 1$. The surface gravity at this inner horizon is larger than that at the cosmological horizon, in particular,

$$\kappa_{bh-}^2 - \kappa_{ch}^2 = 8mc^3 > 0$$

and so, by a result in [88], the Cauchy horizon is unstable. Hence, some of the trajectories in the Reissner-Nordstrom-deSitter metric will be unphysical (in particular when $r < r_{bh-}$). Furthermore the case $4mc > 1$ presents a possible violation of Penrose's cosmic censorship conjecture [89] and may therefore also be unphysical.

It was noted in [90] that, for the Reissner-Nordstrom deSitter metric in the extreme charge equal mass case, the Hawking temperature of the outer black hole horizon is the same, in magnitude, as that of the cosmological horizon endowing a notion of thermodynamic stability among all RNdS solutions.

The plots of projected null geodesics can be obtained in the Kastor-Traschen framework by solving equation (6.23) and making a coordinate transformation $(t, r) \to (T, R)$. As an example, we will perform this calculation for bound orbits in the three-horizon case (when $4mc < 1$). The other curves can be similarly obtained by the reader but do not add much to the discussion. In the following example, we highlight when a trajectory is physical or when it is purely of mathematical interest.

Bound Orbits in Kastor-Traschen with Three Horizons

This case corresponds to the inequalities

$$4m\lambda < 1 \quad , \quad \frac{-1+\sqrt{1+4m\lambda}}{2\lambda} \leq r \leq \frac{1-\sqrt{1-4m\lambda}}{2\lambda}.$$

For bound orbits, any solution of (6.23) will oscillate between the two extremal values for r given above. Beginning with this equation, we can rearrange to get

$$\pm \int \frac{dr}{\sqrt{\lambda^2 \left(r^4 - \left(\frac{r-m}{\lambda}\right)\right)}} = \phi + \gamma, \tag{6.24}$$

where γ is a constant of integration. With the given bounds on r, we can integrate the left hand side of this equation and obtain

$$\mp \frac{2F\left(\arcsin\left(\sqrt{\frac{2\sqrt{1-4m\lambda}(-1-2r\lambda+\sqrt{1+4m\lambda})}{(1-2r\lambda+\sqrt{1-4m\lambda})(-2+\sqrt{1+4m\lambda}+\sqrt{1-4m\lambda})}}\right), \frac{1}{2} - \frac{1}{2\sqrt{1-16m^2\lambda^2}}\right)}{(1-16m^2\lambda^2)^{1/4}} = \phi + \gamma$$

where F is the elliptic integral of the first kind. We can solve this equation for $r = r(\phi)$ to get

$$r(\phi) = \frac{\sin^2(J_{\mp}(\phi))(1+\sqrt{1-4m\lambda})(\sqrt{1+4m\lambda}+\sqrt{1-4m\lambda}-2) + 2\sqrt{1-4m\lambda}(1-\sqrt{1+4m\lambda})}{2\lambda(\sin^2(J_{\mp}(\phi))(\sqrt{1+4m\lambda}+\sqrt{1-4m\lambda}-2) - 2\sqrt{1-4m\lambda})}$$

where

$$J_{\mp}(\phi) = \mathrm{Jac}\left(\mp\frac{(1-16m^2\lambda^2)^{1/4}}{2}(\phi+\gamma), \frac{1}{2} - \frac{1}{2\sqrt{1-16m^2\lambda^2}}\right)$$

and Jac is the Jacobi Amplitude for the elliptic integral (i.e, $F(a,b) = c \Rightarrow a = \mathrm{Jac}(c,b)$). To obtain the oscillatory solution we use $J_{(-1)^n}(\phi)$ whenever

$$-\frac{2F\left(\frac{n-1}{2}\pi, \frac{1}{2} - \frac{1}{2\sqrt{1-16m^2\lambda^2}}\right)}{(1-16m^2\lambda^2)^{1/4}} > \phi+\gamma \geq -\frac{2F\left(\frac{n}{2}\pi, \frac{1}{2} - \frac{1}{2\sqrt{1-16m^2\lambda^2}}\right)}{(1-16m^2\lambda^2)^{1/4}}.$$

We can now plot the static projection of the null geodesics of the RNdS metric. For this purpose, we take the values $m = 1$, $\kappa = \frac{1}{6}$, $c = \frac{1}{8}$ and $\gamma = 0$.

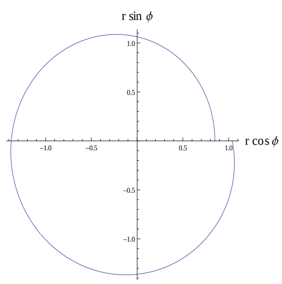

Figure 6.3: Static projection of null geodesic, $0 \leq \phi \leq 2\pi$

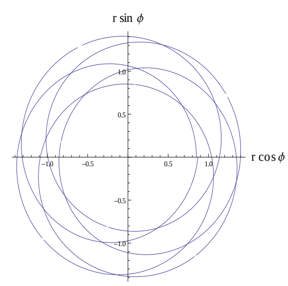

Figure 6.4: $0 \leq \phi \leq 10\pi$

As we mentioned before, it is important that we be careful here with respect to the range of the static time coordinate. In the above plots, the function r will, at several stages, cross a value for which $\Delta_r = 0$ and satisfy $r < r_{bh-}$, where the trajectory is unphysical. This is reflected by the fact that each horizon crossing leads to an infinity in the static time t. For example, if we take a segment of this orbit which passes from the r_{bh-} to r_{bh+}, we obtain the following plots for the geodesic itself and t as a function of ϕ on this range. Using the transformation (6.19) together with the subsequent one for S, we can plot the retraction projection of this null geodesic in the Kastor-Traschen coordinates. Similarly, we can determine the time T as a function of ϕ and we get the following plots: As $R \to 0$, the time $T \to \infty$. However, we see that as R approaches the finite

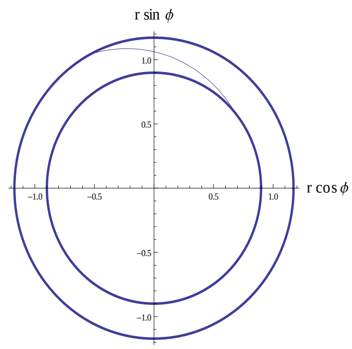

Figure 6.5: Null geodesic traversing between black hole horizons r_{bh-} and r_{bh+}.

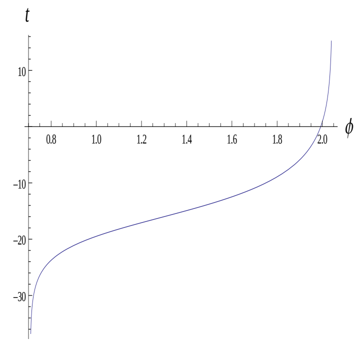

Figure 6.6: Static time t as a function of ϕ on this interval.

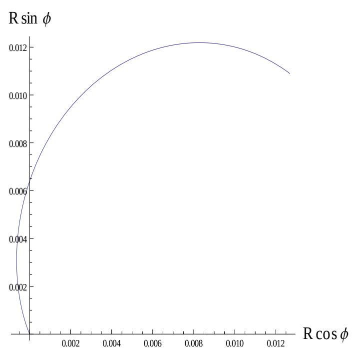

Figure 6.7: Retraction projection of null geodesic of the K-T metric

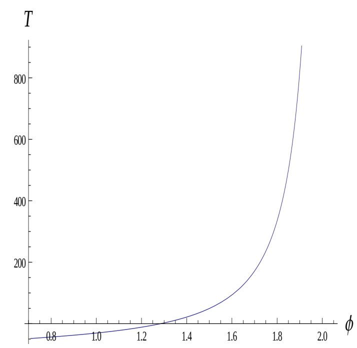

Figure 6.8: Time T as a function of ϕ on this interval.

positive value, T also approaches a finite value. This suggests that we can extend our geodesic in the direction of decreasing T and continue the curve in the retraction projection. As one would expect, this can be done and the extension can be constructed by considering the part of the geodesic in the RNdS coordinates which begins at r_{bh-}, decreases to the minimum value of r and then increases to r_{bh-} again. Of course, this curve lies completely within the Cauchy horizon and so, it is an unphysical extension but we can discuss it mathematically nonetheless.

(**Note:** For this construction to provide the correct extension of the null geodesic, we must set the static time t to be decreasing on this interval).

Hence, we can plot the full retraction projection of this null geodesic in the Kastor-Traschen coordinates. We include a plot of the time T here as a function of ϕ to demonstrate that it is, indeed, the projection of the whole geodesic - here the null geodesic begins at the spatial origin $R = 0$ at $T \to -\infty$, traces the curve in the direction of increasing ϕ and returns to the origin as $T \to +\infty$:

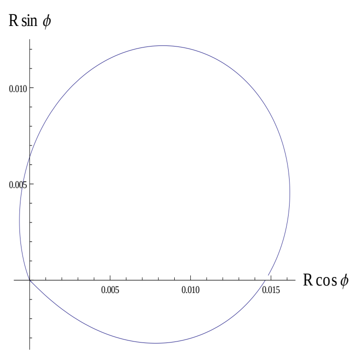

Figure 6.9: Retraction projection of null geodesic in K-T coordinates.

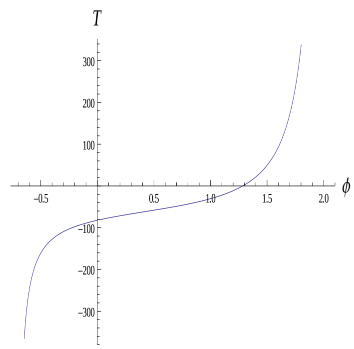

Figure 6.10: Time T as a function of ϕ on this interval.

Conformal Diagram

To gain a better understanding of the geometry here, let us plot the trajectory of this null geodesic in the Penrose-Carter diagram of the spacetime. The authors in [91] have constructed this diagram and have highlighted the region covered by the "cosmological" (Kastor-Traschen) coordinates. We give a copy of this diagram and include a null geodesic which runs from a point at $r < r_{bh-}$ to the outer black hole horizon r_{bh+}.

The authors of [91] make the point that the cosmological coordinates "smoothly cover the entire region from $r = 0$ to $r = \infty$." From the diagram, we see that a single (T, R) chart (bounded by the dotted red line) covers four (t, r) charts encompassing all three horizons. As the light ray crosses the unstable Cauchy

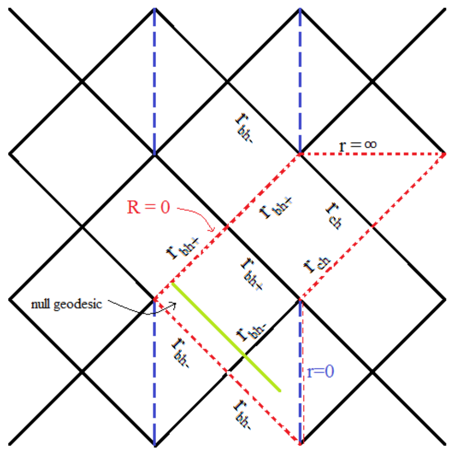

Figure 6.11: Penrose-Carter diagram of extremal Reissner-Nordstrom deSitter spacetime. The region bounded by the red dotted line represents a single (T, R) chart.

horizon, the static time $t \to \infty$. However in the Kastor-Traschen coordinates, we only have $T \to \infty$ as $R \to 0$ and so, mathematically, the Kastor-Traschen metric does not see a time singularity at the inner Cauchy horizon.

6.6 Two-Centre Case

Here, we make some remarks about the Kastor-Traschen metric with two black holes ($N = 2$). Analytic expressions for null geodesics are much more difficult to obtain but we can make some remarks about the general theory. First of all, let us write our Kastor-Traschen metric g in *cylindrical polar* coordinates.

$$-\frac{1}{(V+cT)^2}dT^2 + (V+cT)^2(d\rho^2 + \rho^2 d\phi^2 + dz^2)$$

so that, without loss of generality, the singularities are placed on the z-axis, equidistant from the origin. That is

$$V = \frac{m_1}{(\rho^2 + (z-w)^2)^{1/2}} + \frac{m_2}{(\rho^2 + (z+w)^2)^{1/2}}$$

where m_1 and m_2 are the black hole masses and $2w$ is the distance between the centres. For this metric, we have the following result regarding the retraction projection of null geodesics.

Proposition 6.1. *If the retraction projection of a null geodesic has the property that its initial position and velocity lie in a plane passing through the two centres, then the entire projected null geodesic lies in that plane.*

Proof. Planes which pass through both centres are characterised by the condition $\phi = constant$. Then, the retraction projection of a null geodesic lies in such a plane if and only if $\dot{\phi} = 0$ at all points on the curve. Therefore, we can verify the proposition by showing that, at any point where $\dot{\phi} = 0$, we have $\ddot{\phi} = 0$. But, from equation (6.5), null geodesics of Kastor-Traschen satisfy

$$\ddot{\phi} + \frac{2}{\rho}\dot{\rho}\dot{\phi} = -\frac{2}{V + cT}\left(\frac{\partial V}{\partial \rho}\dot{\rho} + \frac{\partial V}{\partial z}\dot{z}\right)\dot{\phi}$$

and the result follows. □

Now let us impose the additional condition $m_1 = m_2 = M$. Then, we discover another fixed plane of null geodesics.

Proposition 6.2. *If the retraction projection of a null geodesic has the property that its initial position and velocity lie in the plane passing orthogonally through the midpoint of the line segment joining the two centres, then the entire projected null geodesic lies in that plane.*

Proof. Clearly, the plane in question is given, in cylindrical polar coordinates, by $z = 0$ and any null geodesic which lies completely in this plane will satisfy the condition $\dot{z} = 0$ at all points on the curve. Therefore, if the initial conditions $z = 0$, $\dot{z} = 0$ imply that $\ddot{z} = 0$ initially, then the proposition is proved. Again, by equation (6.5), null geodesics of g satisfy

$$\ddot{z} = \frac{2}{V + cT}\left(\frac{\partial V}{\partial z} - \left(\frac{\partial V}{\partial \rho}\dot{\rho} + \frac{\partial V}{\partial z}\dot{z}\right)\dot{z}\right)$$

and the result follows because

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{Ma}{(\rho^2 + w^2)^{3/2}} - \frac{Ma}{(\rho^2 + w^2)^{3/2}} = 0.$$

□

6.6.1 Third Order System describing Null geodesics

In section 6.4, we discovered that the third order system used to describe the retraction projection of null geodesics of the Kastor-Traschen metric was not uniquely defined and by considering a new formulation we could obtain an interpretation of the entire set of integral curves. Here, let us take yet another system of third order ODEs, the integral curves of which contain the projected null geodesics of the one-centre Kastor-Traschen metric by eliminating the x^i term from the system (6.10), using (6.9) and (6.7) to obtain

$$\dddot{x}^i = -|\ddot{\mathbf{x}}|^2 \dot{x}^i + \left[\frac{2mc}{|\mathbf{x}|^3} \left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2} \right) - 3(\mathbf{x}.\dot{\mathbf{x}}) \left(\frac{1}{|\mathbf{x}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}|^2 - (\mathbf{x}.\dot{\mathbf{x}})^2)} \right) \right] \ddot{x}^i. \quad (6.25)$$

Then, we have the following result

Proposition 6.3. *Any integral curve of the system of ODEs (6.25), lies in a plane. Furthermore, if $c \neq 0$, such an integral curve will coincide with a projected null geodesic of the Kastor-Traschen metric g if and only if this plane passes through the origin.*

Proof. We construct the Frenet-Serret frame for a given integral curve of (6.25)

$$\mathbf{T} = \dot{\mathbf{x}} \quad , \quad \mathbf{N} = \frac{\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|} \quad , \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

Then, the Frenet-Serret formulas give

$$\begin{aligned} \dot{\mathbf{T}} &= \kappa \mathbf{N}, \\ \dot{\mathbf{N}} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \end{aligned}$$

where κ and τ are the curvature and torsion of the curve, respectively. The first of these equations gives us $\kappa = |\ddot{\mathbf{x}}|$. Then, if we rewrite our system (6.25) in this frame, we obtain

$$\dot{\mathbf{N}} = -\kappa \mathbf{T}$$

and $\tau = 0$, necessarily. Hence, a given integral curve of this system must lie in a plane.

We have already established that the projected null geodesics in the one-centre case will lie in a plane passing through the origin. To prove the "if" part of the proposition, we note that the initial data of an integral curve of (6.25) which lies on a plane through the origin is specified by six parameters - three for initial position, two for initial velocity (since it is unit in the arclength parametrisation) and one for the acceleration (in the plane of the position and velocity vectors, perpendicular to the velocity). Then, this curve is a projected null geodesic of the Kastor-Traschen metric if there exists a null geodesic with the same initial data. But, we can see that this is the case by analysing equation (6.9). Clearly, we can specify initial position and unit velocity vectors as we please. Then, the initial acceleration vector lies in the same plane perpendicular to the velocity and we can specify its magnitude by choosing the appropriate value of T. $\square$

So, each integral curve of (6.25) lies in some plane but unlike the case of (6.16), these integral curves have no obvious interpretation in terms of the Kastor-Traschen metric unless this plane passes through the origin. However, as we observe, they do arise as the projections of null geodesics for the two-centre case - in particular, those outlined by Proposition 6.3.

To see this, let us consider the original expression for the Kastor-Traschen metric (6.1) with h as the flat metric in Cartesian coordinates with the potential written as follows

$$V = \frac{m_1}{|\mathbf{x} - \mathbf{w}|} + \frac{m_2}{|\mathbf{x} + \mathbf{w}|}$$

where $\mathbf{w} = (w^i)$ is a fixed vector. For ease of notation, let us define the vectors

$$X_1^i = \frac{m_1}{|\mathbf{x} - \mathbf{w}|^3}(x^i - w^i) \quad , \quad X_2^i = \frac{m_2}{|\mathbf{x} + \mathbf{w}|^3}(x^i + w^i).$$

Then, null geodesics of the Kastor-Traschen metric are integral curves of the following system of third order ODEs

$$\begin{aligned}
\dddot{x}^i &= -|\ddot{\mathbf{x}}|^2\dot{x}^i - \frac{3(\mathbf{X}_1+\mathbf{X}_2).\ddot{\mathbf{x}}(\mathbf{X}_1+\mathbf{X}_2).\dot{\mathbf{x}}}{2\,(\mathbf{X}_1+\mathbf{X}_2)\,.\,(\mathbf{X}_1+\mathbf{X}_2-\dot{\mathbf{x}})}\ddot{x}^i + 2c\big(X_1^i+X_2^i-((\mathbf{X}_1+\mathbf{X}_2).\dot{\mathbf{x}})\dot{x}^i\big) \\
&- \frac{(\mathbf{X}_1+\mathbf{X}_2).\ddot{\mathbf{x}}}{(\mathbf{X}_1+\mathbf{X}_2)\,.\,(\mathbf{X}_1+\mathbf{X}_2-\dot{\mathbf{x}})}\left(\frac{3|\mathbf{x}-\mathbf{w}|}{m_1}(\mathbf{X}_1.\dot{\mathbf{x}})\big(X_1^i-(\mathbf{X}_1.\dot{\mathbf{x}})\dot{x}^i\big)\right. \\
&+ \left.\frac{3|\mathbf{x}+\mathbf{w}|}{m_2}(\mathbf{X}_2.\dot{\mathbf{x}})\big(X_2^i-(\mathbf{X}_2.\dot{\mathbf{x}})\dot{x}^i\big)\right).
\end{aligned}$$

This system is difficult to analyze, in general, but now let us assume that $m_1 = m_2 = M$ and restrict attention to null geodesics which lie on the plane passing through the origin, orthogonal to the line between the two centres.

Then $|\mathbf{x}+\mathbf{w}| = |\mathbf{x}-\mathbf{w}|$ and since the acceleration and velocity vectors are perpendicular to $\mathbf{w}$, we have $\mathbf{w}.\dot{\mathbf{x}} = \mathbf{w}.\ddot{\mathbf{x}} = 0$. By making some simplifications using the geodesic equation (6.4), as in the one-centre case, we can replace the system above by

$$\dddot{x}^i = -|\ddot{\mathbf{x}}|^2\dot{x}^i + \left[\frac{4Mc}{|\mathbf{x}-\mathbf{w}|^3}\left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2}\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}-\mathbf{w}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}-\mathbf{w}|^2-(\mathbf{x}.\dot{\mathbf{x}})^2)}\right)\right]\ddot{x}^i. \tag{6.26}$$

Now we notice that this is precisely the system of third order ODEs (6.25) for the single centre case with black hole mass $m = 2M$, where $|\mathbf{x}-\mathbf{w}|$ represents the distance from the centre.

Hence, we have proved the following proposition:

Proposition 6.4. *Every null geodesic of the two-centre Kastor-Traschen metric which lies completely in the plane passing orthogonally through the midpoint of the line segment joining the centres coincides with an integral curve of the system (6.25), with mass $m = 2M$, which lies in the plane a distance w from the origin, where $2w$ is the distance between the centres.*

Remark: This means that every integral curve of (6.25) can be realised as the retraction projection of a null geodesic in either the one-centre or two-centre Kastor-Traschen metric, making all solutions physically relevant.

6.6.2 Perturbation Analysis

Now let us look at the stability of null geodesics in the $z = 0$ plane by applying a small perturbation $\epsilon \ll a$ in the z-direction about the origin so that $\dot{\epsilon}$ and $\ddot{\epsilon}$ are also small. Then substituting $x^3 = z + \epsilon$ into (6.26) gives us the differential equation

$$\begin{aligned}\dddot{\epsilon} &= -\left.|\ddot{\mathbf{x}}|^2\right|_{(z,\dot{z},\ddot{z})=0}\dot{\epsilon} \\ &+ \left.\left[\frac{4Mc}{|\mathbf{x}-\mathbf{w}|^3}\left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2}\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}-\mathbf{w}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}-\mathbf{w}|^2-(\mathbf{x}.\dot{\mathbf{x}})^2)}\right)\right]\right|_{(z,\dot{z},\ddot{z})=0}\ddot{\epsilon}.\end{aligned}$$

We can rewrite this as a coupled system of differential equations by choosing $\eta = \dot{\epsilon}$ and $\mu = \ddot{\epsilon}$ so that

$$\begin{aligned}\frac{d}{ds}\begin{pmatrix}\mu\\ \eta\end{pmatrix} &= \left.\begin{pmatrix}\frac{4Mc}{|\mathbf{x}-\mathbf{w}|^3}\left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2}\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}-\mathbf{w}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}-\mathbf{w}|^2-(\mathbf{x}.\dot{\mathbf{x}})^2)}\right) & -|\ddot{\mathbf{x}}|^2 \\ 1 & 0\end{pmatrix}\right|_{(z,\dot{z},\ddot{z})=0}\begin{pmatrix}\mu\\ \eta\end{pmatrix} \\ &\equiv B\begin{pmatrix}\mu\\ \eta\end{pmatrix}.\end{aligned}$$

The stability of the system under small perturbations is determined by the eigenvalues of B. Given that the determinant of B is positive, $(= |\ddot{\mathbf{x}}|^2)$ we know that both eigenvalues have the same sign. If they are both positive then the system is unstable and if they are both negative then the system is stable. The mutual sign can be obtained from the trace of B and thus, we get the following result,

Proposition 6.5: *For the two-centre equal mass Kastor-Traschen metric, any geodesic which lies in the plane passing orthogonally through the midpoint of the line segment joining the two centres is stable under small perturbations normal to the plane at a point with given initial position, velocity and acceleration data if and only if*

$$\frac{4Mc}{|\mathbf{x}-\mathbf{w}|^3}\left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2}\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}-\mathbf{w}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}-\mathbf{w}|^2-(\mathbf{x}.\dot{\mathbf{x}})^2)}\right) < 0$$

at that point. Otherwise, it is unstable.

6.7 Unparametrised Projection of Null Geodesics in the One-Centre Kastor-Traschen Solution

As final note, we will rewrite the system of third-order ODEs (6.25) in unparametrised form. By doing so, we get a purer notion of the set of projected null geodesics (free of parametrisation) and can make contact with the work in [92] where the author has explicitly derived differential invariants for systems of third order ODEs.

To start with, let us relabel our coordinates $x^i = (z, x^\beta)$ with $\beta = 2, 3$. If we let ' denote differentiation with respect to z then we have

$$\dot{x}^\beta = (x^\beta)'\dot{z} \quad , \quad \ddot{x}^\beta = (x^\beta)''\dot{z}^2 + (x^\beta)'\ddot{z} \quad ,$$
$$\dddot{x}^\beta = (x^\beta)'''\dot{z}^3 + 3(x^\beta)''\dot{z}\ddot{z} + (x^\beta)'\dddot{z}.$$

From the system (6.25), we can eliminate the $\dddot{z}$ term to obtain a pair of third order expressions

$$(x^\beta)'''\dot{z}^3 + 3(x^\beta)''\dot{z}\ddot{z} = \left[\frac{2mc}{|\mathbf{x}|^3}\left(\frac{\mathbf{x}.\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|^2}\right) - 3(\mathbf{x}.\dot{\mathbf{x}})\left(\frac{1}{|\mathbf{x}|^2} + \frac{\mathbf{x}.\ddot{\mathbf{x}}}{2(|\mathbf{x}|^2 - (\mathbf{x}.\dot{\mathbf{x}})^2)}\right)\right](x^\beta)''\dot{z}^2. \tag{6.27}$$

This system can be simplified even further to eliminate the factors of $\dot{z}$ and $\ddot{z}$. First, let us use the following convention

$$\mathbf{x} = \begin{pmatrix} z \\ x^\beta \end{pmatrix} \quad , \quad \mathbf{u} = \begin{pmatrix} 1 \\ (x^\beta)' \end{pmatrix} \quad , \quad \mathbf{a} = \begin{pmatrix} 0 \\ (x^\beta)'' \end{pmatrix}$$

Then, using the arc-length parametrization condition $h_{jk}\dot{x}^j\dot{x}^k = 1$, one can show that

$$|\mathbf{u}|^2\dot{z}^2 = 1 \quad , \quad \ddot{z} = -\frac{\mathbf{u}.\mathbf{a}}{|\mathbf{u}|^2}\dot{z}^2, \tag{6.28}$$

Using this, it can also be shown that

$$|\ddot{\mathbf{x}}|^2 = \left(|\mathbf{a}|^2 - \frac{(\mathbf{u}.\mathbf{a})^2}{|\mathbf{u}|^2}\right)\dot{z}^4 \quad , \quad \mathbf{x}.\ddot{\mathbf{x}} = \left(\mathbf{x}.\mathbf{a} - \frac{(\mathbf{x}.\mathbf{u})(\mathbf{u}.\mathbf{a})}{|\mathbf{u}|^2}\right)\dot{z}^2.$$

Then, we can use the expressions in (6.28) to eliminate $\ddot{z}$ terms and subsequently powers of $\dot{z}$ in (6.27) (This can be done by multiplying each term by the appropriate power of $u^2\dot{z}^2$ to give every term the same "weight" in terms of powers of $\dot{z}$). The resulting expression will give us a pair of third order ODEs whose integral curves are the unparametrized curves of the system (6.25):

$$(x^\beta)''' = \left[3\frac{\mathbf{u.a}}{|\mathbf{u}|^2} + \frac{2mc|\mathbf{u}|^3}{|\mathbf{x}|^3}\left(\frac{|\mathbf{u}|^2(\mathbf{x.a}) - (\mathbf{x.u})(\mathbf{u.a})}{|\mathbf{u}|^2|\mathbf{a}|^2 - (\mathbf{u.a})^2}\right) - 3(\mathbf{x.u})\left(\frac{1}{|\mathbf{x}|^2} + \frac{|\mathbf{u}|^2(\mathbf{x.a}) - (\mathbf{x.u})(\mathbf{u.a})}{2(|\mathbf{u}|^4|\mathbf{x}|^2 - |\mathbf{u}|^2(\mathbf{x.u})^2)}\right)\right](x^\beta)''. \tag{6.29}$$

In the limit $c \to 0$, the second and third terms on the right-hand side of (6.29) vanish coinciding with the unparametrised equation for conformal circles. The Medvedev invariants [92] can be calculated for the system (6.29) and for conformal circles in Mathematica but the results are too long to include here.

Chapter 7

Conclusions and Outlook

Since the important work of T.Y.Thomas [31], [14] the notion of a path geometry has played an important role in both differential geometry and mathematical physics. The ideas of the tractor bundle and tractor connection have allowed authors to approach particular problems in these areas with the aid of invariants from the associated underlying geometry. In [16] the authors develop the tractor technology in the conformal, paraconformal and projective cases with particular focus on the common themes between the three. They also illustrate how to approach some fundamental geometric problems that arise in each area. For example, building on this work, the authors in [93] have constructed conformal invariants the vanishing of which are necessary and sufficient for the existence of an Einstein metric in the conformal class of a given Riemannian metric.

We have seen here how to construct a set of invariants for a given path geometry, the vanishing of which coincide with our set of integral curves being locally diffeomorphic to the set of straight lines in n dimensions. These invariants can be broken into two sets, which we have termed the *Fels* and *Grossman* invariants. The vanishing of either set independently leads to a separate branch of the path geometry theory - the "projective" branch and the "conformal" branch - each with a rich contribution in the theory of differential geometry.

On one hand, the vanishing of the Fels invariants implies the existence of a torsion-free affine connection, the unparametrised geodesics of which coincide with the integral curves of the given path geometry. This connection, with components Γ^a_{bc} is only defined up to a change in the projective equivalence class

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b$$

and so, it's more effective, in this instance, to deal with this class as a single entity - the projective connection. This allows us to deal with the pertinent questions of the unparametrised geodesics of affine connections using the invariant objects of tractor calculus.

Central to the development of the theory of projective connections and the associated tractor geometry has been the search for the solution to the metrisability problem as discussed here. Building on the work of Fels [6], Eastwood and Matveev [8] [17], we have constructed a set of necessary local conditions for a given family of curves on an open set $U \subset \mathbb{R}^n$ to arise as the unparametrised geodesics of some metric on U. By means of several examples in dimension 3, we have shown that this construction enables us to tackle the problem of metrisability more effectively than if one were to try and solve the equations (3.5) directly. We have also shown how this procedure can be used to determine the degree of mobility of a metric g. We have seen that the Thurston metrics Nil and Sol fall into the category of being geodesically rigid whereas as the degree of mobility of a metric in the Levi-Civita class is apparent from the number of constants of integration.

Rather surprisingly, the relevant connection at play here is not the associated tractor connection of the theory but, instead, there is a one-to-one correspondence between metrics in a given projective equivalence class $[\Gamma]$ and parallel sections of the connection (3.17) from which natural invariant obstructions may be determined. Algebraic obstructions to metrisability are obtained by considering the curvature of this connection, as we have demonstrated here. By means of an example, we have seen that new conditions may be derived, algebraic in the unknowns of the system (3.9), at higher orders in the projective connection, by differentiating and referring to the original system. In this way, one could generate all sufficient conditions for a system of paths to be metrisable in dimension n. This algorithm was fulfilled in dimension two [7] but remains open in higher dimensions. Although it has not been carried out to completion here, we suggest that the procedure of repeatedly differentiating to produce new obstructions, algebraic in σ^{ab} will halt quickly due to the apparent gaps in the degree of mobility as noted by [36] in dimension three and [94], more generally, amongst others. Particularly, in dimension three, the degree of mobility of a metric may only take the values 1,2 or 10. Hence, if the Weyl tensor of a given projective structure does not vanish then the co-rank of the associated matrix determined by the conditions (3.11), (3.12) and (3.13) is at most two. Differentiating these conditions we obtain new ones at higher order and if the co-rank of the corresponding matrix for this system does not decrease, we know we have a system with that degree of mobility. In this instance, sufficient conditions should then arise after taking two derivatives but we have not proved that in detail here and the general problem of metrisability remains open.

The tractor approach has also produced some recent additional results in the projective geometry realm. In [95], the authors constructed a set of invariants, the vanishing of which is necessary and sufficient for the existence of an Einstein metric in the projective equivalence class of a given metric g. Furthermore, the authors have produced extra obstructions to the existence of a Levi-Civita connection in the projective class. Specifically, they define a set off Chern-type invariants

$$(p_k)_{a_1 a_2 \ldots a_{2k}} = W_{a_1 a_2}{}^{c_1}{}_{c_2} W_{a_3 a_4}{}^{c_2}{}_{c_3} \ldots W_{a_{2k-1} a_{2k}}{}^{c_k}{}_{c_1}.$$

Then, for a given projective structure to be metrisable, we must have $p_k = 0$ for k odd. This provides a significant improvement to the existing theory developed here. We can obtain similar objects from our theory (i.e, powers of the Weyl tensor with some contractions) by considering the determinants of square submatrices of the matrix Ξ associated to the linear system of equations (3.11). For example, in dimension three, we would consider the six by six determinants, e.g, $|\Xi^{cd}_{23ef}|$. In this case, there are $\left(\binom{10}{6}\right) = 210$ such determinants and it would be useful to know if the complete vanishing of these determinants coincided with the vanishing of a tensor similar to the above but for $k = 6$ and contractions taken differently. We attempted to achieve this, unsuccessfully, but it was discovered, by exhaustive calculation, that no one of the 210 determinants associated to the matrix Ξ may be written as a linear combination of the others, in general. Formally, this means that any such tensor which solves our problem necessarily has 210 components.

Another interesting notion which arises from the study of the metrisability problem is that of the degree of mobility of a metric. Early work by Beltrami [9] gave us insight into the maximum possible degree of mobility of a metric and when it is obtained (by constant curvature metrics). He proposed the problem of determining all normal forms of a pair of geodesically equivalent metrics which are non-proportional. This problem was solved in the Riemannian case by Levi-Civita [10], and is given here as equation (3.26). In the pseudo-Riemannian case, the problem is not so simple. Petrov solved it in dimension three [44] but recent developments in the theory have allowed the authors in [11] and [12] to solve this problem in any dimension. As for the possible values that this number may be in dimension n there has also been significant progress. For $n > 2$, the submaximal degree of mobility, $\frac{(n-2)(n-1)}{2} + 1$ was determined by Sinjukov [96] and Mikes [94] and results concerning the precise range of values that this number can take were then found in [97].

Further work in this area should involve determining the normal forms of metrics with a given degree of geodesic mobility (not just at least 2). These kinds of constructions may give us insight into metrics which are physically interesting and within which the local geometry cannot be determined by the paths of free particles, e.g, Schwarzschild-deSitter. Also, presently, given an arbitrary metric g in n dimensions, it is not always immediately clear if g falls into one of the normal forms given by Levi-Civita or Matveev and Bolsinov, that is, we are not able to determine the degree of mobility of a given metric invariantly, or even, whether or not g is geodesically rigid. This is a relevant problem and an advantageous approach is given by the work here on the metrisability problem. For example, whether or not a metric has maximum mobility is governed by the vanishing of the Weyl tensor. Also, the rank of the matrices determined by the linear algebraic equations (3.11), (3.12) and (3.13) allows us to compute an upper bound for this value so development in both areas is linked.

The second branch of the theory associated with the invariants of a path geometry is the conformal branch, that is, when the Grossman invariants vanish. An n-dimensional path geometry on some open domain U has a $(2n-2)$-dimensional solution space M as demonstrated by the double fibration picture (2.2). If the Grossman invariants vanish, then this gives rise to a Segré structure on M. In particular, for $n = 3$, the moduli space of solutions is endowed with an anti-self-dual conformal structure of signature (2,2) and the integral curves of the path geometry arise as twistor curves corresponding to points in M. Here, we have constructed several examples of systems of two second order ODEs with vanishing Grossman invariants along with the corresponding ASD conformal structure. The maximally symmetric, non-trivial example is a Ricci-flat ASD *pp*-wave with 9-dimensional group of conformal symmetries, and we have given examples of systems with symmetry groups of dimensions between 9 and 4. Some of these examples have a special form

$$y'' = 0 \ , \ \ z'' = G(x, y, z, y', z').$$

This is what Grossman calls 'a weaker form of integrability for the second ruling' [18]. This family gives a surface in the twistor space U, and it is known [56] any ASD (2,2) conformal structure with a conformal null Killing vector admits such structure.

We have seen how to derive the Grossman invariants directly by simply imposing the conformal structure on the moduli space of solutions M. Furthermore, the authors in [20] gave conditions on the path geometry for the corresponding conformal structure to be a Ricci-flat. Further work in this area would involve determining conditions on the path geometry for the corresponding conformal structure to possess a particular geometric property. For example, we may demand that it be Einstein or Kahler. In the complexified case, it is already known that a given ASD conformal structure contains an Einstein metric if and only if the twistor lines have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ with the following additional structure [33]:

1. A projection $\mu : U \to \mathbb{CP}^1$, such that the four parameter family of curves above are sections of μ.
2. An isomorphism $K_M \cong \mu^* \mathcal{O}(-4)$, where K is the line bundle of holomorphic three-forms over M.

Also, we may develop this theory in higher dimensions where the twistor correspondence is between a path geometry and a Segré structure on the moduli space of solutions. Imposing some differential constraints on the path geometry may give rise to some new geometry on the space of solutions. This kind of work would not only give us insight into the geometry of such structures but would be important for understanding correspondences in twistor theory and mathematical physics.

We have seen how the twistor curves can also be viewed as unparametrised geodesics of Finsler structures with scalar flag curvature. In fact, there is a one-to-one correspondence between such structures and torsion-free path geometries. This means that a classification of Finsler metrics with scalar flag curvature could lead to a complete description of path geometries with vanishing Grossman invariants and vice versa and, in turn, we could classify four-dimensional ASD conformal structures of signature (2,2). Seemingly, this is a difficult problem, however and, despite progress for Randers metrics of constant curvature and in some more general case, a general result has yet to be achieved.

The case of a trivial ODE with integral curves being straight lines in U is a starting point for John's integral transform [98]. Given a function $\phi : U \to \mathbb{R}$ with appropriate decay conditions at infinity and a straight line $L \subset U$, define a function $\hat{\phi}$ on the space M of straight lines

$$\hat{\phi}(L) = \int_L \phi.$$

The result of John is that the range of this transform is characterised by an ultra-hyperbolic wave equation on M, where the wave operator is induced by a flat (2,2) metric on M. If the line L is parametrised by $x \to (x, y = \alpha + x\gamma, z = \eta - x\beta)$, then

$$\hat{\phi}(\alpha, \eta, \beta, \gamma) = \int_{\mathbb{R}} \phi(x, \alpha + x\gamma, \eta - x\beta)dx \text{ and } \hat{\phi}_{\alpha\beta} + \hat{\phi}_{\eta\gamma} = 0.$$

As stated in [20], it would be interesting to develop a non-linear version of John's transform, applicable to path geometries with vanishing Grossman invariants. Integral curves of the system

$$y'' = 0 \;\;,\;\; z'' = -2(y')^3$$

would be a good starting point for this construction.

The significance of the metrisability problem and the theory of projective differential geometry, in general, can be seen in its applications to General Relativity. Since gravity is described in terms of a four-dimensional manifold with Lorentzian metric, we find that an equivalence class of unparametrised geodesics can be used to describe the geometry of freely falling particles, an idea that goes back to Weyl. In [11], it is suggested how one might infer the metric on some local patch of spacetime experimentally and in [66], the authors explore, to what extent to which this theory can be applied successfully, yielding a unique picture. In [21], it is suggested that the idea of having a non-trivial degree of mobility for a given set of observed unparametrised geodesics allows a certain freedom in the description of the geometry but, as later emphasised in [22], there is more to GR than projective geometry. Some cosmological observables - for example cosmic jerk, snap and higher order generalisations [99] - are not projectively invariant and thus, depend on the choice of metric in a projective equivalence class.

In Chapter 5, we pushed this theory further by exploring a novel aspect of projective equivalence. Light rays in static spacetimes project down to the unparametrised geodesics of the optical metric on the space of orbits of the timelike Killing vector. The importance of this structure was highlighted in [23] and [69] where properties of the conformal structure of the spacetime can be inferred by properties of the optical metric. This leads to ambiguity if a spacetime is static in more than one way, as non-proportional timelike Killing vectors lead to different optical metrics. We have characterized these multi-static metrics in the generic case here, i.e, when the isometry group generated by any pair of HSO timelike Killing vectors (and their commutators) has two-dimensional orbits in M and found that, in this case, the metric is locally a warped product metric on $M = S_0 \times S_1$ of the form

$$g = e^{w}\gamma_0 + \gamma_1$$

where (S_0, γ_0) is a two-dimensional Lorentzian manifold of constant curvature, (S_1, γ_1) is a two-dimensional Riemannian manifold and $w : S_1 \to \mathbb{R}$ is an arbitrary function. Then optical metrics corresponding to different static Killing vectors are guaranteed to be projectively equivalent only when γ_0 has non-zero curvature. This is the generic case as it admits the smallest number of isometries with the most generic commutation relations. One possible extension of this work would be to use the general form of multistatic metrics and generalize the results about optical equivalence to metrics which are non-generic and admit more than two static Killing vectors. It may also be interesting to regard the same problem in higher dimensions.

The theory of optical metrics has further interesting applications. We have found that the static projection of light rays of a Schwarzschild-Tangherlini metric in n dimensions lie in a plane and coincide with unparametrised geodesics of an optical 2-metric. Physically, these curves also arise as non-relativistic trajectories in a central force. There is a duality due to Bohlin and Arnold between pairs of values of n and we have shown that this duality implies a mapping between the totality of projected null geodesics determined by a 1-parameter family of such metrics.

This idea does not extend cleanly to the charged case - the interpretation of the duality is lost in the spacetime (as n is not an integer ≥ 3) but still remains for the classical particle moving in a central force. It would be interesting to see if such a notion of duality exists for other solutions of Einstein's equations. the role of the cosmological constant in spherically symmetric spacetimes of the form (5.25) is illuminating. Analysis of the dynamics of light rays in such metrics does not shed light on the value of Λ so we require other ways to measure it, as discussed above and in [22]. This again, highlights the importance of the projective equivalence of optical metrics in understanding the properties of light rays in static spacetimes.

The study of optical metrics has demonstrated that physical phenomena observed in Lorentzian metrics which admit a timelike vector field with a certain mathematical property can often be better understood by looking at the projection of null geodesics to the space of orbits of this vector field. The idea of associating a geometric structure on the space of orbits with the conformal structure of the full Lorentzian spacetime had already been succesfully extended to the stationary case in [78]. Here, we pushed this idea to the next step and discussed a specific example of a metric admitting a timelike conformal retraction which is also a solution of the Einstein-Maxwell equations and so, an important GR example. This allowed us to gain significant insight into the structures of a spacetime which admits a timelike conformal retraction.

The third order system (6.8) arises naturally to describe the retraction projection of null geodesics of the Kastor-Traschen metric. We have, however, demonstrated that there is a freedom in the definition of this ODE system and that a more useful analysis is obtained by considering instead the system (6.16) where the totality of integral curves can be interpreted as the projection of null curves in the Kastor-Traschen metric describing a magnetic flow in the background magnetic field. This endows a physical relevance to this system and it would be interesting to probe its relevance further. Using this formulation, we've characterised those integral curves of (6.16) which coincide with the retraction projection of null geodesics.

It is interesting to note here, that unlike the static and stationary cases, the projection of null geodesics could not be identified with the set of integral curves of some second order system of ODEs but required a description via a third order system. This is almost to be expected when we consider the underlying structure. The timelike conformal retraction Θ is endowed with the property that the conformal structure on the space of orbits is preserved when flowed along its integral curves. This hints at the idea that conformal geodesics and not metric geodesics are the objects involved. It would be interesting to determine if such systems do play a more significant role in the theory and whether or not the choice of third order system can be made to coincide with one describing some set of conformal circles.

For the one-centre K-T solution, the projected null geodesics are identified as those which lie on a plane through the origin. However, in this case, there is another third order system (6.25) whose integral curves arise as a distinguished subset of the projected null geodesics of the two-centre metric for some value of the distance between the centres, $2w$, with masses $m_1 = m_2 = M$. This analysis of null geodesics appears to be a step further than has been seen thus far but extracting more analytic solutions is far from easy.

There is a consistent physical intuition here if we consider what happens when $w \to 0$. We should expect to reproduce the retraction projection of some subset of the null geodesics for the one-centre Kastor-Traschen metric with black hole mass $m = 2M$ and this is precisely what happens. In fact, we obtain all of the projected null geodesics because of the inherent spherical symmetry that accompanies this limit.

One final point that we should stress here is that some of the physical properties of null geodesics obvious in the projection along one type of vector field can be obscured in the projection along another. A clear example of this point can be seen in the one-centre Kastor-Traschen metric which we know, via the transformation to extremal RNdS coordinates, admits both a timelike conformal retraction and a timelike static Killing vector field. Light rays project down to unparametrised geodesics of the optical metric associated to the static Killing vector field and it can be shown that for different values of the cosmological constant Λ, the resulting optical metrics are projectively equivalent. One consequence of this is the fact that the differential equations governing light rays in these spacetimes are also independent of Λ. This phenomenon is not evident, however, when we consider the retraction projection of light rays in Kastor-Traschen coordinates. There is a clear c (or Λ) dependence in the system of ODEs (6.25) and this even carries over to the equations governing the unparamterised curves (6.29).

Nonetheless, this is an interesting and physically relevant area of study and our work encourages several open questions. It would be interesting to find analytic solutions to (6.16) in a more general case and to say something more concrete about the Kastor-Traschen metric with arbitrary V. Furthermore, it is an open problem to make an invariant statement on the properties of the null geodesic structure of an arbitrary metric which admits a timelike conformal retraction.

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