Topology and Dynamics of Time Dependent Unitons

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- Based on
  - Reduced dynamics of Ward solitons.
    MD, Nicholas S. Manton, `hep-th/0411068`, Nonlinearity (2005)
  - Topology and Energy of Time Dependent Unitons.
    MD, Prim Plansangkate, `hep-th/0605185`.Proc. Royal. Soc. **A** (2007)
  - Moduli spaces with external fields
    MD, Marcin Kaźmierczak, `hep-th/0610220`.
- Integrable chiral model (Ward)
  $\phi : \mathbb{R}^{2,1} \longrightarrow G$.

$$(\phi^{-1}\phi_t)_t - (\phi^{-1}\phi_x)_x - (\phi^{-1}\phi_y)_y - [\phi^{-1}\phi_t, \phi^{-1}\phi_y] = 0.$$

- Comparison between exact solutions and approximate moduli space dynamics.
- Classical quantisation of total energy of moving solitons.

Energy and Topology

- Allowed energy levels of some physical systems can take only discrete values.
- Quantum Mechanics. Boundary conditions imposed on the wave function imply discrete spectra of the Hamiltonians. (hydrogen atom, harmonic oscillator, ...)
- Classical Field Theory. Smooth field configurations with finite energy. The potential energy of static soliton solutions is proportional to integer homotopy classes of smooth maps. (BPS monopoles, instantons in gauge theory, ...)
- In both cases reasons are global - topology.
- Moving solitons. The total energy is the sum of kinetic and potential terms, and the Bogomolny bound is not saturated. One expects that the moving (non-periodic) solitons will have continuous energy.

- $\phi : \mathbb{R}^{D,1} \longrightarrow (N, h)$

$$L = \int_{\mathbb{R}^D} \Big(\frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\nabla\phi|^2 - U(\phi)\Big) d^D\mathbf{x}$$

Solitons = nonsingular, static, finite energy solutions of the classical field equations.

- Scaling argument (Derrick)

$$E = \int_{\mathbb{R}^D} \Big(\frac{1}{2}|\nabla\phi|^2 + U(\phi)\Big) d^D\mathbf{x} = E_\nabla + E_U.$$

$$\phi_{(c)}(x) = \phi(cx), \qquad \frac{dE}{dc}|_{c=1} = 0.$$

Condition: $(D-2)E_\nabla + DE_U = 0$

– $D = 1$ Kinks with $E_\nabla = E_U$.

– $D = 2$ Sigma models. Solitons possible with $E_U = 0$.

- Harmonic maps into Lie groups. $\phi : \mathbb{R}^2 \longrightarrow G = U(N)$.

$$E(\phi) = -\frac{1}{2}\int_{\mathbb{R}^2} Tr\Big((\phi^{-1}\phi_x)^2 + (\phi^{-1}\phi_y)^2\Big) dxdy.$$

E-L equations: $(\phi^{-1}\phi_x)_x + (\phi^{-1}\phi_y)_y = 0.$

- Grassmanian embeddings. $Gr_K(\mathbb{C}^N)$ = Grassmanian of complex K-planes in $\mathbb{C}^N$. $Gr_K(\mathbb{C}^N) \subset U(N)$.

  K-plane V $\longrightarrow$ unitary transformation $i(\pi_V - \pi_{V^\perp})$.

  e.g. $K = 1$, $\mathbb{CP}^{N-1} \subset U(N)$

$$f = (1, f_1, ..., f_{N-1}) \longrightarrow i\Big(\mathbf{1} - \frac{2f^* \otimes f}{|f|^2}\Big) \in U(N)$$

  'Equatorial condition' $\phi = -\phi^*$ (on top of $\phi\,\phi^* = \mathbf{1}$).

- Finite energy solutions = unitons (Uhlenbeck)

$$\phi(x, y) = M_1 M_2 ... M_n, \qquad M_k \in Gr_K(\mathbb{C}^N).$$

  Uniton number is the smallest integer n for which this factorisation is possible. $n < N$.

- Example. $\phi \in SU(2)$. 1-uniton

$$\phi(x, y) = \frac{i}{1 + |f|^2} \begin{pmatrix} |f|^2 - 1 & -2f \\ -2\overline{f} & 1 - |f|^2 \end{pmatrix}$$

  where $f = f(z)$ is holomorphic in $z = x + iy$.

  Finite energy: $f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ is rational

$$E = \int_{\mathbb{R}^2} \frac{4|f'|^2}{(1 + |f|^2)^2} dxdy = 8\pi \deg(f).$$

- Introduce dynamics. Chiral model ('obvious choice'):

$$(\phi^{-1}\phi_t)_t - (\phi^{-1}\phi_x)_x - (\phi^{-1}\phi_y)_y = 0 \qquad (*).$$

- No (non-trivial) exact solutions. Use geodesic approximation (Manton).

  Field theory $\longrightarrow$ finite dimensional dynamical system on a moduli space $\mathcal{M}$ of static finite energy solutions which are energy minimisers.

  Kinetic energy $\longrightarrow$ Riemannian metric on $\mathcal{M}$.

  $\phi = \phi_S(x, y; \gamma_1, ..., \gamma_{\dim \mathcal{M}})$, where $\gamma \in \mathcal{M}$. Allow $\gamma(t)$.

$$\begin{aligned} T &= -\frac{1}{2}\int_{\mathbb{R}^2} Tr\Big((\phi^{-1}\phi_t)^2\Big)dxdy \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} Tr\Big(\phi^{-1}\frac{\partial\phi}{\partial\gamma_\alpha}\phi^{-1}\frac{\partial\phi}{\partial\gamma_\beta}\Big)\dot{\gamma}_\alpha\dot{\gamma}_\beta dxdy \\ &= \frac{1}{2}g_{\alpha\beta}(\gamma)\dot{\gamma}_\alpha\dot{\gamma}_\beta. \end{aligned}$$

- Exact solution to $(*)$ with small velocity oscillates around a geodesic of $(\mathcal{M}, g)$.
- Problems: Solitons unstable, metric on $\mathcal{M}$ incomplete, exact solutions unknown.

- Integrable chiral model (Ward), $\phi : \mathbb{R}^{2,1} \longrightarrow G$.

$$(\phi^{-1}\phi_t)_t - (\phi^{-1}\phi_x)_x - (\phi^{-1}\phi_y)_y - [\phi^{-1}\phi_t, \phi^{-1}\phi_y] = 0.$$

- $SO(2,1)$ broken to $SO(1,1)$ by $V = dx$.

  Conserved energy

$$E = -\frac{1}{2}\int_{\mathbb{R}^2} Tr\Big((\phi^{-1}\phi_t)^2 + (\phi^{-1}\phi_x)^2 + (\phi^{-1}\phi_y)^2\Big) dxdy.$$

  Boundary conditions

$$\phi(t, r, \theta) = \phi_0 + r^{-1}\phi_1(\theta) + O(r^{-2}).$$

- Completely integrable system: Exact time–dependent solutions, Lax pair, twistor theory, ∞-many conservation laws, so solitons may be stable (?).
- Static solutions = harmonic maps. $(\mathcal{M}, g)$ as before, but there could exist a magnetic field on $\mathcal{M}$ due to the first order term $V \wedge \phi^{-1}d\phi \wedge \phi^{-1}d\phi = [\phi^{-1}\phi_t, \phi^{-1}\phi_y]$.
- Modification not as arbitrary as it seems (chiral model with torsion, symmetry reduction of self–dual Yang–Mills equation in (2, 2) signature by a translation).

- $SU(2)$ static solution, dim $\mathcal{M} = 4Q$

$$f(z) = \frac{(z-p_1)...(z-p_Q)}{(z-p_{Q+1})...(z-p_{2Q})}, \qquad \gamma = (p, \overline{p}).$$

- Moduli space dynamics (M.D - Nick Manton, M.D - Marcin Kaźmierczak). Reduced dynamics on the space of based rational maps: Kähler metric g, and flat $U(1)$ connection A

$$\begin{aligned} g_{\alpha\beta} &= 8 \int_{\mathbb{R}^2} \frac{|\partial_\alpha f \partial_\beta \overline{f}|}{(1+|f|^2)^2} dxdy, \\ A_\alpha &= 4\pi \int_{\mathbb{R}^2} \frac{\mathrm{Re}(\partial_z f \partial_\alpha \overline{f})}{(1+|f|^2)^2} dxdy. \end{aligned}$$

The WZW action $S = S_C + S_M$.

$$\hat{\phi} : \mathbb{R}^{2+1} \times [0,1] \longrightarrow U(N),$$

$$\hat{\phi}(x^\mu, 0) = \mathbf{1}, \qquad \hat{\phi}(x^\mu, 1) = \phi(x^\mu).$$

$$\begin{aligned} S_C &= -\int_{[t_1,t_2]\times\mathbb{R}^2} \mathrm{Tr}(\phi^{-1}d\phi \wedge *\phi^{-1}d\phi) \\ S_M &= \int_{[t_1,t_2]\times\mathbb{R}^2\times[0,1]} \hat{\phi}^*(\mathrm{Torsion}) \wedge V \\ &= \frac{1}{3}\int_{[t_1,t_2]\times\mathbb{R}^2\times[0,1]} \mathrm{Tr}(\hat{\phi}^{-1}d\hat{\phi} \wedge \hat{\phi}^{-1}d\hat{\phi} \wedge \hat{\phi}^{-1}d\hat{\phi} \wedge V). \end{aligned}$$

- $\delta S_M = 0$ so $dA = 0$. Flat connection is still interesting
  - Space of rational maps is not simply connected.
  - Pull back magnetic form. $f : \mathbb{CP}^1 \times \mathcal{M} \longrightarrow \mathbb{CP}^{N-1}$

$$\begin{aligned} f_p &:= f(p, ...) : \mathcal{M} \longrightarrow \mathbb{CP}^{N-1}, \\ f_\gamma &:= f(..., \gamma) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^{N-1} \end{aligned}$$

$$\begin{aligned} g(X, X) &= \int_{\mathbb{R}^2} h((f_{p*}X, (f_p)_* X) dx dy, \qquad X \in T_\gamma \mathcal{M}, \\ A(X) &= \int_{\mathbb{R}^2} h((f_\gamma)_* (\partial/\partial x), (f_p)_* X) dx dy, \end{aligned}$$

where h is the Fubini–Study metric on $\mathbb{CP}^{N-1}$.

- In the moduli space approximation the total energy is close to $8\pi Q$, where $Q \in \mathbb{Z}$. Are there exact solutions with quantised total energy? [Usually potential energy of static solitons $\sim$ integer homotopy classes of smooth maps, but time depencence implies continuous energies].

- Lax pair

$$\begin{aligned} L &= \partial_t + \partial_y + \phi^{-1}(\phi_t + \phi_y) - \lambda\partial_x, \\ M &= \partial_x + \phi^{-1}\phi_x - \lambda(\partial_t - \partial_y). \end{aligned}$$

Modified chiral model is equivalent to $[L, M] = 0$.

Can solve an overdetermined system $L\Psi = M\Psi = 0$ for $\Psi(x, y, t, \lambda)$. Given Ψ, recover $\phi(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0)$.

- Restrict the spectral parameter λ to an equator $S^1 \subset \mathbb{CP}^1$.

$$\Psi \longrightarrow H(x, y, \theta), \qquad \lambda = -\cot(\theta/2).$$

Take the spatial part of the Lax pair on the space-like plane $\mathcal{L} = \lambda L + M$.

- The ODE

$$\mathcal{L}H = 0$$

describes a propagation of H along a line in $\mathbb{R}^2$.

$$H = \psi(x_0 - s\cos\theta, y_0 - s\sin\theta, \theta), \qquad s \in \mathbb{R}.$$

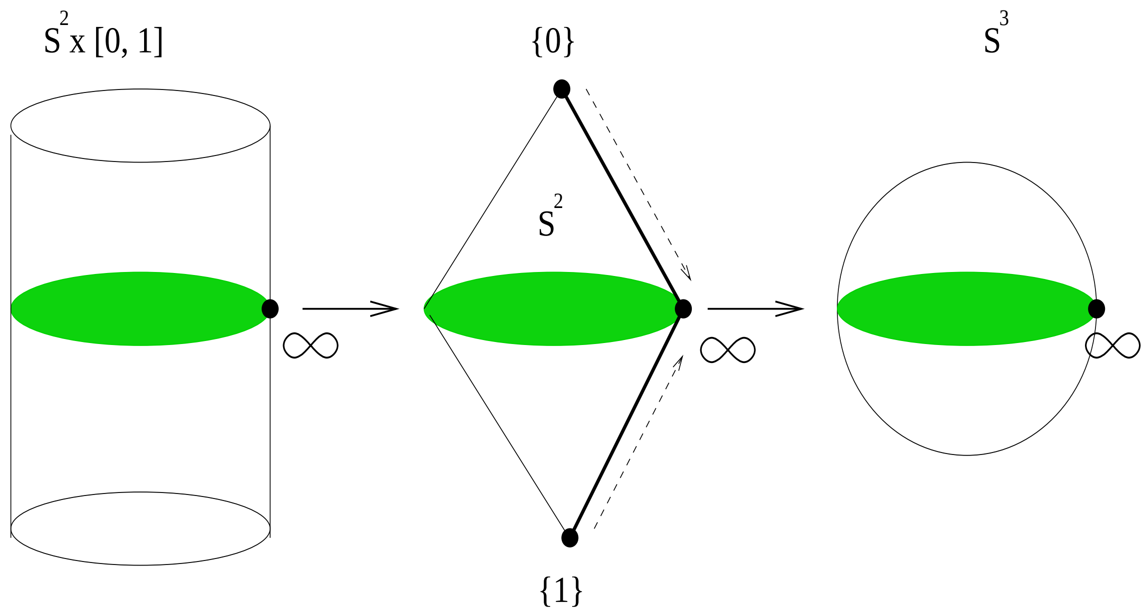
- Gauge choices: $H(x, y, 0) = \mathbf{1}, \quad \lim_{s\to-\infty} H = \mathbf{1}.$

- Impose the 'trivial monodromy' boundary condition

$$\lim_{s\to\infty} H = \mathbf{1}.$$

This is stronger than the requirement $E < \infty$, and picks a finite dimensional family of finite energy solutions.

- H extends to an element of $\mathrm{Map}(S^3, G)$.



- Topological charge

$$[H] = \frac{1}{24\pi^2}\int_{S^3} Tr\Big((H^{-1}dH)^3\Big) \in \pi_3(G) = \mathbb{Z}.$$

Is it related to the total energy?

- Extended solution (Ward, Dai & Terng)

$$\Psi = G_n G_{n-1} \ \dots \ G_1, \qquad \text{where}$$
$$G_k = \Big(\mathbf{1} - \frac{\overline{\mu} - \mu}{\lambda - \mu} \frac{q_k^* \otimes q_k}{||q_k||^2}\Big) \in GL(N, \mathbb{C}),$$

$$q_k = q_k(x^\mu) \in \mathbb{C}^N, k = 1, \dots, n, \qquad \mu = me^{i\phi} \in \mathbb{C}/\mathbb{R}.$$

- Topology (or an explicit calculation of Skyrme). Pointwise group multiplication $\longrightarrow$ addition in $\pi_3(G)$.

$$g_1, g_2 : S^3 \longrightarrow U(N), \qquad [g_1 g_2] = [g_1] + [g_2].$$

$$H = g_n g_{n-1} \ \dots \ g_1$$

$$[H] = \frac{i}{2\pi} \int_{\mathbb{R}^2} \sum_{k=1}^{n} \text{Tr}(R_k[\partial_x R_k, \partial_y R_k]) dx dy, \qquad R_k \equiv \frac{q_k^* \otimes q_k}{||q_k||^2}.$$

- Uniton solutions from first order Bäcklund relations

$$\phi(x, y, t) = M_1 M_2 \dots M_n, \qquad M_k = i\Big(\mathbf{1} - \Big(1 - \frac{\mu}{\bar{\mu}}\Big) R_k\Big).$$

Theorem.[M.D., Prim Plansangkate.] The total energy of the time dependent 'trivial scattering' solitons with $\mu = me^{i\phi}$ is classically quantised

$$E = 4\pi\Big(\frac{1 + m^2}{m}\Big) |\sin(\phi)| \ [H].$$

Twistor theory for $\mathbb{R}^{2+1}$, $\quad ds^2 = dt^2 - dx^2 - dy^2$.

- $$\omega = (t+y) + 2\lambda x + \lambda^2(t-y)$$

Null planes $\leftrightarrow$ *Real* points in $Z = T\mathbb{CP}^1$

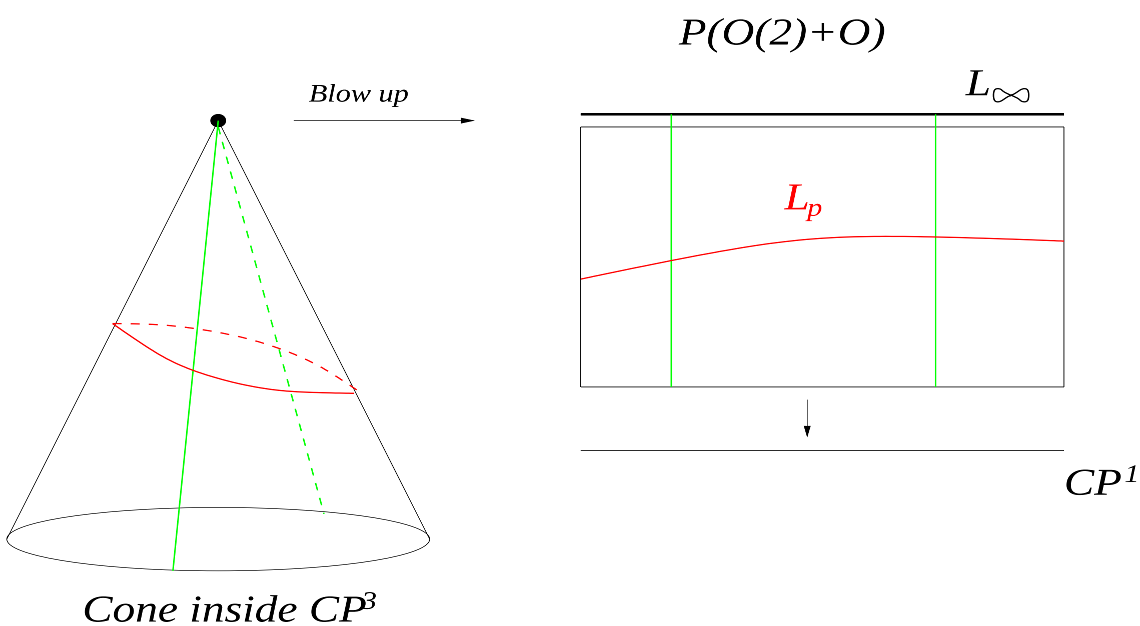
Points $p \in \mathbb{R}^{2,1} \leftrightarrow$ *Real* holomorphic sections L_p of Z

Solutions to ICM $\leftrightarrow$ Holomorphic vector bundles over Z trivial on sections.

- Riemann–Hilbert factorisation of a patching matrix

$$F(\omega, \lambda)|_{L_p} = \tilde{\Theta}\, \Theta^{-1}, \qquad \phi(x, y, t) = \Theta|_{\lambda=0}.$$

- Compactified twistor space $\bar{Z} = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$.



- Holomorphic bundles over compact complex surfaces have finite moduli.

Unitons $\leftrightarrow$ Bundles over $\bar{Z}$

Conclusions

- 2+1 modified chiral model. Not fully Lorentz invariant, but integrable. There exists a magnetic field on the full space of solutions, but it vanishes on the moduli space of static finite energy solutions.
- Comparison of exact solutions with moduli space dynamics. [The moduli space approach to the ordinary $\mathbb{CP}^{N-1}$ model in 2+1 dimensions does not approximate the true dynamics of the model. Rodnianski and Sterbenz `math.AP/0605023` have demonstrated that any solution must blow up in finite time.]
- Classical quantisation of energy of moving solitons.

Questions

- Integrability of geodesic motion on the space of based rational maps (?)
- Stability (?).