§2 DYNAMICS

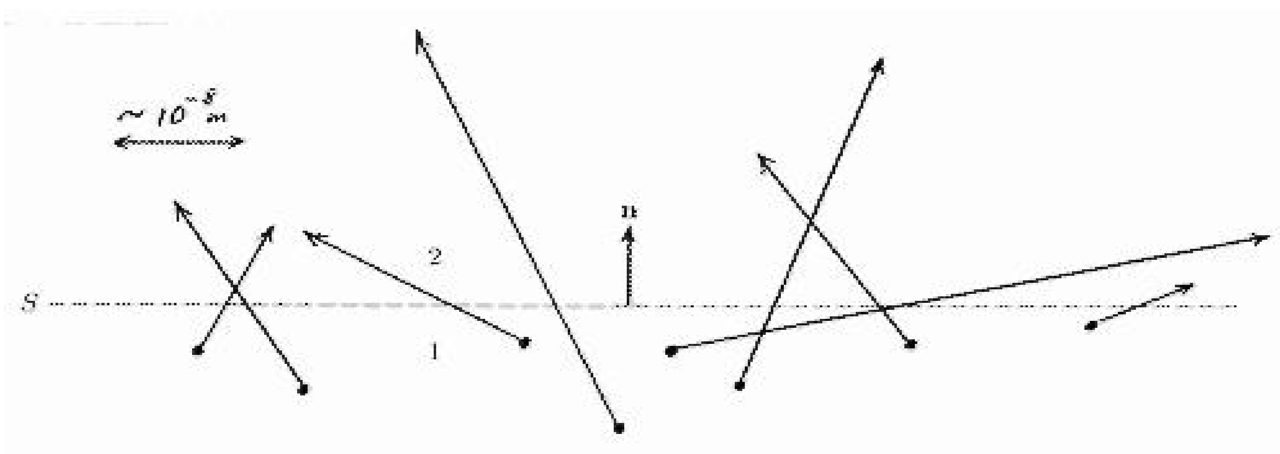
§2.1 Surface vs. body forces, and the concept of pressure

We need to distinguish 'surface' forces from 'body' or 'volume' forces. These are, respectively, forces on a fluid element that are proportional to its

- *surface area* — e.g. pressure or frictional, i.e. viscous, forces, or
- *mass* or *volume* — e.g. weight (gravitational force), or electric or magnetic force.

Surface forces:

We can motivate the idea of surface force by taking the case of a gas, and zooming-in to the molecular scale. Consider molecules of gas crossing a fixed 'control surface' S, in the form of a plane dividing the region occupied by the gas into two sub-regions 1, 2:



Molecules that cross the surface S from region 1 to region 2, as shown, have a positive velocity component in the $\mathbf{n}$ direction, and therefore systematically transfer momentum in the $\mathbf{n}$ direction from region 1 to region 2. Conversely, molecules that cross the surface S from region 2 to region 1 have a negative velocity component in the $\mathbf{n}$ direction, and transfer momentum in the $-\mathbf{n}$ direction from region 2 to region 1. The effects add up, and the average effect is equivalent to a force in $\mathbf{n}$ direction. Region 1 pushes region 2 in the $+\mathbf{n}$ direction, and region 2 pushes region 1 in the $-\mathbf{n}$ direction. This 'pressure force' $\propto$ surface area. It is therefore a surface force.

Pressure forces are strictly speaking *isotropic*, i.e. direction-independent, by definition. This means that they are described by a scalar field $p(\mathbf{x}, t)$ (called the pressure) such that the force exerted across an arbitrarily-oriented surface element δS at $\mathbf{x}$ with unit normal $\mathbf{n}$ is always $\mathbf{n}p\,\delta S$, independent of the orientation chosen. The sign convention is such that $+\mathbf{n}p\,\delta S$ is the force exerted *on* the fluid into which $\mathbf{n}$ points, *by* the fluid away from which $\mathbf{n}$ points. If the fluid as a whole is at rest (i.e. at rest apart from the molecular-scale motions), then the pressure force is the only surface force.

Viscous (i.e. frictional) forces in a moving fluid are also surface forces. In our gas example they are mainly associated with transfer, from region 1 to region 2, of *tangential* momentum, i.e. of momentum components in directions perpendicular to $\mathbf{n}$. We shall neglect viscous forces throughout these lectures, and consider an 'inviscid' (frictionless) fluid only. This is a good model for some purposes, for instance water waves, flow over weirs, flow around rising, oscillating, or collapsing bubbles, and most of the flow field around streamlined bodies like aircraft.

Viscous forces in gases are usually much smaller than pressure forces, because the momentum-transfer effects tend to cancel rather than add up. If the fluid as a whole is at rest, then the cancellation is complete because, for instance, molecules with rightward momentum cross the surface S both ways in equal numbers, on average.

A more sophisticated treatment of surface forces would require us to introduce the *stress tensor*; this is done in Part IIB Fluids, and in Part IIA Theoretical Geophysics.

'Volume' or 'body' forces:

Let $\mathbf{F}$ be the body force per unit mass. (Example: $\mathbf{F} = \mathbf{g}$ where $\mathbf{g}$ = gravitational 'acceleration'.) Then the force per unit volume is $\rho\mathbf{F}$, and the force on volume δV is $\rho\mathbf{F}\delta V$.

The case $\mathbf{F} = \mathbf{g}$ is also an example where the volume force is 'conservative', meaning that $\exists$ a potential Φ with $\mathbf{g} = -\nabla\Phi$, hence $\rho\,\mathbf{F}\,\delta V = \rho\,\mathbf{g}\,\delta V = -\rho\,\nabla\Phi\,\delta V$.

(NB: Some versions of the example sheets use the symbol χ for the force potential, rather than Φ.)

§2.2 Momentum Equation

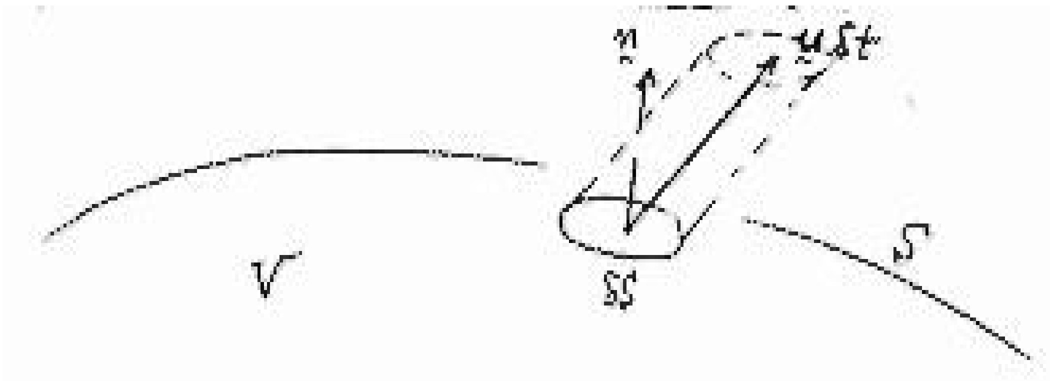
As for mass conservation, consider an arbitrary volume V, fixed in space, and bounded by a surface S, with outward normal $\mathbf{n}$

$$\text{Momentum inside } V \text{ is} \quad \int_V \rho\,\mathbf{u}\,dV$$

How does the momentum inside V change? Need to take account of:

1. volume forces
2. surface forces
3. the fact that fluid entering or leaving takes momentum with it.

Consider, as before, the small volume of a 'slug' or slanted cylinder of fluid swept out by the area element δS in time δt, where δS starts on the surface S and moves with the fluid, i.e. with velocity $\mathbf{u}$:



This time we are interested in momentum rather than mass; the momentum of the slanted cylinder of fluid is $\rho\mathbf{u}\ (\mathbf{u}.\mathbf{n})\,\delta t\ \delta S$. So the change in the total momentum within the (fixed) volume V due to (3) alone is δt times $\int_S \rho(\mathbf{u}.\mathbf{n})\,\mathbf{u}\,dS$. So the rate of change in the total momentum within V, taking all three contributions (1)–(3) into account, is

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = \int_V \rho\mathbf{F}\,dV - \int_S p\,\mathbf{n}\,dS - \int_S \rho\,\mathbf{u}\,(\mathbf{u}.\mathbf{n})\,dS\ .$$

This is one way of writing the integral form of the momentum equation.

Here's another way: take the ith (Cartesian) component:

$$\frac{d}{dt}\int_V \rho\,u_i\,dV = \int_V \rho F_i\,dV - \int_S p n_i\,dS - \int_S \rho u_i u_j n_j\,dS\ .$$

Now use the generalized divergence theorem to convert surface integrals into volume integrals in the above. (For any suffix k, replace n_k by $\partial/\partial x_k$ in the integrands of the surface integrals while replacing surface integration by volume integration.) The result is

$$\frac{d}{dt}\int_V \rho\,u_i\,dV = \int_V \left(\rho F_i - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(\rho u_i u_j)\right) dV.$$

V is fixed, so $(d/dt)\int_V = \int_V(\partial/\partial t)$ as before. Furthermore, V is arbitrary. Hence

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho F_i\ .$$

The left-hand side may be rearranged to give

$$\rho\left\{\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right\} + u_i\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right\} = -\frac{\partial p}{\partial x_i} + \rho F_i\ .$$

The second brace vanishes by mass conservation, even for compressible flow; hence

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho \mathbf{F} \ .$$

This is the *Euler equation.* In this form it can be recognized as a statement of Newton's 2nd law for a inviscid (frictionless) fluid. It says that, for an infinitesimal volume of fluid, mass times acceleration = total force on the same volume, namely force due to pressure gradient plus whatever body forces are being exerted.

If the body forces are all zero, then momentum is *conserved* (again note form $\dfrac{\partial(\ \ )}{\partial t} + \nabla.\{\ \ \} = 0$):

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij}\right) = 0 \ .$$

Here δ_{ij} is the Kronecker delta, i.e. the components of the identity tensor.

Like all other field-theoretic conservation equations, this equation says, as already mentioned, that a four-dimensional divergence vanishes. The 'flux' whose 3-D divergence appears here is a second-rank tensor, $\rho u_i u_j + p\delta_{ij}$, and may be called the total momentum flux. It is customary in parts of the literature to call the 'advective' contribution $\rho u_i u_j$, corresponding to the contribution (3) above, the 'advective' or 'dynamic' momentum flux, or sometimes, confusingly, just 'the' momentum flux.

Together with the mass conservation equation, $\nabla.\mathbf{u} = 0$, and a boundary condition on $\mathbf{u}.\mathbf{n}$, the foregoing equations (expressing the momentum balance and expressing mass conservation) are sufficient to determine the motion, for the incompressible, constant-density flows under consideration.

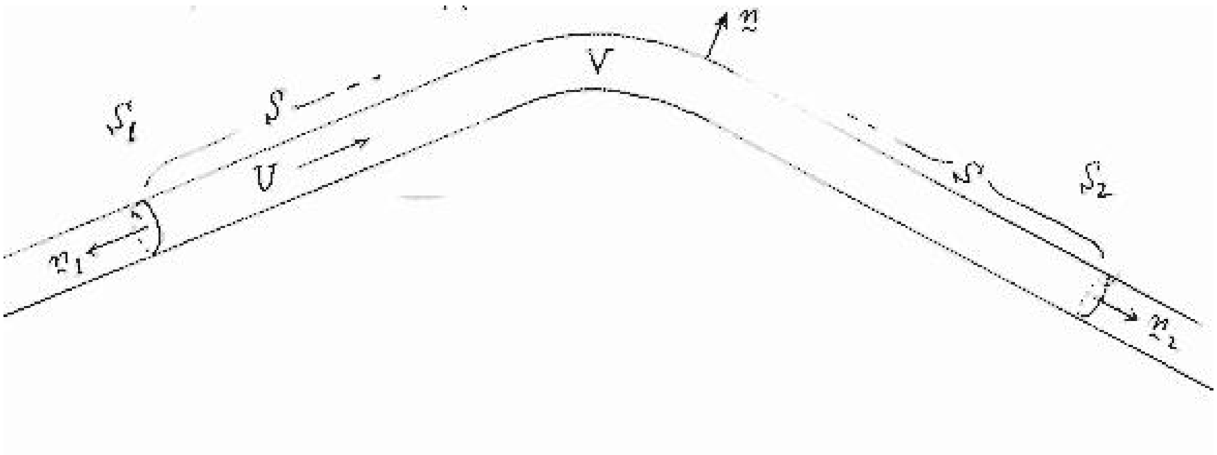
We next turn to some applications of the momentum equation in integral form. One can often make useful deductions *without* solving the whole flow problem, starting with a qualitative knowledge of what the flow is like (perhaps derived from observation or experiment) and then applying integral relations such as the integral forms of the mass and momentum equations.

The first of these applications is to find the force on a curved pipe. This might, for instance, be something you would need to know if you were interested in the safety of some industrial process using high-speed fluid flow.

§2.3 Applications of momentum integral

Pressure force on a curved section of pipe

(Think of liquid sodium rushing round the bend of a pipe in a nuclear power station, or water rushing through a fire hose.) Let pipe have cross-sectional area A. Assume steady flow at speed U. For simplicity, though this is not essential, assume that U is parallel to the pipe, in its straight sections, and uniform across the pipe:



Recall momentum
equation in integral form,
with volume V and surface $S \cup S_1 \cup S_2$ as above:

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = \int_V \rho\mathbf{g}\,dV - \int_{S\cup S_1\cup S_2} \{p\,\mathbf{n} + \rho\,(\mathbf{u}.\mathbf{n})\mathbf{u}\}\,dS\ .$$

($\mathbf{g} = -\nabla\Phi$ is the gravity acceleration)

So total force by fluid on pipe

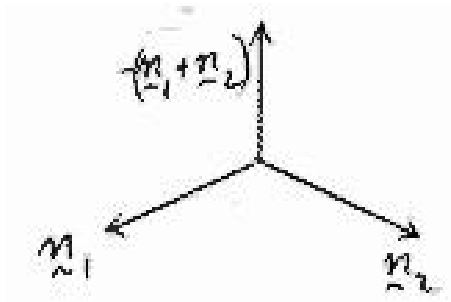
$$= \int_S p\,\mathbf{n}dS \qquad (\mathbf{n} \text{ is normal out of fluid, as before})$$

$$= \int_V \rho\,\mathbf{g}\,dV - \int_{S_1\cup S_2} p\,\mathbf{n}\,dS - \int_{S_1\cup S_2} \rho\,(\mathbf{u}.\mathbf{n})\mathbf{u}\,dS\ ,$$

where use has been made of the boundary condition $\mathbf{u}.\mathbf{n} = 0$ on the pipe wall, assumed impermeable. So the force by the fluid on the pipe is equal to the weight, under gravity, of fluid in the pipe *plus* the two surface integrals over $S_1 \cup S_2$, which give

$$\begin{aligned} &- \quad Ap(\mathbf{n}_1 + \mathbf{n}_2) - (-\rho AU)(-U\mathbf{n}_1) - (\rho AU)(U\mathbf{n}_2) \\ &= \quad -A(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2)\ , \end{aligned}$$

assuming p, as well as U, approximately uniform over the cross-sections S_1, S_2.

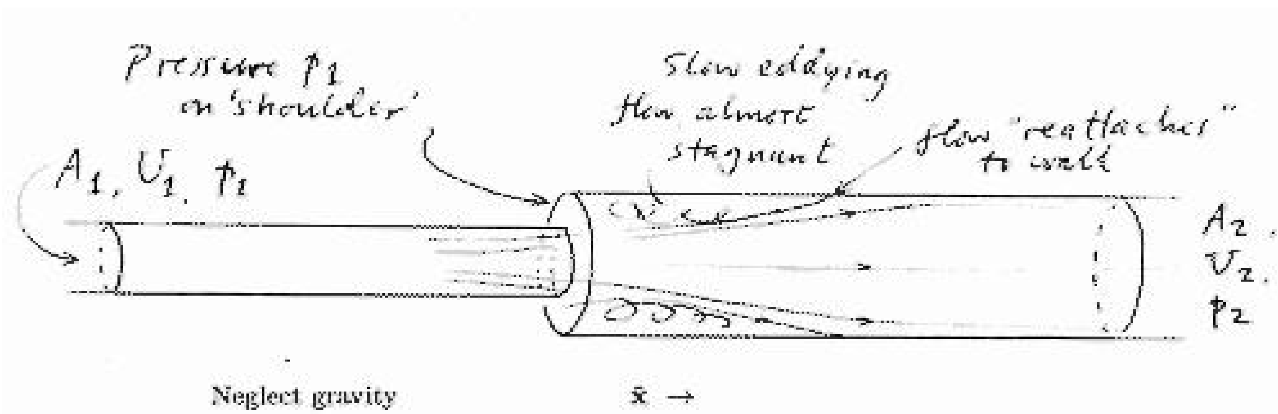


Note that this force depends on the 'background' pressure p (e.g. determined by pumping station or height of reservoir hydrostatic pressure).

The contribution from the advective (dynamic) momentum flux, proportional to velocity squared, can be very important for high-speed flow. (E.g. case of fire hose: $\rho = 10^3$kg m^{-3}; $U = 10\,\mathrm{m\,s^{-1}}$; $A = 0.8 \times 10^{-2}\mathrm{m}^2$ (hose diameter 10 cm) $\Rightarrow$ 800 N force, or $\simeq$ 80 kg wt.

Pressure change at an abrupt change in pipe diameter

If we make plausible assumptions based on the behaviour observed in experiments, then we can again apply the momentum integral to get a useful answer very simply. The observed behaviour is as follows.



Immediately downstream of the junction, there is a complicated eddying behaviour. Ultimately the flow reattaches to the boundary and there is an approximately uniform flow far downstream.

In the 'corner' regions just downstream of the junction, the flow is relatively weak. Therefore there cannot be large pressure gradients across those regions. We assume that pressure gradients across the pipe can be neglected — that just downstream of the junction the pressure is relatively uniform across the pipe, and that it takes approximately the same value, p_1 say, as in the uniform flow upstream of the junction.

The pressure far downstream, p_2 say, will however differ from p_1. This is because a flow of the assumed form has to decelerate.

Apply momentum integral to the entire volume shown in the sketch above, between upstream and downstream cross-sections with areas A_1 and $A_2(> A_1)$, taking component along the pipe (direction $\hat{\mathbf{x}}$ say) and assuming steady, uniform flow at each cross-section. Component along pipe gives

$$\hat{\mathbf{x}}\,.\int \{\rho\,(\mathbf{u}.\mathbf{n})\,\mathbf{u} + p\,\mathbf{n}\}\;dS \quad = \quad 0\;.$$

There is no contribution to the integral from the curved surface since that surface is an impermeable boundary where $\mathbf{u}.\mathbf{n} = 0$. So, remembering the contribution $p_1(A_2 - A_1)$ from the 'shoulder',

$$A_1\rho U_1^2 + A_2 p_1 = A_2\rho U_2^2 + A_2 p_2\;.$$

Mass conservation implies that $A_1U_1 = A_2U_2$. Hence

$$p_1 - p_2 = \rho\left(U_2^2 - U_1^2\frac{A_1}{A_2}\right) = \rho U_1^2\left(\frac{A_1^2}{A_2^2} - \frac{A_1}{A_2}\right) < 0\;.$$

The pressure p increases from upstream to downstream.

This flow model has an interesting property: even though we said nothing about friction or viscosity, the rate of energy input into the volume shown has a well-determined *positive value*. The eddying flow must be dissipating energy at a significant rate, no matter how small the fluid viscosity! We may calculate the rate of rate of energy input as $\{$(pressure force) $\times$ (velocity)$\}$ + (net rate at which kinetic energy is carried into the volume shown); this follows from the energy integral in Sheet I Q9. The kinetic energy per unit mass is $\frac{1}{2}U^2$ for motion with velocity U. The rate at which mass enters and leaves the volume is $\rho U_1 A_1 = \rho U_2 A_2$. So the net energy input rate is

$$\begin{aligned}
& p_1U_1A_1 - p_2U_2A_2 + (\rho U_1A_1)(\tfrac{1}{2}U_1^2) - (\rho U_2A_2)(\tfrac{1}{2}U_2^2) \\
= \quad & \rho U_1^2\left(\frac{A_1^2}{A_2^2} - \frac{A_1}{A_2}\right)U_1A_1 + \tfrac{1}{2}\rho U_1^3 A_1\left(1 - \frac{A_1^2}{A_2^2}\right) \qquad [\,p \text{ terms sum to } (p_1 - p_2)U_1A_1\,] \\
= \quad & \tfrac{1}{2}\rho U_1^3 A_1\left(1 - \frac{A_1}{A_2}\right)^2 > 0\;.
\end{aligned}$$

Energy is being lost at this rate, in the eddying region near the junction. (If the energy input rate had come out negative, then we'd have had to conclude that the flow model had something grossly wrong with it — that we had made a grossly inaccurate or false modelling assumption.)

§2.4 Bernoulli's (streamline) theorem for steady flows with potential forces

We first derive a relevant vector identity:

$$(\mathbf{u}.\nabla)u_i = u_j \left\{ \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right\} + u_j \frac{\partial u_j}{\partial x_i} = -u_j \epsilon_{ijk} (\nabla \times \mathbf{u})_k + \frac{\partial}{\partial x_i} (\tfrac{1}{2} u_j u_j)$$

i.e.

$$\boxed{(\mathbf{u}.\nabla)\mathbf{u} \;=\; (\nabla \times \mathbf{u}) \times \mathbf{u} \;+\; \nabla(\tfrac{1}{2}|\mathbf{u}|^2)}$$

The vector $\nabla \times \mathbf{u}$, $= \boldsymbol{\omega}$ say, has special significance and is called the *vorticity* — see below.

Starting with the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}$$

we assume that the flow is steady, so $\partial \mathbf{u}/\partial t = 0$, and that the body force $\mathbf{F}$ per unit mass is conservative. Therefore $\exists$ scalar field Φ (e.g. gravitational potential) such that $\mathbf{F} = -\nabla\Phi$. The Euler equation (with ρ constant, and using the notation $\boldsymbol{\omega} = \nabla \times \mathbf{u}$) implies that

$$0 \;=\; \boldsymbol{\omega} \times \mathbf{u} \;+\; \nabla(\tfrac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi) \;=\; \boldsymbol{\omega} \times \mathbf{u} \;+\; \nabla H, \quad \text{say.}(*)$$

Notice that the quantity $H \;=\; \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi$ has dimensions of energy per unit mass.

With incompressible flow and a steady pressure field, the field p/ρ can be looked on mathematically as if it were a potential energy per unit mass; but this analogy should not be pushed too far. For instance, if the potential Φ represents gravity under everyday conditions then we can regard the function Φ as given beforehand, but not the function p/ρ, which depends on the flow.

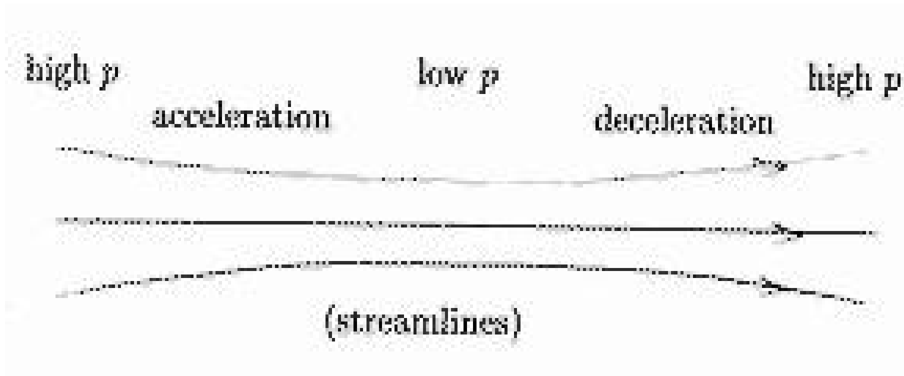
Now take the scalar product of the equation $(*)$ with $\mathbf{u}$ or $\boldsymbol{\omega}$.

$(\mathbf{u}.\nabla)H = 0$, i.e. H constant along streamlines **Bernoulli's** (streamline) **theorem**

$(\boldsymbol{\omega}.\nabla)H = 0$, i.e. H constant along vortex lines, i.e. along integral curves of $\boldsymbol{\omega}(\mathbf{x})$.

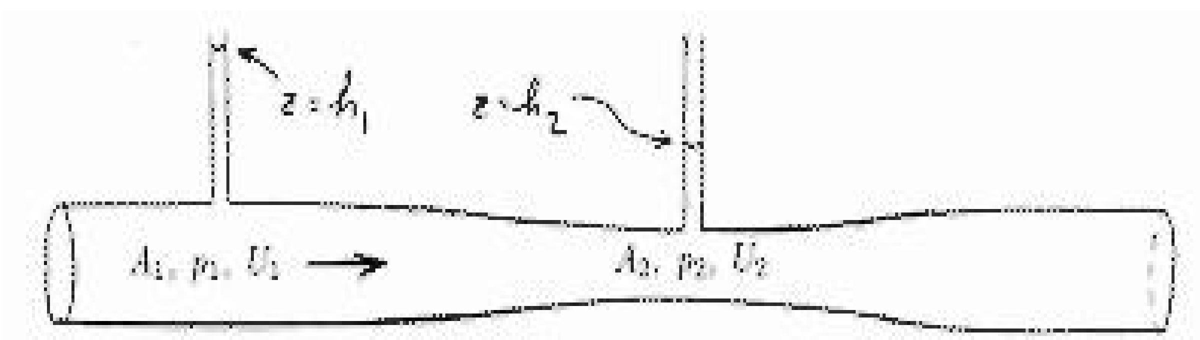
Vortex lines are curves tangent to the vector field $\boldsymbol{\omega}(\mathbf{x})$. Vortex lines are to $\boldsymbol{\omega}(\mathbf{x})$ as streamlines are to $\mathbf{u}(\mathbf{x})$.

Notice that H constant implies that p is low when $|\mathbf{u}|$ is high, and that p is high when $|\mathbf{u}|$ is low. E.g., along streamlines:



§2.5 Applications of Bernoulli's theorem

(1) Venturi meter:



measures flow rate $A_1,\ p_1,\ U_1$ $A_2,\ p_2,\ U_2$

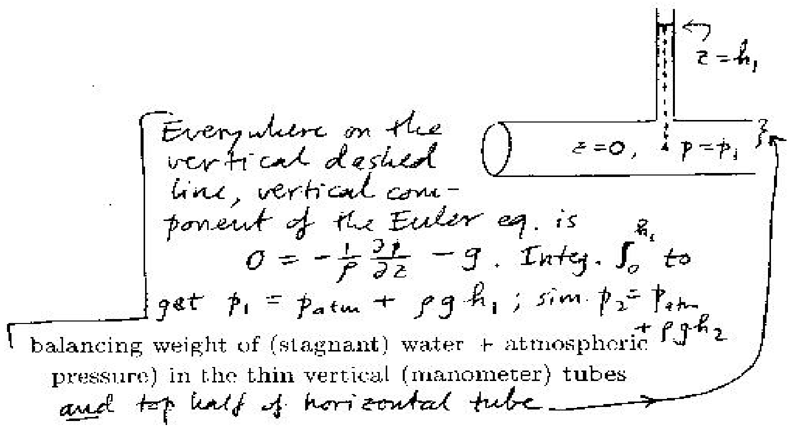
(gentle contraction; uniform tubing upstream and downstream)

so Bernoulli applies if flow is smooth and effectively inviscid — friction negligible:

$$\frac{p_1}{\rho} + \tfrac{1}{2}U_1^2 = \frac{p_2}{\rho} + \tfrac{1}{2}U_2^2 \qquad \text{(Bernoulli)}$$

$$A_1 U_1 = A_2 U_2 \qquad \text{(mass conservation)}$$

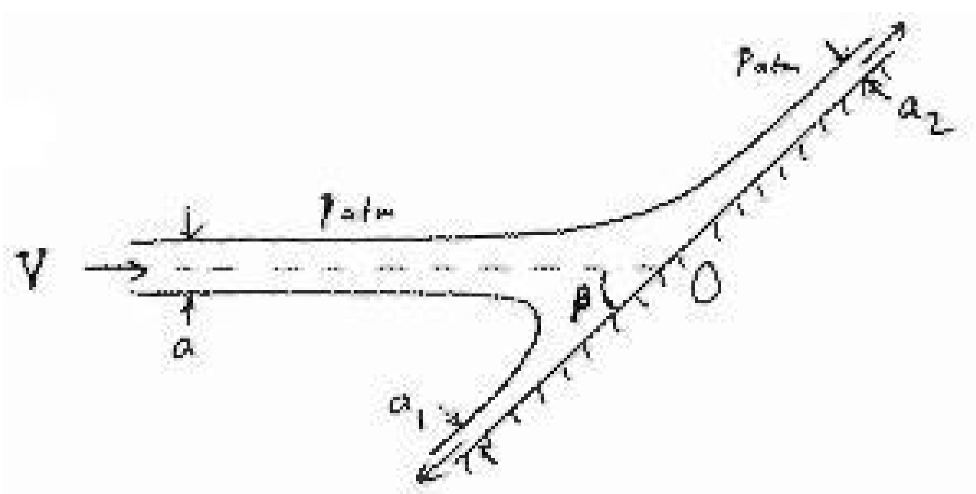
$$\Rightarrow p_1 - p_2 = \tfrac{1}{2}\rho U_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)$$



Use $p_1 - p_{\text{atm}} = \rho g h_1$, $p_2 - p_{\text{atm}} = \rho g h_2$ balancing weight of (stagnant) water + atmospheric pressure) in the thin vertical (manometer) tubes

$$\Rightarrow U_1^2 = \frac{2g(h_1 - h_2)}{(A_1^2/A_2^2 - 1)}$$

(2) 2-D water jet incident on oblique plane:



Apply Bernoulli to free-surface streamline:

$$\frac{p}{\rho} + \tfrac{1}{2}|\mathbf{u}|^2 = \text{ constant} ; \qquad p = p_{\text{atm}} \Rightarrow \tfrac{1}{2}|\mathbf{u}|^2 = \text{ constant}$$

so, far from the point O, the flow speed $V = |\mathbf{u}|$ is uniform.

Mass conservation: $\rho V a = \rho V a_1 + \rho V a_2$

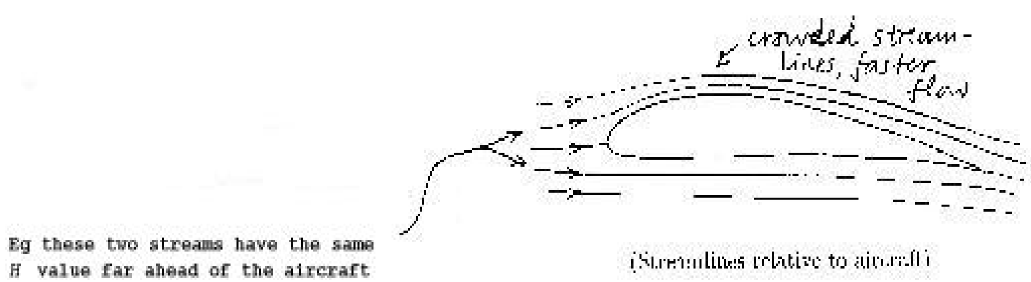
momentum flux $\parallel$ plate: $\rho a V^2 \cos\beta = \rho a_2 V^2 - \rho a_1 V^2$ (all per unit distance into the paper)

$\Rightarrow \quad a_2 = \frac{1}{2}(1+\cos\beta)a \quad , \quad a_1 = \frac{1}{2}(1-\cos\beta)a$ (Note, a_1 can't be zero even for $\beta \ll \pi/2$!)

momentum flux $\perp$ plate: $= \rho a V^2 \sin\beta$ = force on plate (neglecting friction as always)

*If the plate is freely pivoted about the point O, precisely aligned with the axis of the oncoming jet, then the plate feels a torque or couple that tends to rotate it until it is $\perp$ jet, e.g. anticlockwise if $\beta < \pi/2$ as shown. Consider the fluid's angular momentum balance: sum of torques about O (moments about O) of total momentum fluxes, including those due to pressures near O, must be zero. So torque *by* fluid *on* plate = torque *on* fluid *by* (advective) momentum fluxes (which act like pressures, as we saw earlier, hence uniformly over the width of each jet in this model). The far section (a_2) evidently dominates over the near section (a_1), having both a greater momentum flux ($\propto a_2 V^2$) and a greater moment arm about O ($\propto a_2$). Integrating, we get total torque $= \frac{1}{2}\rho(a_2^2 - a_1^2)V^2$ (per unit distance).*

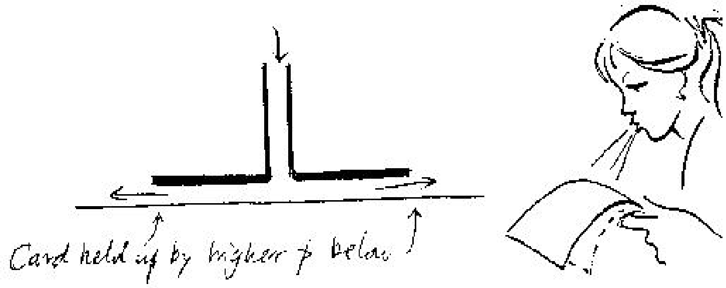
Other applications: Barges ground in Canals
Other applications Refuelling ships collide
Barges ground in canals
Aerofoil lift:



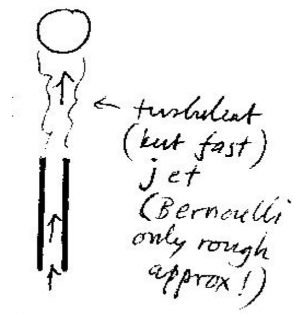
E.g. these two streamlines have the same H values far ahead of the aircraft

Bernoulli party tricks:

Lifting a card by blowing downward (cf barge/ship examples):



pingpong ball on an upward jet (cf aerofoil example):



Bernoulli helps understand stability to sideways displacement. The flow past the ball is faster on the side nearest the pipe axis.

§2.6 Vorticity

Definition: vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ local 'spin', in a generalized sense

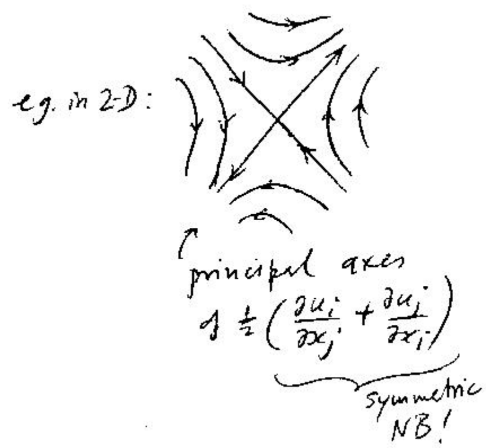
(kth cpt $\epsilon_{klm}\dfrac{\partial u_m}{\partial x_l}$) $\propto$ angular momentum of a spherical fluid parcel about its centre

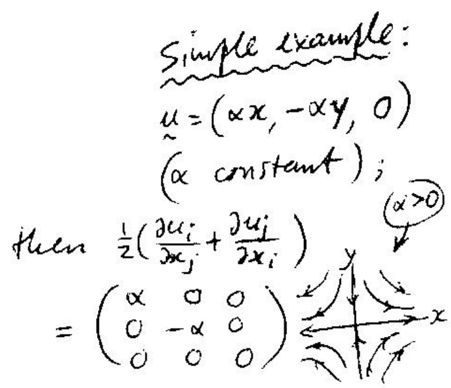
To see this, notice how the velocity field varies locally, according to the first term of a Taylor expansion about a given point $\mathbf{x}_0$:

$$\mathbf{u}(\mathbf{x},t) \simeq \mathbf{u}(\mathbf{x}_0,t) + \underbrace{[(\mathbf{x}-\mathbf{x}_0).\nabla\mathbf{u}]|_{\mathbf{x}=\mathbf{x}_0}}_{\text{linear variation specified by the tensor } \nabla\mathbf{u}} + O(|\mathbf{x}-\mathbf{x}_0|^2)$$

$$\frac{\partial u_j}{\partial x_i} = \underbrace{\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric, trace-free if } \nabla.\mathbf{u}=0} + \underbrace{\tfrac{1}{2}\epsilon_{ijk}\omega_k}_{\text{rotation with angular velocity } \frac{1}{2}\boldsymbol{\omega}}$$

The first term represents a flow pattern known as *pure strain*:





We shall need another vector identity:

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}|_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}\omega_l u_m)$$

$$\begin{aligned} &= (\delta_{il}\delta m_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}(\omega_l u_m) \\ &= \frac{\partial}{\partial x_j}(\omega_i u_j) - \frac{\partial}{\partial x_j}(\omega_j u_i) \\ &= \left\{(\mathbf{u}.\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}(\nabla.\mathbf{u}) - (\boldsymbol{\omega}.\nabla)\mathbf{u} - \mathbf{u}(\nabla.\boldsymbol{\omega})\right\}\Big|_i \end{aligned}$$

note mass conservation $\Rightarrow \nabla.\mathbf{u} = 0$
(incompressible as always),
and $\nabla.\boldsymbol{\omega} = 0$ identically (because div curl = 0)

Hence, by taking the curl of the Euler equation in the form $D\mathbf{u}/Dt = -\rho^{-1}(\nabla p - \rho\mathbf{F})$, and remembering that $(\mathbf{u}.\nabla)\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u} + \nabla(\frac{1}{2}|\mathbf{u}|^2)$ (already used in deriving Bernoulli), we see that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}.\nabla)\boldsymbol{\omega} \quad = \quad (\boldsymbol{\omega}.\nabla)\mathbf{u} + \nabla \times \mathbf{F}$$

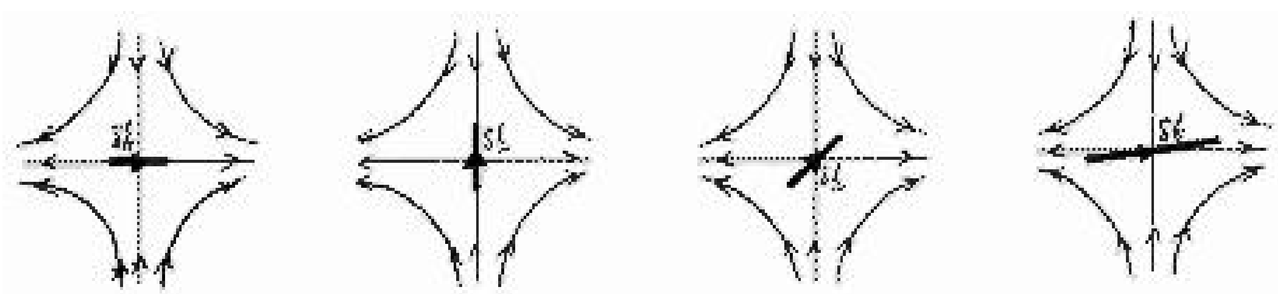
Assume $\mathbf{F}$ conservative; $\quad \Rightarrow \quad \mathbf{F} = \nabla\Phi$ for some function Φ, $\quad \Rightarrow \quad \nabla \times \mathbf{F} = 0$. Then

$$\frac{D\boldsymbol{\omega}}{Dt} \quad = \quad (\boldsymbol{\omega}.\nabla)\mathbf{u}$$

(vorticity equation for frictionless, incompressible flow, conservative $\mathbf{F}$), using $D/Dt = \partial/\partial t + (\mathbf{u}.\nabla)$.

Watching a given fluid particle/parcel we see its $\boldsymbol{\omega}$ value changing at the rate $(\boldsymbol{\omega}.\nabla)\mathbf{u}$. Déjà vu??? Recall equation for a material line element: $\dfrac{D\delta\mathbf{l}}{Dt} = (\delta\mathbf{l}.\nabla)\mathbf{u}$. It's the same equation! (with $\delta\mathbf{l}$ substituted for $\boldsymbol{\omega}$). (Again, strictly speaking, we are regarding $\delta\mathbf{l}$ as a field, of infinitesimal line elements; but all that matters is that D/Dt is the time derivative following a single material element.) So: *Vortex line elements move as if they were material line elements; vortex lines move as if they were material lines.* Or, more succinctly, *'vortex lines move with the fluid'.*

Because the velocity gradient tensor $\nabla\mathbf{u}$ does not usually vanish, material line elements are usually stretched and/or rotated as they are carried along in the flow. This follows from (a) the smallness of $\delta\mathbf{l}$ and (b) the Taylor expansion (mid. p. 18) showing how the velocity field varies locally. To illustrate this, take the same simple example $\mathbf{x}_0 = 0$, $\mathbf{u}(\mathbf{x}_0, t) = 0$, $\mathbf{u} = (\alpha x, -\alpha y,\ 0)$, $\alpha > 0$:



This line element will stretch: $(\delta\mathbf{l}.\nabla)\mathbf{u} = (\alpha, 0, 0)\,|\delta\mathbf{l}|$, so $\delta\mathbf{l} \propto (e^{\alpha t}, 0, 0)$.

This line element will shrink: $(\delta\mathbf{l}.\nabla)\mathbf{u} = (0, -\alpha,\ 0)\,|\delta\mathbf{l}|$, so $\delta\mathbf{l} \propto (0, e^{-\alpha t}, 0)$.

This line element $\delta\mathbf{l} = (\delta l_1,\ \delta l_2,\ 0)$ will rotate clock-wise while stretching. Its behaviour is given by $\delta\mathbf{l} \propto (\delta l_1 e^{\alpha t},\ \delta l_2 e^{-\alpha t},\ 0)$.

Pure-strain flows such as this have no vorticity ($\frac{1}{2}\epsilon_{ijk}\omega_k$ term zero in the Taylor expansion (mid. p. 18)); but *line elements in more general flows still stretch or shrink and/or rotate.*

Then, because the vorticity and line-element equations are the same, as already remarked, vortex lines also change (in this frictionless, incompressible model with conservative $\mathbf{F}$) just as if they, too, were being stretched and rotated. As a reminder of this picture we often say, for brevity, that 'vorticity changes by the stretching and rotation (or twisting, or tilting) of vortex lines'.

An important special case in which the pure-strain and vortical terms in the Taylor expansion are both nonzero is that of pure stretching of vortex lines in axisymmetric flow. Taking the z axis along the symmetry axis, consider a pure-strain velocity field of the form $\mathbf{u}_{\text{strain}} = (-\beta x, -\beta y,\ 2\beta z)$ ($\beta = \text{const.}, > 0$) to which is added another velocity field whose vorticity is nonzero but parallel to the z axis, and given by a (smooth) function $\mathbf{u}_{\text{vort}} = \Omega(r,t)(-y,\ x,\ 0)$, where $r = (x^2 + y^2)^{1/2}$, such that $\Omega(0,t) \neq 0$.

[*Exercise:* show that $\mathbf{u} = \mathbf{u}_{\text{strain}} + \mathbf{u}_{\text{vort}}$ has vorticity $\boldsymbol{\omega} = \left(0,\ 0,\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r^2\Omega)\right), \to \big(0,\ 0,\ 2\Omega(0,t)\big)$ as $r \to 0$.]

In this situation (with $\mathbf{x}_0 = 0$, $\mathbf{u}(\mathbf{x}_0, t) = 0$ as before), material line elements parallel to the z axis undergo pure stretching. Imagine a small material cylinder of length δl and circular cross-sectional area δA, surrounding a material line element on the z axis. As the line element and cylinder are stretched, the cylinder's cross-section remains circular, by axisymmetry. Both the mass and the angular momentum of the cylinder are invariant, again by axisymmetry (pressure force on cylinder surface points toward axis, so has zero moment):

Angular velocity Ω of cylinder $= \Omega(0,t)$;

vorticity $\omega = 2\Omega(0,t)$;

Mass conservation $\Rightarrow$ $\delta l\,\delta A$ invariant;

Angular momentum conservation $\Rightarrow$ $\omega\,\delta A$ invariant

(because angular momentum $\propto$ mass $\times$ angular velocity $\times$ (radius of gyration)^2, and (radius of gyration)^2 $\propto$ cross-sectional area, by axisymmetry).

Therefore

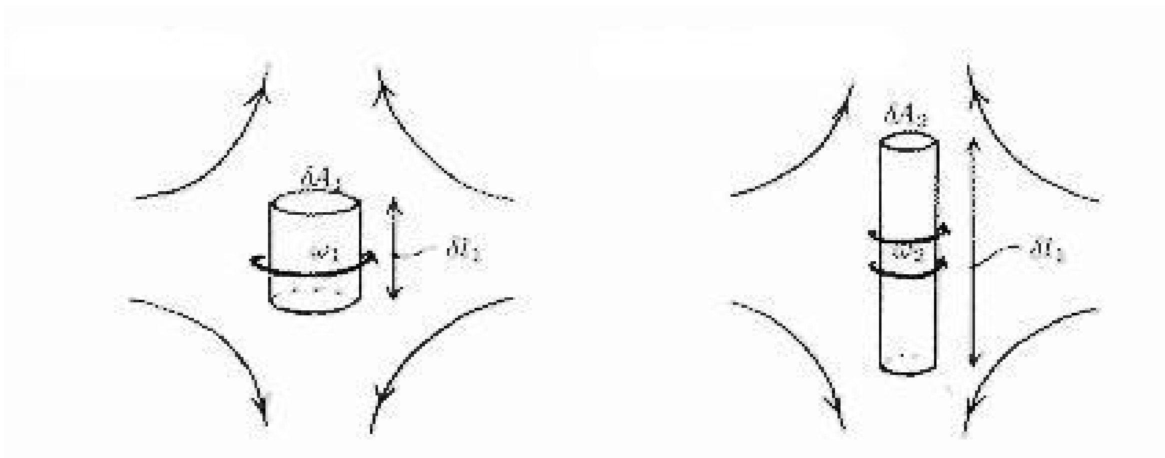
$$\omega \propto \delta l\ ; \qquad \text{i.e., in the following diagrams,} \qquad \frac{\omega_2}{\omega_1} = \frac{\delta l_2}{\delta l_1}\ :$$

At time $t = t_1$:

δl_1

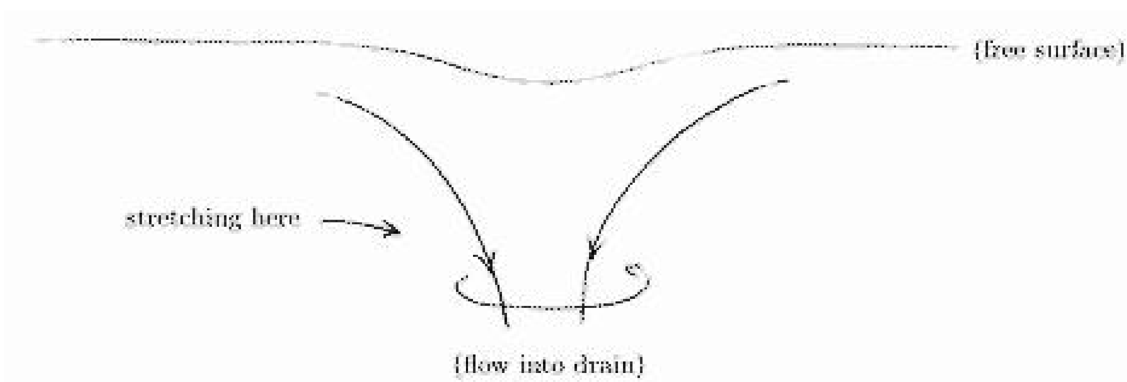
At time $t = t_2 > t_1$:

δl_2



We say that 'stretching amplfies vorticity'. It is also called the 'ballerina effect'. While spinning on your toes, you pull your arms in and spin faster. (Figure skaters do it most spectacularly.)

This is essentially how the familiar 'bathtub vortex' works:

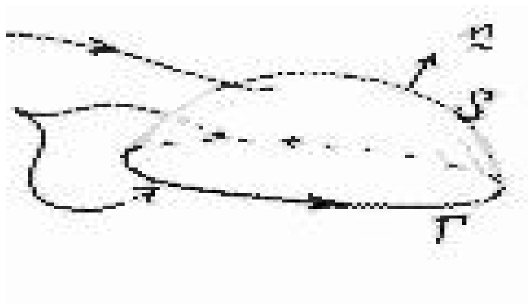


*Atmospheric cyclones and tornadoes likewise depend, in part, on the amplification of vorticity by vertical stretching. The contribution to the vorticity from the Earth's rotation is usually significant in getting such processes started (*unlike* the bathtub case as described in newspapers and dinnertable conversations!).*

§2.7 **Kelvin's circulation theorem** and the **persistence of irrotationality**

Circulation is the integral counterpart of vorticity; Kelvin's circulation theorem (sometimes just called 'the circulation theorem') is the integral counterpart of the vorticity equation. Define the *circulation*, $\mathcal{C}$, around a closed curve Γ by

$$\mathcal{C} = \oint_\Gamma \mathbf{u}.\,d\mathbf{l} = \int_S \boldsymbol{\omega}.\,\mathbf{n}\,dS \qquad \text{where} \quad S \quad \text{spans} \;\; \Gamma$$



(Stokes' theorem). Let Γ be a *material* curve (this is crucial):

$$\frac{d\mathcal{C}}{dt} = \oint_\Gamma \left\{ \frac{D\mathbf{u}}{Dt}.d\mathbf{l} + \mathbf{u}.\frac{D}{Dt}(d\mathbf{l}) \right\}$$

(Here D/Dt means, as before, the material rate of change of the infinitesimal line element $d\mathbf{l}$, previously denoted by $\delta\mathbf{l}$)

$$= \oint_\Gamma \left\{ -\frac{\nabla p}{\rho} \cdot d\mathbf{l} + \mathbf{F}.d\mathbf{l} + \mathbf{u}.(d\mathbf{l}.\nabla)\mathbf{u} \right\} \qquad\qquad = d\mathbf{l}.\nabla(\tfrac{1}{2}|\mathbf{u}|^2)$$

$$= \oint_\Gamma d\mathbf{l}.\left\{ -\frac{\nabla p}{\rho} - \nabla\Phi + \nabla(\tfrac{1}{2}|\mathbf{u}|^2) \right\}$$ (again assume $\mathbf{F} = -\nabla\Phi$, conservative)

$$= \oint_\Gamma d\left(-\frac{p}{\rho} - \Phi + \tfrac{1}{2}|\mathbf{u}|^2 \right) = 0 \text{(because closed curve)}.$$

This is Kelvin's circulation theorem. In words: *for* **inviscid** *(frictionless) fluid of* **uniform density** *with* **conservative forces**, *the circulation around a closed* **material** *curve remains constant.*

(Note consistency with invariance of $\omega\,\delta A$ in the simple vortex-stretching example just considered.)

Alternative derivation, following Acheson, but correcting a slight obscurity, tacitness about the term involving $\frac{1}{2}|\mathbf{u}|^2$: Parametrize Γ as the moving set of points $\Gamma = \{\mathbf{x} \mid \mathbf{x} = \mathbf{X}(a,t),\ 0 \le a < 1\}$, s.t. $a =$ const. $\Rightarrow \mathbf{x} = \mathbf{X}(a,t)$ follows a material particle; thus, in particular, $\partial\mathbf{X}(a,t)/\partial t = \mathbf{u}(\mathbf{X}(a,t),t)$. Then $\mathcal{C} = \int_0^1 \mathbf{u}(\mathbf{X}(a,t),t).(\partial\mathbf{X}(a,t)/\partial a)\,da$. Because da is invariant as the fluid moves, we have $d\mathcal{C}/dt = \int_0^1 \frac{\partial}{\partial t}\{\mathbf{u}(\mathbf{X}(a,t),t).(\partial\mathbf{X}(a,t)/\partial a)\}\,da$. The vanishing of this last integral follows from the standard rules for differentiating products and functions of functions (you should check this!); in particular, $\mathbf{u}.\partial^2\mathbf{X}(a,t)/\partial a\partial t = \mathbf{u}.\partial\mathbf{u}(\mathbf{X}(a,t),t)/\partial a = \partial(\frac{1}{2}|\mathbf{u}(\mathbf{X}(a,t),t)|^2)/\partial a$.

(Notice incidentally that a is a Lagrangian label, albeit a discontinuous one.)

Definition: *irrotational flow*, or *irrotational fluid motion* $\Leftrightarrow$ $\boldsymbol{\omega} = 0$ everywhere.

Corollary of the circulation theorem: *irrotational flow remains irrotational.*

Proof for smooth $\boldsymbol{\omega}(\mathbf{x},t)$*:* Initially irrotational $\Rightarrow$ circulation around all (arbitrary) material circuits initially zero $\Rightarrow$ circulation around all material circuits remains zero $\Rightarrow$ flow remains irrotational.

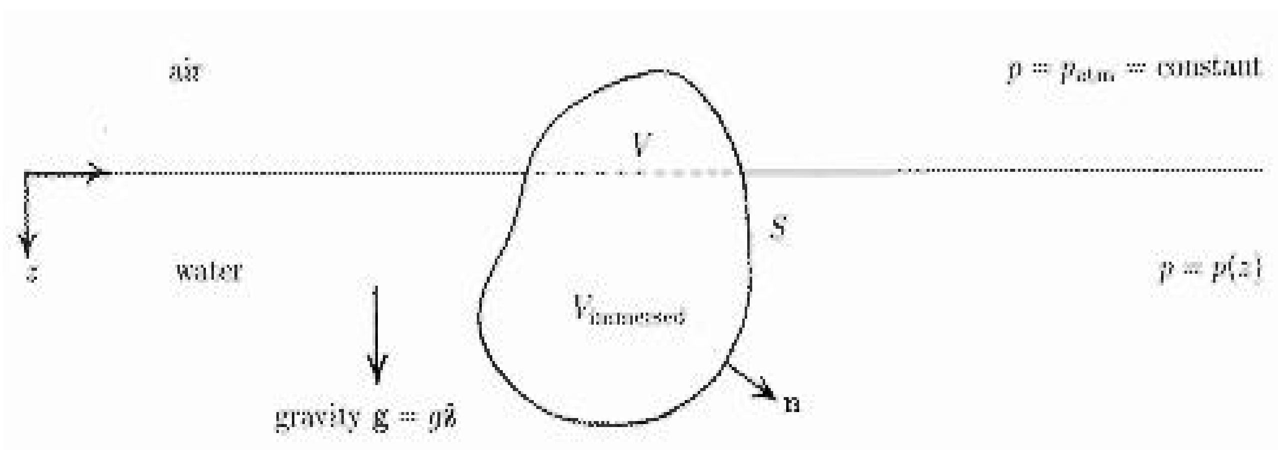
Alternative proof, by reductio ad absurdum, again for smooth $\boldsymbol{\omega}(\mathbf{x},t)$*:* Suppose $\boldsymbol{\omega} \neq 0$ at some point $\mathbf{x}_0$. Then one can find a small material curve Γ, encircling the vortex line through $\mathbf{x}_0$, whose circulation $\mathcal{C} = \oint_\Gamma \mathbf{u}.\,d\mathbf{l} = \int_S \boldsymbol{\omega}.\,\mathbf{n}\,dS \neq 0$. *(It is enough for $\boldsymbol{\omega}$ to be a continuous function of $\mathbf{x}$.)* The non-vanishing of $\mathcal{C}$ is a contradiction, because for the Γ in question $\mathcal{C}$ was zero initially (regardless of where the fluid particles making up Γ were located initially, because initially $\boldsymbol{\omega} = 0$ *everywhere*). So the original supposition is wrong.

This focuses attention on how a real fluid can acquire vorticity — viscous (frictional) forces can allow vorticity to escape from boundaries into interior; & see p. 24. **NB:** Fluid initially at rest is an important special case.

Digression for light relief: ARCHIMEDES' PRINCIPLE

Archimedes' principle states that there is an upward force on a body immersed in fluid equal to the weight of the fluid displaced by the body. It follows that a floating body, for which the upward force balances the weight, displaces an amount of fluid whose weight is equal the weight of the body itself. We can demonstrate this simply as follows; it is a corollary of the momentum integral of §2.

Consider a body of volume V floating in fluid of constant density ρ, with a volume V_{immersed} below the level of the free surface, as shown in the figure below.



The force $\mathbf{F}_{\text{body}}$ on the body is given by the momentum integral:

$$\mathbf{F}_{\text{body}} = -\int_S p\mathbf{n}dS = -\int_S (p - p_{\text{atm}})\mathbf{n}dS$$

where p is the pressure, S is the surface bounding V, and $\mathbf{n}$ is the outward normal to V; p_{atm} is the atmospheric pressure. The second equality follows because p_{atm} is constant and the integral of the normal vector $\mathbf{n}$ over the surface of a closed body is zero, by the generalized divergence theorem (replace $\mathbf{n}$ in surface integral by ∇ in volume integral), as used earlier.

The pressure distribution in the fluid may be deduced by considering the Euler momentum equation. The fluid is at rest, so this reduces simply to

$$-\frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = 0$$

where g is the gravitational acceleration and $\hat{\mathbf{z}}$ is the unit vector in the (downward) vertical direction. This is often referred to as *hydrostatic balance*. It follows, setting $p = p_{\text{atm}}$ on $z = 0$, that (for $z > 0$) the *hydrostatic pressure* distribution is given by

$$p = p_{\text{atm}} + \rho g z.$$

Substituting into the pressure integral, and then noting that there is no contribution from $z < 0$, it follows that

$$\mathbf{F}_{\text{body}} = -\int_{S_{\text{immersed}}} \rho g z \mathbf{n} dS$$

where S_{immersed} is the boundary of V_{immersed}. Then, again using the (generalized) divergence theorem, we have that

$$\mathbf{F}_{\text{body}} = -\int_{V_{\text{immersed}}} \rho g \hat{\mathbf{z}} dV = -\rho g V_{\text{immersed}} \hat{\mathbf{z}} \ .$$

The expression on the right-hand side is simply minus the weight of fluid that would have occupied V_{immersed} in the absence of the body.

The above holds for a (resting) body totally immersed in fluid. Then, of course, $V = V_{\text{immersed}}$.